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A NOTE ON THE ASYMPTOTIC BEHAVIOR OF THE HEIGHT FOR A BIRTH-AND-DEATH PROCESS

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Recently, the asymptotic mean value of the height for a birth-and-death process is given in Videla [Videla, L.A. (2020)]. We consider the asymptotic variance of the height in the case when the number of states tends to infinity. Further, we prove that the heights exhibit a cutoff phenomenon and that the normalized height converges to a degenerate distribution.

Keywords: birth-and-death process, cutoff, height function distribution

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Birth-and-death process is a continuous-time Markov chain, which plays an important role in stochastic processes and queuing theory (see [3,11]). Here, we consider a special birth-and-death process which is related to the mean-field model and the Anick-Mitra-Sondhi model. Let $\{X_t, t \geq 0\}$ be the birth-and-death process with state space $E = \{0, 1, 2, \ldots, N\}$ and the following conservative Q-matrix

$$Q = \begin{pmatrix} -N\nu & N\nu & 0 & \cdots & 0 & 0 & 0 \\ \mu & -\mu-(N-1)\nu & (N-1)\nu & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (N-1)\mu & -(N-1)\mu-\nu & \nu \\ 0 & 0 & 0 & \cdots & 0 & N\mu & -N\mu \end{pmatrix}, \tag{1}$$

where $\mu, \nu > 0$. Recall that a Q-matrix is called conservative, if its row summation is zero.

Let $\rho = \nu/\mu$. Clearly, the chain $\{X_t : t \geq 0\}$ is ergodic and has a stationary distribution

$$\pi_k := \frac{1}{(1+\rho)^N}\binom{N}{k}\rho^k, \quad k \in E. \tag{2}$$

Note that the transition probability matrix of its jump chain is given by

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \dfrac{1}{1+(N-1)\rho} & 0 & \dfrac{(N-1)\rho}{1+(N-1)\rho} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{N-1}{N-1+\rho} & 0 & \dfrac{\rho}{N-1+\rho} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \quad \textbf{(3)}$$

The process $\{X_t, t \geq 0\}$ has been studied in statistical physics as a special mean-field model on a complete graph (see [4–9,13]), which gives the rate of convergence to a stationary distribution, and as the Anick-Mitra-Sondhi model (see [1,10]) which is related to the data-handling system with multiple sources. When $N = 2$ and $N = 3$, $\{X_t, t \geq 0\}$ is also considered as a genomic model in [8].

The present paper focuses on the *height* of $\{X_t : t \geq 0\}$. Write $\{t : X_t > 0\} = \bigcup_{i=1}^{\infty}[\tau_i, \eta_i)$, where $\{[\tau_i, \eta_i), i \in \mathbb{N}\}$ is the family of maximal disjoint random time intervals such that $X_t > 0$ on every interval $[\tau_i, \eta_i)$. We consider the random variables

$$H_N^{(i)} := \max\{X_t, t \in [\tau_i, \eta_i)\},$$

the possible values of $H_N^{(i)}$ may be listed as $1, 2, \ldots, N$. By definition of $H_N^{(i)}$, $X_{\tau_i} = 1$, $H_N^{(i)}$ is the maximal value which X_t can reach, before return to 0. $H_N^{(i)}$ is called the height of $[\tau_i, \eta_i)$. $\{H_N^{(i)}, i \in \mathbb{N}\}$ is independent and identically distributed. This is due to the fact that X_t is Markov chain and $X_{\eta_i} = 0$. The distribution of $H_N^{(i)}$ does not depend on i, $H_N^{(i)}$ is reduced to H_N. H_N is the extremum variable, i.e., the maximum value. In queueing models that have finite buffer capacity, H_N represents the largest queue length in a busy period. It can be regarded as the maximal number of jobs that are served concurrently during a busy period in task-allocation problems, or the maximal number of occupied nodes in a special mean-field model on the complete graph which is related to a birth-death process $\{X_t : t \geq 0\}$ before all nodes are free. The asymptotic behavior of H_N is studied in the case when the number of states tends to infinity.

