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A problem of integer partitions and numerical semigroups

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Let C be a set of positive integers. In this paper, we obtain an algorithm for computing all subsets A of positive integers which are minimals with the condition that if $x_1 + \cdots + x_n$ is a partition of an element in C, then at least a summand of this partition belongs to A. We use techniques of numerical semigroups to solve this problem because it is equivalent to give an algorithm that allows us to compute all the numerical semigroups which are maximals with the condition that has an empty intersection with the set C.

Keywords: integer partitions; numerical semigroups; irreducibles numerical semigroups; Apéry set

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1. Introduction

In how many ways can we write a positive integer as a sum of positive integers? This question appears in correspondence between G. W. Leibniz and J. Bernoulli, in a letter dated 1669 Leibniz asked Bernoulli about the number of 'divulsions' of integers. It is L. Euler in his 'Introductio in analysin infinitorum' who gives the first answers on this subject (see [**7**]). From this small beginning, we are led to a subject with many sides and many applications: The Theory of Partitions. Thenceforth in the literature, one finds many manuscripts devoted to the study the integer partitions (just to mention some of them see [**1**, **6**, **8**]).

A partition of a positive integer c is a way of writing c as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. For example, the five partitions of 4 are 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$.

Let $\mathbb{N}$ be the set of nonnegative integers and let C be a finite subset of $\mathbb{N}\backslash\{0\}$. Our main goal in this paper is to solve the following question:

Question 1.1. To provide an algorithm for computing all subsets A of $\mathbb{N}$ which are minimals with the condition that if $x_1 + \cdots + x_n$ is a partition of an element in C, then at least a summand of this partition belongs to A.

A numerical semigroup is a subset S of $\mathbb{N}$ closed under addition, contains the zero element and has a finite complement in $\mathbb{N}$. This kind of semigroup has been widely treated in the literature (see for example [**3**, **9**, **11**]). In § 2, we started by showing that the problem presented in question 1.1 is equivalent to give an algorithm that allows us compute all the numerical semigroups which are maximals with the condition that has an empty intersection with the set C.

Let $\mathscr{S}$ be the set of all numerical semigroups, $\mathscr{S}(C) = \{S \in \mathscr{S} \mid S \cap C = \emptyset\}$ and $\mathscr{M}(C) = \text{Maximals}(\mathscr{S}(C))$. Considering $C = \{c_1, \ldots, c_l\}$ a set of positive integers, we see that $\mathscr{M}(C) = \text{Maximals}\{S_1 \cap \cdots \cap S_l \mid S_i \in \mathscr{M}(\{c_i\}) \text{ for all } i \in \{1, \ldots, l\}\}$ and furthermore, $\mathscr{M}(\{c_i\})$ is finite for all $i \in \{1, \ldots, l\}$. Hence, in order to introduce a procedure that allows us to compute $\mathscr{M}(C)$, we need the following:

(1) given a positive integer c to build an algorithm that allows us to calculate the set $\mathscr{M}(\{c\})$;

(2) given $A_1, \ldots, A_l$ finite subsets of $\mathscr{S}$ to build an algorithm that allows us to calculate the set $\text{Maximals}\{S_1 \cap \cdots \cap S_l \mid S_i \in A_i \text{ for all } i \in \{1, \ldots, l\}\}$.

Following the terminology used in [**10**] a numerical semigroup is irreducible if it cannot be expressed as an intersection of two numerical semigroups containing it properly. The greatest integer not belonging to a numerical semigroup S is the Frobenius number of S (see [**9**]) and it is denoted by $\mathrm{F}(S)$. In § 3, we see that if c is a positive integer, then $\mathscr{M}(\{c\}) = \{S \mid S \text{ is an irreducible numerical semigroup and } \mathrm{F}(S) = c\}$. In [**4**] is presented as a procedure to get the set of irreducible numerical semigroups with a fixed Frobenius number. Thus we have a procedure to compute $\mathscr{M}(\{c\})$.

