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On permuting cut with contraction

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Dedicated to Jim Lambek on the occasion of his 75th birthday

This paper presents a cut-elimination procedure for intuitionistic propositional logic in which cut is eliminated directly, without introducing the multiple-cut rule mix, and in which pushing cut above contraction is one of the reduction steps. The presentation of this procedure is preceded by an analysis of Gentzen's mix-elimination procedure, made in the perspective of permuting cut with contraction. We also show that in the absence of implication, pushing cut above contraction does not pose problems for directly eliminating cut.

1. Introduction

The structural rule of contraction poses special problems for cut elimination. It is because of contraction that Gentzen in the cut-elimination procedure of Gentzen (1935) replaced his rule

$$\frac{\Gamma \vdash \Theta, A \qquad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda} \text{ (Gentzen's cut)}$$

by a rule derived from cut, contraction and interchange, called *mix* (*Mischung* in German),

$$\frac{\Gamma \vdash \Theta \qquad \Delta \vdash \Lambda}{\Gamma, \Delta^* \vdash \Theta^*, \Lambda}$$

where Θ and Δ are sequences of formulae, each of which contains at least one instance of a formula A, called the *mix formula*, and Θ^* and Δ^* are obtained from Θ and Δ respectively, by deleting all occurrences of A. The fact that cut can be eliminated is then demonstrated by eliminating mix.

Mix also solves a problem involving the structural rule of interchange. Namely, we cannot permute (Gentzen's cut) with an interchange above the cut involving the cut formula A, because Gentzen required that the cut formula be the last formula in the sequence on the right-hand side of the left-hand premise and the first formula on the left-hand side of the right-hand premise. However, this problem is easily solved by replacing (Gentzen's cut) with

$$\frac{\Gamma \vdash \Theta_1, A, \Theta_2 \qquad \Delta_1, A, \Delta_2 \vdash \Lambda}{\Delta_1, \Gamma, \Delta_2 \vdash \Theta_1, \Lambda, \Theta_2}\ \text{(cut)}$$

which in the context of classical and intuitionistic logic does not represent an essential departure from the original systems of Gentzen. We can always obtain the effect of the rule (cut) with the help of (Gentzen's cut) and interchanges preceding and following this cut.

The special problem brought to cut elimination by contraction, which led to the introduction of mix, occurs when we have to permute a cut with contraction above the cut involving the cut formula A, that is, when we push a cut above such a contraction. If a figure with a topmost cut

$$(*) \qquad \frac{\Gamma \vdash \Theta, A \qquad \dfrac{A, A, \Delta \vdash \Lambda}{A, \Delta \vdash \Lambda}\ \text{contraction}}{\Gamma, \Delta \vdash \Theta, \Lambda}\ \text{cut}$$

is replaced by the figure

$$(**) \qquad \frac{\dfrac{\Gamma \vdash \Theta, A \qquad \dfrac{\Gamma \vdash \Theta, A \qquad A, A, \Delta \vdash \Lambda}{\Gamma, A, \Delta \vdash \Theta, \Lambda}\ \text{cut}}{\Gamma, \Gamma, \Delta \vdash \Theta, \Theta, \Lambda}\ \text{cut}}{\dfrac{\cdots}{\Gamma, \Delta \vdash \Theta, \Lambda}}\ \text{interchanges and contractions}$$

we have two cuts with the same cut formula A replacing a single cut with this cut formula. Of these two cuts, the upper cut has lower rank and can be eliminated by the induction hypothesis, but after this elimination is made, the remaining, lower, cut, which has now become topmost, need not have lower rank than the original cut.

On the other hand, if the following figure with a topmost mix

$$\frac{\Gamma \vdash \Theta \qquad \dfrac{A, A, \Delta \vdash \Lambda}{A, \Delta \vdash \Lambda}\ \text{contraction}}{\Gamma, \Delta^* \vdash \Theta^*, \Lambda}\ \text{mix}$$

is replaced by the figure

$$\frac{\Gamma \vdash \Theta \qquad A, A, \Delta \vdash \Lambda}{\Gamma, \Delta^* \vdash \Theta^*, \Lambda}\ \text{mix}$$

the new, single, application of mix is topmost and has lower rank than the original mix.

It is sometimes assumed that Gentzen's cut-elimination procedure is based on replacing

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(∗) by (∗∗) (see Carbone (1997, page 285)). When mix is reconstructed in terms of cut and other structural rules, Gentzen's procedure does indeed involve pushing cut above contraction, but only as part of more complicated steps, as we shall show in Section 2 below. It might even be said that in some of these steps cut is pushed below contraction, in the opposite direction.

When some forty years ago Lambek undertook in Lambek (1958) to eliminate cut in a contractionless sequent system, he did not need to bother with mix, and could eliminate cut directly. Of course, one also does not need to rely on mix in other contractionless systems of substructural logics that have been introduced since: namely, systems of BCK logic and linear logic.

In Szabo (1978), Szabo attempted to systematize the cut-elimination algorithm so that it can apply to a number of systems, with and without contraction. In this algorithm, when contraction is present, cut is permuted with contraction by passing from a figure like (∗) to a figure like (∗∗) (see Szabo (1978, Appendix C, C.19.3, page 234, C.38.3, page 239)). To demonstrate that the figure of (∗∗) is somehow simpler, Szabo introduced in Szabo (1978, pages 242–243) a measure of complexity counting the number of contractions above a cut. However, Szabo's measure fails to show that the lower cut in (∗∗) will have a smaller measure of complexity, as can be seen in a counterexample presented in detail in the third section of Borisavljević (1999)[†].

Actually, one cannot push a cut above both a contraction on the left and a contraction on the right, as the following simple counterexample shows. The figure

$$\dfrac{\text{contraction}\ \dfrac{\Gamma\vdash\Theta,A,A}{\Gamma\vdash\Theta,A} \qquad \dfrac{A,A,\Delta\vdash\Lambda}{A,\Delta\vdash\Lambda}\ \text{contraction}}{\Gamma,\Delta\vdash\Theta,\Lambda}\ \text{cut}$$

is not replaceable by a figure where all the cuts will be above all the contractions. Szabo does not eschew problems posed by this figure, though he requires in Szabo (1978, page 234) that the right rank of the cut be 1 if we want to diminish the left rank. This is because, after permuting the cut in the figure with the contraction on the left above the right-hand premise, we obtain

$$\dfrac{\dfrac{\text{contraction}\ \dfrac{\Gamma\vdash\Theta,A,A}{\Gamma\vdash\Theta,A} \qquad \dfrac{\text{contraction}\ \dfrac{\Gamma\vdash\Theta,A,A}{\Gamma\vdash\Theta,A} \qquad A,A,\Delta\vdash\Lambda}{\Gamma,A,\Delta\vdash\Theta,\Lambda}\ \text{cut}}{\dfrac{\Gamma,\Gamma,\Delta\vdash\Theta,\Theta,\Lambda}{\cdots}}\ \text{cut}}{\Gamma,\Delta\vdash\Theta,\Lambda}\ \text{interchanges and contractions}$$

where the upper cut may be of the lowest possible right rank. When we next permute this

† We are grateful to Andreja Prijatelj for pointing out to one of us a long time ago that Szabo's treatment of the matter is unsatisfactory.

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upper cut with contraction on the right over the left-hand premise, we obtain the figure

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A}\ \text{contraction} \qquad \cfrac{\cfrac{\cfrac{\cfrac{\Gamma \vdash \Theta, A, A \qquad A, A, \Delta \vdash \Lambda}{\Gamma, A, \Delta \vdash \Theta, A, \Lambda}\ \text{cut} \qquad A, A, \Delta \vdash \Lambda}{\Gamma, A, \Delta, A, \Delta \vdash \Theta, \Lambda, \Lambda}\ \text{cut}}{\cdots}}{\Gamma, A, \Delta \vdash \Theta, \Lambda}\ \text{interchanges and contractions}}{\Gamma, \Gamma, \Delta \vdash \Theta, \Theta, \Lambda}\ \text{cut}}{}}{\cdots}}{\Gamma, \Delta \vdash \Theta, \Lambda}\ \text{interchanges and contractions}
$$

where the lowest cut is in the same position as the initial one.

However, this does not exclude the possibility that Szabo's complexity measure could be replaced by another measure, presumably more complicated, which would show that the algorithm he envisaged would terminate if we have only contraction on the left in a system close to Gentzen's *LJ* of Gentzen (1935), that is, in a system for intuitionistic logic. (In Girard *et al.* (1992) something like this measure is computed, but the elimination of all cuts is not sought; in particular, some difficult cuts with contracted cut formulae are not eliminated.) These matters are very much tied to the particular formulation of a system. Zucker shows in Zucker (1974, Section 7) that if in a system for intuitionistic logic one replaces Gentzen's 'additive', that is, *lattice*, rules for disjunction by 'multiplicative' rules, a procedure such as envisaged by Szabo would not terminate.

Actually, it is not difficult to find such a measure in the absence of implication, as we show below in Section 4. The presence of implication poses special problems, for which we shall devise a cut-elimination procedure that involves permuting contractions with other rules, and not only with cut. Such permutations of contraction were studied in Kleene (1952), Zucker (1974), Minc (1996) and Dyckhoff and Pinto (1997), but we are not aware that they have been integrated before into a cut-elimination procedure. (Among these papers only Zucker's envisages permuting contraction with cut.)

The goal of this paper is to present a cut-elimination procedure for intuitionistic propositional logic, in which cut is directly eliminated, without passing via mix, and in which pushing cut above contraction, that is, passing from $(*)$ to $(**)$, is a reduction step. The cut-elimination procedure of Borisavljević (1999) also eliminates cut directly, and it involves pushing cut above contraction, but it is different and more entangled than the procedure we are going to present here. In a procedure envisaged by Carbone (1997), reminiscent of Curry's mix-elimination procedure (see Curry (1963, Chapter 5, D2)), cut should be directly eliminated, but without pushing it above contraction.

Although we assume that our procedure could be extended to the whole of intuitionistic predicate logic, we restrict ourselves to the propositional case, to make the exposition simpler. Anyway, our result is of more theoretical than practical interest. If one is just interested in eliminating cut, and does not care exactly how this is done, Gentzen's solution based on mix is simpler – it is probably optimal.

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However, our procedure may perhaps come in handy in studies of the complexity of proofs. It exhibits more clearly than Gentzen's procedure that contraction is the culprit for the hyperexponential growth of proofs in cut elimination.

Anyway, it seems worth knowing that cut can be eliminated by pushing it above contraction. If for no other reason than to block the inept criticism that would confuse 'I do not know how to eliminate cut by pushing it above contraction' with 'Cut cannot be so eliminated'.

Our procedure consists of three phases. In the first phase we push contractions below all rules, including cut, except for the rule of introduction of implication on the right. Proofs where this has been accomplished are called 'W-normal'. In the second phase, to reduce the rank, we push cuts above other rules, among which, because of W-normality, we do not have any more troublesome applications of contraction, like those in (∗). This phase essentially involves permuting cuts with cuts, which is a matter only implicitly and incompletely present in Gentzen's procedure (see the comments below (2^*) in Section 2, and cases (2.4), (3.7) and (3.8) in the proof of Theorem 6.1; see also the passage from (3P) to (3^*P) in Section 2, the end of Section 2 and the beginning of Section 7). However, this permuting is prominent in categorial proof theory: it corresponds to associativity of composition and to bifunctoriality equalities. In the third phase, we reduce cuts to cuts of lower degree. Then we re-enter the first phase of W-normalizing, and then repeat the second phase, and so on. The last phase will be a second phase where only cuts with axioms remain, which are then eliminated.

Before describing this procedure precisely, we consider in the next section (Section 2) what Gentzen's mix-elimination procedure has to say about permuting cut with contraction when the mix rule is reconstructed in terms of cut, contraction and interchange. In Section 3 we formally introduce our variant of Gentzen's sequent system *LJ* of intuitionistic propositional logic, which we call $\mathcal{G}$. The main difference between *LJ* and $\mathcal{G}$ is that in the latter we have rules like (cut) above, instead of (Gentzen's cut). In Section 4 we show by a simple argument that in implicationless $\mathcal{G}$ we can eliminate cut by freely pushing cuts above contractions. Perhaps, as Szabo supposed, such a free policy of pushing cut above contraction leads to cut elimination in $\mathcal{G}$ even in the presence of implication, but we have been unable to show that this is indeed the case.

In the last two sections we present our cut-elimination procedure. Section 5 is devoted to W-normalizing, and Section 6 to the remaining phases of the procedure. In Section 7 we make some concluding comments.