The asymptotic mean value of H_N is considered in [12]:

Theorem 1.1 [12, Thm,1]: *For $\rho \in (0, 1)$, let $\alpha := \alpha(\rho)$ be the unique solution of the equation $x^x(1-x)^{1-x} = \rho^x$, let*

$$f(\rho) = \begin{cases} \alpha, & 0 < \rho < 1, \\ 1, & \rho \geq 1. \end{cases} \quad \textbf{(4)}$$

Then,

$$\lim_{N\to\infty} \frac{\mathbb{E}(H_N)}{N} = f(\rho). \quad \textbf{(5)}$$

In the present paper, following the work of [12], we study the fluctuations of H_N. Firstly, we have the following asymptotic behavior of the variance of H_N.

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Theorem 1.2: *Let $f(\rho)$ is given by (4), then*

$$\lim_{N\to\infty} \frac{Var(H_N)}{N} = \frac{f^2(\rho)}{\rho} \tag{6}$$

and

$$\lim_{N\to\infty} \frac{H_N}{N} = f(\rho) \text{ in } L^2. \tag{7}$$

Secondly, we give a upper bound to the fluctuation of H_N as follows.

Theorem 1.3: *Suppose $\varphi(x)$ satisfies that $\lim_{x\to\infty} \log x/\varphi(x) = 0$, then*

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{H_N - \mathbb{E}(H_N)}{\varphi(N)} \le x\right) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \tag{8}$$

Remark 1.1: *Theorem 1.3 indicates that the fluctuation of H_N is upper bounded by $\log N$. In the case when $\varphi(N) = \sqrt{Var(H_N)}$, by (6), Eq. (8) shows that the sequence of r.v.'s $\{H_N : N \ge 1\}$ admits a cutoff phenomenon. For the cutoff phenomenon, see [2].*

2. PROOFS OF THEOREMS

Before giving proofs, we introduce some useful notations. For any $x \in \mathbb{R}$, denote by $[x]$ the integer part of x. For positive series $\{a_n : n \ge 1\}$ and $\{b_n : n \ge 1\}$, write $b_n = O(a_n)$ if there exists some constant $C > 0$ such that $b_n \le Ca_n$ for all large enough n; write $b_n = \Theta(a_n)$, if $b_n = O(a_n)$ and $a_n = O(b_n)$.

By the law of total probability and iteration, the distribution of H_N is given in [12] as follows:

Lemma 2.1 [12, Lemma 1]:

$$\mathbb{P}(H_N \ge k) = \frac{1}{\sum_{i=0}^{k-1} \frac{1}{\rho^i \binom{N-1}{i}}}, \quad k = 1, 2, \dots, N. \tag{9}$$

Let $r_{\rho,n}(i) := \rho^{-i}\binom{n-1}{i}^{-1}$, $i = 1, 2, \dots, n-1$. It is straightforward to check that $r_{\rho,n}(i)$ decreases strictly in i when $i < [n/(\rho+1)]$ and increases strictly in i otherwise. For $0 < \rho < 1$, it was proved in [12] that α is the unique solution of equation $x^x(1-x)^{1-x} = \rho^x$ and $\rho < \alpha < 1$. Let $h_n = [\alpha(n-1)]$, then by Stirling's formula, we have

$$r_{\rho,n}(h_n) = \Theta(\sqrt{n}). \tag{10}$$

Before giving proofs to the theorems, we shall give the following lemmas.

Lemma 2.2: *Let $\rho \in (0,1)$, then for constants $C_1 = 2/(\log\alpha - \log\rho(1-\alpha))$ and $C_2 = 3/(\log\alpha - \log\rho)$, we have*

$$r_{\rho,n}(h_n + [C_1 \log n]) \ge r_{\rho,n}(h_n) n^2, \tag{11}$$

$$r_{\rho,n}(h_n - [C_2 \log n]) \le r_{\rho,n}(h_n) n^{-3}. \tag{12}$$

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Proof: First, we prove (11). By the definition of $r_{\rho,n}(i)$,

$$\frac{r_{\rho,n}(h_n+[C_1\log n])}{r_{\rho,n}(h_n)} = \prod_{i=1}^{[C_1\log n]} \frac{r_{\rho,n}(h_n+i)}{r_{\rho,n}(h_n+i-1)} = \prod_{i=1}^{[C_1\log n]} \rho^{-1}\frac{h_n+i}{n-h_n-i-1}$$
$$\geq \left(\frac{\alpha}{\rho(1-\alpha)}\right)^{[C_1\log n]} \cdot \prod_{i=1}^{[C_1\log n]} \frac{1+\frac{i-1}{\alpha n}}{1-\frac{i}{(1-\alpha)(n-1)}}$$
$$\geq \left(\frac{\alpha}{\rho(1-\alpha)}\right)^{[C_1\log n]} = n^2.$$

