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On the effective, nef, and semi-ample monoids of blowups of Hirzebruch surfaces at collinear points

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Dedicated to Professor Brian Harbourne on the occasion of his 65th birthday

Abstract. This paper is devoted to determine the geometry of a class of smooth projective rational surfaces whose minimal models are the Hirzebruch ones; concretely, they are obtained as the blowup of a Hirzebruch surface at collinear points. Explicit descriptions of their effective monoids are given, and we present a decomposition for every effective class. Such decomposition is used to confirm the effectiveness of some divisor classes when the Riemann–Roch theorem does not give information about their effectiveness. Furthermore, we study the nef divisor classes on such surfaces. We provide an explicit description for their nef monoids, and, moreover, we present a decomposition for every nef class. On the other hand, we prove that these surfaces satisfy the anticanonical orthogonal property. As a consequence, the surfaces are Harbourne–Hirschowitz and their Cox rings are finitely generated. Finally, we prove that the complete linear system associated with any nef divisor is base-point-free; thus, the semi-ample and nef monoids coincide. The base field is assumed to be algebraically closed of arbitrary characteristic.

1 Introduction

In his Ph.D. thesis [36] of 1978, Rosoff studied the following question: given a smooth algebraic variety defined over an algebraically closed field K, is the effective monoid of X finitely generated? Here, the *effective monoid* associated with X is the set of all effective divisors modulo algebraic equivalence and we denote it by $\mathrm{Eff}(X)$. In particular, Rosoff focused on the case when X is the blowup of the projective plane $\mathbb{P}^2_K$ in at most eight points in general position, he gave a positive answer in this case by providing the minimal generating set that generates the effective monoid. Such results were published after in [37]. Nowadays, such problem is still open even in the case of surfaces, some contributions in this direction are [3, 4, 8–15, 17–19, 21, 30–34].

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In this work, we study this problem for a family of smooth projective rational surfaces obtained as the blowups of Hirzebruch surfaces at *collinear points*. Moreover, we give the minimal generating sets for such effective monoids and a decomposition for every effective class (see Theorem 2.1). We would like to emphasize that such decomposition is useful when the Riemann–Roch theorem does not imply the effectivity of a divisor class (see Section 3). Furthermore, we study the nef classes on such surfaces in order to compute the dimensions of all complete linear systems, to conclude that their *Cox rings* are finitely generated and to conclude that their semi-ample and nef monoids are equal (see Section 4).

Here, we recall some basic notions of Hirzebruch surfaces that we will need later on. For a fixed nonnegative integer n, the *nth Hirzebruch surface* Σ_n is the rational ruled surface defined by the locally free sheaf $\mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(-n)$ of rank two on the projective line $\mathbb{P}^1_K$ over an algebraically field K of arbitrary characteristic. It is well known that $\{\mathfrak{C}_n, \mathfrak{F}\}$ is a minimal generating set of the Néron–Severi group $\mathrm{NS}(\Sigma_n)$ of Σ_n as $\mathbb{Z}$-module, where $\mathfrak{C}_n$ is the class of a section C_n of Σ_n (unique if n is positive; in this case, such section is usually called the *negative section*) and $\mathfrak{F}$ is the class of a fiber F of Σ_n. The intersection form on Σ_n is given by the three equalities ${\mathfrak{C}_n}^2 = -n$, $\mathfrak{F}^2 = 0$, and $\mathfrak{C}_n \cdot \mathfrak{F} = 1$ (for more details, see, for example, [25, Chapter V, Section 2]).

The notion of *collinear points* for Σ_n is motivated by the following two facts. Let r be a positive integer:

a) Consider r collinear points $p_1, \dots, p_r$ contained in a line L of the projective plane $\mathbb{P}^2_K$. The surface obtained as the blowup of $\mathbb{P}^2_K$ at p_1 is the Hirzebruch surface Σ_1, where the exceptional divisor corresponding to p_1 is the negative section C_1 of Σ_1 and the strict transform $\tilde{L}$ of L is a fiber of Σ_1 that contains the points $p_2, \dots, p_r$. In this way, we can think the blowup of $\mathbb{P}^2_K$ at r collinear points as the blowup of Σ_1 at $r-1$ points contained in a fiber of Σ_1 with none of them belonging to C_1.
b) Consider a point p in $\mathbb{P}^2_K$, take r points infinitely near to p in the first infinitesimal neighborhood, and let S be the surface obtained as the blowup of $\mathbb{P}^2_K$ at such points. As in the previous case, the obtained surface when we blow up the point p is Σ_1 and the r infinitely near points are contained in the negative section C_1. So, we can obtain S as the blowup of Σ_1 at r points lying on the curve C_1.