2. Cut elimination via mix elimination in *LJ*

Gentzen's mix rule is derivable in the presence of the structural rules of cut, contraction and interchange. However, for any mix of Gentzen's system *LJ* of Gentzen (1935)

$$\frac{\Gamma \vdash A \qquad \Delta \vdash \Lambda}{\Gamma, \Delta^* \vdash \Lambda}\ (\text{mix})$$

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where A happens to occur in Δ more than once, there is no unique way to reconstruct it in terms of cut, contraction and interchange. For example, the following instance of (mix)

$$\dfrac{B, C \vdash A \qquad A, A, D, A \vdash E}{B, C, D \vdash E}\ \text{mix}$$

can be reconstructed either as a number of cuts and interchanges followed by contractions:

$$\dfrac{\dfrac{B, C \vdash A \qquad \dfrac{\dfrac{\dfrac{B, C \vdash A \qquad \dfrac{\dfrac{\dfrac{B, C \vdash A \qquad A, A, D, A \vdash E}{B, C, A, D, A \vdash E}\ \mathit{LJ}\ \text{cut}}{\cdots}\ \text{interchanges}}{A, B, C, D, A \vdash E}}{B, C, B, C, D, A \vdash E}\ \mathit{LJ}\ \text{cut}}{\cdots}\ \text{interchanges}}{A, B, C, B, C, D \vdash E}}{B, C, B, C, B, C, D \vdash E}\ \mathit{LJ}\ \text{cut}}{\cdots}\ \text{interchanges and contractions}}{B, C, D \vdash E}$$

or as interchanges and contractions followed by a single cut:

$$\dfrac{B, C \vdash A \qquad \dfrac{\dfrac{\dfrac{A, A, D, A \vdash E}{A, A, A, D \vdash E}\ \text{interchange}}{\cdots}\ \text{contractions}}{A, D \vdash E}}{B, C, D \vdash E}\ \mathit{LJ}\ \text{cut}$$

or in many other ways intermediate between these two extremes, such as

$$\dfrac{\dfrac{B, C \vdash A \qquad \dfrac{\dfrac{\dfrac{B, C \vdash A \qquad \dfrac{A, A, D, A \vdash E}{A, D, A \vdash E}\ \text{contraction}}{B, C, D, A \vdash E}\ \mathit{LJ}\ \text{cut}}{\cdots}\ \text{interchanges}}{A, B, C, D \vdash E}}{B, C, B, C, D \vdash E}\ \mathit{LJ}\ \text{cut}}{\cdots}\ \text{interchanges and contractions}}{B, C, D \vdash E}$$

We call the first of these reconstructions, with many cuts, *polytomic*, while the second, with a single cut, will be *monotomic*. Note that in the polytomic reconstruction, and in

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the intermediate third reconstruction, the left-hand premise of mix appears more than once. To pass from such reconstructions to the mix reconstructed, we have to apply a contraction principle of higher level, which permits us to omit repetitions among the sequents that make the premises of a rule.

Note also that the polytomic reconstruction of a mix is not unique: one such reconstruction may be obtained from another by introducing interchanges and by permuting *LJ* cuts with other *LJ* cuts. The order of contractions in the bottom of the reconstruction is also not uniquely determined. It is possible to make this reconstruction unique by introducing an order among the rules involved in the reconstruction, the shortest way being to attack first the leftmost formula. However, there is something arbitrary in this order.

Whether Gentzen's mixes of *LJ* will be reconstructed polytomically, monotomically or in some other, intermediate, way is a matter of choice. This choice is of no consequence if the goal is just to eliminate cut by whatever means. However, if we are interested in describing the cut-elimination procedure exactly, and wish to reconstruct this procedure from the mix-elimination procedure, we will not end up with the same algorithm if we reconstruct mix always polytomically or always monotomically.

Let us now investigate when cut has to be pushed above contraction involving the cut formula in the uniform polytomic and uniform monotomic reconstructions; namely, in the reconstruction where mixes are always reconstructed polytomically and in the reconstruction where mixes are always reconstructed monotomically. We shall only consider these uniform reconstructions. (Note that passing from the monotomic to the polytomic reconstruction of a mix may itself be thought of as obtained by pushing cut above contraction.)

If the right rank of a mix is equal to 1, then this mix is just an *LJ* cut. So we only have to consider cases where the right rank of the mix is greater than 1 (see Gentzen (1935, Section III.3121)). The first interesting case for us is when we have

$$\text{(1)}\qquad \dfrac{\Gamma \vdash A \qquad \dfrac{A, A, \Delta \vdash \Lambda}{A, \Delta \vdash \Lambda}\ \text{contraction}}{\Gamma, \Delta^{*} \vdash \Lambda}\ \text{mix}$$

and there are n occurrences of A in Δ. Polytomically, (1) is reconstructed as

$$\text{(1P)}\qquad \dfrac{\dfrac{\Gamma \vdash A \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash A \qquad \dfrac{A, A, \Delta \vdash \Lambda}{A, \Delta \vdash \Lambda}\ \text{contraction}}{\Gamma, \Delta \vdash \Lambda}\ LJ\ \text{cut}}{\cdots}}{A, \Gamma, \ldots, \Gamma, \Delta^{*} \vdash \Lambda}\ \begin{matrix}\text{interchanges and } n-1\\ \text{applications of } LJ \text{ cut}\end{matrix}}{\Gamma, \Gamma, \ldots, \Gamma, \Delta^{*} \vdash \Lambda}\ LJ\ \text{cut}}{\dfrac{\cdots}{\Gamma, \Delta^{*} \vdash \Lambda}}\ \begin{matrix}\text{interchanges and}\\ \text{contractions}\end{matrix}$$

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and monotomically as

(1M)
$$\frac{\Gamma\vdash A \qquad \dfrac{\dfrac{\dfrac{A,A,\Delta\vdash\Lambda}{A,\Delta\vdash\Lambda}\;\text{contraction}}{\dots}\;\text{interchanges and contractions}}{A,\Delta^*\vdash\Lambda}}{\Gamma,\Delta^*\vdash\Lambda}\;LJ\text{ cut}$$

In Gentzen (1935, III.3.121.21), Gentzen transforms (1) into

(1*)
$$\frac{\Gamma\vdash A \qquad A,A,\Delta\vdash\Lambda}{\Gamma,\Delta^*\vdash\Lambda}\;\text{mix}$$

Polytomically, (1*) is reconstructed as

(1*P)
$$\dfrac{\dfrac{\Gamma\vdash A \qquad \dfrac{\dfrac{\Gamma\vdash A \qquad \dfrac{\dfrac{\dfrac{\Gamma\vdash A \qquad A,A,\Delta\vdash\Lambda}{\Gamma,A,\Delta\vdash\Lambda}\;LJ\text{ cut}}{\dots}\;\text{interchanges}}{A,\Gamma,\Delta\vdash\Lambda}}{\Gamma,\Gamma,\Delta\vdash\Lambda}\;LJ\text{ cut}}{\dfrac{\dots}{A,\Gamma,\Gamma,\dots,\Gamma,\Delta^*\vdash\Lambda}}\;\text{interchanges and } n-1 \text{ applications of } LJ\text{ cut}}{\Gamma,\Gamma,\Gamma,\dots,\Gamma,\Delta^*\vdash\Lambda}\;LJ\text{ cut}}{\dfrac{\dots}{\Gamma,\Delta^*\vdash\Lambda}}\;\text{interchanges and contractions}$$

Transforming (1P) into (1*P) involves pushing cut above contraction. The monotomic reconstruction (1*M) of (1*) is obtained from (1M) by permuting interchanges with contractions, and transforming (1M) into (1*M) does *not* involve pushing cut above contraction.

The next interesting case is when we have

(2)
$$\frac{\Gamma\vdash A \qquad \dfrac{\Psi,\Delta\vdash\Lambda_1}{A,\Delta\vdash\Lambda_2}\;\text{R}}{\Gamma,\Delta^*\vdash\Lambda_2}\;\text{mix}$$

where A does not occur in Γ and either R is introduction of $\wedge$ on the left, in which case A is of the form $A_1\wedge A_2$, while Ψ is either A_1 or A_2, and Λ_1 is equal to Λ_2, or R is introduction of $\neg$ on the left, in which case A is of the form $\neg A_1$, while Ψ and Λ_2 are

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empty and Λ_1 is A_1. Polytomically, (2) is reconstructed as

(2P)
$$\dfrac{\Gamma\vdash A \qquad \dfrac{\dfrac{\dfrac{\Gamma\vdash A \qquad \dfrac{\Psi,\Delta\vdash\Lambda_1}{A,\Delta\vdash\Lambda_2}\,\mathrm{R}}{\Gamma,\Delta\vdash\Lambda_2}\,LJ\ \text{cut}}{\cdots}\,\text{interchanges and } LJ \text{ cuts}}{A,\Gamma,\ldots,\Gamma,\Delta^*\vdash\Lambda_2}}{\dfrac{\dfrac{\Gamma,\Gamma,\ldots,\Gamma,\Delta^*\vdash\Lambda_2}{\cdots}\,\text{interchanges and contractions}}{\Gamma,\Delta^*\vdash\Lambda_2}}\,LJ\ \text{cut}$$

and monotomically as

(2M)
$$\dfrac{\Gamma\vdash A \qquad \dfrac{\dfrac{\dfrac{\Psi,\Delta\vdash\Lambda_1}{A,\Delta\vdash\Lambda_2}\,\mathrm{R}}{\cdots}\,\text{interchanges and contractions}}{A,\Delta^*\vdash\Lambda_2}}{\Gamma,\Delta^*\vdash\Lambda_2}\,LJ\ \text{cut}$$

In Gentzen (1935, III.3.121.22 and 3.121.222), Gentzen transforms (2) into

(2^*)
$$\dfrac{\Gamma\vdash A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma\vdash A \qquad \Psi,\Delta\vdash\Lambda_1}{\Gamma,\Psi^*,\Delta^*\vdash\Lambda_1}\,\text{mix}}{\cdots}\,\text{thinning or interchanges}}{\Psi,\Gamma,\Delta^*\vdash\Lambda_1}}{A,\Gamma,\Delta^*\vdash\Lambda_2}\,\mathrm{R}}{\dfrac{\dfrac{\Gamma,\Gamma,\Delta^*\vdash\Lambda_2}{\cdots}\,\text{interchanges and contractions}}{\Gamma,\Delta^*\vdash\Lambda_2}}\,\text{mix (that is, } LJ \text{ cut)}$$

When the upper mix of (2^*) is reconstructed polytomically, the result of the reconstruction being called (2^*P), transforming (2P) into (2^*P) involves permuting *LJ* cuts with *LJ* cuts and with R. (This permuting of cut with cut corresponds to (3.8) of the proof of Theorem 6.1 below, and not to (2.4) and (3.7).) It also involves pushing contraction above cut, but it does *not* involve pushing cut above contraction.

When, on the other hand, the upper mix of (2^*) is reconstructed monotomically, the result of the reconstruction being called (2^*M), transforming (2M) into (2^*M) involves, among other things, pushing cut above contraction.

The final interesting case is when we have

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\Delta \vdash B \qquad C, \Theta \vdash \Lambda}{B \to C, \Delta, \Theta \vdash \Lambda}\,{\to}\mathrm{L}}{\Gamma, (B \to C)^*, \Delta^*, \Theta^* \vdash \Lambda}\,\text{mix} \tag{3}$$

where A does not occur in Γ, while $(B \to C)^*$ stands either for the empty sequence or for $B \to C$, depending on whether A is $B \to C$ or not, and A occurs in both Δ and Θ. Polytomically, (3) is reconstructed as

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \dfrac{\dfrac{\Gamma \vdash A \qquad \dfrac{\dfrac{\dfrac{\Delta \vdash B \qquad C, \Theta \vdash \Lambda}{B{\to}C, \Delta, \Theta \vdash \Lambda}\,{\to}\mathrm{L}}{\ldots}\,\text{interchanges}}{A, A, \ldots, A, B{\to}C, \Delta^*, \Theta^* \vdash \Lambda}}{\Gamma, A, \ldots, A, B{\to}C, \Delta^*, \Theta^* \vdash \Lambda}\,LJ\text{ cut}}{\dfrac{\ldots}{A, \Gamma, \ldots, \Gamma, (B{\to}C)^*, \Delta^*, \Theta^* \vdash \Lambda}\,\text{interchanges and } LJ \text{ cuts}}}{\Gamma, \Gamma, \ldots, \Gamma, (B{\to}C)^*, \Delta^*, \Theta^* \vdash \Lambda}\,LJ\text{ cut}}{\dfrac{\ldots}{\Gamma, (B{\to}C)^*, \Delta^*, \Theta^* \vdash \Lambda}\,\text{interchanges and contractions}} \tag{3P}$$

and monotomically as

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\dfrac{\dfrac{\Delta \vdash B \qquad C, \Theta \vdash \Lambda}{B{\to}C, \Delta, \Theta \vdash \Lambda}\,{\to}\mathrm{L}}{\ldots}\,\text{interchanges and contractions}}{A, (B{\to}C)^*, \Delta^*, \Theta^* \vdash \Lambda}}{\Gamma, (B{\to}C)^*, \Delta^*, \Theta^* \vdash \Lambda}\,LJ\text{ cut} \tag{3M}$$

In Gentzen (1935, III.3.121.233.1), Gentzen transforms (3) into

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta^* \vdash B}\,\text{mix} \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash A \qquad C, \Theta \vdash \Lambda}{\Gamma, C^*, \Theta^* \vdash \Lambda}\,\text{mix}}{\ldots}\,\text{thinning or interchanges}}{C, \Gamma, \Theta^* \vdash \Lambda}}{B{\to}C, \Gamma, \Delta^*, \Gamma, \Theta^* \vdash \Lambda}\,({\to}\mathrm{L}) \tag{3*}$$

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which if $B \to C$ is A, is continued by

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash A \qquad B{\to}C, \Gamma, \Delta^*, \Gamma, \Theta^* \vdash \Lambda}{\Gamma, \Gamma, \Delta^*, \Gamma, \Theta^* \vdash \Lambda}\ \text{mix (that is, } LJ \text{ cut)}}{\dots}\ \text{interchanges and contractions}}{\Gamma, \Delta^*, \Theta^* \vdash \Lambda}$$

and if $B \to C$ is not A, is continued by

$$\cfrac{\cfrac{B{\to}C, \Gamma, \Delta^*, \Gamma, \Theta^* \vdash \Lambda}{\dots}\ \text{interchanges and contractions}}{\Gamma, B{\to}C, \Delta^*, \Theta^* \vdash \Lambda}$$

When the two top mixes of (3*) are reconstructed polytomically, the result of the reconstruction being called (3*P), transforming (3P) into (3*P), which is analogous to the transformation of (2P) into (2*P), does *not* involve pushing cut above contraction. When, on the other hand, the two top mixes of (3*) are reconstructed monotomically, the result of the reconstruction being called (3*M), transforming (3M) into (3*M) involves pushing cut above contraction.