Second, we obtain (12) as follows:

$$\frac{r_{\rho,n}(h_n-[C_2\log n])}{r_{\rho,n}(h_n)} = \prod_{i=1}^{[C_2\log n]} \frac{r_{\rho,n}(h_n-i)}{r_{\rho,n}(h_n-i+1)} = \prod_{i=1}^{[C_2\log n]} \rho\frac{n-h_n+i-2}{h_n-i+1}$$
$$\leq \left(\frac{\rho}{\alpha}\right)^{[C_2\log n]} \prod_{i=1}^{[C_2\log n]} \frac{1-\alpha+\frac{i}{n-1}}{1-\frac{i-1}{\alpha(n-1)}}$$
$$\leq \left(\frac{\rho}{\alpha}\right)^{[C_2\log n]} \leq n^{-3}.$$

■

Lemma 2.3: *For $0<\rho<1$, $C_3=(\alpha(3+\rho))/\rho^2$ and C_2 as given in Lemma 2.2, we have*

$$[\alpha N]-[C_2\log N]-C_3 \leq \mathbb{E}(H_N) \leq [\alpha N]+1 \tag{13}$$

for N large enough. For $\rho \geq 1$, we have

$$N-4 \leq \mathbb{E}(H_N) \leq N \tag{14}$$

for N large enough.

Proof: First, we prove the lower bound part of (13). For $\rho \in (0,1)$, by (12), $r_{\rho,n}(h_n - [C_2\log n]) \leq r_{\rho,n}(h_n)n^{-3}$. For $2 \leq i \leq h_n-[C_2\log n]$, we have

$$r_{\rho,n}(i) \leq \frac{2}{\rho^2(n-1)(n-2)},$$

then

$$\sum_{i=0}^{h_n-[C_2\log n]} r_{\rho,n}(i) \leq 1+\frac{1}{\rho(n-1)}+\frac{2}{\rho^2(n-1)(n-2)}\cdot(h_n-[C_2\log n])$$
$$\leq 1+\frac{1}{\rho(n-1)}+\frac{2}{\rho^2(n-2)}$$
$$\leq \frac{\rho(n-1)+1+3/\rho}{\rho(n-1)}.$$

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Hence,

$$\begin{aligned}\mathbb{P}(H_N \geq h_N - [C_2 \log N]) &= \frac{1}{\sum_{i=0}^{h_N - [C_2 \log N]} r_{\rho,N}(i)} \\ &\geq \frac{\rho(N-1)}{\rho(N-1) + 1 + 3/\rho} \\ &\geq 1 - \frac{3+\rho}{(N-1)\rho^2}. \qquad (15)\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathbb{E}(H_N) = \sum_{i=1}^{N} \mathbb{P}(H_N \geq i) &\geq \sum_{i=1}^{h_N - [C_2 \log N]} \mathbb{P}(H_N \geq i) \\ &\geq (h_N - [C_2 \log N])\mathbb{P}(H_N \geq h_N - [C_2 \log N]) \\ &\geq (h_N - [C_2 \log N])(1 - \frac{3+\rho}{(N-1)\rho^2}) \\ &\geq h_N - [C_2 \log N] - C_3. \qquad (16)\end{aligned}$$

Second, we prove the upper bound part of (13). For $i \geq h_n$, then $i > [(n-1)/(1+\rho)]$. Noticing the fact that $r_{\rho,n}(i)$ strictly increases in i, we have $r_{\rho,n}(i) \geq r_{\rho,n}(h_n)$. Hence, for $k \geq 1$, we have

$$\sum_{i=0}^{h_n+k-1} r_{\rho,n}(i) \geq \sum_{i=h_n}^{h_n+k-1} r_{\rho,n}(i) \geq k r_{\rho,n}(h_n).$$