Let S be a numerical semigroup and let $n \in S\backslash\{0\}$. The Apéry set (named so in honour [**2**]) of n in S is $\mathrm{Ap}(S, n) = \{w(0), w(1), \ldots, w(n-1)\}$, where $w(i)$ is the least element of S congruent with i modulo n for all $i \in \{0, \ldots, n-1\}$. In § 4, we show how we can compute the Apery set of an intersection of finitely many numerical semigroups, consequently, we have a procedure to compute $\mathscr{M}(C)$.

Finally, in § 5, we obtain an alternative algorithm to the previous one for the case we need to compute only one element in $\mathscr{M}(C)$.

2. Translate the problem to numerical semigroups

Let C be a set of positive integers. We use the following notation:

$$P(C) = \{x_1 + \cdots + x_n \mid x_1 + \cdots + x_n \text{ is a partition of } c \text{ for some } c \in C\},$$

$$G(C) = \{\{x_1, \ldots, x_n\} \mid x_1 + \cdots + x_n \in P(C)\},$$

$$L(C) = \{A \subseteq \mathbb{N}\backslash\{0\} \mid A \cap X \neq \emptyset \text{ for all } X \in G(C)\} \text{ and}$$

$$R(C) = \text{Minimals}(L(C)).$$

With this notation, note that to solve the question 1.1 is equivalent to get an algorithm for computing the set $R(C)$. Our aim in this section is to prove that $A \in R(C)$ if and only if $\mathbb{N}\backslash A$ is a maximal numerical semigroup with the condition that has an empty intersection with the set C.

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LEMMA 2.1. *Let C be a set of positive integers and $A \in R(C)$. Then*

(1) $C \subseteq A \subseteq \{1, \ldots, \max(C)\}$.

(2) *If $a \in A$ then there exists $X \in G(C)$ such that $A \cap X = \{a\}$.*

(3) *If x and y are positive integers and $x + y \in A$ then $\{x, y\} \cap A \neq \emptyset$.*

(4) *If $x, y \in \mathbb{N} \backslash A$ then $x + y \in \mathbb{N} \backslash A$.*

Proof.

(1) Trivial.

(2) If $A \in R(C)$ and $a \in A$ then it is clear that $A \backslash \{a\} \notin L(C)$. Then there exists $X \in G(C)$ such that $(A \backslash \{a\}) \cap X = \emptyset$. Since $A \in R(C)$, we have that $A \in L(C)$ and so $A \cap X \neq \emptyset$. Consequently, $A \cap X = \{a\}$

(3) Assume that x and y are positive integers and $x + y \in A$. By applying (2), we deduce that there exist $c \in C$ and $x_1 + \cdots + x_n$ a partition of c such that $\{x_1, \ldots, x_n\} \cap A = \{x + y\}$. Without loss of generality we can assume that $x_1 = \cdots = x_r = x + y$ and $x_i \neq x + y$ for all $i \in \{r + 1, \ldots, n\}$. Then

$$\underbrace{x + y + \cdots + x + y}_{r} + x_{r+1} + \cdots + x_n$$

is a partition of c. Therefore, $\{x, y, x_{r+1}, \ldots, x_n\} \cap A \neq \emptyset$ and so $\{x, y\} \cap A \neq \emptyset$.

(4) It is a reformulation of (3). □

As an immediate consequence of (1) and (4) of the previous lemma, we have the following result.

PROPOSITION 2.2. *Let C be a set of positive integers. If $A \in R(C)$ then $\mathbb{N} \backslash A$ is a numerical semigroup. Moreover, $(\mathbb{N} \backslash A) \cap C = \emptyset$.*

PROPOSITION 2.3. *Let C be a set of positive integers and $S \in \mathscr{S}(C)$. Then $\mathbb{N} \backslash S \in L(C)$.*

Proof. Assume that $x_1 + \cdots + x_n$ is a partition of $c \in C$. Then, we have that $x_1 + \cdots + x_n = c \notin S$ and thus there exists $i \in \{1, \ldots, n\}$ such that $x_i \notin S$, because S is a numerical semigroup. Therefore, $\{x_1, \ldots, x_n\} \cap (\mathbb{N} \backslash S) \neq \emptyset$. Consequently, $(\mathbb{N} \backslash S) \cap X \neq \emptyset$ for all $X \in G(C)$ and so $\mathbb{N} \backslash S \in L(C)$. □

