The Sum of the Parts Can Violate the Whole

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We develop a geometric approach to identify all possible profiles that support specified votes for separate initiatives or for a bundled bill. This disaggregation allows us to compute the likelihood of different scenarios describing how voters split over the alternatives and to offer new interpretations for pairwise voting. The source of the problems—an unanticipated loss of available information—also explains a variety of other phenomena, such as Simpson's paradox (a statistical paradox in which the behavior of the "parts" disagrees with that of the "whole") and Arrow's theorem from social choice.

At least since the late 1700s, when Condorcet introduced his voting cycles, it has been understood that even if voters vote sincerely over pairs of outcomes, the final conclusion may be supported by very few or even none of them. This troubling phenomenon, often described in terms of shifting majorities (the voters who define the majority shift with the issue), suggests that a basic tool of democracy can subvert the intent of the voters. As such, it is understandable why bothersome versions of this behavior continue to arise in political science. As it will be shown here, the Anscombe (1976) paradox (also see Nurmi 1999), which shows a majority of the voters can be frustrated on a majority of the issues, is a version of the Condorcet cycle. Ostrogorski ([1910] 1970; Nurmi 1999) uses this behavior to question the meaning of the "dominant party"; the choice can change by emphasizing the party liked by most voters for most of its stands on issues, or the party that wins elections over a majority of the issues. In his influential work on voting cycles, Kramer (1977) suggests that this troubling behavior can be avoided with supermajority q rules. (The "stable" choice of the q rule, where q votes are needed to win, depends on the number of issues; see McKelvey 1979, Nakamura 1978, Saari 1997, Schofield 1983, and Slutsky 1979.) In the special setting of "yes-no" elections, Brams, Kilgour, and Zwicker (1998) provide examples from initiatives in California elections.

This article originated with and focuses on conjectures developed by Sieberg while doing the statistical analysis for the Brams, Kilgour, and Zwicker article. For instance, it is reasonable to believe that the oddity whereby the totality of the parts can violate the intent of the whole extends to more general political science behavior. Moreover, the statistical analysis from the California initiatives suggests that shifting majority peculiarities are not isolated anomalies but occur with a reasonable likelihood. (Indeed, out of 28 propositions on the November 1990 California ballot, the Brams, Kilgour, and Zwicker article reports that not a single voter voted for the winning combination.) In verifying these conjectures, we discover that a common interpretation of what pairwise voting means is incorrect when there are several pairs. Alternative interpretations are offered.

To indicate why this odd behavior should occur in settings other than voting, we treat pairwise voting as a decentralized decision process that attempts to capture the voters' wishes by determining their societal choices for the pairs. If this decentralization causes the pairwise voting problems (it does), then we should anticipate similar oddities whenever other aggregation tools—from probability, statistics, or even strategies for game theory—decentralize and concentrate on disconnected parts.

We illustrate with a statistical example and recall that, during the impeachment process of President Clinton, various groups wished to gauge the level of public support for the three specific proposals advanced. The results of exit polls conducted by CNN and the major networks taken during the November 1998 U.S. elections uniformly reported that American voters by overwhelming percentages opposed (1) impeaching Clinton, (2) censuring him, and (3) imposing a fine. These survey outcomes suggest that most voters wished to excuse the president, but that is misleading, because an overwhelming majority (on that election day) also felt that he should be punished. It is arguable, of course, that this example illustrates the dangers of using poor public opinion methods, that is, these conflicting conclusions may reflect an inappropriate choice of questions. But such an argument further underscores our point that the part-whole conflict extends beyond voting to pose concerns for the proper use of statistical tools. "More appropriate" questions arise by identifying and incorporating issues that connect the parts.

BUNDLED VOTING

At the opposite extreme from voting on individual issues is the gathering of issues into one bill. The advantages of this approach are understood. For instance, it is accepted that logrolling and bundled voting can avoid the shifting majority obstacle by providing

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structure to legislation. But, as can most tools, bundled voting can be used in many ways; for example, party leaders may use it to pass undesired bills or pork by bundling them with more popular alternatives. One way to measure the use of this tool is to determine whether the bundled parts form an intended, coherent, connected whole. An analysis of bundled voting requires understanding how legislators can split over component parts of the legislation. To do so, we develop an approach to determine what passage of the bill means relative to the legislators' views about the individual parts. A worrisome outcome occurs, for instance, if most lawmakers disagree with much, or a particular part, of the bill that they just passed. More encouraging conclusions that support a positive intent of logrolling emerge when the disaggregation indicates a balance among the legislators' goals.

A different kind of example is bundled voting that may frustrate the intent of the individual parts. Such a situation, also during the Clinton impeachment proceedings, occurred when White House Counsel Charles Ruff (McLoughlin 1999, 284) claimed in his opening defense statement to the Senate that the impeachment articles were "constitutionally deficient." His argument derived from principles of criminal justice, whereby "lumping multiple offenses together in one charging document [is prohibited because it] creates a risk that a verdict may be based not on a unanimous finding . . . but instead may be composed of multiple individual judgments." Because the House bundled multiple alleged offenses into one article, Ruff argued, it was possible that even if two-thirds of the senators would not find Clinton guilty of any individual charge, the necessary two-thirds vote for conviction could emerge if some found him guilty on some charges and others found him guilty on other charges in the bundled indictment article. In this case the worry is not whether the sum of the individual parts violates the whole, but whether support of the whole violates the intent of having a two-thirds vote on each of the individual parts.

Ruff's concern is closely related to the principles of criminal justice. The troubling possibility is that a person may be convicted by unanimous vote even though the intended unanimity (of the jury members) on any specific issue is not assured. Are these highly unlikely situations? With the high thresholds imposed by "unanimity" and a "two-thirds" vote, does not the passage of a bill ensure there is "essentially" the required support? Or is it likely that only a few voters accept any part of the total bill? Our approach for finding all disaggregations of a vote provides a tool to analyze a variety of issues, ranging from "pork" and even to objectives such as logrolling, which can have generally positive outcomes.

SEPARATION LOSES INFORMATION

Beyond explaining these various phenomena, we emphasize why they occur. A main conclusion is that these paradoxical behaviors arise because the separation of inputs into disconnected parts can cause a concomitant loss of available and crucial information. If so, then we also know how to resolve these difficulties: Identify the nature of the lost information and then discover how to reincorporate it into the process. This approach is compatible in spirit with the earlier comments about logrolling and the design of appropriate survey questions.

By identifying and then measuring the lost information, we show that these oddities are not rare, essentially concocted settings but are surprisingly likely. Indeed, our approach makes it apparent that increasing the number of separated parts significantly increases the likelihood of paradoxical conclusions. Similarly, with bundled voting we show, for instance, that Ruff's concerns have merit because it is uncomfortably likely for a bundled bill to pass with a two-thirds or even a unanimous vote, even though only a few voters approve of any individual part. The conflict between disconnected parts and the whole must be taken seriously.

For expositional reasons we emphasize voting, but it must be stressed that with many (if not all) aggregation procedures, odd behavior can occur whenever "parts" are disconnected. Unexpected behavior arises if the decentralization loses important, available information or when aggregations (e.g., bundled voting) connect inappropriate parts. Thus, expect similar examples to arise whether the tools involve game theory, statistics, general decision theory, and so forth. To underscore this point, the following examples explain certain oddities by identifying the information lost by the parts.

Simpson's Paradox

Suppose that in Atlanta and Boston the recovery outcome of groups using an experimental drug is compared with the recovery of control groups. In Atlanta, the experimental group enjoyed a two-thirds recovery rate compared to only one-half for the control group, and in Boston the respective figures were one-third and one-quarter. Although the data from both locales support the experimental drug, the combined data could indicate that the control approach is more successful. In other words, the information from the parts need not predict the behavior for the whole. (For political science implications of Simpson's [1951] paradox see Nurmi [1998, 1999, 2000].)

This reversal occurs when the whole goes beyond the information about the parts to use also the number of subjects in each of the four groups. To illustrate, let d_A^e, d_A^c, d_B^e, and d_B^c denote the number of subjects in the Atlanta and Boston experimental and control groups. Then, $(2/3)d_A^e$ and $(1/3)d_B^e$ are the number of recovered Atlanta and Boston subjects with the experimental treatment. Examples are generated by finding d_j^i values, so that

$$\frac{(2/3)d_A^e + (1/3)d_B^e}{d_A^e + d_B^e} < \frac{(1/2)d_A^c + (1/4)d_B^c}{d_A^c + d_B^c}. \quad \textbf{(1)}$$

To create examples, set the left-hand side equal to any fraction, say, 9/24, that is between the two recovery fractions of 1/3 and 2/3. Solving

$$\frac{(2/3)d_A^e + (1/3)d_B^e}{d_A^e + d_B^e} = \frac{9}{24}$$

leads to $7d_A^e = d_B^e$, so let $d_A^e = 30$, $d_B^e = 210$. An example is created by setting the right-hand side of equation 1 equal to a value larger than 9/24 and between the two control group recovery values. For instance, choosing 11/24 leads to $d_A^c = 5d_B^c$, so the values $d_B^c = 8$, $d_A^c = 40$ suffice.

So, the 20 recoveries from the 30 subjects in the Atlanta experimental group has a better record than the 20 recoveries from the 40 subjects in the control group. As is often the case, in Boston, 210 subjects volunteered for the experimental study, while only 8 joined the control group. Here, 70 of the experimental group recovered compared to only 2 from the control group. Using the above analysis, examples as varied as desired can be constructed.

Game Theory; Lotteries

Separate lotteries illustrate how conflicting outcomes can arise in game theory, or even individual decision making. To show that information is lost by considering only disjoint parts, a new strategy admitted by the whole is identified.

Suppose Paul is a professional gambler who has \$70 to wager on a basketball game between the Lakers and the Knicks. Bob is so confident of the Lakers that he offers Paul 2 to 1 odds. (If the Lakers lose, Bob pays \$2 for each dollar bet with him.) Later Paul meets Sue; she is sufficiently confident of the Knicks to offer 3 to 1 odds. When considered individually, Paul's strategy and decision about whether and how much to bet depends upon information gathered about the two teams. Whatever choice is made, a bet with Bob and/or with Sue involves an element of randomness and risk.

By considering the two parts as a connected whole, Paul's newly admitted strategy is to bet against both teams in a manner to remove all risk. For instance, betting \$40 on the Knicks (with Bob) and \$30 on the Lakers (with Sue) eliminates all risk (and demonstrates the radically different nature of the connected game), as Paul is guaranteed a \$50 profit no matter which team wins. For success, Paul must be sure that Bob and Sue are unaware of the odds the other is offering.

Arrow's Impossibility Theorem

Arrow's theorem ([1952] 1963), one of the more troubling results in decision theory, provides a third type of difference in structure between the parts and the whole. Because Arrow's independence of irrelevant alternative (IIA) assumption forces the decision process for the societal rankings of the parts—each pair—to be totally disconnected from one another, such a procedure can service a voter with cyclic preferences as well as one with transitive preferences. It turns out that with a sufficiently heterogeneous society (Saari 1994, 1995, 1998), IIA loses the crucial information that the voters have transitive preferences. As argued in these three references, the Borda count is a way to include the information and resolve the difficulty. As pairwise voting satisfies IIA, versions of the Borda count resolve the problems we examine below.

GEOMETRIC INTERPRETATION

Our geometric approach is designed to explain why decentralizing societal outcomes into disconnected parts loses information and causes paradox. An advantage of the geometry is that it also displays the robust nature of this phenomenon.

It is natural to use a graph to represent the relationships between the inputs and outputs, but there is a serious obstacle; social science problems typically involve so many variables that the many dimensions required by a standard graph would make it impossible to visualize and use. Alternative representations are needed. Economists resolved this difficulty with the Edgeworth box (see any introductory textbook on microeconomics). On the same rectangle, the preferences and initial endowments of two agents for two commodities are simultaneously graphed. This representation allows the group outcomes and properties of, say, the Pareto sets and price equilibria to be geometrically determined and described. The power of the Edgeworth box comes from its clever, simple representation of the independent and dependent variables in a single, lower dimensional setting; it is a form of a graph. Moreover, the Edgeworth box makes it possible to predict what happens with more agents and/or commodities. Our goal is to create a related geometric tool to address the decision concerns of political science.