So that

$$\mathbb{P}(H_N \geq h_N + k) \leq \frac{1}{k r_{\rho,N}(h_N)}, \qquad (17)$$

and

$$\begin{aligned}\mathbb{E}(H_N) &= \sum_{i=1}^{N} \mathbb{P}(H_N \geq i) \\ &\leq h_N + \sum_{i=h_N+1}^{N} \mathbb{P}(H_N \geq i) \\ &\leq h_N + \sum_{k=1}^{N-h_N} \frac{1}{k r_{\rho,N}(h_N)}.\end{aligned}$$

By the relation between harmonic series and natural logarithm, we have

$$\lim_{N\to\infty} \left[\sum_{k=1}^{N} \frac{1}{k} - \log N\right] = \gamma,$$

where γ is Euler–Mascheroni constant. By (10), we have

$$\sum_{k=1}^{N-h_N} \frac{1}{k r_{\rho,N}(h_N)} = O\left(\frac{\log N}{\sqrt{N}}\right).$$

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For N large enough, then

$$\mathbb{E}(H_N) \leq h_N + 1. \tag{18}$$

The inequality (13) follows from (16) and (18).

For $\rho \geq 1$, $r_{\rho,n}(i) \leq \binom{n-1}{i}^{-1}$, then

$$\sum_{i=0}^{n-2} r_{\rho,n}(i) \leq \sum_{i=0}^{n-2} \binom{n-1}{i}^{-1} \leq 1 + \frac{3}{n-1},$$

and then,

$$\mathbb{P}(H_N \geq N-1) = \frac{1}{\sum_{k=0}^{N-2} r_{\rho,N}(k)} \geq \frac{N-1}{N+2}. \tag{19}$$

By (19), we obtain (14) and finish the proof of the lemma as follows:

$$N \geq \mathbb{E}(H_N) = \sum_{i=1}^{N} \mathbb{P}(H_N \geq i) \geq (N-1)\mathbb{P}(H_N \geq N-1) \geq N-4.$$

■

Proof of Theorem 1.2: For $\rho \geq 1$, first we have

$$\begin{aligned}\mathrm{Var}(H_N) &= \sum_{i=1}^{N} (i - \mathbb{E}(H_N))^2 \mathbb{P}(H_N = i) \\ &\geq (1 - \mathbb{E}(H_N))^2 \mathbb{P}(H_N = 1),\end{aligned}$$

by (9) and (14), we have

$$\mathrm{Var}(H_N) \geq \frac{(N-3)^2}{1 + \rho(N-1)}. \tag{20}$$

Second, let $c = c(\rho)$ be the constant such that $r_{\rho,N}(i)N^2 \leq c/N$ for all $3 \leq i \leq N-4$. Using the fact that

$$\mathbb{P}(H_N = i) = \mathbb{P}(H_N \geq i) - \mathbb{P}(H_N \geq i+1) \leq r_{\rho,N}(i), \tag{21}$$

we have

$$\begin{aligned}\mathrm{Var}(H_N) &= \sum_{i=1}^{N} [i - \mathbb{E}(H_N)]^2 \mathbb{P}(H_N = i) \\ &\leq N^2 r_{\rho,N}(1) + N^2 r_{\rho,N}(2) + \sum_{i=3}^{N-4} N^2 r_{\rho,N}(i) + \sum_{i=N-3}^{N} (N-i)^2 \\ &\leq \frac{N}{\rho} + \frac{3c}{\rho^2} + 13.\end{aligned} \tag{22}$$

Then, Eq. (6) follows from (20) and (22).