So we can conclude that in the polytomic reconstruction pushing cut above contraction is involved in the first case, while in the monotomic reconstruction in the second and third case. If in the first case we favour the monotomic reconstruction, while non-uniformly, in the second and third case we favour the polytomic reconstruction, we shall never have to push cut above contraction in order to perform the steps of Gentzen's procedure, but we shall need this pushing to pass from a monotomic reconstruction to the corresponding polytomic reconstruction.

It is worth remarking that in the polytomic reconstruction, in the second and third case not only do we not push cut above contraction, but, in fact, we push contraction above cut.

In the second case and in the third case when the mix formula A is $B \to C$, let us call the lowest mix in (2*) and (3*), which is in fact an *LJ* cut, the *critical mix* of the transformation. The specificity of the critical mix is that it is the lowest mix in the figure and that its right rank is 1. In the monotomic reconstruction, the critical mix, that is, *LJ* cut, originates from one of the two cuts obtained by pushing a cut above a contraction. In this pushing, which is the relativization to *LJ* of the transformation of (∗) into (∗∗) of Section 1, we must ensure that the critical mix originates in the lower cut of (∗∗). Otherwise, we would need also to permute cut with cut to ensure that the critical mix ends up as the lowest cut.

3. A sequent system for intuitionistic propositional logic

Our propositional language will have the propositional constant $\bot$ and the binary connectives $\wedge$, $\vee$ and $\to$. We use $A, B, C, \dots, A_1, \dots$ as schematic letters for formulae and $\Gamma, \Delta, \Theta, \dots, \Gamma_1, \dots$ as schematic letters for finite, possibly empty, sequences of formulae. As usual, $\neg A$ can be defined as $A \to \bot$. Sequents are expressions of the form $\Gamma \vdash A$.

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The sequent system $\mathscr{G}$ has as postulates the postulates of the sequent system $\mathscr{G}^r$ below with all superscripts omitted. The postulates of $\mathscr{G}$ are named by the same names as in $\mathscr{G}^r$ except that the superscript r is always omitted. (It would be wasteful to write these postulates twice, once for $\mathscr{G}$ without superscripts, and once again, just a little bit further down, for $\mathscr{G}^r$ with the superscripts added.)

In Gentzen's original rules of Gentzen (1935) the sequence Θ in the postulates of $\mathscr{G}$ is always empty, both in the structural rules and in the rules for connectives. Our, more general, rules are derivable from Gentzen's rules in the presence of the structural rule of interchange. We have already replaced (Gentzen's cut) by the present form of cut in Section 1, in order to be able to permute cut with the structural rule of interchange. We likewise replace the other rules of Gentzen by the present more general forms in order to be able to permute contraction with other rules, and, also, for the sake of uniformity.

As usual, we call an application of (cut) in a proof of $\mathscr{G}$ *a cut*. With this form of speech it should be kept in mind that our cuts are applications of the rule (cut) of $\mathscr{G}$, and not of (Gentzen's cut).

The *degree* of a cut is, as usual, the number of binary connectives in the cut formula A. The *degree of a proof* in $\mathscr{G}$ is the maximal degree among the degrees of the cuts in this proof. A proof of degree 0 can have only cuts whose cut formulae are atomic. A proof without cuts has degree 0.

To compute the rank of a cut, we introduce an auxiliary sequent system, which we call $\mathscr{G}^r$. In the sequents of $\mathscr{G}^r$ we do not have ordinary formulae, but indexed formulae A^n where A is an ordinary formula and the *rank index* $n \geqslant 1$ is a natural number. To formulate the postulates of $\mathscr{G}^r$, we introduce the following conventions. If Γ^i is a sequence of indexed formulae, then Γ^{i+1} is the sequence of indexed formulae obtained by increasing by 1 every rank index in Γ^i. (Note that here the subscript i does not stand for a single natural number: it is a schema for any natural number in the rank indices of the sequence Γ^i.) We use Γ^i and Γ^j for sequences of indexed formulae that may differ only in the rank indices. When for Γ^i and Γ^j we write $i \leqslant j$, we mean that if in Γ^i we find A^n and at the same place in Γ^j we find A^m, then $n \leqslant m$. Starting from Γ^i and Γ^j, we obtain the sequence of indexed formulae $\Gamma^{max(i,j)+1}$ as follows: if in Γ^i we find A^n and at the same place in Γ^j we find A^m, then at the same place in $\Gamma^{max(i,j)+1}$ we put $A^{max(n,m)+1}$.

We can now give the postulates of $\mathscr{G}^r$, which are just indexed variants of the postulates of $\mathscr{G}$:

axioms

$(1^r)\quad A^1 \vdash A^1$

$(\bot^r)\quad \bot^1 \vdash A^1$

structural rules

$$(\mathrm{C}^r)\quad \frac{\Delta^i, A^n, B^m, \Gamma^j \vdash C^k}{\Delta^{i+1}, B^{m+1}, A^{n+1}, \Gamma^{j+1} \vdash C^{k+1}}$$

$$(\mathrm{W}^r)\quad \frac{\Theta^i, A^n, A^m, \Gamma^j \vdash C^k}{\Theta^{i+1}, A^{max(n,m)+1}, \Gamma^{j+1} \vdash C^{k+1}}$$

$$(\mathrm{K}^r)\quad \frac{\Theta^i, \Gamma^j \vdash C^k}{\Theta^{i+1}, A^1, \Gamma^{j+1} \vdash C^{k+1}}$$

$$(\mathrm{cut}^r)\quad \frac{\Delta^i \vdash A^n \qquad \Theta^j, A^m, \Gamma^h \vdash C^k}{\Theta^{j+1}, \Delta^{i+1}, \Gamma^{h+1} \vdash C^{k+1}}$$

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rules for connectives

$$(\wedge\mathrm{L}^r)\ \frac{\Theta^i, A^n, \Gamma^j \vdash C^k}{\Theta^{i+1}, A \wedge B^1, \Gamma^{j+1} \vdash C^{k+1}} \qquad \frac{\Theta^i, B^n, \Gamma^j \vdash C^k}{\Theta^{i+1}, A \wedge B^1, \Gamma^{j+1} \vdash C^{k+1}}$$

$$(\wedge\mathrm{R}^r)\ \frac{\Gamma^i \vdash A^n \qquad \Gamma^j \vdash B^m}{\Gamma^{max(i,j)+1} \vdash A \wedge B^1}$$

$$(\vee\mathrm{L}^r)\ \frac{\Theta^i, A^n, \Gamma^j \vdash C^k \qquad \Theta^l, B^m, \Gamma^h \vdash C^q}{\Theta^{max(i,l)+1}, A \vee B^1, \Gamma^{max(j,h)+1} \vdash C^{max(k,q)+1}}$$

$$(\vee\mathrm{R}^r)\ \frac{\Gamma^i \vdash A^n}{\Gamma^{i+1} \vdash A \vee B^1} \qquad \frac{\Gamma^i \vdash B^n}{\Gamma^{i+1} \vdash A \vee B^1}$$

$$(\rightarrow \mathrm{L}^r)\ \frac{\Delta^i \vdash A^n \qquad \Theta^j, B^m, \Gamma^h \vdash C^k}{\Theta^{j+1}, \Delta^{i+1}, A \rightarrow B^1, \Gamma^{h+1} \vdash C^{k+1}} \qquad (\rightarrow \mathrm{R}^r)\ \frac{A^n, \Gamma^i \vdash B^m}{\Gamma^{i+1} \vdash A \rightarrow B^1}$$

Take a cut in $\mathcal{G}$, and perform again in $\mathcal{G}^r$ the proofs of the two premises $\Delta \vdash A$ and $\Theta, A, \Gamma \vdash C$ of this cut exactly as they are done in $\mathcal{G}$, except that in $\mathcal{G}^r$ rank indices are taken into account. Let these two proofs in $\mathcal{G}^r$ prove $\Delta^i \vdash A^n$ and $\Theta^j, A^m, \Gamma^h \vdash C^k$. Then the *rank* of our cut is $n + m$. The *left rank* of this cut is n, and the *right rank* is m.

A cut in $\mathcal{G}$ is *topmost* iff there are no cuts above it. Gentzen computed rank for topmost mixes only, that is, those above which there are no mixes, and his notion of rank coincides with our notion of rank for topmost cuts. However, for our cut-elimination procedure (see Section 6 below) we need the more general notion of rank that we have just introduced, which applies to any cut, and not only topmost cuts.

4. Cut elimination in implicationless $\mathcal{G}$

For every proof in $\mathcal{G}$ in which the connective of implication $\rightarrow$ does not occur, there is a simple procedure of cut elimination, which eliminates cut directly, not via mix, and involves pushing cut above contraction. To describe this procedure, we introduce the following auxiliary implicationless sequent system called $\mathcal{G}^z$. (The index z stands for 'Zucker', from whose indexing of sequents in Zucker (1974), the indices of $\mathcal{G}^z$ are derived; a measure analogous to these indices may be found in Borisavljević (1999).)

On the left-hand sides of the sequents of $\mathcal{G}^z$ we do not have ordinary formulae, but indexed formulae A^α where A is an ordinary implicationless formula and the *contraction index* $\alpha \geqslant 1$ is a natural number. To formulate the postulates of $\mathcal{G}^z$, we use conventions analogous to those we used for $\mathcal{G}^r$ in the preceding section.

The postulates of $\mathcal{G}^z$ are the following indexed variants of the postulates of $\mathcal{G}$ minus

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the rules for implication:

axioms

(1^z) $A^1 \vdash A$

$(\bot^z)$ $\bot^1 \vdash A$

structural rules

$$(\mathrm{C}^z)\ \frac{\Delta^i, A^\alpha, B^\beta, \Gamma^j \vdash C}{\Delta^i, B^\beta, A^\alpha, \Gamma^j \vdash C}$$

$$(\mathrm{W}^z)\ \frac{\Theta^i, A^\alpha, A^\beta, \Gamma^j \vdash C}{\Theta^i, A^{\alpha+\beta}, \Gamma^j \vdash C}$$

$$(\mathrm{K}^z)\ \frac{\Theta^i, \Gamma^j \vdash C}{\Theta^i, A^1, \Gamma^j \vdash C}$$

$$(\mathrm{cut}^z)\ \frac{\Delta^i \vdash A \qquad \Theta^j, A^\alpha, \Gamma^h \vdash C}{\Theta^j, \Delta^{i\alpha}, \Gamma^h \vdash C}$$

rules for connectives

$$(\wedge\mathrm{L}^z)\ \frac{\Theta^i, A^\alpha, \Gamma^j \vdash C}{\Theta^i, A \wedge B^\alpha, \Gamma^j \vdash C} \qquad \frac{\Theta^i, B^\alpha, \Gamma^j \vdash C}{\Theta^i, A \wedge B^\alpha, \Gamma^j \vdash C}$$

$$(\wedge\mathrm{R}^z)\ \frac{\Gamma^i \vdash A \qquad \Gamma^j \vdash B}{\Gamma^{max(i,j)} \vdash A \wedge B}$$

$$(\vee\mathrm{L}^z)\ \frac{\Theta^i, A^\alpha, \Gamma^j \vdash C \qquad \Theta^l, B^\beta, \Gamma^h \vdash C}{\Theta^{max(i,l)}, A \vee B^{max(\alpha,\beta)}, \Gamma^{max(j,h)} \vdash C}$$

$$(\vee\mathrm{R}^z)\ \frac{\Gamma^i \vdash A}{\Gamma^i \vdash A \vee B} \qquad \frac{\Gamma^i \vdash B}{\Gamma^i \vdash A \vee B}$$

As we did for applications of (cut) in $\mathcal{G}$, we now use the word *cut* to mean application of (cut^z) in $\mathcal{G}^z$.

We shall prove the following theorem by eliminating cut directly and by pushing cut above contractions.