THEOREM 2.4. *Let C be a finite set of positive integers. Then $A \in R(C)$ if and only if $\mathbb{N} \backslash A \in \mathscr{M}(C)$.*

Proof. Necessity. Proposition 2.2 states that $\mathbb{N} \backslash A \in \mathscr{S}(C)$. Consider that $S \in \mathscr{S}(C)$ such that $\mathbb{N} \backslash A \subseteq S$. This means that $\mathbb{N} \backslash S \subseteq A$. In addition, from proposition 2.3 we deduce that $\mathbb{N} \backslash S \in L(C)$. Since A belongs to $\mathrm{Minimals}\,(L(C))$, then $\mathbb{N} \backslash S = A$. Therefore, $\mathbb{N} \backslash A \in \mathrm{Maximals}\,(\mathscr{S}(C))$ and consequently, $\mathbb{N} \backslash A \in \mathscr{M}(C)$.

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Sufficiency. From proposition 2.3, we know that $A \in L(C)$. Let $B \in R(C)$ such that $B \subseteq A$. We have that $\mathbb{N}\backslash A \subseteq \mathbb{N}\backslash B$. Moreover, by applying proposition 2.2 we know that $\mathbb{N}\backslash B \in \mathscr{S}(C)$. As $\mathbb{N}\backslash A \in \mathscr{M}(C)$, it follows that $\mathbb{N}\backslash A = \mathbb{N}\backslash B$. Therefore, we conclude that $A = B \in R(C)$. □

As a consequence of the previous theorem, we have the following result.

COROLLARY 2.5. *If C is a finite set of positive integers, then*

$$R(C) = \{\mathbb{N}\backslash S \mid S \in \mathscr{M}(C)\}.$$

Given a finite set of positive integers C, our next aim is to describe an algorithm for computing $\mathscr{M}(C)$. For this purpose, irreducible numerical semigroups play an important role in this study.

3. Irreducible numerical semigroups with a given Frobenius number

Let c be a positive integer. Our aim in this section is to describe an algorithm for computing $\mathscr{M}(\{c\})$ which is essential for computing $\mathscr{M}(C)$.

The following result appears in [10, theorem 1].

PROPOSITION 3.1. *Let S be a numerical semigroup. The following conditions are equivalent:*

(1) *S is irreducible,*

(2) *S is maximal in the set of all numerical semigroups with Frobenius number* $\mathrm{F}(S)$*,*

(3) *S is maximal in the set of all numerical semigroups that do not contain* $\mathrm{F}(S)$*.*

As an immediate consequence of the proposition 3.1, we have the following result.

COROLLARY 3.2. *If c is a positive integer, then*

$$\mathscr{M}(\{c\}) = \{S \mid S \textit{ is an irreducible numerical semigroup and } \mathrm{F}(S) = c\}.$$

Note that if S is a numerical semigroup with $\mathrm{F}(S) = c$, then $\{0, c+1, \rightarrow\} \subseteq S$. Hence, there exists a set $X \subseteq \{1, \ldots, c-1\}$ such that $S = \{0, c+1, \rightarrow\} \cup X$. Consequently, we can conclude that the set of all numerical semigroups with given Frobenius number c has finitely many elements. From corollary 3.2, we have that the set $\mathscr{M}(\{c\})$ has finitely many elements. In [4] is presented a procedure to obtain the set of irreducible numerical semigroups with a fixed Frobenius number which is implemented in `IrreducibleNumericalSemigroupsWithFrobeniusNumber( )` of [5]. Therefore, by applying again corollary 3.2, we get a procedure to compute $\mathscr{M}(\{c\})$.

Below we shall give an example, but first, we need to introduce and establish some notation and concepts.