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For $0 < \rho < 1$, first, by the lower bound given in (13), we have

$$\mathrm{Var}(H_N) \geq [1 - \mathbb{E}(H_N)]^2 \mathbb{P}(H_N = 1) \geq ([\alpha N] - [C_2 \log N] - C_3)^2 \cdot \frac{1}{1+\rho N}. \tag{23}$$

Second, by (13) and (21), we have

$$\begin{aligned}
&\sum_{i=1}^{h_N - C_2 \log N} [i - \mathbb{E}(H_N)]^2 \mathbb{P}(H_N = i) \\
&\quad \leq \alpha^2 N^2 r_{\rho,N}(1) + \alpha^2 N^2 r_{\rho,N}(2) + \sum_{i=3}^{h_N - C_2 \log N} \alpha^2 N^2 r_{\rho,N}(i) \\
&\quad \leq \frac{\alpha^2 N}{\rho} + \frac{3\alpha^2}{\rho^2} + \frac{13\alpha^2}{N}, \\
&\sum_{i=h_N - C_2 \log N}^{h_N + C_1 \log N} [i - \mathbb{E}(H_N)]^2 \mathbb{P}(H_N = i) \leq (C_1 + C_2)^2 (\log N)^2,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=h_N + C_1 \log N}^{N} [i - \mathbb{E}(H_N)]^2 \mathbb{P}(H_N = i) &\leq N^2 \sum_{i=h_N + C_1 \log N}^{N} \mathbb{P}(H_N = i) \\
&= N^2 \mathbb{P}(H_N \geq h_N + C_1 \log N) \\
&\leq N^2 \frac{1}{r_{\rho,N}(h_N + C_1 \log N)} \\
&\leq O\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

Note that last inequality follows from (10) and (11). Thus,

$$\begin{aligned}
\mathrm{Var}(H_N) &= \sum_{i=1}^{N} [i - \mathbb{E}(H_N)]^2 \mathbb{P}(H_N = i) \\
&\leq \frac{\alpha^2 N}{\rho} + \frac{3\alpha^2}{\rho^2} + \frac{13\alpha^2}{N} + (C_1 + C_2)^2 (\log N)^2 + O\left(\frac{1}{\sqrt{N}}\right).
\end{aligned} \tag{24}$$

Then, Eq. (6) follows from (23) and (24).
Finally, by properties of variance, we prove (7). Actually,

$$\mathbb{E}\left[\left(\frac{H_N}{N} - f(\rho)\right)^2\right] = \left[\frac{\mathrm{Var}(H_N)}{N^2} + \left(\frac{\mathbb{E}(H_N)}{N} - f(\rho)\right)^2\right].$$

Then, by (5) and (6), we have

$$\lim_{N \to \infty} \left[\frac{\mathrm{Var}(H_N)}{N^2} + \left(\frac{\mathbb{E}(H_N)}{N} - f(\rho)\right)^2\right] = 0.$$

Thus,

$$\frac{H_N}{N} \to f(\rho) \quad \text{as } N \to \infty$$

in L^2. ■

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Proof of Theorem 1.3: By lemma 2.3 and the condition that $\lim_{N\to\infty} \log N/\varphi(N) = 0$, for any $x > 0$ and N large enough, we have $x\varphi(N) \geq [C_2 \log N] + C_3$, then

$$\mathbb{P}(H_N \leq \mathbb{E}(H_N) + x\varphi(N)) \geq \mathbb{P}(H_N \leq h_N - [C_2 \log N] - C_3 + x\varphi(N)) \geq \mathbb{P}(H_N \leq h_N).$$

By (10) and (17), we have

$$\lim_{N\to\infty} \mathbb{P}(H_N \leq h_N) = 1.$$

then

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{H_N - \mathbb{E}(H_N)}{\varphi(N)} \leq x\right) = 1.$$

For any $x < 0$, for N large enough, $x\varphi(N) \leq -[C_2 \log N] - 1$, then

$$\mathbb{P}(H_N \leq \mathbb{E}(H_N) + x\varphi(N)) \leq \mathbb{P}(H_N \leq h_N + x\varphi(N)) \leq \mathbb{P}(H_N \leq h_N - [C_2 \log N] - 1).$$

By (15), we have

$$\lim_{N\to\infty} \mathbb{P}(H_N \leq h_N - [C_2 \log N] - 1) = 0,$$

then

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{H_N - \mathbb{E}(H_N)}{\varphi(N)} \leq x\right) = 0.$$

■

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