Theorem 4.1. Every proof of $\Pi^i \vdash C$ in $\mathcal{G}^z$ can be reduced to a cut-free proof of $\Pi^j \vdash C$ where $j \leqslant i$.

Proof. We proceed by an induction on triples $\langle d, z, r\rangle$, lexicographically ordered, where d is the degree of a cut, z is the contraction index of the cut formula in the right-hand premise of (cut^z) and r is the rank of the cut (rank is defined for $\mathcal{G}^z$ as it is defined for $\mathcal{G}$, via $\mathcal{G}^r$). We show that every proof of $\Pi^i \vdash C$ with a single cut, which is the last rule of the proof, can be reduced to a cut-free proof of $\Pi^j \vdash C$ where $j \leqslant i$.

(1) Suppose the rank of our cut is 2. Then our cut is covered by at least one of the following cases:

(1.1)
$$\frac{A^1 \vdash A \qquad \overset{\pi}{\Theta^j, A^\alpha, \Gamma^h \vdash C}}{\Theta^j, A^\alpha, \Gamma^h \vdash C}\ \mathrm{cut}^z$$

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We replace this proof by the cut-free proof π of the right-hand premise of cut^z.

$$\frac{\perp^1 \vdash A \qquad \overset{\pi}{\Theta^j, A^\alpha, \Gamma^h \vdash C}}{\Theta^j, \perp^\alpha, \Gamma^h \vdash C}\ \mathrm{cut}^z \tag{1.2}$$

We replace this proof by

$$\frac{\dfrac{\perp^1 \vdash C}{\cdots}\ \begin{array}{l}\text{applications}\\ \text{of } (\mathrm{K}^z)\end{array}}{\Theta^1, \perp^1, \Gamma^1 \vdash C}$$

$$\frac{\overset{\pi}{\Gamma^i \vdash C} \qquad C^1 \vdash C}{\Gamma^i \vdash C}\ \mathrm{cut}^z \tag{1.3}$$

We replace this proof by the cut-free proof π of the left-hand premise of cut^z.

$$\frac{\overset{\pi_1}{\Gamma^i \vdash A} \qquad \dfrac{\overset{\pi_2}{\Theta^h, \Delta^j \vdash C}}{\Theta^h, A^1, \Delta^j \vdash C}\ \mathrm{K}^z}{\Theta^h, \Gamma^i, \Delta^j \vdash C}\ \mathrm{cut}^z \tag{1.4}$$

We replace this proof by the following proof

$$\frac{\dfrac{\overset{\pi_2}{\Theta^h, \Delta^j \vdash C}}{\cdots}\ \begin{array}{l}\text{applications}\\ \text{of } (\mathrm{K}^z)\end{array}}{\Theta^h, \Gamma^1, \Delta^j \vdash C}$$

$$\frac{\dfrac{\overset{\pi_1}{\Gamma^i \vdash A} \qquad \overset{\pi_2}{\Gamma^j \vdash B}}{\Gamma^{max(i,j)} \vdash A \wedge B}\ \wedge\mathrm{R}^z \qquad \dfrac{\overset{\pi}{\Theta^h, A^\alpha, \Delta^l \vdash C}}{\Theta^h, A \wedge B^\alpha, \Delta^l \vdash C}\ \wedge\mathrm{L}^z}{\Theta^h, \Gamma^{max(i,j)\alpha}, \Delta^l \vdash C}\ \mathrm{cut}^z \tag{1.5}$$

We replace this proof by

$$\frac{\overset{\pi_1}{\Gamma^i \vdash A} \qquad \overset{\pi}{\Theta^h, A^\alpha, \Delta^l \vdash C}}{\Theta^h, \Gamma^{i\alpha}, \Delta^l \vdash C}\ \mathrm{cut}^z$$

We proceed analogously when A^α is replaced by B^α.

$$(1.6)\qquad \cfrac{\cfrac{\overset{\pi}{\Gamma^i \vdash A}}{\Gamma^i \vdash A \vee B}\,\vee\mathrm{R}^z \qquad \cfrac{\overset{\pi_1}{\Theta^h, A^\alpha, \Delta^l \vdash C} \qquad \overset{\pi_2}{\Theta^u, B^\beta, \Delta^v \vdash C}}{\Theta^{max(h,u)}, A \vee B^{max(\alpha,\beta)}, \Delta^{max(l,v)} \vdash C}\,\vee\mathrm{L}^z}{\Theta^{max(h,u)}, \Gamma^{i\,max(\alpha,\beta)}, \Delta^{max(l,v)} \vdash C}\,\mathrm{cut}^z$$

We replace this proof by

$$\cfrac{\overset{\pi}{\Gamma^i \vdash A} \qquad \overset{\pi_1}{\Theta^h, A^\alpha, \Delta^l \vdash C}}{\Theta^h, \Gamma^{i\alpha}, \Delta^l \vdash C}\,\mathrm{cut}^z$$

We proceed analogously when π ends with $\Gamma^i \vdash B$.

(2) Suppose the left rank of our cut is greater than 1. Then we have the following cases.

$$(2.1)\qquad \cfrac{\cfrac{\overset{\pi}{\Delta^i \vdash D}}{\Phi^k \vdash D}\,\mathrm{R} \qquad \overset{\pi_2}{\Theta^j, D^\gamma, \Gamma^h \vdash C}}{\Theta^j, \Phi^{k\gamma}, \Gamma^h \vdash C}\,\mathrm{cut}^z$$

where R is C^z, W^z, K^z or $\wedge\mathrm{L}^z$. We replace this proof by

$$\cfrac{\cfrac{\overset{\pi}{\Delta^i \vdash D} \qquad \Theta^j, D^\gamma, \Gamma^h \vdash C}{\Theta^j, \Delta^{i\gamma}, \Gamma^h \vdash C}\,\mathrm{cut}^z}{\Theta^j, \Phi^l, \Gamma^h \vdash C}\,\mathrm{R}$$

When R is C^z, W^z or $\wedge\mathrm{L}^z$, we have $l = k\gamma$, and when R is K^z, we have one index γ of $\Phi^{k\gamma}$ replaced by 1 in Φ^l.

$$(2.2)\qquad \cfrac{\cfrac{\overset{\pi_1}{\Theta^i, A^\alpha, \Gamma^j \vdash D} \qquad \overset{\pi_2}{\Theta^l, B^\beta, \Gamma^h \vdash D}}{\Theta^{max(i,l)}, A \vee B^{max(\alpha,\beta)}, \Gamma^{max(j,h)} \vdash D}\,\vee\mathrm{L}^z \qquad \overset{\pi}{\Delta^u, D^\gamma, \Xi^v \vdash C}}{\Delta^u, \Theta^{max(i,l)\gamma}, A \vee B^{max(\alpha,\beta)\gamma}, \Gamma^{max(j,h)\gamma}, \Xi^v \vdash C}\,\mathrm{cut}^z$$

We replace this proof by

$$\cfrac{\cfrac{\overset{\pi_1}{\Theta^i, A^\alpha, \Gamma^j \vdash D} \qquad \overset{\pi}{\Delta^u, D^\gamma, \Xi^v \vdash C}}{\Delta^u, \Theta^{i\gamma}, A^{\alpha\gamma}, \Gamma^{j\gamma}, \Xi^v \vdash C}\,\mathrm{cut}^z \qquad \cfrac{\overset{\pi_2}{\Theta^l, B^\beta, \Gamma^h \vdash D} \qquad \overset{\pi}{\Delta^u, D^\gamma, \Xi^v \vdash C}}{\Delta^u, \Theta^{l\gamma}, B^{\beta\gamma}, \Gamma^{h\gamma}, \Xi^v \vdash C}\,\mathrm{cut}^z}{\Delta^{max(u,u)}, \Theta^{max(i\gamma,l\gamma)}, A \vee B^{max(\alpha\gamma,\beta\gamma)}, \Gamma^{max(j\gamma,h\gamma)}, \Xi^{max(v,v)} \vdash C}\,\vee\mathrm{L}^z$$

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(3) Suppose the right rank of our cut is greater than 1. Then we have the following cases.

$$\dfrac{\begin{matrix}\pi_1\\ \Delta^i \vdash D\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \Theta^j, D^\gamma, \Gamma^h \vdash E\end{matrix}}{\Phi^l, D^\gamma, \Xi^k \vdash C}\,\mathrm{R}}{\Phi^l, \Delta^{i\gamma}, \Xi^k \vdash C}\,\mathrm{cut}^z \tag{3.1}$$

where R is C^z, W^z, K^z, $\wedge\mathrm{L}^z$ or $\vee\mathrm{R}^z$. We replace this proof by

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Delta^i \vdash D\end{matrix} \qquad \begin{matrix}\pi_2\\ \Theta^j, D^\gamma, \Gamma^h \vdash E\end{matrix}}{\Theta^j, \Delta^{i\gamma}, \Gamma^h \vdash E}\,\mathrm{cut}^z}{\Phi^l, \Delta^{i\gamma}, \Xi^k \vdash C}\,\mathrm{R}$$

except when R is C^z, and when in the transformed proof R can be a number of applications of (C^z).

$$\dfrac{\begin{matrix}\pi\\ \Delta^i \vdash D\end{matrix} \qquad \dfrac{\begin{matrix}\pi_1\\ \Theta^j, D^\alpha, \Gamma^h \vdash C_1\end{matrix} \qquad \begin{matrix}\pi_2\\ \Theta^k, D^\beta, \Gamma^l \vdash C_2\end{matrix}}{\Theta^{max(j,k)}, D^{max(\alpha,\beta)}, \Gamma^{max(h,l)} \vdash C_1 \wedge C_2}\,\wedge\mathrm{R}^z}{\Theta^{max(j,k)}, \Delta^{i\,max(\alpha,\beta)}, \Gamma^{max(h,l)} \vdash C_1 \wedge C_2}\,\mathrm{cut}^z \tag{3.2}$$

We replace this proof by

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \Delta^i \vdash D\end{matrix} \qquad \begin{matrix}\pi_1\\ \Theta^j, D^\alpha, \Gamma^h \vdash C_1\end{matrix}}{\Theta^j, \Delta^{i\alpha}, \Gamma^h \vdash C_1}\,\mathrm{cut}^z \qquad \dfrac{\begin{matrix}\pi\\ \Delta^i \vdash D\end{matrix} \qquad \begin{matrix}\pi_2\\ \Theta^k, D^\beta, \Gamma^l \vdash C_2\end{matrix}}{\Theta^k, \Delta^{i\beta}, \Gamma^l \vdash C_2}\,\mathrm{cut}^z}{\Theta^{max(j,k)}, \Delta^{max(i\alpha,i\beta)}, \Gamma^{max(h,l)} \vdash C_1 \wedge C_2}\,\wedge\mathrm{R}^z$$

$$\dfrac{\begin{matrix}\pi\\ \Delta^i \vdash D\end{matrix} \qquad \dfrac{\begin{matrix}\pi_1\\ \Theta_1^{j_1}, A^\gamma, \Theta_2^{j_2}, D^\alpha, \Gamma^h \vdash C\end{matrix} \qquad \begin{matrix}\pi_2\\ \Theta_1^{k_1}, B^\delta, \Theta_2^{k_2}, D^\beta, \Gamma^l \vdash C\end{matrix}}{\Theta_1^{max(j_1,k_1)}, A \vee B^{max(\gamma,\delta)}, \Theta_2^{max(j_2,k_2)}, D^{max(\alpha,\beta)}, \Gamma^{max(h,l)} \vdash C}\,\vee\mathrm{L}^z}{\Theta_1^{max(j_1,k_1)}, A \vee B^{max(\gamma,\delta)}, \Theta_2^{max(j_2,k_2)}, \Delta^{i\,max(\alpha,\beta)}, \Gamma^{max(h,l)} \vdash C}\,\mathrm{cut}^z \tag{3.3}$$

We replace this proof by

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \Delta^i \vdash D\end{matrix} \qquad \begin{matrix}\pi_1\\ \Theta_1^{j_1}, A^\gamma, \Theta_2^{j_2}, D^\alpha, \Gamma^h \vdash C\end{matrix}}{\Theta_1^{j_1}, A^\gamma, \Theta_2^{j_2}, \Delta^{i\alpha}, \Gamma^h \vdash C}\,\mathrm{cut}^z \qquad \dfrac{\begin{matrix}\pi\\ \Delta^i \vdash D\end{matrix} \qquad \begin{matrix}\pi_2\\ \Theta_1^{k_1}, B^\delta, \Theta_2^{k_2}, D^\beta, \Gamma^l \vdash C\end{matrix}}{\Theta_1^{k_1}, B^\delta, \Theta_2^{k_2}, \Delta^{i\beta}, \Gamma^l \vdash C}\,\mathrm{cut}^z}{\Theta_1^{max(j_1,k_1)}, A \vee B^{max(\gamma,\delta)}, \Theta_2^{max(j_2,k_2)}, \Delta^{max(i\alpha,i\beta)}, \Gamma^{max(h,l)} \vdash C}\,\vee\mathrm{L}^z$$

We proceed analogously when π_1 ends with $\Theta^j, D^\alpha, \Gamma_1^{h_1}, A^\gamma, \Gamma_2^{h_2} \vdash C$ and π_2 ends with