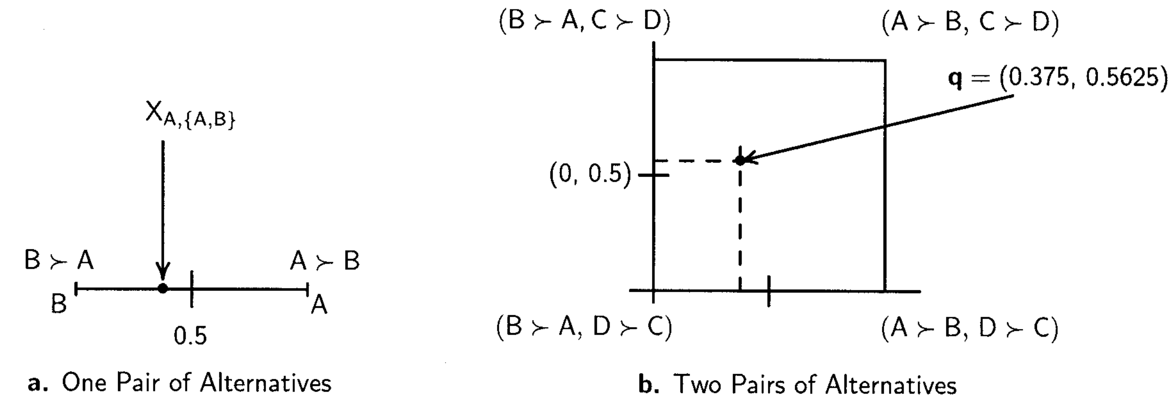
A particular difficulty in analyzing voting issues is the inability to represent profiles (i.e., a listing of the voters' rankings of the issues) and outcomes in a single sketch. To overcome this difficulty, we extend the geometric approach developed by Saari (1994, 1995) to analyze three and N-alternative elections.

Representing Elections of Pairs

To represent geometrically the outcome for a single pair of alternatives, $\{A, B\}$, we let $X_{A,\{A,B\}}$ be the fraction of the total vote received by A. So, $X_{B,\{A,B\}} = 1 - X_{A,\{A,B\}}$, and the values $X_{A,\{A,B\}} = 1, \frac{1}{2}, 0$ represent, respectively, these outcomes: A wins unanimously, A and B are tied, and B wins unanimously (because A received no votes).

When the election point is plotted on a unit interval, its location represents both the division of voter preferences and the majority vote tally. The support for a particular candidate is graphically represented by the distance from the other candidate's unanimity point. For instance, a point 3/8 of the way from B to A, as represented by the • in Figure 1a, indicates with 100% certainty that 37.5% of the voters preferred A to B and that A received 37.5% of the vote. Conversely, the

FIGURE 1. Representation Lines and Squares



point is ⅝ of the distance from A to B, so B received 62.5% the vote.

The outcomes for the two pairs $\{A, B\}$ and $\{C, D\}$ are simultaneously represented by the • located at the point $(X_{A,\{A,B\}}, X_{C,\{C,D\}})$ in the Figure 1b square. The vertical and horizontal dashed lines indicate each pair's level of support. For instance, the horizontal dashed line is 56.25% of the distance above the axis—where D has a unanimous vote—so C beats D with 56.25% of the vote. Outcomes for $N \geq 2$ pairs are similarly represented with a point in an N-dimensional cube. Although actual geometric representations are impossible to draw for $N \geq 4$, as is true with the Edgeworth box, basic properties of election outcome can be found.

The issue is to determine what this Figure 1b outcome—B beats A (with 62.5% of the vote) and C beats D (with 56.25% of the vote)—means about the voters' preferences. With certainty, 62.5% of the voters prefer $B > A$, and 56.25% prefer $C > D$. But what can we say about the individual voter rankings over both pairs? Does this result mean, for instance, that most voters prefer both $B > A$ and $C > D$? Or is there a shifting majority effect whereby a sizeable majority of the voters is unhappy with at least parts of this total outcome?

To illustrate with numbers, we let the Figure 1b outcomes represent an election held with 80 voters. With certainty, 50 of them prefer $B > A$, and 45 prefer $C > D$, but this could mean that (1) more than half, 45, of these voters prefer both the ($B > A$ and $C > D$) outcomes, 5 show partial support by preferring ($B > A$ but $D > C$), and the remaining 30 are opposed to both outcomes, or (2) only 15 voters approve of both outcomes, whereas the sizeable number of 65 voters disagree with parts of the combined outcome, because 35 prefer ($B > A$ but $D > C$) and the remaining 30 prefer ($C > D$ but $A > B$). Both choices generate the same combined election outcome, but they admit different interpretations. For instance, the first scenario indicates support for the joint outcome, but the second creates doubt about its appropriateness because more than 80% of the voters disapprove of parts of the societal outcome.

Why does it matter which profile represents the voters? Imagine 80 legislators trying to cope with a rise in the number of school children. Suppose the first vote is between A = (raise teachers' salaries) and B = (leave the salaries at the current level). The second ballot imposes a choice between C = (increase the size of classes) and D = (keep the same class size).

Suppose a distinct minority of the legislators, 15, believe that the teachers can cope with larger classes on their current salary, so they choose B on the first vote and C on the second. The remaining legislators wish to do something, but none of them wants the dual expense of higher salaries and the same class size. Suppose 30 of the legislators favor compensating teachers with higher salaries for overcrowded classrooms, so they vote for A and C. The remaining 35 believe the current class size is already too large. Rather than raise salaries, they prefer to find extra classrooms for the additional students. They choose B on the first vote and D on the second.

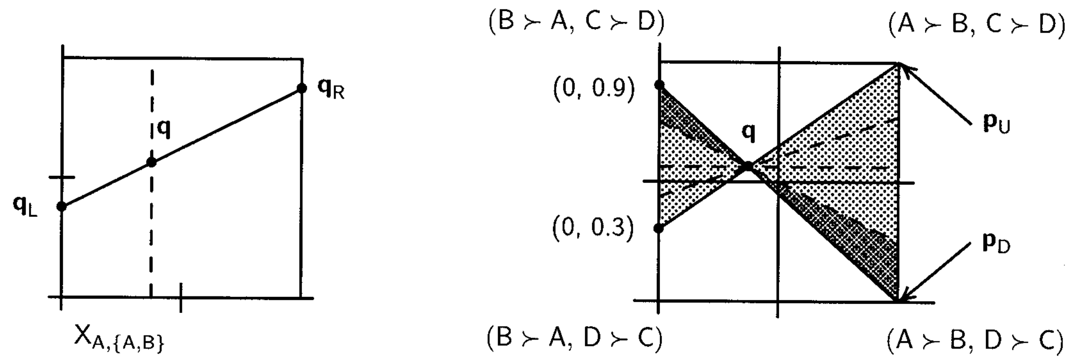
Although more than 80% (65/80) of these legislators would like to resolve the problem by helping the teachers, their goal is frustrated as B and C win with, respectively, votes of 50/80 and 45/80; the teachers are forced to cope with larger classes without added compensation. Without a disaggregation of the vote, this outcome may appear to be the wish of most legislators. In other words, there is no way to distinguish between these two scenarios, which have a different intent and offer distinctly different messages.

Profile Line

We now develop a geometric representation of voter preferences and election outcomes on the same diagram, Figure 2. Two pairs of alternatives define four voter types according to how they rank each pair. Let $\nu(A, C)$ be the fraction of all voters who prefer $\{A > B$ and $C > D\}$; similar definitions hold for $\nu(A, D)$, $\nu(B, C)$, $\nu(B, D)$. Thus, a profile $\mathbf{p}$ becomes the four-vector

$$\mathbf{p} = (\nu(A, C), \nu(A, D), \nu(B, C), \nu(B, D)).$$

FIGURE 2. Profile Representations



a. Profile line

b. All supporting profiles

The above two profiles are, respectively, $\mathbf{p}_1 = (0, 30/80, 45/80, 5/80)$ and $\mathbf{p}_2 = (30/80, 0, 15/80, 35/80)$.

To identify $\mathbf{p}$ with the pairwise outcomes $\mathbf{q} = (q_1, q_2)$, note that since $X_{A,\{A,B\}}$ is the fraction of all voters who prefer $A > B$, we have that $X_{A,\{A,B\}} = \nu(A, C) + \nu(A, D)$. Thus, the coordinates of $\mathbf{q}$ in Figure 1b are

$$\mathbf{q} = (X_{A,\{A,B\}}, X_{C,\{C,D\}}) = (\nu(A, C) + \nu(A, D), \nu(A, C) + \nu(B, C)). \quad \textbf{(2)}$$

To motivate what follows, we treat the $\{A, B\}$ rankings as dividing the voters into two parties. We further divide the "leftists," defined by their $B > A$ belief, according to their preferences over C and D; this division is given by the number of leftists with preferences associated with the two vertices on the left edge of the Figure 1b square. Similarly, the "rightists," with their $A > B$ preference, are subdivided by their C and D preferences as represented by the two vertices on the square's right edge.

The division of each party according to its $\{C, D\}$ preferences can be shown by using a Figure 1a type of representation. To do so, define and then plot the points $\mathbf{q}_L$ and $\mathbf{q}_R$ on the respective vertical edges. For instance, by letting $\mathbf{q}_R$ represent the fraction of the rightists who prefer $C > D$, the division is given by $\nu(A, C)$ (the fraction of voters who prefer $A > B$—a ranking needed to be a rightist—and $C > D$) divided by $X_{A,\{A,B\}} = [\nu(A, D) + \nu(A, C)]$ (the fraction of all voters who are rightist). Thus, $\mathbf{q}_R$ is at height $\nu(A, C)/[\nu(A, D) + \nu(A, C)]$ on the representation square's right edge. Each point's coordinates, as plotted on the appropriate edges of the Figure 2a square, are

$$\mathbf{q}_L = \left(0, \frac{\nu(B, C)}{\nu(B, C) + \nu(B, D)}\right),$$

$$\mathbf{q}_R = \left(1, \frac{\nu(A, C)}{\nu(A, C) + \nu(A, D)}\right), \quad \textbf{(3)}$$

or, equivalently,

$$\mathbf{q}_L = \left(0, \frac{\nu(B, C)}{1 - X_{A,\{A,B\}}}\right), \qquad \mathbf{q}_R = \left(1, \frac{\nu(A, C)}{X_{A,\{A,B\}}}\right). \quad \textbf{(4)}$$

Intuitively, the combined $\mathbf{q}$ outcome should involve the relative strength of the rightist and leftist parties. This happens because, according to equation 4,

$$\mathbf{q} = (1 - X_{A,\{A,B\}})\mathbf{q}_L + X_{A,\{A,B\}}\mathbf{q}_R. \quad \textbf{(5)}$$

But equation 5 is the equation of a straight line, so the combined election outcome $\mathbf{q}$ is found by plotting the $\mathbf{q}_L$ and $\mathbf{q}_R$ outcomes for the left-hand and right-hand groups. Next, connect these points with a straight line. The actual election outcome $\mathbf{q}$ is the intersection of this line with the vertical line defined by $x = X_{A,\{A,B\}}$ (the dashed line in Figure 2a). In this manner, a profile is uniquely represented by a line and the point $\mathbf{q}$ on the line.

A specified profile $\mathbf{p}$ uniquely determines $\mathbf{q}_L = (0, y_L)$, $\mathbf{q}_R = (1, y_R)$, $\mathbf{q}$, and the line. Conversely, if a line is specified, then the defining profile can be recovered by use of algebra. As the specified $\mathbf{q}_L$, $\mathbf{q}_R$, and $\mathbf{q}$ values of a line uniquely determine a profile $\mathbf{p}$, we have the following definition.