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Given a set of positive integers X, we will denote by $\langle X\rangle$ the submonoid of $(\mathbb{N},+)$ generated by X, that is,

$$\langle X\rangle = \left\{\sum_{i=1}^{n}\lambda_i x_i \mid n\in\mathbb{N}\backslash\{0\}, \lambda_i\in\mathbb{N}, x_i\in X \text{ for all } i\in\{1,\dots,n\}\right\}.$$

It is well known (see [**11**, lemma 2.1]) that $\langle X\rangle$ is a numerical semigroup if and only if $\gcd(X)=1$. If S is a numerical semigroup and $S=\langle X\rangle$ then we say that X is a system of generators of S. Moreover, if $S\neq\langle Y\rangle$ for all $Y\subsetneq X$, then we say that X is a minimal system of generators of S. Every numerical semigroup admits a unique minimal system of generators, which is finite (see [**11**, theorem 2.7]).

Example 3.3. Let us calculate $\mathscr{M}(\{11\})$. First, we compute all irreducible numerical semigroups with Frobenius number 11.

```
gap> I:=IrreducibleNumericalSemigroupsWithFrobeniusNumber(11);;
gap> List(I,MinimalGeneratingSystem);
[ [2,13],[3,7],[4,5],[4,6,9],[5,7,8,9],[6,7,8,9,10] ].
```

Using corollary 3.2, we have that

$$\mathscr{M}(\{11\}) = \{\langle 2,13\rangle, \langle 3,7\rangle, \langle 4,5\rangle, \langle 4,6,9\rangle, \langle 5,7,8,9\rangle, \langle 6,7,8,9,10\rangle\}.$$

4. Algorithm for the general case

Our aim in this section is to provide an algorithm to compute $\mathscr{M}(C)$.

Lemma 4.1. *Let $C=\{c_1,\dots,c_l\}$ be a set of positive integers. Then*

$$\mathscr{S}(C) = \{S_1\cap\cdots\cap S_l \mid S_i\in\mathscr{S}(\{c_i\}) \text{ for all } i\in\{1,\dots,l\}\}.$$

Proof. Clearly, if $S_i\in\mathscr{S}(\{c_i\})$ for all $i\in\{1,\dots,l\}$, then $S_1\cap\cdots\cap S_l\in\mathscr{S}(C)$. For the other inclusion, if $S\in\mathscr{S}(C)$ then $S\in\mathscr{S}(\{c_i\})$ for all $i\in\{1,\dots,l\}$. By considering $S=S_1=\cdots=S_l$, we have that $S=S_1\cap\cdots\cap S_l$. □

Observe that if c is a positive integer and $S\in\mathscr{S}(\{c\})$, then $S'=S\cup\{c+1,\rightarrow\}\in\mathscr{S}(\{c\})$, $\mathrm{F}(S')=c$ and $S\subseteq S'$. Furthermore, the set of all numerical semigroups with given Frobenius number c is finite. Hence, there exists a numerical semigroup $\overline{S}$ such that $S'\subseteq\overline{S}$ and $\overline{S}$ is maximal in the set of all numerical semigroups with Frobenius number c. By applying proposition 3.1 and corollary 3.2, we deduce the following result.

Lemma 4.2. *If c is a positive integer and $S\in\mathscr{S}(\{c\})$ then there exists $\overline{S}\in\mathscr{M}(\{c\})$ such that $S\subseteq\overline{S}$.*

As a consequence of lemma 4.1, if $C=\{c_1,\dots,c_l\}$ is a set of positive integers then we have that

$$\mathscr{M}(C) = \text{Maximals}\{S_1\cap\cdots\cap S_l \mid S_i\in\mathscr{S}(\{c_i\}) \text{ for all } i\in\{1,\dots,l\}\}.$$

The next theorem improves this result which is fundamental to achieve our goal.