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$\Theta^k, D^\beta, \Gamma_1^{l_1}, B^\delta, \Gamma_2^{l_2} \vdash C$.

$$(3.4)\qquad \dfrac{\begin{matrix}\pi_1\\ \Delta^i \vdash D\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \Theta^j, D^\alpha, D^\beta, \Gamma^h \vdash C\end{matrix}}{\Theta^j, D^{\alpha+\beta}, \Gamma^h \vdash C}\,\mathrm{W}^z}{\Theta^j, \Delta^{i(\alpha+\beta)}, \Gamma^h \vdash C}\,\mathrm{cut}^z$$

We replace this proof by

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Delta^i \vdash D\end{matrix} \qquad \dfrac{\begin{matrix}\pi_1\\ \Delta_i \vdash D\end{matrix} \qquad \begin{matrix}\pi_2\\ \Theta^j, D^\alpha, D^\beta, \Gamma^h \vdash C\end{matrix}}{\Theta^j, \Delta^{i\alpha}, D^\beta, \Gamma^h \vdash C}\,\mathrm{cut}^z}{\Theta^j, \Delta^{i\alpha}, \Delta^{i\beta}, \Gamma^h \vdash C}\,\mathrm{cut}^z}{\ldots}}{\Theta^j, \Delta^{i\alpha+i\beta}, \Gamma^h \vdash C}\ \text{applications of } (\mathrm{C}^z) \text{ and } (\mathrm{W}^z)$$

In the transformed proof, the cut formula of the upper cut^z has the same degree as in the original cut, but it has a lower contraction index in the right-hand premise (even the rank has decreased, but this is not now essential). Hence, by the induction hypotheses, we have a cut-free proof of $\Theta^k, \Delta^l, D^\gamma, \Gamma^n \vdash C$ with $k \leqslant j$, $l \leqslant i\alpha$, $\gamma \leqslant \beta$ and $n \leqslant h$. So we obtain

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Delta^i \vdash D\end{matrix} \qquad \begin{matrix}\pi\\ \Theta^k, \Delta^l, D^\gamma, \Gamma^n \vdash C\end{matrix}}{\Theta^k, \Delta^l, \Delta^{i\gamma}, \Gamma^n \vdash C}\,\mathrm{cut}^z}{\ldots}}{\Theta^k, \Delta^{l+i\gamma}, \Gamma^n \vdash C}\ \text{applications of } (\mathrm{C}^z) \text{ and } (\mathrm{W}^z)$$

where the cut formula of cut^z is again of the same degree as in the original cut, but has a lower contraction index in the right-hand premise (its rank has perhaps increased). □

The contraction indices of $\mathcal{G}^z$ are not the only possible indices that we could have chosen. For example, we could replace $\alpha + \beta$ by $max(\alpha, \beta) + 1$ in (W^z). Whereas the original contraction index measures the number of contractions in the clusters, this new index would measure the height of clusters. (For the notion of cluster (*Bund* in German), see Gentzen (1938, Section 3.41) – see also Došen and Petrić (1999) and references therein.) A rationale for the maximum function in the indices of Γ and Θ in ($\wedge$R^z) and ($\vee$L^z) may be found in the proofs of Lemma 5.3 and Theorem 5.5 below.

5. W-normal form

To formulate our new cut-elimination procedure for $\mathcal{G}$, we need to introduce the following notion of normal form. (Note that W is sometimes used as a label for thinning, also called 'weakening', while our use of this label for contraction is suggested by combinatory logic.

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So our terminology should not be confused with the terminology of some other authors, who may use the same terms to designate other things; *cf.*, for example, Mints (1996).)

A proof in $\mathcal{G}$ is called W-*normal* iff every application of (W) in this proof is either the last rule of the proof, or it has only applications of (W) below it, or it is the upper rule in the following contexts:

$$\cfrac{\cfrac{\Theta, A, A, A, \Gamma \vdash C}{\Theta, A, A, \Gamma \vdash C}\ \mathrm{W}}{\Theta, A, \Gamma \vdash C}\ \mathrm{W}$$

$$\cfrac{\cfrac{A, A, \Gamma \vdash B}{A, \Gamma \vdash B}\ \mathrm{W}}{\Gamma \vdash A \rightarrow B}\ {\rightarrow}\mathrm{R}$$

We also need the following terminology.

We say that an application of (W) in a proof

$$\begin{array}{c}\pi\\ \Gamma \vdash C\end{array}$$

is *tied* to an occurrence G of a formula in Γ iff the principal formula (that is, contracted formula) of this application of (W) belongs to the cluster of G in π. (For more information about the notion of cluster, see Gentzen (1938, Section 3.41).)

A contraction in a proof π is *engaged* iff it is tied to the cut formula of the right-hand premise of some cut in π. If the corresponding cut is immediately below the engaged contraction, then we call such a contraction *directly engaged*. A contraction in π that is not engaged is called *neutral*.

We shall now prove a series of lemmata leading to the proof of the theorem that every proof can be reduced to a W-normal proof of the same degree of the same sequent. This theorem covers the first phase of our cut-elimination procedure.

Lemma 5.1. Every segment of a proof π of the form

$$\cfrac{\cfrac{\Phi \vdash C}{\cdots}}{\Psi \vdash C}\quad \begin{array}{l}e+n \text{ applications of (W) followed by}\\ \text{applications of (C)}\end{array}$$

can be transformed into a segment of the form

$$\cfrac{\cfrac{\Phi \vdash C}{\cdots}}{\Psi \vdash C}\quad \begin{array}{l}\text{applications of (C) followed by } e+n\\ \text{applications of (W)}\end{array}$$

where e is the number of engaged contractions of π and n is the number of neutral contractions of π that occur in the figures above. The degree of the transformed proof is the same as the degree of π.

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Proof. By induction on the lexicographically ordered couples $\langle e+n, i\rangle$, where i is the number of applications of (C) in the initial segment $\mathscr{S}$. We have the following cases.

(a) The segment $\mathscr{S}$ is of the form

$$
\begin{array}{cl}
\Phi \vdash C & \\ \hline
\cdots & e+n-1 \text{ applications of (W)} \\ \hline
\Gamma, A, A, D, \Delta \vdash C & \\ \hline
\Gamma, A, D, \Delta \vdash C & \text{W} \\ \hline
\Gamma, D, A, \Delta \vdash C & \text{C} \\ \hline
\cdots & i-1 \text{ applications of (C)} \\ \hline
\Psi \vdash C &
\end{array}
$$

By transforming the segment beginning with $\Gamma, A, A, \Delta \vdash C$ and ending with $\Gamma, D, A, \Delta \vdash C$, the whole segment $\mathscr{S}$ is transformed into

$$
\begin{array}{cl}
\Phi \vdash C & \\ \hline
\cdots & e+n-1 \text{ applications of (W)} \\ \hline
\Gamma, A, A, D, \Delta \vdash C & \\ \hline
\Gamma, A, D, A, \Delta \vdash C & \text{C} \\ \hline
\Gamma, D, A, A, \Delta \vdash C & \text{C} \\ \hline
\Gamma, D, A, \Delta \vdash C & \text{W} \\ \hline
\cdots & i-1 \text{ applications of (C)} \\ \hline
\Psi \vdash C &
\end{array}
$$

This transformation preserves the engagement or neutrality of the lowest contraction.

Since $\langle e+n-1, 2\rangle < \langle e+n, i\rangle$, by the induction hypothesis, the segment beginning with $\Phi \vdash C$ and ending with $\Gamma, D, A, A, \Delta \vdash C$ can be transformed so that our whole segment becomes

$$
\begin{array}{cl}
\Phi \vdash C & \\ \hline
\cdots & \text{applications of (C)} \\ \hline
\Phi' \vdash C & \\ \hline
\cdots & e+n-1 \text{ applications of (W)} \\ \hline
\Gamma, D, A, A, \Delta \vdash C & \\ \hline
\Gamma, D, A, \Delta \vdash C & \text{W} \\ \hline
\cdots & i-1 \text{ applications of (C)} \\ \hline
\Psi \vdash C &
\end{array}
$$

Since $\langle e+n, i-1\rangle < \langle e+n, i\rangle$, by the induction hypothesis, the segment beginning with $\Phi' \vdash C$ and ending with $\Psi \vdash C$ can be transformed so that the whole segment is brought into the desired form.

(b) The segment $\mathscr{S}$ is of the form

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Phi \vdash C}{\cdots}\ e+n-1 \text{ applications of (W)}}{\Gamma, A, A, \Delta \vdash C}}{\Gamma, A, \Delta \vdash C}\ \text{W}}{\Gamma', A, \Delta' \vdash C}\ \text{C}}{\cfrac{\cdots}{\Psi \vdash C}\ i-1 \text{ applications of (C)}}$$

By transforming the segment beginning with $\Gamma, A, A, \Delta \vdash C$ and ending with $\Gamma', A, \Delta' \vdash C$, the whole segment $\mathscr{S}$ is transformed into

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Phi \vdash C}{\cdots}\ e+n-1 \text{ applications of (W)}}{\Gamma, A, A, \Delta \vdash C}}{\Gamma', A, A, \Delta' \vdash C}\ \text{C}}{\Gamma', A, \Delta' \vdash C}\ \text{W}}{\cfrac{\cdots}{\Psi \vdash C}\ i-1 \text{ applications of (C)}}$$

The remaining steps are analogous to the steps in (a). □

A W-normal proof is called *tailless* iff its last rule is not an application of (W). Let π_1 and π_2 be tailless. We define the class of proofs $\mathscr{C}(\pi_1, \pi_2)$ inductively as follows:

(i) The proof π_2 belongs to $\mathscr{C}(\pi_1, \pi_2)$.

(ii) If π belongs to $\mathscr{C}(\pi_1, \pi_2)$, then the proof

$$\cfrac{\cfrac{\cfrac{\pi_1 \qquad \pi}{\Phi \vdash B}\ \text{cut}}{\cdots}\ \text{applications of (C)}}{\Psi \vdash B}$$

belongs to $\mathscr{C}(\pi_1, \pi_2)$, provided that there is no occurrence of a formula in a subproof π_1 of π that belongs to the cluster of the cut formula in the right-hand premise of the cut noted in the figure.

(iii) If π belongs to $\mathscr{C}(\pi_1, \pi_2)$, then π followed by an application of (W) belongs to $\mathscr{C}(\pi_1, \pi_2)$.

The application of (W) in (iii) in the definition of $\mathscr{C}(\pi_1, \pi_2)$ is called *mobile*. The *height*

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of a mobile application of (W) is the number of applications of (W) and (cut) below it in the proof (we do not count applications of (C)).

It is easy to verify that an application of (W) in a tailless subproof of a proof cannot be engaged in this proof. This fact will be useful in the proof of the following lemma.

Lemma 5.2. For every pair of tailless proofs π_1 and π_2, every proof from $\mathscr{C}(\pi_1, \pi_2)$ can be transformed into a W-normal proof of the same degree.

Proof. We make an induction on the lexicographically ordered pairs $\langle \kappa, \lambda \rangle$, where κ is the number of engaged applications of (W) in the proof and λ is the sum of the heights of all mobile applications of (W) in the proof.

By the definition of $\mathscr{C}(\pi_1, \pi_2)$, if there is no mobile application of (W) followed immediately by a cut in a proof from $\mathscr{C}(\pi_1, \pi_2)$, then this proof is W-normal.

(a) Suppose our proof is of the form

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash C\end{array} \qquad \dfrac{\begin{array}{c}\pi\\ \Theta, C, C, \Delta \vdash B\end{array}}{\Theta, C, \Delta \vdash B}\ \text{W directly engaged}}{\Theta, \Gamma, \Delta \vdash B}\ \text{cut}}{\dots}\ \begin{array}{l}\text{applications of (C) followed by}\\ \text{applications of (W)}\end{array}}{\begin{array}{c}\Xi \vdash B\\ \vdots\end{array}}
$$

for π in $\mathscr{C}(\pi_1, \pi_2)$.

By pushing the directly engaged application of (W) below cut, this proof is transformed into

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash C\end{array} \qquad \dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash C\end{array} \qquad \begin{array}{c}\pi\\ \Theta, C, C, \Delta \vdash B\end{array}}{\Theta, \Gamma, C, \Delta \vdash B}\ \text{cut}}{\Theta, \Gamma, \Gamma, \Delta \vdash B}\ \text{cut}}{\dots}\ \begin{array}{l}\text{applications of (C) followed by}\\ \text{neutral applications of (W)}\end{array}}{\dfrac{\Theta, \Gamma, \Delta \vdash B}{\dots}\ \begin{array}{l}\text{remaining applications of (C)}\\ \text{followed by applications of (W)}\end{array}}
$$

$$
\begin{array}{c}\Xi \vdash B\\ \vdots\end{array}
$$

The neutral applications of (W) mentioned above, which contract formulae from Γ, are neutral by the proviso in (ii) of the definition of $\mathscr{C}(\pi_1, \pi_2)$.