Definition 1. *A profile line with distinguished point* $\mathbf{q}$ *in a representation square is given by a straight line connecting points on the vertical edges and a distinguished point,* $\mathbf{q}$*, on the line. The points on the vertical edges represent how these two sets of voters rank the alternatives along the vertical edges. The distinguished point* $\mathbf{q}$ *is determined by the fraction of all voters who prefer one or the other of the two alternatives represented by the horizontal axis;* $\mathbf{q}$ *represents the election outcomes.*

To summarize, a profile with $\mathbf{q}$ as an outcome defines a profile line with distinguished point $\mathbf{q}$. Conversely, *each line that connects opposite edges and passes through* $\mathbf{q}$ *defines a profile line (i.e., it uniquely defines a profile) with outcome* $\mathbf{q}$. By using this representation, we now can identify all possible profiles that support a specified election outcome $\mathbf{q}$. This set of all possible disaggre-

TABLE 1. Extreme Profiles

Profile	Approve All	Approve Only $B > A$	Approve Only $C > D$	Approve None
$\mathbf{p}_D$	56.25%	6.25%	0	37.5%
$\mathbf{p}_U$	18.75%	43.75%	37.5%	0

gated outcomes is the cone of all possible lines that connect the vertical edges and pass through $\mathbf{q}$.

To analyze the cone of profile lines represented by Figure 2b, we start with the two extreme profiles; they are the two lines that include a vertex on the right edge. As two points ($\mathbf{q}$ and the appropriate vertex) define a line, the equations for the two extreme profile lines are

$$y = \frac{1-q_2}{1-q_1}x + \frac{q_2-q_1}{1-q_1} \text{ and } y = \frac{q_2}{1-q_1}(-x+1). \quad \textbf{(6)}$$

A word of caution is necessary. The two extreme profile lines in Figure 2b involve vertices on the right edge only because this particular $\mathbf{q}$ satisfies

$$q_1 \leq 1/2 \left(\text{so } \mathbf{q} \text{ is to the left of } x = \frac{1}{2}\right),$$

$$\text{and } q_1 + q_2 \leq 1,\ q_2 \geq q_1 \quad \textbf{(7)}$$

(the last conditions require the two vertices on the right edge to define the extreme profiles). If $\mathbf{q}$ were closer to the top edge, then an extreme profile line might involve a top left vertex rather than the bottom right vertex. Unless rightist and leftist define actual coalitions, however, these terms are just artifacts that assist the analysis. By changing which pair is on the vertical or horizontal axis, and which edge has lines ending in a vertex, all choices of $\mathbf{q}$ can be converted into a form that satisfies the equation 7 conditions with a Figure 2b type representation. (In what follows, we purposely vary the $\mathbf{q}$ location to illustrate how to handle other settings.)

The two extreme profiles supporting $\mathbf{q}$ in Figure 2b are $\mathbf{p}_D$ and $\mathbf{p}_U$, which are, respectively, the extreme downward and upward sloping profile lines that end in a vertex. (For the earlier 80-voter illustration, $\mathbf{p}_D$ and $\mathbf{p}_U$ are, respectively, geometric representations of the $\mathbf{p}_1$ and $\mathbf{p}_2$.) How do we recover the profile representation from $\mathbf{p}_D$? As ⅝ of the voters are leftist and as the y_L component on the left edge is 0.9, it follows that $0.9 \times 5/8 = 0.5625$, or 56.25% of all voters (all leftists) agree with both outcomes. In this algebraic manner, the extreme profiles are characterized relative to the joint outcome as shown in Table 1. Notice how Table 1 shows that the profiles supporting $\mathbf{q}$ vary in their support for both outcomes (the $\nu(B, C)$ term) from only 18.75% of all voters (with $\mathbf{p}_U$) to 56.25% (with $\mathbf{p}_D$). The levels of disagreement with at least one outcome ranges from 43.25% to a surprisingly large 81.25% of the voters. Complete disapproval (the $\nu(A, D)$ term) ranges from 0% to 37.5%.

The profiles that indicate a lack of support for the final combined conclusion underscore a cost of considering separate, disjoint parts: the potential loss of information about what the voters want for the total package. Instead of knowing the actual $\mathbf{p} = (\nu(A, C), \nu(A, D), \nu(B, C), \nu(B, D))$, we merely know the outcome $\mathbf{q}$. The problem, as in the school example, is that $\mathbf{q}$ can be the outcome for a surprisingly large variety of profiles with contrary interpretations. Thus, by ignoring the relevant information about how the voters connect the pairs, we introduce the danger that a majority of the voters do not embrace the combined outcomes.

With two pairs, however, there always must be voters who approve of both outcomes. For a proof, notice that if $\mathbf{q}$ is in the upper left quarter, then no supporting profile line can meet the (0, 0) vertex. This geometry requires $y_L > 0$ for all profile lines, so it means that any supporting profile must include voters who approve of both election conclusions.

Likelihood Estimates

Analyzing a single profile can, at best, illustrate a particular unusual behavior. The $\mathbf{p}_U$ profile, for instance, proves that a substantial percentage of the voters can disapprove of parts of the $\mathbf{q}$ outcome even though each victory is by a substantial vote. A more ambitious and crucial question is to understand whether examples such as $\mathbf{p}_U$ are isolated oddities or identify a troubling behavior that must be addressed. We now tackle this more general concern.

With a single $\{A, B\}$ election (which could be a "yes-no" election, or a choice between two alternatives), it is certain that the winning alternative is supported by at least 50% of the voters. To understand what replaces this certainty assertion when there are two pairs, we determine the likelihood that at least 50% of all voters agree with both $\mathbf{q}$ outcomes of Figure 2a and b.

The first step is to find all profiles that satisfy the 50% approval level. Since ⅝ of the voters are leftist (with the winning $B > A$ outcome), at least half of all voters will approve both outcomes only if $y_L \times 5/8 \geq 1/2$, or if $y_L \geq 0.8$ of the leftists prefer $C > D$. To find all profiles that satisfy this condition, we draw the profile line defined by the $y_L = 0.8$ value on the left edge and $\mathbf{q}$. As all profile lines with $y_L \geq 0.8$ have the desired 50% approval behavior, these profiles are represented by the profile lines in the heavier shaded region of Figure 2b. Notice that by computing where the defining line with left endpoint $y = 0.8$ hits the right edge, it follows that the cost of achieving this level of agreement is that at least 31.25% of the voters directly oppose both outcomes.

It should be noted that not all endpoints define profile lines. In the 80-voter example, as 30 of the voters are rightist (the smaller of 30 and 50), divide the regions on the right and left edges into 30 equally distributed regions, or 31 equally distributed points, starting at the lower point of each region and ending at the upper one. Each point defines a profile. If there are 8,000 voters, then each edge is divided into 3,001 points. As the number of voters increases, the better the continuum represents the possible voter views.

The likelihood estimates of voter satisfaction, or dissatisfaction, with the joint outcome now can be extracted from the figure. These values, however, depend upon assumptions made about the voters. We illustrate the approach with three scenarios; many others are possible.

Strict Partisan Vote. If the voters adopt a strict party line, then the party's vote is unanimous for each pair. Here, the profile line must start from a vertex on the appropriate edge. This means for our example that leftists do *not* enforce a strict party vote (as no profile line with the **q** outcome ends at a vertex on the left edge), and with certainty the profile is described by either $\mathbf{p}_U$ or $\mathbf{p}_D$. Thus, with certainty, either more than half of all voters prefer the joint outcome (when $\mathbf{p}_D$ is the profile), or a sizeable percentage of the voters disapprove of parts of the **q** outcome (with profile $\mathbf{p}_U$).

Uniform Distribution. Since the Figure 2b geometry captures all profiles supporting **q**, it is reasonable to assume that the likelihood of this 50% approval property is the ratio of the number of profiles with the desired property divided by number of profiles with the **q** outcome. This suggests use of the ratio of the heavily shaded area (the profiles satisfying this 50% property) relative to the fully shaded area (the set of profiles supporting **q**). With this assumption, the Figure 2b geometry (the darkly shaded region is much smaller than the shaded region) shows that a 50% approval of both outcomes is fairly unlikely. That is, with this assumption, likelihood estimates of various behaviors follow almost directly from the geometry. (This assertion holds even when the continuum is replaced by an equally distributed number of points for leftists and rightists.)

A simple way to compute this likelihood (see the Appendix) is to use one of the edges. The likelihood equals the ratio of the length of the heavily shaded region on the edge to the length of the fully shaded region on the same edge. Thus, the likelihood of a specified behavior reduces to computing the length of the segment that defines the behavior divided by the length of the segment of all possible outcomes.

To illustrate this computation with the Figure 2b example, we note that the segment of endpoints on the left edge satisfying the 50% agreement property is $0.8 \leq y \leq 0.9$, so it has length $0.9 - 0.8 = 0.1$. The segment of left endpoints where the profile has this **q** outcome is $0.3 \leq y \leq 0.9$; it has length $0.9 - 0.3 = 0.6$. The ratio of these lengths, only $1/6$, defines the proportion of profiles that support this 50% approval property of both outcomes. Consequently, under the assumption that each profile line is equally likely, we reach the surprising conclusion that, although each Figure 2b pairwise election is won with a strong majority vote, the likelihood that more than half the voters are dissatisfied with at least part of the outcome is the surprisingly large $5/6$, or 83%.

As further illustration of the geometric approach, we determine the likelihood that at least one-quarter of the voters dislike both outcomes. These voters are represented by the $\nu(A, D)$ value, or the vertex in the lower right-hand edge, so our interest is in profiles whose right edge point is below height y_R, where $(1 - y_R) \times 3/8 \geq 1/4$, or $y_R \leq 1/3$. (The $1 - y_R$ value determines the distance from the C unanimity vote to the D unanimity vote.) It follows immediately from Figure 2b that (assuming all profile lines are equally likely), with probability of $1/3$, at least one-quarter of all voters disagree with both outcomes.

The geometry, then, allows us to determine quickly the likelihood of various behaviors. For instance, a joint outcome in which each pair barely wins a majority vote has **q** near the completely tied point of $(1/2, 1/2)$. The associated profile cone comes close to being described by lines connecting diametrically opposite vertices. In turn, those profile lines for which at least 50% of the voters approve of both outcomes come close to requiring an almost unanimous support from the leftists. This means that the line segment of a heavily shaded region must be very small, and the segment with a shaded region is nearly the full edge. The geometry, then, immediately proves that such support is highly unlikely. By using this computational approach, we obtain the following more general assertion.

THEOREM 1. *For two pairs, suppose the winning alternative wins with the majority m_j of the vote, $m_j > 1/2$, $j = 1, 2$, where $m_1 \geq m_2$. Assume all profiles, as represented by their endpoints on an edge, are equally likely. The likelihood that at least the fraction α of all voters prefer both outcomes is the smaller of unity or*

$$Prob(\alpha) = \max\left(\frac{m_2 - \alpha}{1 - m_1}, 0\right). \quad \textbf{(8)}$$

Similarly, the likelihood that at least β of all voters dislike both outcomes is

$$Prob(\beta) = \max\left(\frac{1 - m_1 - \beta}{1 - m_1}, 0\right). \quad \textbf{(9)}$$

To illustrate this result with numbers, we recall that with one pair it is certain that the winning alternative enjoys at least 50% of the vote. To find the parallel conclusion for two pairs—that is, conditions in which we can say with certainty that at least 50% of the voters prefer both outcomes—then (according to equation 8, with $\alpha = 1/2$ and Prob(α) set equal to unity) it must be that $m_1 + m_2 \geq 3/2$. Since $m_1 \geq m_2$, this conclusion requires the first election to be won with at least a 75% vote, and the second victory must have nearly as strong a vote; "certainty" for two pairs requires surprisingly strong votes. Using more commonly observed election outcomes, such as $m_1 = 0.52$ and $m_2 = 0.51$, equation 8 establishes that it is highly unlikely for even half the voters ($\alpha = 0.5$) to approve of both outcomes. (The likelihood is only $0.01/0.48 \approx 0.02$). But there is about a 54% chance that at least one-quarter ($\alpha = 1/4$) of all voters approve both outcomes. These are disturbing results.