Considering these facts, we introduce the concept of collinearity for a Hirzebruch surface.

Definition 1.1 Let n be a nonnegative integer. A finite number of points on Σ_n are *collinear* if all of them belong to a fiber or all of them are contained in C_n.

Note that in the case when $n \geq 1$, one has to distinguish between two cases depending whether there exists a point in the negative section or not. While in the case $n = 0$, there is always a fiber in the second ruling containing each of the points. Hence, for collinear points $p_1, \dots, p_r$ on Σ_n, the following cases occur:

Case a) $n = 0$ and the points are contained in a fiber of Σ_0. Note that in this case, there is always a curve of the second ruling passing through each point.
Case b) $n > 0$ and all the points are contained in a fiber of Σ_n. In such a situation, one of the following occurs:

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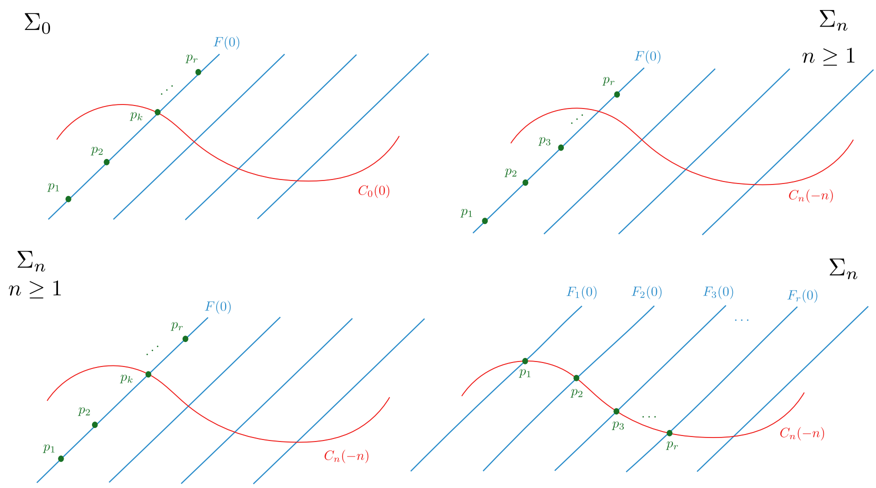

Figure 1: Configurations of collinear points in Σ_n.

Case b.1) none of them lie on C_n,
Case b.2) p_k lies on C_n for a unique $k = 1, \dots, r$.

Case c) $n \geq 0$ and all the points are contained on C_n.

Thus, the possible configurations for collinearity in Σ_n are illustrated in Figure 1.

Fix a nonnegative integer n and a positive integer r. Consider r collinear points $p_1, \dots, p_r$ on Σ_n. We denote by Y_n^r the blowup of Σ_n at $p_1, \dots, p_r$. Our main result regarding the finite generation of the effective monoid of Y_n^r is the following.

Theorem 1.1 (See Theorem 2.1) *Let Y_n^r be the blowup of the Hirzebruch surface Σ_n at r collinear points. Then the effective monoid* $\mathrm{Eff}(Y_n^r)$ *of the surface Y_n^r is finitely generated. Moreover, an explicit decomposition for every effective class is given in the proof of Theorem* 2.1.

The technique used to achieve the above result is purely geometric based on the intersection theory and some special divisors on Hirzebruch surfaces. See the proof of Theorem 2.1 for the explicit decomposition.