By Lemma 5.1, the segment beginning with $\Theta, \Gamma, \Gamma, \Delta \vdash B$ and ending with $\Xi \vdash B$ can

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be transformed so that our whole proof becomes

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma\vdash C\end{array} \qquad \dfrac{\begin{array}{c}\pi_1\\ \Gamma\vdash C\end{array} \quad \begin{array}{c}\pi\\ \Theta,C,C,\Delta\vdash B\end{array}}{\Theta,\Gamma,C,\Delta\vdash B}\,\text{cut}}{\Theta,\Gamma,\Gamma,\Delta\vdash B}\,\text{cut}}{\cdots}\,\begin{array}{l}\text{applications of (C) followed by}\\ \text{applications of (W)}\end{array}}{\Xi\vdash B}$$
$$\vdots$$

which belongs to $\mathscr{C}(\pi_1,\pi_2)$ and has one engaged application of (W) less than the original proof. (Here we use the fact that no application of (W) in π_1 can be engaged in the proof above.) So the measure of the transformed proof is $\langle\kappa-1,\lambda\rangle<\langle\kappa,\lambda\rangle$. By the induction hypothesis, this proof can be transformed into a W-normal proof.

(b) Suppose our proof is of the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma\vdash C\end{array} \qquad \dfrac{\begin{array}{c}\pi\\ \Theta,C,\Delta\vdash B\end{array}}{\Theta',C,\Delta'\vdash B}\,\text{W neutral or not directly engaged}}{\Theta',\Gamma,\Delta'\vdash B}\,\text{cut}}{\cdots}\,\begin{array}{l}\text{applications of (C) followed by}\\ \text{applications of (W)}\end{array}}{\Xi\vdash B}$$
$$\vdots$$

By pushing the distinguished application of (W), which immediately follows π, below cut, this proof is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma\vdash C\end{array} \quad \begin{array}{c}\pi\\ \Theta,C,\Delta\vdash B\end{array}}{\Theta,\Gamma,\Delta\vdash B}\,\text{cut}}{\Theta',\Gamma,\Delta'\vdash B}\,\text{W}}{\cdots}\,\begin{array}{l}\text{applications of (C) followed by}\\ \text{applications of (W)}\end{array}}{\Xi\vdash B}$$
$$\vdots$$

where the distinguished applications of (W) in the original figure and in the transformed figure are either both neutral or both engaged. By Lemma 5.1, the segment beginning with $\Theta,\Gamma,\Delta\vdash B$ and ending with $\Xi\vdash B$ can be transformed so that our whole proof

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becomes

$$\cfrac{\cfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash C\end{array} \qquad \begin{array}{c}\pi\\ \Theta, C, \Delta \vdash B\end{array}}{\Theta, \Gamma, \Delta \vdash B}\,\text{cut}}{\cfrac{\cdots}{\Xi \vdash B}}\;\begin{array}{l}\text{applications of (C) followed by}\\ \text{applications of (W)}\end{array}$$

$$\vdots$$

This proof belongs to $\mathscr{C}(\pi_1, \pi_2)$ and has the same number of engaged applications of (W), but its λ has decreased by 1. Since $\langle \kappa, \lambda - 1 \rangle < \langle \kappa, \lambda \rangle$, by the induction hypothesis, our proof can be transformed into a W-normal proof. □

The following lemma is covered by Lemma 12 in Kleene (1952). However, Kleene's sequent system is not quite the same: interchange is only implicit in it, and his proof does not cover all of the details that we need to cover. (In his proof on page 24, in the third illustration, Kleene assumes that the n_1 contractions above $A, A, \Gamma \to \Theta$ are all tied to the first A, whereas we cannot assume that. We could only assume it after introducing a new reduction step that transforms sequences of contractions tied to the same occurrence of a formula.)

Lemma 5.3. A proof of the form

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\pi_1\\ \Phi \vdash C\end{array}}{\cdots}}{\Theta, A, \Gamma \vdash C}\;\begin{array}{l}\text{applications}\\ \text{of (W)}\end{array} \qquad \cfrac{\cfrac{\begin{array}{c}\pi_2\\ \Psi \vdash C\end{array}}{\cdots}}{\Theta, B, \Gamma \vdash C}\;\begin{array}{l}\text{applications}\\ \text{of (W)}\end{array}}{\Theta, A \vee B, \Gamma \vdash C}\,\vee\text{L}$$

where π_1 and π_2 are tailless, can be transformed into a W-normal proof, of the same degree, of the form

$$\cfrac{\cfrac{\begin{array}{c}\pi\\ \Xi \vdash C\end{array}}{\cdots}}{\Theta, A \vee B, \Gamma \vdash C}\;\text{applications of (W)}$$

where π is tailless, and for every occurrence G of a formula in Θ, if above the left-hand premise of $\vee$L in the former figure there are k_1 applications of (W) tied to G, and if above the right-hand premise of $\vee$L in the former figure there are k_2 applications of (W) tied to this same G, then in the latter figure there are $max(k_1, k_2)$ applications of (W) tied to G. The same holds for occurrences of formulae in Γ.

Proof. Let n be the number of applications of (W) tied to A in the left-hand premise of $\vee$L and m be the number of applications of (W) tied to B in the right-hand premise

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of $\vee$L in the figure of the initial proof. We prove the lemma by induction on $n+m$. Our proof is first transformed into

$$\pi_1 \quad \Phi \vdash C \quad \text{(applications of (K))} \quad \Theta', A, \ldots, A, \Gamma' \vdash C \quad \text{(n applications of (W))} \quad \Theta', A, \Gamma' \vdash C$$

$$\pi_2 \quad \Psi \vdash C \quad \text{(applications of (K))} \quad \Theta', B, \ldots, B, \Gamma' \vdash C \quad \text{(m applications of (W))} \quad \Theta', B, \Gamma' \vdash C$$

$$\frac{\Theta', A, \Gamma' \vdash C \qquad \Theta', B, \Gamma' \vdash C}{\Theta', A \vee B, \Gamma' \vdash C}\ \vee\text{L}$$

$$\Theta', A \vee B, \Gamma' \vdash C \quad \text{(applications of (W))} \quad \Theta, A \vee B, \Gamma \vdash C$$

Note that this step involves permuting applications of (W) one with another. In the sequence of applications of (W) below the sequent $\Theta', A \vee B, \Gamma' \vdash C$ there are $max(k_1, k_2)$ applications of (W) tied to G from Θ, where G, k_1 and k_2 are as in the formulation of the lemma.

If $n+m=0$, then this proof is W-normal.

If $n>0$, then our proof is transformed into

$$\pi_1 \quad \Phi \vdash C \quad \text{(applications of (K))} \quad \Theta', A, \ldots, A, \Gamma' \vdash C \quad \text{($n-1$ appl. of (W))} \quad \Theta', A, A, \Gamma' \vdash C$$

$$\pi_2 \quad \Psi \vdash C \quad \text{(applications of (K))} \quad \Theta', B, \ldots, B, \Gamma' \vdash C \quad \text{(K)} \quad \Theta', A, B, \ldots, B, \Gamma' \vdash C \quad \text{(m appl. of (W))} \quad \Theta', A, B, \Gamma' \vdash C$$

$$\frac{\Theta', A, A, \Gamma' \vdash C \qquad \Theta', A, B, \Gamma' \vdash C}{\Theta', A, A \vee B, \Delta' \vdash C}\ \vee\text{L}$$

$$\pi_2 \quad \Psi \vdash C \quad \text{(applications of (K))} \quad \Theta', B, \ldots, B, \Gamma' \vdash C \quad \text{(K)} \quad \Theta', B, \ldots, B, A \vee B, \Gamma' \vdash C \quad \text{(m appl. of (W))} \quad \Theta', B, A \vee B, \Gamma' \vdash C$$

$$\frac{\Theta', A, A \vee B, \Delta' \vdash C \qquad \Theta', B, A \vee B, \Gamma' \vdash C}{\Theta', A \vee B, A \vee B, \Gamma' \vdash C}\ \vee\text{L}$$

$$\frac{\Theta', A \vee B, A \vee B, \Gamma' \vdash C}{\Theta', A \vee B, \Gamma' \vdash C}\ \text{W} \quad \text{(applications of (W))} \quad \Theta, A \vee B, \Gamma \vdash C$$

Consider the subproof whose endsequent is $\Theta', A, A \vee B, \Gamma' \vdash C$. Its measure is $n_2 + m$, where n_2 is the number of applications of (W) tied to the right-hand A in the left-hand premise of the last rule of this subproof. The number of applications of (W) tied to the left-hand A of the same sequent is n_1 and we have $n_1 + n_2 = n - 1$. We apply the induction

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hypothesis to this subproof, and therefore our proof is transformed into

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \Xi \vdash C\end{matrix}}{\dfrac{\cdots}{\Theta', A, A \vee B, \Gamma' \vdash C}}\ \text{applications of (W)} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\pi_2\\ \Psi \vdash C\end{matrix}}{\dfrac{\cdots}{\Theta', B, \ldots, B, \Gamma' \vdash C}}\ \text{applications of (K)}}{\Theta', B, \ldots, B, A \vee B, \Gamma' \vdash C}\ \mathrm{K}}{\dfrac{\cdots}{\Theta', B, A \vee B, \Gamma' \vdash C}}\ m\ \text{applications of (W)}}{\Theta', A \vee B, A \vee B, \Gamma' \vdash C}\ \vee\mathrm{L}}{\dfrac{\Theta', A \vee B, \Gamma' \vdash C}{\dfrac{\cdots}{\Theta, A \vee B, \Gamma \vdash C}}\ \text{applications of (W)}}\ \mathrm{W}
$$

where in the subproofs whose endsequent is $\Theta', A, A \vee B, \Gamma' \vdash C$ there are only n_1 applications of (W) tied to A and there are no applications of (W) tied to any occurrence of a formula in Θ' and Γ'. Consider the subproof whose endsequent is $\Theta', A \vee B, A \vee B, \Gamma' \vdash C$. Its measure is $n_1 + m$, and by applying the induction hypothesis, our proof is transformed into a W-normal proof with $max(k_1, k_2)$ applications of (W) tied to G from Θ in its endsequent.

We proceed quite analogously if $m > 0$. □

Lemma 5.4. A proof of the form

$$
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Delta' \vdash A\end{matrix}}{\dfrac{\cdots}{\Delta \vdash A}}\ \text{applications of (W)} \qquad \dfrac{\begin{matrix}\pi_2\\ \Phi \vdash C\end{matrix}}{\dfrac{\cdots}{\Theta, B, \Gamma \vdash C}}\ \text{applications of (W)}}{\Theta, \Delta, A \to B, \Gamma \vdash C}\ \to\mathrm{L}
$$

where π_1 and π_2 are tailless, can be transformed into a W-normal proof, of the same degree, of the form

$$
\dfrac{\begin{matrix}\pi\\ \Xi \vdash C\end{matrix}}{\dfrac{\cdots}{\Theta, \Delta, A \to B, \Gamma \vdash C}}\ \text{applications of (W)}
$$

where π is tailless, and for every occurrence G of a formula in Θ, if above the right-hand premise of $\to$L in the former figure there are k applications of (W) tied to G, then in the latter figure there are k applications of (W) tied to G. The same holds for occurrences of formulae in Γ.

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Proof. Let n be the number of applications of (W) tied to B in the right-hand premise of $\rightarrow$L in the figure of the initial proof, and let the total number of applications of (W) above this premise be l. We prove the lemma by induction on n. Our proof is first transformed into

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Delta' \vdash A\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\pi_2\\ \Phi \vdash C\end{matrix}}{\cdots}\;\begin{matrix}n{-}1 \text{ applications of}\\ \text{(W) tied to } B\end{matrix}}{\Theta', B, B, \Gamma' \vdash C}}{\Theta', B, \Gamma' \vdash C}\,\mathrm{W}}{\Theta', \Delta', A \rightarrow B, \Gamma' \vdash C}\,\rightarrow\mathrm{L}}{\cdots}\;\begin{matrix}\text{applications of (W) including } l{-}n \text{ ap-}\\ \text{plications of (W) tied to formulae in}\\ \Theta \text{ and } \Gamma\end{matrix}}{\Theta, \Delta, A \rightarrow B, \Gamma \vdash C}
$$

Note that this step involves permuting applications of (W) one with another. The transformed proof is next transformed into

$$
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Delta' \vdash A\end{matrix} \qquad \dfrac{\begin{matrix}\pi_1\\ \Delta' \vdash A\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \Phi \vdash C\end{matrix}}{\cdots}\;\begin{matrix}n{-}1 \text{ applications}\\ \text{of (W)}\end{matrix}}{\Theta', B, B, \Gamma' \vdash C}}{\Theta', \Delta', A \rightarrow B, B, \Gamma' \vdash C}\,\rightarrow\mathrm{L}}{\dfrac{\Theta', \Delta', A \rightarrow B, \Delta', A \rightarrow B, \Gamma' \vdash C}{\cdots}\;\begin{matrix}\text{applications of (C) and (W) includ-}\\ \text{ing } l-n \text{ applications of (W) tied to}\\ \text{formulae in } \Theta \text{ and } \Gamma\end{matrix}}\,\rightarrow\mathrm{L}
$$

$$
\overline{\Theta, \Delta, A \rightarrow B, \Gamma \vdash C}
$$

By the induction hypothesis, there is a W-normal proof

$$
\dfrac{\dfrac{\begin{matrix}\pi\\ \Psi \vdash C\end{matrix}}{\cdots}\;\text{applications of (W)}}{\Theta', \Delta', A \rightarrow B, B, \Gamma' \vdash C}
$$

where π is tailless, and where there are $m \leqslant n-1$ applications of (W) tied to B in the endsequent, and no application of (W) tied to formulae in Θ' and Γ'.