By setting $\text{Prob}(\alpha) = 1$ in equation 8, we obtain the interesting relationship that, with certainty, the small proportion of $\alpha = m_1 + m_2 - 1$ of all voters approve of both outcomes. Using the Figure 1b example, this means that, with certainty, $0.625 + 0.5625 - 1 = 0.1875$ of the voters prefer both outcomes. But for $m_1 = 0.52$, $m_2 = 0.51$, all we can say with certainty is that at least 3% of the voters prefer both outcomes. Similar results using equation 9 determine the fraction of all voters who do not like either result. Using $m_1 = 0.52$, $m_2 = 0.51$, equation 9 shows the likelihood that at least β of all voters disapprove of both outcomes is $[0.48 - \beta]/0.48$. For instance, there is about a 50% chance that 24% of the voters disapprove of both outcomes.

Normal Type Distributions. The results change dramatically and become much more troubling if the distribution of profile lines uses the binomial or normal distributions. For instance, with the 80-voter example, if each rightist is equally likely to prefer C or D, then the likelihood that none of these voters prefer C is $(1/2)^{30}$, but the likelihood that 10 of them prefer C is $(30/10)(1/2)^{30}$, or more than 30 million times more likely.

Because the binomial and normal distributions concentrate most of the probability around the expected value, these distributions make it highly unlikely that a sizeable number of voters approve of the combined outcome. For instance, using the binomial distribution in which the 400 leftists have probability 0.6 of voting for C over D (the 0.6 value is the midpoint of the region of the left edge), standard computations with the central limit theorem prove that the 50% approval outcome occurs with probability essentially zero. (It is several standard deviations above the mean of 0.6.) Such distribution assumptions place a high emphasis on profile lines near the mean.

Interpreting Pairwise Elections over Several Pairs

When two pairs are voted upon separately, it may be that only a few voters approve of both outcomes; these are the profiles that create the paradoxes described in the literature. We adopt a radically different interpretation. We argue that these are "paradoxes" only if we incorrectly believe that the pairwise votes should accurately reflect the views of the voters over *all outcomes*. This is not the case; these difficulties reflect the information lost about how the voters connect the two issues. Rather than paradoxes, the real problem is our mistaken interpretation of what pairwise votes mean. But if standard interpretations are faulty, new ones are needed.

Any alternative definition must use the fact that the pairwise vote strips from the profile all information about the voters' preferences over all pairs. Except in a unanimity setting, the pairwise vote cannot distinguish a specified profile from any other profile in the profile cone. Consequently, more accurate interpretations must emphasize general aspects about the set of profiles in the cone. We illustrate this argument with two different "statistical" interpretations that describe the pairwise outcomes as capturing reasonable features of "most" supporting profiles.

We use Figure 2b to motivate the first interpretation and note that $\mathbf{q}$ is a reasonable outcome for the extreme profile line $\mathbf{p}_D$. After all, more than half the $\mathbf{p}_D$ voters are leftists, and in this profile about 90% of the leftists prefer the elected $C > D$. Aspects of this example are emphasized with the following definition, which is specifically designed to address the Anscombe (1976) and Ostrogorski ([1910] 1970) concerns.

DEFINITION 2. *Profile* $\mathbf{p}$ *justifies the pairwise outcomes* $\mathbf{q} = (q_1, q_2)$ *in a "party dominant" way if*

- *the outcome for the pair defining the parties reflects the wishes of the larger party and*
- *the outcome for the pair describing a division within the two parties represents the majority wish of the majority party.*

The geometry of the representation square and profile lines shows how to find all profiles that agree with the $\mathbf{q}$ outcome and all profiles—such as $\mathbf{p}_U$ in Figure 2b—that cast doubt about the appropriateness of $\mathbf{q}$'s combined outcomes. In a very real sense, then, the $\mathbf{q}$ outcome does not reflect the actual profile but, rather, the likelihood $\mathbf{q}$ is an "appropriate outcome" for some portion of the supporting profiles. To use definition 2 to capture this sense and to interpret the pairwise vote, we note that, in addition to $\mathbf{p}_D$, all profiles with left endpoint $y_L > 1/2$ justify $\mathbf{q}$ in a party-dominant manner. The next theorem asserts that this party-dominant property is satisfied by most profiles that support any $\mathbf{q}$. Thus, one "statistical" interpretation for the pairwise vote for two pairs is that it is the "party-dominant" outcome of most profiles with the $\mathbf{q}$ outcome. Notice how this result provides a positive statistical response for the Anscombe and Ostrogorski paradoxes.

THEOREM 2. *Let* $\mathbf{q} = (q_1, q_2)$ *be pairwise outcomes over two pairs when the outcome for each pair is by a strict majority. Most (more than half) profile lines supporting* $\mathbf{q}$ *justify* $\mathbf{q}$ *in a party-dominant manner.*

Theorem 2 can be extended to involve any number of pairs ("most" is replaced with "the largest proportion"), but extensions are easier to make with the following alternative interpretation of pairwise voting. Again, notice from Figure 2b that all profile lines with a left endpoint satisfying $y_L > 1/2$ have $\nu(B, C)$ as the largest value in the profile; that is, $\nu(B, C) > \nu(A, C)$, $\nu(B, D)$, $\nu(A, D)$. For these profiles, the $B > A$ and $C > D$ outcomes make sense, as they reflect the profile's dominant $\nu(B, C)$ component. Not all supporting profiles allow this dominant component interpretation; for example, the largest $\mathbf{p}_U$ entry is $\nu(B, D)$. Indeed, it is precisely because $\nu(B, D)$ is the largest $\mathbf{p}_U$ component that it can be argued—with help from the tacit but incorrect assumption that pairwise votes reflect properties of the given profile—that B and D should be the $\mathbf{p}_U$ "winners."

To convert this "dominant component" argument into a "statistical" interpretation of pairwise voting, which shifts the emphasis from a specific profile to properties of the set of profiles, we note from the profile cone of Figure 2b that most profile lines supporting $\mathbf{q}$ have $\nu(B, C)$ as the largest component in the profile. Since the pairwise vote cannot distinguish which profile from the cone is the actual one, another statistical interpretation of $\mathbf{q}$ is that it identifies which entry of the supporting profiles is the largest in most of them.

THEOREM 3. *For two pairs, the combined pairwise outcome agrees with the largest component of most profiles when the probability distribution is the uniform distribution or the binomial distribution.*

Again, the flawed sense that the widely used pairwise outcomes reflect the properties of the actual profile can be replaced with a more accurate interpretation that the outcome represents a particular property of most supporting profiles.

Bundled Issues: Impeachment, Legislation, and Criminal Justice

As asserted, bundled voting lies at the other extreme from voting separately on issues. In bundled voting, several issues are combined into one vote. The many examples include the impeachment concerns of Ruff, what can happen in juries when charges are bundled, and even the standard legislative technique to ensure a bill will pass—bundling items of particular interest to different legislators.

Bundled voting, as in some examples of logrolling, may be an explicit attempt to overcome shifting majority difficulties by reintroducing information and connections that can be lost by voting on individual issues. In the following, these settings are captured by profile lines that indicate balance in the approval of parts of the final conclusion. Probability estimates for such situations require (as with the distribution used in the earlier special case of a straight party-line vote) using appropriate probability distributions that emphasize profile lines with balanced outcomes.

As also asserted, bundled voting can be misused by combining inappropriate parts. For instance, it is not an appropriate tool if the intent is to preserve the integrity of separate items. Ruff's statement about the impeachment proceedings specifically addresses these concerns: "a verdict may be based not on a unanimous finding of guilt as to any particular charge." This problem had arisen, Ruff noted, in the trial of Judge Nixon, when Senator Herbert Kohl (D-WI) had protested: "Please do not bunch up your allegations.... Charge each act of wrongdoing in a separate count ... and allow for a cleaner vote on guilt or innocence." In the Clinton case, Ruff continued, "the managers ... have searched every nook and cranny of the grand jury transcript and sent forward to you a shopping list of alleged misstatements, obviously in the hope that among them you will find one with which you disagree. But ... the record simply will not support a finding that the president perjured himself before the grand jury" (McLoughlin 1999, 284).

To analyze bundled voting, we need to discover how to disaggregate the outcome. As to the specific mechanism, a voter must choose either to approve a package of A and C or to disapprove of it. (As this could be a "yes-no" vote, it includes voting for a bill versus the status quo. Also, we changed the alternatives to be approved to A and C to demonstrate a different geometric setting.) So that the analysis does not become overly technical, we assume that a voter will opt for the package if s/he likes at least one of its options; that is, a voter can vote for a bill that offers a combination (A and C) without approving of both options. (A more realistic but technically more complicated assumption is that the voter compares the utility of voting for the bill versus that of the status quo.) This means that if a majority of voters support either option of the bundled bill, it will pass.

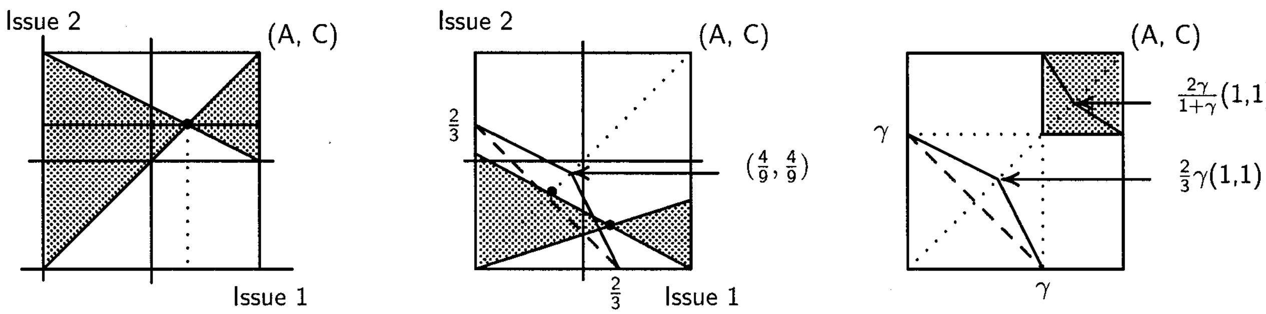
We now show why a bill can pass even if a majority of the voters do not approve of one part, say, C. This is because, similar to voting on separate issues, an inappropriately designed bundled vote can lose crucial information. In forcing a choice between A and C, or B and D, we lose the $(\nu(A, C), \nu(A, D), \nu(B, C), \nu(B, D))$ information about how the voters rank various combinations. As demonstrated next, unless a bundled bill is carefully designed, bundled voting can lose even more of this information.

Only minor modifications of the geometric approach developed for pairwise voting are needed to analyze what can happen when the whole consists of bundled issues. For purposes of comparison, Figure 3a represents a nonbundled setting in which each issue is voted upon separately, and passage of each issue requires a two-thirds vote. The profile cone (the shaded region of profile lines) immediately proves that, for most supporting profiles, less than two-thirds of the voters approve of both outcomes. In this example, both issues pass by the bare two-thirds vote, but only one profile (the extreme upward sloping line indicating a straight party vote) shows that two-thirds of all voters support both outcomes. Thus, other than a strict partisan vote, such an outcome is highly unlikely.

Bundled Bills with a Two-Thirds Vote. Now consider the bundled effect for a bill, or an impeachment verdict, or a trial verdict if the voter accepts at least one of the two bundled issues. As shown later, as long as $\mathbf{q} = (q_1, q_2)$ satisfies

$$q_1 + q_2 \geq \frac{2}{3}, \tag{10}$$

FIGURE 3. Voting with Two Bundled Issues



a. All supporting profiles **b.** Impeachment via bundling **c.** Likelihood estimates

there are profiles that support **q** and that allow the bundled bill to pass with the two-thirds vote. To appreciate the flexibility offered by this condition, consider the downward sloping dashed line in Figure 3b, which is the graph of $q_1 + q_2 = 2/3$. Thus, all **q** above this dashed line admit at least one profile that satisfies the two-thirds requirement.

A bundled bill with a two-thirds vote can significantly diminish the level of support needed for passage. The bullet on the Figure 3b dotted line represents **q** = (0.4, 0.4), which, by being above the dashed line, has supporting profiles that allow the passage of the bundled bill. Yet, because $q_1 = q_2 = 0.4 < 1/2$, the bundled bill would satisfy the two-thirds vote barrier, even though neither issue would receive even a majority vote. Indeed, unless **q** is in the upper right-hand quarter, at least one issue would fail to receive a majority vote, even though the bundled bill could pass with a two-thirds vote. Because the geometry proves that it is quite likely for **q** not to be in this upper right-hand region, it is likely for a bundled bill to pass with a two-thirds vote, even though at least one, or both, individual issues would not receive even a majority vote.