Another question related to the finite generation of the effective monoid is the finite generation of the Cox ring. In the case of a smooth projective variety X defined over an algebraically closed field K such that the linear and numerical equivalence are the same, the Cox ring of X is the K-algebra $\mathrm{Cox}(X)$ given by

$$\mathrm{Cox}(X) = \bigoplus_{(n_1,\dots,n_t)\in\mathbb{Z}^t} H^0(X, \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}),$$

where $\mathcal{L}_1, \dots, \mathcal{L}_t$ form a basis of the Picard group $\mathrm{Pic}(X)$ of X. One of the most interesting problems nowadays is the classification of smooth projective varieties whose Cox rings are finitely generated and also to determine explicitly the generators and relations for such K-algebras, this is justified from the point of view of the

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birational geometry classification of varieties. Indeed, Hu and Keel proved in [28] that there is an equivalence between the finite generation of the Cox ring of X and the fact that one is able to run the Minimal Model Program for any divisor on X. In the two-dimensional case, there are some results that ensure the finite generation of the Cox ring (for example, [1, 2, 6, 7, 16, 26, 29, 40]. However, there does not exist a complete and concrete classification of smooth projective rational surfaces whose Cox rings are finitely generated.

One way to achieve the finite generation of the Cox ring for an anticanonical rational surface (that is, a smooth projective rational surface whose anticanonical class is effective) is by means of the finite generation of the effective monoid and the so-called *anticanonical orthogonal property*.

Definition 1.2 A smooth projective surface S has the *anticanonical orthogonal property* whenever every nef divisor on S orthogonal to an anticanonical divisor is the zero divisor.

Here, a *nef divisor* D is a divisor on S such that $D \cdot E \geq 0$ for every effective divisor E on S. Thus, according to Definition 1.2, it is interesting the study of nef classes. This notion was introduced first in [13] in 2018. Then it was used in the context of anticanonical rational surfaces in [8, 14], and more generally studied in the context of regular surfaces in [5].

In our context, the surface Y_n^r is anticanonical (see Proposition 4.2), and then we will prove that anticanonical orthogonal property is satisfied in order to conclude the finite generation of the Cox ring.

The set of all classes of nef divisors on S will be denoted by $\mathrm{Nef}(S)$, and obviously it has an algebraic structure of a monoid. By a *nef class*, we mean the class of a nef divisor. Our main result regarding the nef monoid of Y_n^r is the following.

Theorem 1.2 (See Theorem 4.1) *Let Y_n^r be the blowup of the Hirzebruch surface Σ_n at r collinear points. Then the nef monoid $\mathrm{Nef}(Y_n^r)$ of the surface Y_n^r is finitely generated. Moreover, an explicit decomposition for every nef class is given in the proof of Theorem 4.1.*

In the proof of Theorem 4.1, we present the explicit decomposition for every nef class. This result along with the one in Theorem 1.1 generalizes the results obtained by Ottem in [35] regarding the finite generation of the effective and nef monoids.

On the other hand, another interesting problem is to determine the dimensions of the complete linear systems on a smooth projective surface (see, for example, [20, 23, 27, 39] when the surface is $\mathbb{P}^2_K$ and the points are in general position, and [22] when the points may be not in general position). In this direction, in [13], the following notion was introduced.

Definition 1.3 A smooth projective surface S is a *Harbourne–Hirschowitz surface* if for every effective and nef divisor H on S, the $\mathbb{Z}$-module $H^1(S, \mathcal{O}_S(H))$ vanishes.

The importance of the vanishing of the first cohomology groups for nef divisors is that one is able to compute the dimension of the complete linear systems. In particular, any anticanonical rational surface that satisfies the anticanonical orthogonal property is a Harbourne–Hirschowitz one (see [13, Theorem 2.5]). Thus, it turns out that our surfaces Y_n^r are Harbourne–Hirschowitz. Moreover, the complete linear system of every nef divisor on Y_n^r is base-point-free (see Theorem 4.6). Thus, every nef divisor is semi-ample; here, a *semi-ample* divisor D on a surface S is a divisor such that for sufficiently large s, the complete linear system $|sD|$ associated with sD is base-point-free.