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THEOREM 4.3. *Let $C = \{c_1, \ldots, c_l\}$ be a set of positive integers. Then*

$$\mathscr{M}(C) = \text{Maximals}\left\{S_1 \cap \cdots \cap S_l \mid S_i \in \mathscr{M}\left(\{c_i\}\right) \text{ for all } i \in \{1, \ldots, l\}\right\}.$$

Proof. If S belongs to $\mathscr{M}(C)$ then it is in $\mathscr{S}(C)$ and thus, from lemma 4.1, for each $i \in \{1, \ldots, l\}$ there exists $S_i \in \mathscr{S}(\{c_i\})$ such that $S = S_1 \cap \cdots \cap S_l$. By lemma 4.2, we obtain that for each $i \in \{1, \ldots, l\}$ there exists $\overline{S}_i \in \mathscr{M}(\{c_i\})$ such that $S_i \subseteq \overline{S}_i$. Assume that $\overline{S} = \overline{S}_1 \cap \cdots \cap \overline{S}_l$. Then $S \subseteq \overline{S}$. In addition, from lemma 4.1, we conclude that $\overline{S} \in \mathscr{S}(C)$. Since S belongs to $\mathscr{M}(C)$, we have that $S = \overline{S}$.

In view of the previous paragraph we get that

$$\mathscr{M}(C) \subseteq \{S_1 \cap \cdots \cap S_l \mid S_i \in \mathscr{M}(\{c_i\}) \text{ for all } i \in \{1, \ldots, l\}\}.$$

Moreover, from lemma 4.1 it follows that $\{S_1 \cap \cdots \cap S_l \mid S_i \in \mathscr{M}\left(\{c_i\}\right)$ for all $i \in \{1, \ldots, l\}\} \subseteq \mathscr{S}(C)$. Consequently,

$$\mathscr{M}(C) = \text{Maximals}\left\{S_1 \cap \cdots \cap S_l \mid S_i \in \mathscr{M}\left(\{c_i\}\right) \text{ for all } i \in \{1, \ldots, l\}\right\}. \qquad \square$$

In order to build an algorithm that allows us to calculate the numerical semigroup $S_1 \cap \cdots \cap S_l$, from the numerical semigroups $S_1, \ldots, S_l$, we introduce the following concept.

Given two integers a and b with $b \neq 0$, we denote by $a \bmod b$ the remainder of the division of a by b. The knowledge of $\text{Ap}(S, n)$ for some $n \in S\backslash\{0\}$ gives us enough information about S. In fact, we have that an integer x belongs to S if and only if $x \geqslant w(x \bmod n)$ and that $S = \langle \text{Ap}(S, n) \cup \{n\}\rangle$. It is easy to prove the next result.

PROPOSITION 4.4. *Let $S_1, \ldots, S_r$ be numerical semigroups, $n \in (S_1 \cap \cdots \cap S_r)\backslash\{0\}$ and $\text{Ap}(S_k, n) = \{w_k(0), w_k(1), \ldots, w_k(n-1)\}$ for all $k \in \{1, \ldots, r\}$. Then $\text{Ap}(S_1 \cap \cdots \cap S_r, n) = \{w(0), w(1), \ldots, w(n-1)\}$ where $w(i) = \max\{w_1(i), \ldots, w_r(i)\}$ for all $i \in \{0, \ldots, n-1\}$.*

Given a numerical semigroup S and $n \in S\backslash\{0\}$, we denote by $\theta_n(S) = (w(1), \ldots, w(n-1))$. Assume that $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_k) \in \mathbb{N}^k$ and denote by $(x_1, \ldots, x_k) \vee (y_1, \ldots y_k) = (\max\{x_1, y_1\}, \ldots, \max\{x_k, y_k\})$. As a consequence of proposition 4.4, we obtain the following.

COROLLARY 4.5. *Let n a positive integer and let $\mathscr{S}_n = \{S \in \mathscr{S} \mid n \in S\}$. Then*

(1) *$\theta_n : \mathscr{S}_n \to \mathbb{N}^{n-1}$ is a injective map;*

(2) *if $(x_1, \ldots, x_{n-1}) \in Im(\theta_n)$, then $S = \langle x_1, \ldots, x_{n-1}, n\rangle \in \mathscr{S}_n$ and $\theta_n(S) = (x_1, \ldots, x_{n-1})$;*

(3) *if $S, T \in \mathscr{S}_n$, then $S \subseteq T$ if and only if $\theta_n(T) \leqslant \theta_n(S)$ (with $\leqslant$ is the product order on $\mathbb{N}^{n-1}$);*

(4) *if $S, T \in \mathscr{S}_n$, then $\theta_n(S \cap T) = \theta_n(S) \vee \theta_n(T)$.*

Let $C = \{c_1, \ldots, c_l\}$ be a set of positive integers and $n = \max(C) + 1$. From corollary 3.2, we can deduce that for all $i \in \{1, \ldots, l\}$ and $S_i \in \mathscr{M}(\{c_i\})$ then $n \in S_i$.