Most Profiles. Although equation 10 establishes conditions for a profile to pass the bundled bill with a two-thirds vote, not all profiles with the **q** outcome can pass the bill. Our earlier computational methods, for instance, show that only a small portion of the upwardly sloping profile lines in the cone supporting **q** = (0.4, 0.4) allow a two-thirds vote (as most profile lines would have $\nu(B, D) > 1/3$).

This phenomenon further supports our statistical interpretation of the pairwise vote; rather than represent a specified profile, the pairwise vote represents a specified property of **q** that holds for "most" of the supporting profiles. Where do we find the **q**'s in which at least half the profiles in the profile cone pass the bundled bill? Our computational approach and algebra (see the Appendix) reveal that these **q** are above the Figure 3b solid line going from (0, 2/3) (where all voters are leftists, and two-thirds of them approve of issue 2) to (4/9, 4/9) to (2/3, 0) (where no one supports issue 2, but two-thirds of all voters are rightists who support issue 1).

The farther **q** is above the solid line, the larger is the percentage of profiles that provide a two-thirds vote. To illustrate, **q** = (3/5, 2/15) is on the line. The $q_1 = 3/5$ value means that 60% of all voters are rightists, and, because they support issue 1, all of them vote for the bundled bill. To ensure passage, the extra 6.66%, or 1/15 of all voters, must be leftists. They disagree with issue 1, so only those leftists who support issue 2 vote for the bundled bill. Thus, a profile leading to passage must have at least y_L of the leftists who support issue 2, where $y_L \times (2/5) \geq 1/15$, or $y_L \geq 1/6$. By computing the **q** profile cone, we find that the left-hand endpoints of profile lines are in the interval [0, 1/3]. Thus, precisely half of these profile lines (those with endpoints satisfying $1/6 \leq y_L \leq 1/3$) permit the two-thirds passage of the bundled bill.

Compare this outcome with the point **q** = (3/5, 1/6) plotted in the right-hand lower corner of Figure 3b. As this **q** is above the solid line, we should expect that more than half the profile lines ensure passage of the bundled bill. Verification only involves determining the profile cone indicated in Figure 3b. The extreme profile lines that pass through (3/5, 1/6) connect (0, 0) with (1, 5/18) and (0, 5/12) with (1, 0). As above, since 60% of the voters are rightists (who vote for the bundled bill because of issue 1), only $y_L \geq 1/6$ of the leftists need to approve of issue 2 for passage.

From Figure 3b and computations, we have the following conclusions about **q** = (3/5, 1/6). First, 60% (given by [(5/12) − (1/6)]/5/12) of the profile lines defined by **q** = (3/5, 1/6) lead to passage of the bundled bill with a two-thirds vote. Second, neither issue considered separately would pass with a two-thirds vote. Indeed, issue 2 could not even pass with a majority vote. Third, for only one profile (the most extreme upwardly sloping line) will $3/5 \times 5/18 = 1/6$ of all voters agree with both issues. Thus, even though the bundled bill passes with a two-thirds vote, with a uniform distribution of profile lines it is with probability zero that 1/6 or more of all voters approve of both issues. Finally, the likelihood that between 1/10 and 1/6 of all voters prefer both issues is about 40%. With a binomial or normal probability distribution, it becomes much more unlikely that even small fractions of all voters accept both issues. Ruff's concerns are very realistic.

Indeed, because our argument assumes a neutral stance by treating bundled voting as a tool, it includes a range of possibilities: from voters, jurists, or legislators unwittingly voting for a combination of outcomes when a majority does not support at least one of the bundled issues all the way to strategic behavior. Legislators are obviously well aware of this opportunity to pass bills that may not gain a majority on their own by bundling in other issues. The 1991 congressional "corn for porn" deal is a prime example.

Appropriations Bill HR2686, the fiscal year 1992 appropriations bill, was frozen in conference as the conferees debated two key issues. The first pertained to an amendment to the bill that Senator Jesse Helms, R-NC, proposed to prevent the National Endowment for the Arts from funding projects that depicted "in a patently offensive way sexual or excretory activities or organs" (*Congressional Quarterly Almanac* 1991, 566). This amendment had been approved by the House 286-135 on October 16, 1991, and 287-133 on October 17. The amendment was also adopted by the Senate by a majority of 68 senators. Western House members, many of whom had supported the Helms amendment, were frustrated, however, by another amendment to the bill, to raise grazing fees, that had passed the House floor with a majority vote of 232 legislators on June 25, 1991. Initially, each side to the debate refused to concede. It became clear, however, that action on the appropriations bill would be impossible to complete if both amendments remained.

The intent of House and Senate conferees was to find a compromise in order to pass the bill with majority support in the House and Senate. The impasse was finally broken by Representative AuCoin, D-OR, who suggested a trade in which eastern House negotiators would drop the grazing fee increase in return for conservatives' abandonment of the amendment regarding NEA funding. "Seizing the compromise as the only viable way to complete action on the bill, House conferees voted 7-2 for it . . . Senate conferees quickly followed suit" (*Congressional Quarterly Almanac* 1991, 567). The deal was labeled "corn for porn" in a catchy reference to a substitute choice of feed for livestock and the explicit art supposedly encouraged by the endowment. The bill, stripped of the two amendments, later passed the House, and, despite initial objections, passed the Senate as well, keeping the "corn for porn" deal intact.

The deal demonstrates that by bundling votes—or in this case by "bundling out" votes to negate previously approved changes—issues that could not obtain a majority on their own may be passed. In particular, conservatives in favor of the Helms amendment could vote for the final bill by referring to the removal of the portion describing grazing fees. Indeed, this compromise explicitly passed in the Senate for this reason. "Western conservatives, usually Helms' allies, seemed keenly aware of what that vote [in favor of the Helms' amendment] might portend—a vengeful Rep. Yates. Yates was likely . . . to reopen the grazing fees issue if the corn for porn deal was broken [Senator] Byrd warned" (*Congressional Quarterly Almanac* 1991, 568). Representatives not from the western states, who would find it difficult to vote against an increase in grazing fees, could support the bill by emphasizing their opposition to the Helms amendment. Thus, although neither the Helms amendment nor the grazing fee increase could have been defeated individually, by linking the issues, both were successfully removed from the 1992 appropriations bill. It is possible to argue either that this deal was beneficial, or that it was detrimental. In fact, both arguments were made, "An outraged [Representative] Dannemeyer called the conference deal 'arrogance of the worse order.' . . . But the compromise drew applause from Western reachers" (*Congressional Quarterly Almanac* 1991, 566). In this case, the tradeoff and preferences were explicit; in other situations, voters may not be aware that a bundled vote is allowing a minority position to emerge victorious.

In a well-known article on logrolling, Ferejohn (1986) attributes the power of committees to their ability to bundle issues. In describing why food stamps became an agricultural program, he explains that this program is often cited as an example of logrolling between two groups—congressmen from urban and from agricultural districts—who favored two unconnected policies, but he maintains the logrolling hypothesis is inappropriate. Ferejohn (p. 223) cites collective choice results as evidence for his claim: "If a logroll is required to enact some set of bills, then there can be no package of bills that could win a majority against every other package." He further points out that logrolls are unstable because they are vulnerable to counteroffers from those who are excluded.

But if the food stamp program was not a logroll, what explains the deal? According to Ferejohn (1986, 225),

> the power of congressional committees in the legislative process provides opportunities for exchanges of support that span different stages of congressional action. . . . This exchange of support should not be understood as a symmetric logroll organized in the Congress as a whole. Rather the agriculture committees bundle the two programs, one popular in committee and the other on the floor, into a single legislative package which is popular enough to survive the whole process.

Ferejohn's analysis adds strength to our argument that, as a tool, bundled voting can be used in a variety of ways.

Bundled Votes and Other Thresholds. Two other interesting settings involve a majority vote (say, for legislation) and a unanimous vote (say, for a jury verdict). In the majority vote setting, at least one profile leads to passage for any **q** satisfying $q_1 + q_2 > ½$. In the case of unanimity votes, at least one profile leads to passage for any **q** above or on the line $q_1 + q_2 = 1$; this line connects the (0, 1) and (1, 0) vertices. To illustrate, we suppose that the unanimity vote over the bundled indictment is achieved with a (¾, ¼) vote (so, 75% of the voters agree with issue 1). The associated profile cone—the extreme profile lines connect (0, 0) with (1, ⅓) and (1, 0) with (0, 1)—shows that, even with the unanimous vote, at most ¼ of these voters agree with both bundled issues, and the likelihood that between ⅛ and ¼ of all voters agree with both issues is ½.

The general assertion for these and some other concerns are described in the following theorem. In nonmathematical terms, it states that whenever a uniform distribution of profile lines is a reasonable assumption, there is serious doubt as to whether bundled outcomes reflect accurately the views of the voters.

THEOREM 4. *Suppose a bundled bill of two issues needs at least* γ *of all votes to pass. Suppose a voter votes for the bill if s/he approves of at least one issue. For* $\mathbf{q} = (q_1, q_2)$ *to ensure the existence of at least one profile that leads to passage of the bill, it is necessary and sufficient that* $q_1 + q_2 \geq \gamma$.

Assume that each profile line is equally likely. Suppose α, $0 \leq \alpha \leq 1$, *is the proportion of all profile lines that support* $\mathbf{q}$ *and that pass the bundled bill with a* γ *vote. If* $\alpha \leq 2(1 - \gamma)$, *then it is necessary and sufficient that* $\mathbf{q} = (q_1, q_2)$ *satisfy*

$$\begin{cases} q_1 + (1-\alpha)q_2 \geq \gamma & \text{if } q_1 \geq q_2 \\ q_2 + (1-\alpha)q_1 \geq \gamma & \text{if } q_2 > q_1. \end{cases} \tag{11}$$

If $\alpha > 2(1 - \gamma)$, *then a necessary and sufficient condition is for* $\mathbf{q}$ *to be on, or above, the collection of line segments in the representation square that connects* $(\gamma, 0)$ *to* $((\gamma + \alpha - 1)/\alpha, (1 - \gamma)/\alpha)$ *to* $((\gamma + \alpha - 1)/\alpha, (\gamma + \alpha - 1)/\alpha)$ *to* $((1 - \gamma)/\alpha, (\gamma + \alpha - 1)/\alpha)$ *to* $(0, \gamma)$.

For $\mathbf{q}$ *to admit at least one profile (but perhaps only one) in which* γ *of the voters approve of both issues, then* $q_1, q_2 \geq \gamma$.

A necessary and sufficient condition for $\mathbf{q}$ *to have at least* $\beta > 0$ *of all profiles in which* γ *of the voters support both issues is*

$$\begin{cases} q_2 + \beta(q_1 - 1) \geq \gamma & \text{if } q_1 \geq q_2 \\ q_1 + \beta(q_2 - 1) \geq \gamma & \text{if } q_2 > q_1. \end{cases} \tag{12}$$

The special case $q_1 + q_2 \geq \gamma + 1$ *represents the situation in which all* $(\beta = 1)$ *supporting profiles for* $\mathbf{q}$ *have* γ *of all voters who support both issues.*

The different situations described in theorem 4 are captured in Figure 3c. In the region above the downward slanting dashed line are the $\mathbf{q}$ outcomes with at least one profile that supports passage of the bundled bill. The region above and on the solid line bent at the dotted diagonal represents equation 11, that is, at least 50% ($\alpha = ½$) of the profiles supporting $\mathbf{q}$ pass the bill. The shaded square consists of all $\mathbf{q}$ with at least one supporting profile in which at least γ of the voters approve both issues. Finally, in the region above the bent line in the shaded region, representing equation 12, $\beta = \gamma$ of all supporting profiles have at least γ of all voters who support both issues. The region in which all $(\beta = 1)$ supporting profiles have at least γ of all voters who support both issues is the straight line connecting the endpoints of this bent line.