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We apply again the induction hypothesis to

$$\dfrac{\overset{\pi_1}{\Delta' \vdash A} \qquad \dfrac{\dfrac{\overset{\pi}{\Psi \vdash C}}{\cdots}\ \text{applications of (W)}}{\Theta', \Delta', A \to B, B, \Gamma' \vdash C}}{\Theta', \Delta', A \to B, \Delta', A \to B, \Gamma' \vdash C}\ \to\text{L}$$

and we use Lemma 5.1 to obtain a W-normal proof of $\Theta, \Delta, A \to B, \Gamma \vdash C$. In the final proof there are still only $l - n$ applications of (W) tied to formulae in Θ and Γ. □

We can now prove the theorem that covers the first phase of our cut-elimination procedure.

Theorem 5.5. Every proof of a sequent in $\mathscr{G}$ can be reduced to a W-normal proof of the same degree of the same sequent.

Proof. We proceed by induction on the length of the proof of our sequent in $\mathscr{G}$.

If our proof is just an axiom, then this proof is W-normal.

If our sequent is proved by the following proof

$$\dfrac{\overset{\pi}{\Delta, A, B, \Gamma \vdash C}}{\Delta, B, A, \Gamma \vdash C}\ \text{C}$$

then, by the induction hypothesis, there is a W-normal proof

$$\dfrac{\dfrac{\overset{\pi'}{\Lambda \vdash C}}{\cdots}\ \text{applications of (W)}}{\Delta, A, B, \Gamma \vdash C}$$

where π' is tailless. Then we apply Lemma 5.1.

If our sequent is proved by the following proof

$$\dfrac{\overset{\pi}{\Theta, A, A, \Gamma \vdash C}}{\Theta, A, \Gamma \vdash C}\ \text{W}$$

then, by the induction hypothesis, there is a W-normal proof

$$\dfrac{\overset{\pi'}{\Theta, A, A, \Gamma \vdash C}}{\Theta, A, \Gamma \vdash C}\ \text{W}$$

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If our sequent is proved by the following proof

$$\dfrac{\begin{matrix}\pi \\ \Theta, \Gamma \vdash C\end{matrix}}{\Theta, A, \Gamma \vdash C}\ \mathrm{K}$$

then, by the induction hypothesis, there is a W-normal proof

$$\dfrac{\dfrac{\begin{matrix}\pi' \\ \Lambda \vdash C\end{matrix}}{\cdots}}{\Theta, \Gamma \vdash C}\ \text{applications of (W)}$$

where π' is tailless. Applications of (W) can be permuted with K as follows:

$$\dfrac{\dfrac{\Theta, \Gamma \vdash C}{\Theta', \Gamma' \vdash C}\ \mathrm{W}}{\Theta', A, \Gamma' \vdash C}\ \mathrm{K} \qquad \dfrac{\dfrac{\Theta, \Gamma \vdash C}{\Theta, A, \Gamma \vdash C}\ \mathrm{K}}{\Theta', A, \Gamma' \vdash C}\ \mathrm{W}$$

And, by induction on the number of applications of (W) below π', we prove that there is a W-normal proof of $\Theta, A, \Gamma \vdash C$.

If our sequent is proved by the following proof

$$\dfrac{\begin{matrix}\pi \\ \Delta \vdash A\end{matrix} \qquad \begin{matrix}\rho \\ \Theta, A, \Gamma \vdash C\end{matrix}}{\Theta, \Delta, \Gamma \vdash C}\ \mathrm{cut}$$

then, by the induction hypothesis, we have a proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi' \\ \Lambda \vdash A\end{matrix}}{\cdots}}{\Delta \vdash A}\ \begin{matrix}\text{applications}\\ \text{of (W)}\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\rho' \\ \Phi \vdash C\end{matrix}}{\cdots}}{\Theta, A, \Gamma \vdash C}\ \begin{matrix}\text{applications}\\ \text{of (W)}\end{matrix}}{\Theta, \Delta, \Gamma \vdash C}\ \mathrm{cut}$$

where π' and ρ' are tailless. We push below cut all the applications of (W) below π' so as

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to obtain

$$\dfrac{\begin{matrix}\pi' \\ \Lambda \vdash A\end{matrix} \qquad \begin{matrix}\rho' \\ \dfrac{\dfrac{\Phi \vdash C}{\cdots}\ \text{applications of (W)}}{\Theta, A, \Gamma \vdash C}\end{matrix}}{\dfrac{\dfrac{\Theta, \Lambda, \Gamma \vdash C}{\cdots}\ \text{applications of (W)}}{\Theta, \Delta, \Gamma \vdash C}}\ \text{cut}$$

Then this proof belongs to $\mathscr{C}(\pi', \rho')$, and we can apply Lemma 5.2.

If our sequent is proved by the following proof

$$\begin{matrix}\pi \\ \dfrac{\Theta, A, \Gamma \vdash C}{\Theta, A \wedge B, \Gamma \vdash C}\ \wedge\text{L}\end{matrix}$$

then, by the induction hypothesis, there is a W-normal proof

$$\begin{matrix}\pi \\ \dfrac{\dfrac{\Lambda \vdash C}{\cdots}\ \text{applications of (W)}}{\Theta, A, \Gamma \vdash C}\end{matrix}$$

where π is tailless. Applications of (W) can be permuted with ($\wedge$L) as follows

$$\dfrac{\dfrac{\Theta, A, \Gamma \vdash C}{\Theta', A, \Gamma' \vdash C}\ \text{W}}{\Theta', A \wedge B, \Gamma' \vdash C}\ \wedge\text{L} \qquad \dfrac{\dfrac{\Theta, A, \Gamma \vdash C}{\Theta, A \wedge B, \Gamma \vdash C}\ \wedge\text{L}}{\Theta', A \wedge B, \Gamma' \vdash C}\ \text{W}$$

$$\dfrac{\dfrac{\Theta, A, A, \Gamma \vdash C}{\Theta, A, \Gamma \vdash C}\ \text{W}}{\Theta, A \wedge B, \Gamma \vdash C}\ \wedge\text{L} \qquad \dfrac{\dfrac{\dfrac{\Theta, A, A, \Gamma \vdash C}{\Theta, A \wedge B, A, \Gamma \vdash C}\ \wedge\text{L}}{\Theta, A \wedge B, A \wedge B, \Gamma \vdash C}\ \wedge\text{L}}{\Theta, A \wedge B, \Gamma \vdash C}\ \text{W}$$

And, by an induction analogous to that in the proof of Lemma 5.1, we show that there is a W-normal proof of $\Theta, A \wedge B, \Gamma \vdash C$. We proceed analogously for the other ($\wedge$L) rule, involving B.

If our sequent is proved by the following proof

$$\dfrac{\begin{matrix}\pi \\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\rho \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \wedge B}\ \wedge\text{R}$$

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then, by the induction hypothesis, there are W-normal proofs

$$\begin{array}{c}\pi' \\ \Gamma' \vdash A \\ \hline \cdots \\ \hline \Gamma \vdash A\end{array}\ \begin{array}{l}\text{applications}\\ \text{of (W)}\end{array} \qquad\qquad \begin{array}{c}\rho' \\ \Gamma'' \vdash B \\ \hline \cdots \\ \hline \Gamma \vdash B\end{array}\ \begin{array}{l}\text{applications}\\ \text{of (W)}\end{array}$$

where π' and ρ' are tailless. Then we have the W-normal proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi' \\ \Gamma' \vdash A\end{array}}{\cdots}}{\Gamma''' \vdash A}\ \begin{array}{l}\text{applications}\\ \text{of (K)}\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\rho' \\ \Gamma'' \vdash B\end{array}}{\cdots}}{\Gamma''' \vdash B}\ \begin{array}{l}\text{applications}\\ \text{of (K)}\end{array}}{\dfrac{\dfrac{\Gamma''' \vdash A \wedge B}{\cdots}}{\Gamma \vdash A \wedge B}\ \text{applications of (W)}}\ \wedge\text{R}$$

If our sequent is proved by the following proof

$$\dfrac{\begin{array}{c}\pi \\ \Theta, A, \Gamma \vdash C\end{array} \qquad \begin{array}{c}\rho \\ \Theta, B, \Gamma \vdash C\end{array}}{\Theta, A \vee B, \Gamma \vdash C}\ \vee\text{L}$$

we apply the induction hypothesis to π and ρ, and next we apply Lemma 5.3.

If our sequent is proved by the following proof

$$\dfrac{\begin{array}{c}\pi \\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \vee B}\ \vee\text{R}$$

we apply the induction hypothesis to π, and we push applications of (W) below $\vee$R as follows

$$\dfrac{\dfrac{\Theta \vdash A}{\Theta' \vdash A}\ \text{W}}{\Theta' \vdash A \vee B}\ \vee\text{R} \qquad\qquad \dfrac{\dfrac{\Theta \vdash A}{\Theta \vdash A \vee B}\ \vee\text{R}}{\Theta' \vdash A \vee B}\ \text{W}$$

Of course, we proceed analogously with the other ($\vee$R) rule, involving B.

If our sequent is proved by the following proof

$$\dfrac{\begin{array}{c}\pi \\ \Delta \vdash A\end{array} \qquad \begin{array}{c}\rho \\ \Theta, B, \Gamma \vdash D\end{array}}{\Theta, \Delta, A \to B, \Gamma \vdash D}\ \to\text{L}$$

we apply the induction hypothesis to π and ρ, and next we apply Lemma 5.4.

If our sequent is proved by the following proof

$$\dfrac{\begin{matrix}\pi \\ A, \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \rightarrow B}\ \rightarrow\text{R}$$

we apply the induction hypothesis to π to obtain the W-normal proof

$$\dfrac{\dfrac{\begin{matrix}\pi' \\ \Phi \vdash B\end{matrix}}{\cdots}}{A, \Gamma \vdash B}\ \text{applications of (W)}$$

We next push below $\rightarrow$R each of the applications of (W) not tied to A in $A, \Gamma \vdash B$. □

6. Maximal cuts

A cut in a proof of $\mathscr{G}$ will be called *maximal* iff its rank is 2 and none of its premises is an axiom. A proof will be called *maximalized* iff all cuts in it are maximal. We can prove the following theorem, which covers the second phase of our cut-elimination procedure.

Theorem 6.1. Every W-normal proof of a sequent in $\mathscr{G}$ can be reduced to a maximalized W-normal proof, of the same or of a lower degree, of the same sequent.

Proof. It is enough to consider a W-normal proof of the form

$$\dfrac{\begin{matrix}\pi \\ \Delta \vdash A\end{matrix} \qquad \begin{matrix}\rho \\ \Theta, A, \Gamma \vdash C\end{matrix}}{\Theta, \Delta, \Gamma \vdash C}\ \text{cut}$$

where the cut noted in this figure is not maximal and all cuts in π and ρ are maximal. The rank of such a proof is the rank of the nonmaximal cut. We show by induction on rank that this proof can be reduced to a maximalized W-normal proof of the same degree of $\Theta, \Delta, \Gamma \vdash C$.

Suppose the rank of our nonmaximal cut is 2. This means that one of its premises is an axiom. Then we eliminate this cut by standard reduction steps, like those in (1.1)–(1.3) of the proof of Theorem 4.1. At this point the degree of the proof may decrease.

Suppose now that the rank of our nonmaximal cut is greater than 2. In order to decrease the rank of the proof, we introduce a number of reduction steps that decrease the left rank first. When this rank is 1, we introduce other reduction steps that decrease the right rank. (This is opposite to Gentzen's procedure, where the right rank is first reduced to 1. However, the matter is not essential, and we could proceed as Gentzen did. Gentzen need not have reduced rank to 1 on one side, before reducing the rank on the other side – he could just as well have worked in a zig-zag manner, passing from one side to another before reaching 1. However, for us it is essential that the rank on one side has fallen to 1 before we attack the rank on the other side.)