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Moreover, from (3) of the corollary 4.5, we know that the map θ_n inverts the orders inclusion and product. From this remark and theorem 4.3, we get the following result.

Corollary 4.6. *Let* $C = \{c_1, \dots, c_l\}$ *be a set of positive integers and* $n = \max(C) + 1$. *Then*

$$\mathscr{M}(C) = \left\{ S \in \mathscr{S}_n \;\middle|\; \begin{array}{c} \theta_n(S) \in \text{Minimals}\{\theta_n(S_1) \vee \cdots \vee \theta_n(S_l)\} \\ \textit{with } S_i \in \mathscr{M}(\{c_i\}) \textit{ for all } i \in \{1, \dots, l\} \end{array} \right\}.$$

If S is a numerical semigroup and $n \in S\backslash\{0\}$, then the function `AperyListOfNumericalSemigroupWRTElement (S,n)` of [**5**] gives us a list $[x_0, x_1, \dots, x_{n-1}]$ such that $\text{Ap}(S, n) = \{w(0) = x_0, w(1) = x_1, \dots, w(n-1) = x_{n-1}\}$. Therefore, we have an algorithm that allows us to calculate $\theta_n(S)$ from an element n in S and a system of generators of S.

Example 4.7. Let $S = \langle 5, 7, 9\rangle$. Let us calculate $\theta_5(S)$ using [**5**].

```
gap> S:=NumericalSemigroups(5,7,9);;
gap> AperyListOfNumericalSemigroupWRTElement (S,5);
[0,16,7,18,9 ]
```

Hence, $\theta_5(S) = (16, 7, 18, 9)$.

Gathering what we have seen so far, we get the result announced at the beginning of this section.

Algorithm 1. *Input:* $C = \{c_1, \dots, c_l\}$ *be a set of positive integers.*
Output: The set $\mathscr{M}(C)$.

(1) *By using* `IrreducibleNumericalSemigroupsWithFrobeniusNumber`(c_i) *of* [**5**], *we compute* $\mathscr{M}(\{c_i\})$ *for all* $i \in \{1, \dots, l\}$.

(2) $n = \max(C) + 1$.

(3) *By using* `AperyListOfNumericalSemigroupWRTElement`(S, n), *we calculate* $A_i = \{\theta_n(S) \mid S \in \mathscr{M}(\{c_i\})$ *for all* $i \in \{1, \dots, l\}\}$.

(4) $A = \{\alpha_1 \vee \cdots \vee \alpha_n \mid \alpha_i \in A_i$ *for all* $i \in \{1, \dots, l\}\}$.

(5) $B = \text{Minimals}(A)$.

(6) *Return* $\{\langle n, x_1, x_2, \dots, x_{n-1}\rangle \mid (x_1, x_2, \dots, x_{n-1}) \in B\}$.

Next, we give an example that illustrates the algorithm 1.

Example 4.8. Let us compute the set $\mathscr{M}(\{8, 11\})$.

(1) $\mathscr{M}(\{8\}) = \{\langle 3, 7, 11\rangle, \langle 5, 6, 7, 9\rangle\}$
$\mathscr{M}(\{11\}) = \{\langle 2, 13\rangle, \langle 3, 7\rangle, \langle 4, 5\rangle, \langle 4, 6, 9\rangle, \langle 5, 7, 8, 9\rangle, \langle 6, 7, 8, 9, 10\rangle\}$.

(2) $n = 12$.