As for the $\mathbf{q}$ conditions in which α of all supporting profiles pass the bundled bill, it turns out that when $\alpha \leq 2(1 - \gamma)$, the boundary lines of equation 11 meet on the $y = x$ diagonal below the $x + y = 1$ diagonal. For larger α values, the two equation 11 boundary equations meet the $x + y = 1$ diagonal. At this stage, passage of the bill at the indicated level requires a minimal number of voters of particular types. So, the two lines are continued either horizontally or vertically until they meet on the $y = x$ diagonal at the point where both values are $(\gamma + \alpha - 1)/\alpha$.

It is not necessary to provide estimates because the geometry already supplies the basic message: Whenever assumptions about $\mathbf{q}$ and profiles hold, such as a uniform distribution of profile lines, it is highly unlikely that a bundled bill will satisfy this γ property. Instead, the geometry proves that the likelihood an outcome allows even one profile with this γ property is small (it is the ratio of the area of the shaded square to the area of the region above the lower curved line), and this ratio approaches zero as γ approaches unity, the unanimous vote.

THREE AND MORE ALTERNATIVES

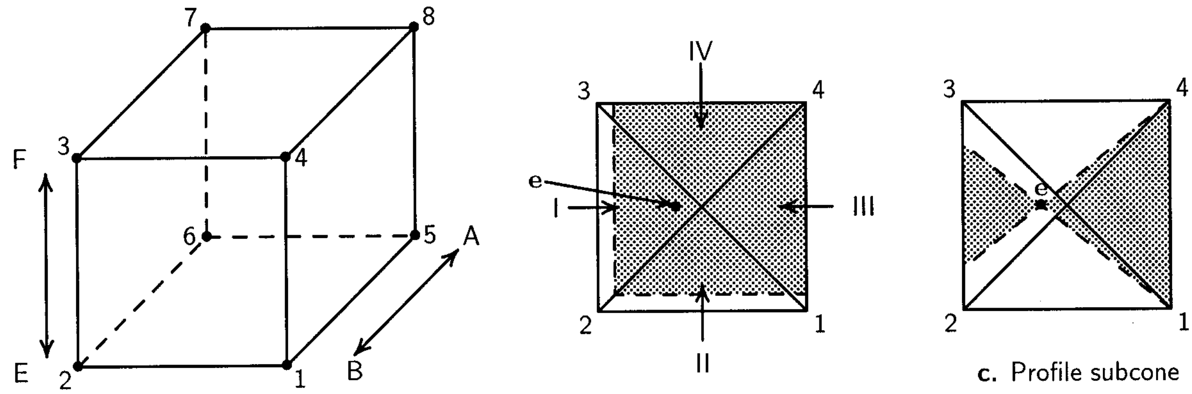
These conclusions extend far beyond the troubling properties of pairs. Only slight modifications are needed to prove that similar problems occur in any decentralized setting with disconnected parts. Related difficulties occur when the societal election outcomes are determined with triplets, or sets of four, or any number of alternatives. This is because with disconnected sets of alternatives a similar geometry illustrates the same loss of "connecting" information with similar disturbing behavior. To see this, divide a four-alternative problem into two subsets of three alternatives and rank each subset with the plurality vote or any other procedure. With minor modifications of the above, examples can be constructed in which the final outcome is supported by very few voters. Here, however, "statistical" arguments identify which ranking procedure (the Borda count) reduces the likelihood of conflict.

These comments suggest that a way to exacerbate the difficulties is to increase the number of parts. Adding pairs—or any other disjoint parts—significantly increases the already large likelihood of a conflict. To review: With certainty more than 50% of the voters approve the majority vote outcome of a single pair. For two pairs with majority votes m_1, m_2, the best we can say with certainty is that the fraction of voters who approve the combined outcome is $[m_1 + m_2] - 1$, a value that can be surprisingly small. As we show next, with more pairs the final outcome can fail to reflect the views of any voter. The arguments for $n \geq 3$ pairs mimic those for two pairs, so we indicate only differences in the geometry and certain outcomes.

Three Pairs

The three-pair setting uses the representation cube of Figure 4a. Vertex 4 at point (1, 1, 1), for instance, represents $F > E$, $D > C$, $B > A$ unanimous votes. Let these rankings be the societal outcome for $\mathbf{q} = (q_1,$

FIGURE 4. Representation Cube



a. Three pair cube **b.** The (0.52, 0.52, 0.52) outcome **c.** Profile subcone

q_2, q_3). The geometric representation of profiles iterates our earlier approach. Again, use the $\{A, B\}$ ranking to divide the voters into two parties. The "frontists" prefer $B > A$, and their preferences are represented by the four vertices on the cube's front face. The "backists" prefer $A > B$, and their preferences are represented by vertices 5, 6, 7, and 8.

The geometric representation of a profile involves three line segments. The "primary" profile line segment passes through the "outcome-distinguished point" $\mathbf{q} = (q_1, q_2, q_3)$, with endpoints on the cube's front and back faces; call these the profile's primary endpoints. (The figure does not show q because of the difficulty of showing its location in a three-dimensional cube.) The front primary endpoint describes the pairwise outcomes of the frontist vote on the two pairs $\{C, D\}$ and $\{E, F\}$. Therefore, in precisely the manner described above for two pairs, disaggregate this primary endpoint with a "secondary" profile line; this line segment lies in the front face, passes through the primary front endpoint, and has its secondary endpoints on the side edges. This secondary line can be thought of as subdividing the frontists into leftist and rightist subparties. A similar decomposition describes a secondary profile line in the back face, which passes through the back face's primary endpoint.

Using the same arguments and techniques as above, these three profile line segments (the primary and two secondaries) uniquely represent a profile. So, a profile is represented by (1) a primary profile line passing through $\mathbf{q}$, with its (primary) endpoints on the front and back faces; (2) a secondary profile line in the front face, which connects the side edges and passes through the front face primary endpoint; and (3) another secondary profile line in the back face, which connects the side edges and passes through the back face primary endpoint. Similarly, any collection of three line segments that satisfies this geometry uniquely defines a profile with $\mathbf{q}$ as its combined election outcome.

The set of all profiles that support $\mathbf{q}$ consists of all admissible combinations of these three line segments; as such, this set involves a complicated arrangement of three cones. First, the set of all admissible primary profile lines defines the primary profile cone. For example, Figure 4b depicts the primary endpoints on the front face defined by $\mathbf{q} = (q_1, q_2, q_3)$; the victorious alternative ($F > E, D > D, B > A$, a vertex 4 ranking outcome) in each pair receives 52% of the vote. To compute this region, find those extreme primary lines that pass through $\mathbf{q} = (q_1, q_2, q_3)$ and a vertex on the front or back face. As with the two-pair analysis, if $q_2, q_3, 1 - q_2, 1 - q_3 \leq q_1$, then the back primary endpoint can be anywhere on the back face, and the corresponding front primary endpoint is in the rectangle defined by the four "front point" vertices of Table 2. In this table, which identifies the corresponding front and back primary endpoints of the four extreme primary lines, the extreme primary profile line starting at vertex 6 (on the back) defines the front face primary endpoint, which is toward vertex 4; the one starting at vertex 7 is toward vertex 1, and so on (see Figure 4).

TABLE 2. Front and Back Vertices

Back Vertex	Front Pt.	Back Vertex	Front Pt.
6	$\left(1, \frac{q_2}{q_1}, \frac{q_3}{q_1}\right)$ toward 4	7	$\left(1, \frac{q_2}{q_1}, \frac{q_1+q_3-1}{q_1}\right)$ toward 1
5	$\left(1, \frac{q_1+q_2-1}{q_1}, \frac{q_3}{q_1}\right)$ toward 3	8	$\left(1, \frac{q_1+q_2-1}{q_1}, \frac{q_1+q_3-1}{q_1}\right)$ toward 2

Recall that the frontist party is subdivided into leftists and rightists. Thus, each primary endpoint on the front face, $\mathbf{e} = (1, e_1, e_2)$, defines a profile subcone (Figure 4c). This subcone is the set of all secondary profile lines that lie in the front face and pass through this particular primary endpoint; it captures all divisions of the four voter types within the frontist party whose vote over the two pairs is **e**. A similar subcone is defined in the back face for each back face primary endpoint. So, each primary profile line, with its two primary endpoints, defines a pair of subcones.

The various consequences that can be extracted from the subcone geometry depend on which Figure 4b triangular region, $I, \ldots, IV$, contains the primary endpoint **e** represented by the •. For instance, the **e** given by the • in region I of Figure 4b defines the Figure 4c profile subcone for the frontist party. The geometry of this cone admits the extreme Figure 4c subprofile line that ends in vertex 1; for this profile, no voter supports all three elected outcomes. All remaining subprofiles (i.e., secondary profile lines) in this subcone, however, must include voters who accept all three outcomes. The same conclusion holds for all primary endpoints $\mathbf{e} \in II$. The subcones for $\mathbf{e} \in III \cup IV$, in contrast, have their extreme profiles connecting vertex 3 with a point on the right edge, and vertex 4 with a point on the left edge. Thus, all profiles with such a primary **e** include voters who approve of the vertex 4 combined outcome.

This three-pair geometry indicates why adding a pair loses more information about voter preferences. The subcone defined by each primary endpoint **e** is a two-pair profile cone, so the three-pair setting loses at least as much information as with two pairs. (This is one-dimensional because the profile line is defined by a point on an edge and the distinguished outcome.) But with three pairs, each **e** defines a new subcone with the associated lost information; for example, each point in the shaded region of Figure 4b defines a new subcone. (By dimension counting, two dimensions of indistinguishable information are represented by the primary lines, and another dimension for each subcone, which leads to a four-dimensional set of profiles that defines the same outcome.) Namely, the surprising variety of allowed **e** positions on the front and back faces demonstrates the enormous dimensional increase in lost information.

As a brief aside, this iterative process of describing profiles proves that adding more pairs radically escalates the loss of information about voters. For instance, with four pairs, the rankings of one pair can be used to define two parties. The primary profile lines pass through $\mathbf{q} = (q_1, q_2, q_3, q_4)$, with its endpoints in two representation cubes—one for each party. Each primary endpoint defines this party's election outcomes over the remaining three pairs. Thus, each primary endpoint defines a three-pair subdivision of the above type. The fact that the four-pair setting uses a three-pair analysis for each of many possible choices of primary endpoints demonstrates the serious escalation in the amount and kind of lost information. (With n pairs, this disaggregation approach [see the Appendix] shows that there is a $2^n - (n + 1)$ dimensional set of profiles with the same election outcome. For four pairs, then, the eleven dimensions indicate a potentially serious erosion of information.)

Geometric Consequences

We now show that with three or more candidates the erosion of information about the voters allows settings in which no one accepts all parts of a combined outcome. For three pairs, we characterize all profiles with this behavior. As **q** has a type 4 (i.e., vertex 4) ranking, this phenomenon occurs if and only if the profile has no type 4 voters. All such profiles must use the extreme subprofile line in an **e** subcone (see Figure 4c) with a vertex 1 endpoint. This, in turn, forces the front face primary endpoint **e** to be in regions I and II or on their boundary (see Figure 4b).

We defer probability comments to a later section, but our earlier arguments make it clear that with a uniform probability distribution on profile lines these extreme profiles are rare. (At most, one subprofile line comes from each subcone in the front face, so these profiles are in a three-dimensional set defined by those primary profile lines with an appropriate front endpoint and the associated subcones in the back face.) A rough sense of the likelihood of this behavior comes from the number of **e** endpoints that allow this extreme subcone line. According to Figure 4b, one such profile is associated with each **e** in the shaded region below or on the diagonal connecting vertices 1 and 3. So, those **q** that define a larger region of primary endpoints below the diagonal are more likely to admit such a profile.