This paper is organized as follows: In Section 2, we prove the finite generation of the effective monoid of Y_n^r, and we give an explicit decomposition for any effective class. Such decomposition is used in Section 3 to prove that some divisor classes are effective when the Riemann–Roch theorem is not able to give such information. Finally, in Section 4, we present a study of nef classes on Y_n^r; concretely, we provide the minimal generating set for the nef monoid and an explicit decomposition for any nef class, and we prove that the anticanonical orthogonal property is satisfied and that the complete linear systems of the nef divisors are base-point-free. The latter implies that the semi-ample and nef monoids of Y_n^r are equal.

2 The minimal generating set of the effective monoid

Recall that for a fixed positive integer r and nonnegative integer n, the surface Y_n^r is the blowup of the Hirzebruch surface Σ_n at r collinear points $p_1, \dots, p_r$. So, by construction, we have a birational morphism $\pi : Y_n^r \to \Sigma_n$ and Y_n^r is a smooth projective rational surface whose Picard number $\rho(Y_n^r)$ is equal to $r+2$. A minimal generating set for the Néron–Severi group $\mathrm{NS}(Y_n^r)$ of Y_n^r as $\mathbb{Z}$-module is given by $\{\mathcal{C}_n, \mathcal{F}, -\mathcal{E}_1, -\mathcal{E}_2, \dots, -\mathcal{E}_r\}$, where $\mathcal{C}_n$ is the class of the total transform of C_n by π, $\mathcal{F}$ is the class of the total transform of a fiber F of Σ_n not containing any of the points by π, and $\mathcal{E}_j$ is the class of the exceptional divisor corresponding to p_j for every $j = 1, \dots, r$. The intersection form on Y_n^r is given by the following equalities:

- $\mathcal{C}_n^2 = -n$,
- $\mathcal{F}^2 = 0$,
- $\mathcal{C}_n \cdot \mathcal{F} = 1$,
- $\mathcal{C}_n \cdot \mathcal{E}_j = \mathcal{F} \cdot \mathcal{E}_j = 0$ for all $j = 1, \dots, r$, and
- $\mathcal{E}_i \cdot \mathcal{E}_j = -\delta_{ij}$ for all $i, j = 1, \dots, r$, where δ_{ij} stands for the Kronecker delta.

In order to prove Theorem 1.1, we have to consider all the possible configurations of collinear point on Σ_n as it was given on page 2.

Theorem 2.1 *With notation as above, the effective monoid* $\mathrm{Eff}(Y_n^r)$ *is finitely generated and its minimal generating set* $\mathcal{M}$ *is given by the following:*

Case a)
1. $\mathcal{C}_0 - \mathcal{E}_i$ *for all* $i = 1, \dots, r$,
2. $\mathcal{F} - \sum_{j=1}^r \mathcal{E}_j$,
3. $\mathcal{E}_i$ *for all* $i = 1, \dots, r$.

Case b.1)
1. $\mathcal{C}_n$,
2. $\mathcal{F} - \sum_{j=1}^r \mathcal{E}_j$,
3. $\mathcal{E}_i$ *for all* $i = 1, \dots, r$.

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Case b.2) 1. $\mathcal{C}_n - \mathcal{E}_k$ *for the unique index* k,
2. $\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j$,
3. $\mathcal{E}_i$ *for all* $i = 1, \dots, r$.

Case c) 1. $\mathcal{C}_n - \sum_{j=1}^{r} \mathcal{E}_j$,
2. $\mathcal{F} - \mathcal{E}_i$ *for all* $i = 1, \dots, r$,
3. $\mathcal{E}_i$ *for all* $i = 1, \dots, r$.

Proof Let $\mathcal{M}$ be the set described in Theorem 2.1. It is clear that in each case, $\mathcal{M}$ is contained in $\mathrm{Eff}(Y_n^r)$. Conversely, let $\mathcal{D}$ be an element of $\mathrm{Eff}(Y_n^r)$. So, there exist integer numbers $a, b, c_1, \dots, c_r$ such that $\mathcal{D} = a\mathcal{C}_n + b\mathcal{F} - c_1\mathcal{E}_1 - \cdots - c_r\mathcal{E}_r$. Without loss of generality, we assume that $\mathcal{D}$ is irreducible and different from the elements of $\mathcal{M}$ in each case.