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(3) $A_1 = \{(13, 14, 3, 16, 17, 6, 7, 20, 9, 10, 11), (13, 14, 15, 16, 5, 6, 7, 20, 9, 10, 11)\}$
$A_2 = \{(13, 2, 15, 4, 17, 6, 19, 8, 21, 10, 23), (13, 14, 3, 16, 17, 6, 7, 20, 9, 10, 23), (13, 14, 15, 4, 5, 18, 19, 8, 9, 10, 23), (13, 14, 15, 4, 17, 6, 19, 8, 9, 10, 23), (13, 14, 15, 16, 5, 18, 7, 8, 10, 23), (13, 14, 15, 16, 17, 6, 7, 8, 9, 10, 23)\}$.

(4) $A = \{(13, 14, 15, 16, 17, 6, 19, 20, 21, 10, 23), (13, 14, 3, 16, 17, 6, 7, 20, 9, 10, 23), (13, 14, 15, 16, 17, 18, 19, 20, 9, 10, 23), (13, 14, 15, 16, 17, 6, 19, 20, 9, 10, 23), (13, 14, 15, 16, 17, 18, 7, 20, 9, 10, 23), (13, 14, 15, 16, 17, 6, 7, 20, 9, 10, 23), (13, 14, 15, 16, 17, 6, 19, 20, 21, 10, 23), (13, 14, 15, 16, 17, 6, 7, 20, 9, 10, 23), (13, 14, 15, 16, 5, 18, 19, 20, 9, 10, 23), (13, 14, 15, 16, 17, 6, 19, 20, 9, 10, 23), (13, 14, 15, 16, 5, 18, 7, 20, 9, 10, 23), (13, 14, 15, 16, 17, 6, 7, 20, 9, 10, 23)\}$.

(5) $B = \{(13, 14, 3, 16, 17, 6, 7, 20, 9, 10, 23), (13, 14, 15, 16, 5, 18, 7, 20, 9, 10, 23)\}$.

(6) $\mathscr{M}(\{8, 11\}) = \{\langle 12, 13, 14, 3, 16, 17, 6, 7, 20, 9, 10, 23\rangle, \langle 12, 13, 14, 15, 16, 5, 18, 7, 20, 9, 10, 23\rangle\} = \{\langle 3, 7\rangle, \langle 5, 7, 9, 13\rangle\}$.

Recall that the main goal of the present work is to give an algorithm that allows to calculate $R(C)$ and, by corollary 2.5, we know that $R(C) = \{\mathbb{N}\backslash S \mid S \in \mathscr{M}(C)\}$. Summarizing the results obtained so far in this section, we have the following algorithm.

Algorithm 2. *Input: A finite set C of positive integers.*
Output: The set $R(C)$.

(1) *Applying algorithm 1 computes the set $\mathscr{M}(C)$.*

(2) *Return $\{\mathbb{N}\backslash S \mid S \in \mathscr{M}(C)\}$.*

Note that algorithm 1 computes $\theta_n(S)$ for all $S \in \mathscr{M}(C)$ with $n = \max(C) + 1$. Observe also that if $\theta_n(S) = (x_1, \ldots, x_{n-1})$ then $\mathrm{Ap}(S, n) = \{0, x_1, \ldots, x_{n-1}\}$ and furthermore for each $i \in \{1, \ldots, n-1\}$ there exists $q_i \in \mathbb{N}$ such that $x_i = q_i n + i$. Consequently, we have that $\mathbb{N}\backslash S = \{kn + i \mid i \in \{1, \ldots, n-1\},\ k \in \mathbb{N}$ and $0 \leqslant k \leqslant q_i - 1\}$.

Example 4.9. Let us compute the set $R(\{8, 11\})$.

(1) $\mathscr{M}(\{8, 11\}) = \{\langle 3, 7\rangle, \langle 5, 7, 9, 13\rangle\}$.

(2) $R(\{8, 11\}) = \{\mathbb{N}\backslash\langle 3, 7\rangle, \mathbb{N}\backslash\langle 5, 7, 9, 13\rangle\}$.
Algorithm 1 computes $\mathscr{M}(C)$ and gives us
$\theta_{12}(\langle 3, 7\rangle) = (13, 14, 3, 16, 17, 6, 7, 20, 9, 10, 23)$ and
$\theta_{12}(\langle 5, 7, 9, 13\rangle) = (13, 14, 15, 16, 5, 18, 7, 20, 9, 10, 23)$.
Consequently, from previous remark, we have that
$\mathbb{N}\backslash\langle 3, 7\rangle = \{1, 2, 4, 5, 8, 11\}$ and $\mathbb{N}\backslash\langle 5, 7, 9, 13\rangle = \{1, 2, 3, 4, 6, 8, 11\}$.