Because **e** is above the bottom edge connecting vertices 1 and 2, and because the extreme profile starts from vertex 1, the subprofile line in the front must end on the left edge above vertex 2. This means the profile always must include voters of type 3. Similarly, because the front primary endpoint **e** must be below (or on) the diagonal connecting vertices 1 and 3, the back endpoint must be above the diagonal connecting vertices 5 and 7. Based on our two-pair subcone arguments, all back face subprofile lines include type 8 voters. Therefore, all profiles in which no voter agrees with the type 4 outcome have two properties. First, the profile must involve (and be dominated by) voters of three types: those whose preferences agree with the rankings of vertices 1, 3, and 8. For instance, if **e** requires the profile line to have only a few type 3 voters, then **e** is near the bottom edge in the front face. In turn, the geometry requires the endpoint on the back face to be close to the top edge and vertex 8; this requires the profile to have more voters of type 8. Second, because only one subprofile from each **e** subcone is used, this behavior is very rare with a binomial, normal, or uniform probability distribution of profile lines. Yet, the likelihood is represented by the size of the portion of the rectangle of **e** values that is below the diagonal connecting vertices 1 and 3.

To illustrate, we let a type 4 outcome consist of "yes" votes over three issues. Specifically, B, D, F represent

TABLE 3. Where No Voter Is Happy

Voter Type	$\{A, B\}$	$\{C, D\}$	$\{E, F\}$
1	Yes	Yes	No
3	Yes	No	Yes
8	No	Yes	Yes
Outcome	Yes	Yes	Yes

"yes" stands on the three issues (pairs), and A, C, E are "no" positions. Using this geometry and only voters of the three types that must be represented, in Table 3 we present a profile in which no one completely approves of the final vote. This behavior, caused by a loss of crucial information about how the voters connect the parts, may be called a paradox. According to the above geometric argument, however, or our earlier statistical interpretation of pairwise votes over several pairs, such profiles become anomalies. Since the iterative construction of profile lines for more pairs must involve the above three-pair construction, versions of these statements extend.

All three-issue pairwise voting examples that manifest this unexpected behavior must have the above characteristics, so it is easy to construct and analyze examples (Nurmi 1998) that exhibit the Anscombe and Ostrogorski paradoxes. To do so, just add two more voters. For instance, by adding two type 6 voters, Table 3 is augmented by Table 4. The final outcome is "no," by a 3:2 vote, on each of the three issues. If the "yes" and "no" parties are defined, respectively, by taking a positive and negative stand on all issues, then a majority of the voters (1, 3, 8) support the "yes" party on a majority of the issues, even though the "no" party wins over all issues. Similarly, each voter in a majority (voters 1, 3, 8) is frustrated on two of the three issues. Again, rather than a paradox, these examples reflect the loss of crucial information about the voters. Again, from the statistical interpretation, such profiles are in the minority.

As further support that such behavior is in the minority, we describe the $\mathbf{q}$ choices in which all supporting profiles have voters who approve of the combined outcome. Such a profile cannot have a subprofile with vertex 1, so the condition holds if and only if the rectangle of admissible $\mathbf{e}$ values is strictly above the diagonal connecting vertices 1 and 3; that is, the lower left vertex (coming from vertex 8; Table 2) of the rectangle must be above this diagonal. The line is $y + z = 1$, so the sum of the appropriate components of this vertex must be greater than unity, or (after collecting terms)

$$q_1 + q_2 + q_3 > 2.$$

TABLE 4. Adding Voters

Voter Type	$\{A, B\}$	$\{C, D\}$	$\{E, F\}$
6	No	No	No
6	No	No	No

TABLE 5. Assigning Rankings to Vertices

Vertex	Ranking	Vertex	Ranking
1	$\mathcal{A} > \mathcal{B} > \mathcal{C}$	7	$\mathcal{C} > \mathcal{B} > \mathcal{A}$
2	$\mathcal{A} > \mathcal{C} > \mathcal{B}$	8	$\mathcal{B} > \mathcal{C} > \mathcal{A}$
3	$\mathcal{C} > \mathcal{A} > \mathcal{B}$	5	$\mathcal{B} > \mathcal{A} > \mathcal{C}$

This inequality means that if $\mathbf{q}$ is such that $q_1 + q + 2 + q_3 = 2$, then $\mathbf{q}$ has precisely one profile in which no voter supports the outcome. To explain, this $\mathbf{q}$ allows only one primary profile line to have a front face endpoint on this diagonal. Then, the extreme front face subprofile line connects vertices 1 and 3, which are the only allowed frontist voter types. Similarly, because $\mathbf{e}$ is a vertex of the rectangle of endpoints in the front face, its associated back endpoint (see Table 2) is vertex 8. Consequently, the profile consists strictly of voters of types 1, 3, 8. For instance, according to Table 2, the outcome $\mathbf{q} = (2/3, 2/3, 2/3)$ places this particular front face vertex at the midpoint of the diagonal. By use of algebra, it now follows that the only profile supporting $\mathbf{q}$ that has no type 4 voters is

$$\nu(B, D, E) = \nu(B, C, F) = \nu(A, D, F) = \frac{1}{3}, \qquad \textbf{(13)}$$

or that of Table 3. No voter approves the joint outcome, but the many remaining profiles that support $\mathbf{q}$, which use all other remaining choices of $\mathbf{e}$ and their subcones, include type 4 voters. Thus, the Table 3, equation 13 profile is rare and unusual among those defining the election outcome $\mathbf{q} = (2/3, 2/3, 2/3)$; rather than a paradox, this profile should be treated as an anomaly among those defining this outcome.

Voting Paradoxes and Arrow's Theorem

To continue our geometric description, we relate the loss of information caused by pairwise voting to cyclic voting outcomes. To do so, we treat the Figure 4 pairs as the pairs of the three alternatives $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. By selecting the $\{\mathcal{A}, \mathcal{B}\}$, the $\{\mathcal{B}, \mathcal{C}\}$, and the $\{\mathcal{C}, \mathcal{A}\}$ rankings to be represented, respectively, on the x, y, and z axis of a standard coordinate representation. We show in Table 5 the Figure 4 vertices that represent the rankings. The remaining two vertices, 4 and 6, represent the cyclic preferences of $\mathcal{A} > \mathcal{B}$, $\mathcal{B} > \mathcal{C}$, $\mathcal{C} > \mathcal{A}$ and $\mathcal{B} > \mathcal{A}$, $\mathcal{C} > \mathcal{B}$, $\mathcal{A} > \mathcal{C}$. The individual rationality assumption, whereby each voter must have transitive preferences, restricts the voter types to those listed in Table 5.

The above discussion shows how to design all transitive profiles that cause cyclic outcomes. Equation 13 defines a profile with an equal number of voters of the three types,

$$\mathcal{A} > \mathcal{B} > \mathcal{C}, \quad \mathcal{B} > \mathcal{C} > \mathcal{A}, \quad \mathcal{C} > \mathcal{A} > \mathcal{B}, \qquad \textbf{(14)}$$

and the result is a cyclic outcome in which each pair wins by a two-thirds vote. This profile is the Condorcet triplet, and we now know why it causes difficulties.

TABLE 6. Finding Probabilities

Condition	Prob.	Condition	Prob.
$e_1 \leq 1 - e_2, e_2$, (in *I*)	$\frac{q_1e_1-\alpha}{q_1e_1}$	$e_2 \leq e_1 \leq 1 - e_2$, (in *II*)	$\frac{q_1e_2-\alpha}{q_1e_2}$
$1 - e_2, e_2 \leq e_1$, (in *III*)	$\frac{q_1e_2-\alpha}{q_1(1-e_1)}$	$1 - e_2 \leq e_1 \leq e_2$, (in *IV*)	$\frac{q_1e_1-\alpha}{q_1(1-e_2)}$

Specifically, even though voters are explicitly assumed to have transitive preferences, the pairwise vote loses all this connecting information—it loses the rationality assumption. Instead, as far as the pairwise vote is concerned, not only are voters with cyclic preferences admissible, but also the procedure treats them as being the more likely voter types. Thus, the discussion following equation 13 applies: As far as the pairwise vote procedure is concerned, it is highly unlikely for the profile in equation 14 to define the outcome; most profiles with this particular outcome involve cyclic voters. Stated in another manner, the appropriate probability distribution for the pairwise vote is either the binomial or uniform distribution of profile lines. The standard assumption of transitive voter preferences imposes a probability distribution that conflicts with the procedure because it does not allow voters to have preferences of type 4 or 6. (Essentially, the same explanation holds for Arrow's theorem [Saari 1998] because the IIA condition also drops "connecting" rationality information about pairs.)

A resolution requires finding a way to include this rationality assumption. As described in Saari (1995, 1999), this is the Borda count, that is, 2, 1, and 0 points are assigned to a voter's first, second, and third preferences. For the same reasons, a similar approach holds in general for all the difficulties described here.

Finding Likelihood Estimates

As a final illustration, we compute the likelihood that at least α of all voters approve of the combined outcome for $\mathbf{q} = (0.52, 0.52, 0.52)$ and the Figure 4b endpoint $\mathbf{e} = (1, e_1, e_2) = (1, 0.375, 0.5)$. Because q_1 of all voters are frontist, e_1 of the frontists are rightist, and y_R of all rightists prefer all three outcomes, the condition requires $q_1e_1y_R \geq \alpha$, or $y_R \geq \alpha/q_1e_1$. The full right edge of the front face is in the profile subcone, so the uniform distribution likelihood is $1 - (\alpha/q_1e_1) = (q_1e_1 - \alpha)/q_1e_1$. The numerator never can be negative, so with the specified values $q_1 = 0.52$, $e_1 = 0.375$, no more than 19.5% of the voters agree with all outcomes. Yet, with probability 0.73 at least 5% ($\alpha = 0.05$) of all voters accept all outcomes. A general statement is found in the same manner.

THEOREM 5. *If* $\mathbf{e} = (1, e_1, e_2)$ *is a front face endpoint for* $\mathbf{q}$, *where* $q_1, q_2, q_3 > 0.5$, *and if the lines and profile sublines are uniformly distributed, the likelihood that* α *of all voters agree with all three outcomes is given in Table 6. The* α *values are restricted to* $0 \leq \alpha \leq \min(q_1e_1, q_1e_2)$.

Thus, "paradoxes" are surprisingly likely, but that label implies the combined pairwise outcomes should accurately describe what voters want; this is false, as the best we can expect is a probabilistic interpretation. Here, theorem 3 extends to any number of pairs to assert that if profiles are divided according to which component is the largest, the largest proportion of profiles have agreement between the rankings of the largest component and $\mathbf{q}$. Pairwise rankings must be interpreted in a statistical rather than an absolute sense.

Much more is possible. For example, by integrating Table 6 values over all primary $\mathbf{e}$ positions, we can determine the likelihood that at least α of the voters prefer the combined outcome. The resulting complicated expressions, however, do not provide quick insight. Still, for purposes of completeness, we indicate how to perform the computations. If we assume that $q_3 \leq q_2 \leq q_1$, that each admitted $\mathbf{e}$ is equally likely, and that each subprofile line in each subcone is equally likely, then the likelihood of at least α of all voters preferring the combined outcome is determined by integrating the Table 6 values over the regions determined by Table 2. For instance, the likelihood over the region I portion is $\int_{(q_1+q_2-1)/q_1}^{1/2} \int_x^{1-x} (q_1x - \alpha)/q_1x \, dy \, dx = 1/q_1^2[((1 - q_2 - q_1)/2)(q_1 + 2\alpha) - q_1^2/4 + (q_1 + q_2 - 1)^2 + q_1\alpha \ln((q_1 + q_2 - 1)/q_1)]$. Because the expressions for the other three regions are equally complicated, for many purposes it is more reasonable to use the geometry supported by intuition derived from two-pair computations.

CONCLUSION

Even though pairwise and bundled voting seems attractive and intuitive in many political settings, there can be problems. In both procedures, it is possible for a significant percentage of the population to be dissatisfied with at least part of the overall outcome. We have shown why this can occur as well as the likelihood that any given percentage of the voters will be frustrated. The intuition developed from our analysis suggests that, to obtain reliable outcomes, these approaches should be replaced. But with what?

Both procedures become problematic by losing information about the profile $\mathbf{p}$. For comparison, a Borda count procedure allows voters to rank combinations in order of preference. By assigning the usual 3, 2, 1, 0 points to these priorities, information about the voters is reintroduced into the process. In particular, since this method requires voters to compare one option with all other combinations, it becomes possible to say, with certainty, that the selected options have support from the voters.