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Suppose now that the left rank of the nonmaximal cut above is greater than 1. Then in addition to the standard reduction steps like those considered in (2) of the proof of Theorem 4.1, we have the following additional reduction steps

$$\frac{\dfrac{\overset{\pi_1}{\Delta_2 \vdash B} \qquad \overset{\pi_2}{\Delta_1, C, \Delta_3 \vdash A}}{\Delta_1, \Delta_2, B \to C, \Delta_3 \vdash A}\,{\to}\mathrm{L} \qquad \overset{\rho}{\Theta, A, \Gamma \vdash C}}{\Theta, \Delta_1, \Delta_2, B \to C, \Delta_3, \Gamma \vdash C}\,\mathrm{cut} \tag{2.3}$$

is reduced to

$$\frac{\overset{\pi_1}{\Delta_2 \vdash B} \qquad \dfrac{\overset{\pi_2}{\Delta_1, C, \Delta_3 \vdash A} \qquad \overset{\rho}{\Theta, A, \Gamma \vdash C}}{\Theta, \Delta_1, C, \Delta_3, \Gamma \vdash C}\,\mathrm{cut}}{\Theta, \Delta_1, \Delta_2, B \to C, \Delta_3, \Gamma \vdash C}\,{\to}\mathrm{L}$$

The cut in the lower figure has lower rank and we may apply the induction hypothesis to it.

$$\frac{\dfrac{\overset{\pi_1}{\Delta_2 \vdash B} \qquad \overset{\pi_2}{\Delta_1, B, \Delta_3 \vdash A}}{\Delta_1, \Delta_2, \Delta_3 \vdash A}\,\mathrm{cut} \qquad \overset{\rho}{\Theta, A, \Gamma \vdash C}}{\Theta, \Delta_1, \Delta_2, \Delta_3, \Gamma \vdash C}\,\mathrm{cut} \tag{2.4}$$

is reduced to

$$\frac{\overset{\pi_1}{\Delta_2 \vdash B} \qquad \dfrac{\overset{\pi_2}{\Delta_1, B, \Delta_3 \vdash A} \qquad \overset{\rho}{\Theta, A, \Gamma \vdash C}}{\Theta, \Delta_1, B, \Delta_3, \Gamma \vdash C}\,\mathrm{cut}}{\Theta, \Delta_1, \Delta_2, \Delta_3, \Gamma \vdash C}\,\mathrm{cut}$$

By the induction hypothesis, the subproof of the reduced proof ending with the right-hand premise of the lower cut can be reduced to a maximalized W-normal proof, of the same or of a lower degree, of the same sequent. The first step of this reduction, which is one of the reduction steps (2.1)–(2.3), makes the lower cut maximal, and subsequent steps leave it so. We must apply (2.1)–(2.3) because the left rank of the upper cut in the lower figure is greater than 1 (the proof π_2 cannot be an axiom, and the right rank of the upper cut in the first figure is 1), and, moreover, π_2 cannot end with a cut. Note that in the reduction step (2.1) the rule R cannot be (W).

Suppose now that the left rank of our cut is 1 and the right rank is greater than 1. Then in addition to the standard reduction steps like those considered in (3) of the proof of Theorem 4.1 (except for (3.1) with R being (W), and (3.4), which we do not have because of W-normality), we have the following additional cases.

$$\dfrac{\begin{matrix}\pi\\ \Delta\vdash A\end{matrix}\qquad \dfrac{\begin{matrix}\rho_1\\ \Theta_2,A,\Gamma_1\vdash B\end{matrix}\qquad\begin{matrix}\rho_2\\ \Theta_1,D,\Gamma_2\vdash C\end{matrix}}{\Theta_1,\Theta_2,A,\Gamma_1,B\to D,\Gamma_2\vdash C}\,{\to}\mathrm{L}}{\Theta_1,\Theta_2,\Delta,\Gamma_1,B\to D,\Gamma_2\vdash C}\,\mathrm{cut} \tag{3.5}$$

is reduced to

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \Delta\vdash A\end{matrix}\qquad\begin{matrix}\rho_1\\ \Theta_2,A,\Gamma_1\vdash B\end{matrix}}{\Theta_2,\Delta,\Gamma_1\vdash B}\,\mathrm{cut}\qquad\begin{matrix}\rho_2\\ \Theta_1,D,\Gamma_2\vdash C\end{matrix}}{\Theta_1,\Theta_2,\Delta,\Gamma_1,B\to D,\Gamma_2\vdash C}\,{\to}\mathrm{L}$$

We have analogous reduction steps when A in the initial proof is in Θ_1 or Γ_2.

$$\dfrac{\begin{matrix}\pi\\ \Delta\vdash A\end{matrix}\qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\rho\\ C_1,\dots,C_1,\Gamma_1,A,\Gamma_2\vdash C_2\end{matrix}}{\dots}}{C_1,\Gamma_1,A,\Gamma_2\vdash C_2}\,\text{applications of (W)}}{\Gamma_1,A,\Gamma_2\vdash C_1\to C_2}\,{\to}\mathrm{R}}{\Gamma_1,\Delta,\Gamma_2\vdash C_1\to C_2}\,\mathrm{cut} \tag{3.6}$$

provided ρ is tailless, is reduced to

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \Delta\vdash A\end{matrix}\qquad\begin{matrix}\rho\\ C_1,\dots,C_1,\Gamma_1,A,\Gamma_2\vdash C_2\end{matrix}}{C_1,\dots,C_1,\Gamma_1,\Delta,\Gamma_2\vdash C_2}\,\mathrm{cut}}{\dots}}{C_1,\Gamma_1,\Delta,\Gamma_2\vdash C_2}\,\text{applications of (W)}}{\Gamma_1,\Delta,\Gamma_2\vdash C_1\to C_2}\,{\to}\mathrm{R}$$

$$\dfrac{\begin{matrix}\pi\\ \Delta\vdash A\end{matrix}\qquad \dfrac{\begin{matrix}\rho_1\\ \Theta_2,A,\Gamma_1\vdash B\end{matrix}\qquad\begin{matrix}\rho_2\\ \Theta_1,B,\Gamma_2\vdash C\end{matrix}}{\Theta_1,\Theta_2,A,\Gamma_1,\Gamma_2\vdash C}\,\mathrm{cut}}{\Theta_1,\Theta_2,\Delta,\Gamma_1,\Gamma_2\vdash C}\,\mathrm{cut} \tag{3.7}$$

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is reduced to

$$\dfrac{\dfrac{\overset{\pi}{\Delta \vdash A} \qquad \overset{\rho_1}{\Theta_2, A, \Gamma_1 \vdash B}}{\Theta_2, \Delta, \Gamma_1 \vdash B}\,\text{cut} \qquad \overset{\rho_2}{\Theta_1, B, \Gamma_2 \vdash C}}{\Theta_1, \Theta_2, \Delta, \Gamma_1, \Gamma_2 \vdash C}\,\text{cut}$$

By the induction hypothesis, the subproof of the reduced proof ending with the left-hand premise of the lower cut can be reduced to a maximalized W-normal proof, of the same or of a lower degree, of the same sequent. The first step of this reduction, which is one of the reduction steps (3.1)–(3.6), makes the lower cut maximal, and subsequent steps leave it so. We must apply (3.1)–(3.6) because the left rank of the upper cut in the lower figure is equal to 1 and the right rank is greater than 1 (the proof ρ_1 cannot be an axiom, and the left rank of the upper cut in the first figure is 1), and, moreover, ρ_1 cannot end with a cut.

(3.8)
$$\dfrac{\overset{\pi}{\Delta \vdash A} \qquad \dfrac{\overset{\rho_1}{\Theta_3 \vdash B} \qquad \overset{\rho_2}{\Theta_1, A, \Theta_2, B, \Gamma \vdash C}}{\Theta_1, A, \Theta_2, \Theta_3, \Gamma \vdash C}\,\text{cut}}{\Theta_1, \Delta, \Theta_2, \Theta_3, \Gamma \vdash C}\,\text{cut}$$

is reduced to

$$\dfrac{\overset{\rho_1}{\Theta_3 \vdash B} \qquad \dfrac{\overset{\pi}{\Delta \vdash A} \qquad \overset{\rho_2}{\Theta_1, A, \Theta_2, B, \Gamma \vdash C}}{\Theta_1, \Delta, \Theta_2, B, \Gamma \vdash C}\,\text{cut}}{\Theta_1, \Delta, \Theta_2, \Theta_3, \Gamma \vdash C}\,\text{cut}$$

and we reason as for (3.7), by applying the induction hypothesis to the subproof of the reduced proof ending with the right-hand premise of the lower cut. We have an analogous reduction step when A in the initial proof is in Γ. □

In terms of categories, the reduction steps (2.4) and (3.7) in the proof above correspond to associativity of composition, whereas (3.8) corresponds to bifunctoriality equalities.

We can now finally go into the third phase of our cut-elimination procedure, which is covered by the following theorem.

Theorem 6.2. Every maximalized proof of degree greater than 0 of a sequent of $\mathcal{G}$ can be reduced to a proof of lower degree of the same sequent.

Proof. Take a maximalized proof of $\mathcal{G}$ of degree greater than 0, and starting from the top of the proof apply to every maximal cut of the initial proof either the standard reduction steps like those of (1.5) and (1.6) of the proof of Theorem 4.1, or the standard

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reduction step that consists in replacing

$$\dfrac{\dfrac{A,\Gamma \vdash B}{\Gamma \vdash A \rightarrow B}\rightarrow\!R \qquad \dfrac{\Delta \vdash A \qquad \Theta, B, \Xi \vdash C}{\Theta, \Delta, A \rightarrow B, \Xi \vdash C}\rightarrow\!L}{\Theta, \Delta, \Gamma, \Xi \vdash C}\text{cut}$$

by

$$\dfrac{\dfrac{\Delta \vdash A \qquad A,\Gamma \vdash B}{\Delta, \Gamma \vdash B}\text{cut} \qquad \Theta, B, \Xi \vdash C}{\Theta, \Delta, \Gamma, \Xi \vdash C}\text{cut}$$

The result is a proof whose degree has decreased. □

By applying successively the first phase, the second phase and the third phase of our procedure, that is, Theorems 5.5, 6.1 and 6.2, and then again the first phase, the second phase, and so on, we must obtain after one second or third phase a proof of degree 0. If this phase was a second phase, then there are no cuts in this proof, whereas if this phase was a third phase, then there are cuts in the proof and all of them have atomic cut formulae. By applying in the latter case the first and second phase once more, we will end up with a cut-free proof, because there are no maximal cuts of degree 0.

7. Concluding comments

It is instructive to compare Gentzen's cut-elimination procedure with ours at the place where Gentzen has *critical mixes* (see the end of Section 2). These critical mixes correspond to the maximal cuts whose reduction we postpone until the third phase of our procedure. Gentzen's separation of a critical mix out of a mix, and leaving it below, corresponds to something achieved in the first and second phase of our procedure. When in the first phase a cut is pushed above a contraction and is replaced by two cuts, the second phase will ensure that the maximal cut that corresponds to the critical mix will be at its proper place below other cuts.

To work in the presence of the lattice connectives $\wedge$ and $\vee$, our procedure presupposes the presence of thinning (see the proofs of Lemma 5.3 and Theorem 5.5, case with ($\wedge$R)). So this procedure as it is formulated here cannot be transferred to relevant logic, which has contraction but lacks thinning, except if in this logic we omit the 'additive', that is, lattice, connectives and restrict ourselves to 'multiplicative' connectives.

The problem with the lattice connectives $\wedge$ and $\vee$ is that in the rules ($\wedge$R) and ($\vee$L) there are implicit contractions: in terms of a multiplicative rule, ($\wedge$R) could be reconstructed as

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma, \Gamma \vdash A \wedge B}\text{multiplicative rule}}{\cdots}\text{applications of (C) and (W)}}{\Gamma \vdash A \wedge B}$$

while for ($\vee$L) there is no such simple reconstruction, but similar contractions are involved.

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The W-normalization of the first phase of our procedure does not take care of these implicit contractions; that is, these are not pushed below other rules as far as they can go. Because of that, we can say that when in the second phase of that procedure, in cases (2.2) and (3.2) of the proof of Theorem 6.1 (which are taken over from the proof of Theorem 4.1), there is an increase in size in the transformed proof, this increase is again due to contraction. Contraction is, of course, to blame for the increase in size that occurs in the first phase of the procedure.

All the steps of our cut-elimination procedure are covered by equalities of bicartesian closed categories, which is not the case for all the steps of Gentzen's procedure. The categorially unjustified steps of Gentzen (1935) are like the following step, licensed by 3.121.1 in which

$$\dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A, A \to A \vdash A}\,{\to}\mathrm{L} \qquad \dfrac{A \vdash A \qquad A \vdash A}{A, A \to A \vdash A}\,{\to}\mathrm{L}}{A, A \to A, A \to A \vdash A}\ \text{mix, that is, cut}$$

is replaced by

$$\dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A, A \to A \vdash A}\,{\to}\mathrm{L}}{A, A \to A, A \to A \vdash A}\ \text{thinning and interchange}$$

Another problem is that Gentzen's mix

$$\dfrac{\Gamma \vdash A \qquad \Delta \vdash C}{\Gamma, \Delta^* \vdash C}$$

is strict in the sense that in Δ^* we must omit *all* the occurrences of A, whereas a 'liberal' mix where in Δ^* we must omit *some*, but not necessarily all, occurrences of A is better justified categorially. In terms of Gentzen's strict mix the following cut

$$\dfrac{\Gamma \vdash A \qquad A, A \vdash C}{\Gamma, A \vdash C}\ \text{cut}$$

is reconstructed as

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A \qquad A, A \vdash C}{\Gamma \vdash C}\ \text{mix}}{\cdots}\ \text{thinning and interchanges}}{\Gamma, A \vdash C}$$

which is not always justified. However, it is possible to mend Gentzen's mix-elimination procedure so that all of its steps are justified by equalities of bicartesian closed categories.

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