Observe that as a consequence of example 4.9 cardinality of the elements in $R(C)$ is not necessarily the same.

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5. A relaxation of the problem

Let C be a finite set of positive integers. Our aim in this section is to give an algorithm that calculates a subset A of positive integers which is minimal with the condition that if $x_1 + \cdots + x_n$ is a partition of an element in C, then at least a summand of this partition belongs to A. That is, we are interested in computing an element in $R(C)$. As this is equivalent to compute an element in $\mathscr{M}(C)$ the algorithm 1 solves it. Our goal will be to give an alternative algorithm to solve this problem.

The next results are proposition 17 and corollary 18 of [**12**], respectively.

Proposition 5.1. *Let S be a numerical semigroup and $C \subseteq \mathbb{N}\backslash S$. If there exists $h = \max\{x \in \mathbb{N}\backslash S \mid 2x \in S,\ c - x \notin S \text{ for all } c \in C\}$, then $S \cup \{h\}$ is a numerical semigroup and $(S \cup \{h\}) \cap C = \emptyset$.*

Proposition 5.2. *Let S be a numerical semigroup and $C \subseteq \mathbb{N}\backslash S$. Then $S \in \mathscr{M}(C)$ if and only if $\{x \in \mathbb{N}\backslash S \mid 2x \in S,\ c - x \notin S \text{ for all } c \in C\}$ is the empty set.*

Let S be a numerical semigroup such that $S \cap C = \emptyset$ (for example, we can consider $S = \{0, \max(C) + 1, \rightarrow\}$). We define recursively the following sequence of numerical semigroups:

· $S_0 = S$,
· $S_{n+1} = S_n \cup \{h(S_n)\}$, where

$$h(S_n) = \max\{x \in \mathbb{N}\backslash S_n \mid 2x \in S_n, c - x \notin S_n \text{ for all } c \in C\}.$$

From propositions 5.1 and 5.2, we have a finite chain of numerical semigroups $S_0 \subset S_1 \subset \cdots \subset S_k$ such that $S_i \in \mathscr{S}(C)$ for all $i \in \{0, \ldots, k\}$ and $\{x \in \mathbb{N}\backslash S_k \mid 2x \in S_k, c - x \notin S_k \text{ for all } c \in C\}$ is the empty set. As a consequence of the proposition 5.2, we get that $S_k \in \mathscr{M}(C)$. Thus we can enunciate the following algorithm.

Algorithm 3. *Input: A finite set C of positive integers.*
Output: An element in $\mathscr{M}(C)$.

(1) $S = \{0, \max(C) + 1, \rightarrow\}$.

(2) $A = \{x \in \mathbb{N}\backslash S \mid 2x \in S,\ c - x \notin S \text{ for all } c \in C\}$.

(3) *If $A = \emptyset$ returns S.*

(4) $S = S \cup \{\max(A)\}$ *and go to Step (1).*

Next, we give an example that illustrates the previous algorithm.

Example 5.3. Let us compute an element in $\mathscr{M}(\{8, 11\})$

. $S = \{0, 12, \rightarrow\}$.

. $A = \{6, 7, 9, 10\}$.

. $S = \{0, 10, 12, \rightarrow\}$.

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. $S = \{0, 9, 10, 12, \rightarrow\}$.

. $S = \{0, 7, 9, 10, 12, \rightarrow\}$.

. $S = \{0, 6, 7, 9, 10, 12, \rightarrow\}$.

. $S = \{0, 3, 6, 7, 9, 10, 12, \rightarrow\}$.

. $A = \emptyset$.

. $S = \{0, 3, 6, 7, 9, 10, 12, \rightarrow\} = \langle 3, 7\rangle \in \mathscr{M}(\{8, 11\})$.

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