In order to be politically feasible, the procedure should be restricted to forming combinations of realistic size among relevant alternatives. (Asking voters to rank combinations of 20 or more unrelated choices is time consuming, unrealistic, and unnecessary.) This comment finds support in the Brams, Kilgour, and Zwicker (1998) article, which revealed that the most frequently chosen combination in voting for 52 initiatives (in 1992) was "all abstain."

Interestingly, in the case of bundled voting, the restriction on votes over related issues might hamper the legislative tendency to combine issues that, on their own, would not receive majority support. This includes unrelated issues important to certain legislators but sufficiently minor to others that they will not affect passage of the bundled bill. Restrictions on the relevance of alternatives in the ranking of combinations should discourage such practices.

This is an important concern because a basic goal of democracy is a government and mechanisms representative of the wishes of the populace. When we discover flaws in our decision tools, we should highlight and correct them to ensure that outcomes are not dictated by unrepresentative minorities. Indeed, legislators intuitively understand and manipulate the kinds of results demonstrated here. Amendments are routinely added to popular bills with full understanding on the part of all involved that these issues would not survive a vote if considered on their own. The awareness of the effects of bundling is so widespread that it is acknowledged in the *Congressional Quarterly Almanac* (1998, 2-112) description of the fiscal 1999 Omnibus Appropriations Bill: "Democrats felt shut out of the process, but they won concessions that they would not have obtained had the 13 bills [included in the appropriations bill package] been negotiated individually." These arrangements are made in the spirit of compromise—special favors are given in response to support for other issues. Nonetheless, these comments illustrate the familiarity of legislators with the manipulative opportunities afforded by the bundled vote.

APPENDIX: PROOFS

We provide here the proofs of the various assertions and more complicated computations. To begin, we indicate why, for probability computations, the edge lengths suffice. In Figure 2b, the three lines are the extreme profile lines, and the third is the boundary line for the heavily shaded region. The area of the heavily shaded region on the left side is half the distance from the left edge to $\mathbf{q}$ times the length of the region's base on the left edge. Similarly, the area of the shaded region is half the distance from the left edge to $\mathbf{q}$ times the length of its base along the left edge. Thus, the ratio of these two areas is the ratio of the lengths of their bases on the left edge. That either the left or right edge can be used in probability computations follows from similar triangles. Because the heavily shaded triangle on the left is similar to the one on the right, and because the same statement holds for the shaded triangles, it follows that the values of the ratios are the same if computed on the right or on the left edge.

Proof of Theorem 1

According to the assumptions of the theorem, the two pairs are, respectively, $\{A, B\}$ and $\{C, D\}$, where $B > A$ with m_1 of the vote, and $C > D$ with m_2 of the vote. Thus, in equation 6, $q_1 = 1 - m_1$, $q_2 = m_2$, and the profile lines have endpoints on the left edge on the interval $[(m_1 + m_2 - 1)/m_1, m_2/m_1]$. The only way at least α of all voters can prefer both outcomes is if α of all voters are leftist with preferences listed by the upper vertex of Figure 2b. Thus, we are only interested in profile lines with a left endpoint that satisfies $y \times m_1 \geq \alpha$, or $y \geq \alpha/m_1$. The likelihood is given by the fraction $[m_2/m_1 - \alpha/m_1]/[m_2/m_1 - (m_1 + m_2 - 1)/m_1] = (m_2 - \alpha)/1 - m_1$.

Similarly, if at least β of all voters dislike both outcomes, they all are rightist with preferences on the lower corner. Because $(1 - m_1)$ of all voters are rightist, we need profile lines with a right endpoint that is y satisfying $(1 - m_1)(1 - y) \geq \beta$, or $1 - m_1 - \beta/1 - m_1 \geq y$. The conclusion follows.

Proof of Theorem 2

If $\mathbf{q}$ is in one of the quadrants, then the outcome for the pair defining the two parties must correspond to the wishes of the larger party. If $\mathbf{q}$ is above the horizontal line, then so are most profile lines; this proves the second result.

Proof of Theorem 4

A profile line with left and right endpoints $\mathbf{q}_L = (0, y_L)$ and $\mathbf{q}_R = (1, y_R)$, has the representation

$$(1 - t)(0, y_L) + t(1, y_R), \qquad 0 \leq t \leq 1. \tag{15}$$

Since $\mathbf{q}$ is the point on this line when $t = q_1$, we have

$$\begin{aligned}(1 - q_1)(0, y_L) + q_1(1, y_R) &= (q_1, (1 - q_1)y_L + q_1 y_R)\\ &= (q_1, q_2),\end{aligned} \tag{16}$$

or

$$(1 - q_1)y_L + q_1 y_R = q_2. \tag{17}$$

All of the rightists, who constitute q_1 of all voters, vote for the bundled bill. Thus, the $\gamma - q_1$ extra voters needed to pass the bundled bill must come from the leftists. In turn, this requires y_L to satisfy the inequality

$$y_L(1 - q_1) \geq (\gamma - q_1), \text{ or } y_L \geq (\gamma - q_1)/(1 - q_1). \tag{18}$$

There are two situations. The first occurs when an extreme profile line passes through $\mathbf{q}$ and the (1, 0) vertex; this is true if and only if $q_1 + q_2 \leq 1$. From the use of $y_R = 0$, $y_L = (\gamma - q_1)/(1 - q_1)$, and equation 16, it follows that a supporting profile exists for $\mathbf{q}$, $q_1 + q_2 \leq 1$, which passes the bundled bill if and only if $(1 - q_1)[(\gamma - q_1)/(1 - q_1)] \leq q_2$, or if and only if $q_1 + q_2 \geq \gamma$.

The remaining setting of $q_1 + q_2 > 1$ has a profile line that passes through the (0, 1) vertex, which represents where all leftists, $(1 - q_1)$ of all voters, vote for the bill. Here, the $y_L = 1$ value always satisfies equation 18, so the bill always passes.

Since $q_1 + q_2 \geq 1$ always satisfies $q_1 + q_2 \geq \gamma$, this second inequality suffices to describe the conditions.

Next, we determine when α, $0 \leq \alpha \leq 1$, of the profile lines supporting $\mathbf{q}$ pass the bundled bill. This requires finding conditions on q_1, q_2, so that at least α of these profile lines satisfy equation 18. The two diagonal lines of the square, $y = x$ and $x + y = 1$, divide the representation square into four triangles; the four settings of $\mathbf{q}$ in each triangle are considered separately. If $q_1 \leq q_2 \leq 1 - q_1$ (this triangle has the square's left edge as a leg), then the extreme profile lines for the profile cone for $\mathbf{q}$ are defined by the two vertices on the right edge. Thus, the extreme profile lines $\mathbf{p}_U$ and $\mathbf{p}_D$ are defined, respectively, by $y_R = 1$ and $y_R = 0$. According to equation 17, the two respective left endpoints are $y_L = (q_2 - q_1)/(1 - q_1)$ and $y_L = q_2/(1 - q_1)$. Because the internal $(q_2 - q_1)/(1 - q_1) \leq y_L \leq q_2/(1 - q_1)$ include all possible left endpoints, α of these endpoints are above the value $y_L = \alpha[(q_2 - q_1)/(1 - q_1)] + (1 - \alpha)[q_2/(1 - q_1)]$. According to equation 18, these left endpoints define profile lines that pass the bundled bill if and only if $\alpha[(q_2 - q_1)/(1 - q_1)] + (1 - \alpha)[q_2/(1 - q_1)] \geq (\gamma - q_1)/(1 - q_1)$, or if and only if

$$q_2 + (1 - \alpha)q_1 \geq \gamma. \qquad \textbf{(19)}$$

The triangular region with the square's bottom edge as a leg is where $q_2 < q_1 < 1 - q_2$. One extreme profile line has $y_R = 0$, so, according to equation 17, $y_L = q_2/(1 - q_1)$. As the other extreme profile line has $y_L = 0$, the top α of the left endpoints are given by $y_L \geq (1 - \alpha)q_2/(1 - q_1)$. Thus, according to equation 18, at least α of the supporting profiles lead to passage of the bundled bill if and only if $(1 - \alpha)q_2/(1 - q_1) \geq (\gamma - q_1)/(1 - q_1)$. Thus, replacing equation 19 is $(1 - \alpha)q_2 + q_1 \geq \gamma$.

The final two triangles satisfy $q_1 + q_2 > 1$. The triangle with $q_1 > q_2$ has the square's right edge as a leg. Here, more than half the voters are rightist, and the extreme profile lines involve the two vertices of the left edge. Thus, at least α of these lines support the bundled bill when $y_L \geq 1 - \alpha$. According to equation 18, the bundled bill passes if

$$q_1 \geq [\gamma + \alpha - 1]/\alpha. \qquad \textbf{(20)}$$

If $\gamma + \alpha - 1 \leq 0$ (so the equation 20 restriction is $q_1 \geq 0$), just the fact that $\mathbf{q}$ is in this triangular sector suffices to ensure that at least α of the supporting profiles lead to the passage of the bundled bill. In fact, since $q_1 > 1/2$, it follows (by setting the left-hand side of equation 20 equal to $1/2$) that equation 20 imposes a restriction only if $\alpha > 2(1 - \gamma)$.

Similarly, in the upper triangular region, the y_L values satisfy $(q_2 - q_1)/(1 - q_1) \leq y_L \leq 1$, so the condition for α of the supporting profiles to allow passage now is

$$\left[(1 - \alpha) + \alpha\,\frac{q_2 - q_1}{1 - q_1}\right](1 - q_1) + q_1 \geq \gamma.$$

Here, equation 20 is replaced by $q_2 \geq [\gamma + \alpha - 1]/\alpha$ with the same comments that these restrictions have meaning only when $\alpha \geq 2(1 - \gamma)$.

By graphing the restrictions for the four regions when $\alpha > 2(1 - \gamma)$, the description of the theorem is obtained.

For a profile to have at least γ of the voters who prefer both issues, $\nu(A, C) \geq \gamma$. Thus, $y_R q_1 \geq \gamma$, or $y_R \geq \gamma/q_1$. If $q_2 \geq q_1$, then the profile line most favorable for this condition has the vertex (1, 1). Here, $y_R = 1$, so the condition is $q_2 \leq q_1 \leq \gamma$. If $q_1 > q_2$, then the most favorable profile line has (0, 0) as a vertex. Thus, using equation 17 with $y_L = 0$ leads to the condition $y_R = q_2/q_1$. This means that $q_2/q_1 \geq \gamma/q_1$. Consequently, $q_1 \geq q_2 \geq \gamma$.

With $q_1, q_2 \geq \gamma \geq 1/2$, if $q_1 \geq q_2$, the extreme profile lines have $y_L = 1$ and $y_L = 0$ leading to $(q_1 + q_2 - 1)/q_1 \leq y_R \leq q_2/q_1$. The β portions of largest y_R values satisfy $y_R \geq \beta[(q_1 + q_2 - 1)/q_1] + (1 - \beta)[q_2/q_1]$. When this value is at least as large as the required γ/q_1 and when the resulting inequality is solved, we have the condition that, when $q_1 \geq q_2$, $q_2 + \beta(q_1 - 1) \geq \gamma$. The only difference when $q_2 \geq q_1$ is that the profile lines satisfy $(q_1 + q_2 - 1)/q_1 \leq y_R \leq 1$. Thus, the threshold for the top β values is $y_R = \beta[(q_1 + q_2 - 1)/q_1] + (1 - \beta) \geq \gamma/q_1$. Therefore, when $q_2 \geq q_1$, the condition is $q_1 + \beta(q_2 - 1) \geq \gamma$.

Proof of Theorem 5

This is a direct algebraic computation.

Finally, a quick way to derive the comments concerning the dimensions of the set of profiles defining a specified outcome for n pairs is to use algebra. With n pairs, there are two ways for a voter to vote for each issue, so there are 2^n different voter types. Our choice of a profile describes the fraction of all voters who are of each type, so the space of profiles is $2^n - 1$ dimensions. The outcome $\mathbf{q} = (q_1, \ldots, q_n)$ is a point in n-dimensional space. Thus, an election can be viewed as a set of n linear equations with $2^n - 1$ variables. Standard rules from algebra state that the inverse set for any specified outcome $\mathbf{q}$ usually has dimension $2^n - 1 - n = 2^n - (n + 1)$.

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