Case a) Note that the integer numbers $a - \sum_{j=1}^{r} c_j$, c_i, and $b - c_i$ are nonnegative since the intersection numbers $\mathcal{D} \cdot \left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right)$, $\mathcal{D} \cdot \mathcal{E}_i$, and $\mathcal{D} \cdot (\mathcal{C}_0 - \mathcal{E}_i)$ are greater than or equal to zero for all $i = 1, \dots, r$. Hence, we can write $\mathcal{D}$ in the following way:

$$\mathcal{D} = \sum_{j=1}^{r-1} c_j \left(\mathcal{C}_0 - \mathcal{E}_j\right) + \left(a - \sum_{j=1}^{r-1} c_j\right)(\mathcal{C}_0 - \mathcal{E}_r) + b\left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + b\sum_{j=1}^{r-1} \mathcal{E}_j + \left(a + b - \sum_{j=1}^{r} c_j\right)\mathcal{E}_r.$$

Case b.1) In this case, we may write $\mathcal{D}$ as

$$\mathcal{D} = a\mathcal{C}_n + b\left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{j=1}^{r} \left(b - c_j\right)\mathcal{E}_j,$$

and every coefficient is nonnegative since $\mathcal{D} \cdot \left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) = a - \sum_{j=1}^{r} c_j$, $\mathcal{D} \cdot \mathcal{C}_n = b - na$, and $\mathcal{D} \cdot \mathcal{E}_i = c_i$ are nonnegative for all $i = 1, \dots, r$.

Case b.2) The conditions $\mathcal{D} \cdot \left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) \geq 0$, $\mathcal{D} \cdot (\mathcal{C}_n - \mathcal{E}_k) \geq 0$, and $\mathcal{D} \cdot \mathcal{E}_i \geq 0$ imply that $a - \sum_{j=1}^{r} c_j$, $b - na - c_k$, and c_i are nonnegative for each $i = 1, \dots, r$. Then we have that

$$\mathcal{D} = a(\mathcal{C}_n - \mathcal{E}_k) + b\left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{\substack{j=1 \\ j \neq k}}^{r} \left(b - c_j\right)\mathcal{E}_j + (b + a - c_k)\mathcal{E}_k.$$

Case c) Finally, in this case, we can consider the next decomposition of $\mathcal{D}$:

$$\mathcal{D} = a\left(\mathcal{C}_n - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{j=1}^{r-1} c_j\left(\mathcal{F} - \mathcal{E}_j\right) + \left(b - \sum_{j=1}^{r-1} c_j\right)(\mathcal{F} - \mathcal{E}_r) + a\sum_{j=1}^{r-1} \mathcal{E}_j + \left(a + b - \sum_{j=1}^{r} c_j\right)\mathcal{E}_r.$$

Indeed, the integer numbers $\mathcal{D} \cdot \left(\mathcal{C}_n - \sum_{j=1}^{r} \mathcal{E}_j\right) = b - na - \sum_{j=1}^{r} c_j$, $\mathcal{D} \cdot \mathcal{E}_i = c_i$, and $\mathcal{D} \cdot (\mathcal{F} - \mathcal{E}_i) = a - c_i$ are nonnegative for each $i = 1, \dots, r$.

This completes the proof. ∎

Remark 2.2 This result generalizes the one obtained by Rosoff in [38] in the case when we are considering the ruled surface over the projective line $\mathbb{P}^1_K$.

3 Dimensions of complete linear systems without the use of the Riemann–Roch theorem

In this section, we illustrate some examples of effective classes on Y_n^r in each of the cases that we are considering. In all of them, it is given an effective class, fact that cannot be deduced from the Riemann–Roch theorem, but the decomposition exhibited in the proof in Theorem 2.1 will do. To clarify the notation below, if $\mathcal{D}$ is a divisor class on Y_n^r, then $h^i(Y_n^r, \mathcal{D})$ denotes the dimension of the ith cohomology group of the invertible sheaf associated with a divisor in the class of $\mathcal{D}$ for $i = 0, 1, 2$.

The strategy in all the examples below is as follows: we begin with a divisor class $\mathcal{D}$ such that $\mathcal{K}_{Y_n^r} - \mathcal{D}$ is not an effective class; this follows from the fact that $\mathcal{F}$ is a nef class and $\mathcal{F} \cdot (\mathcal{K}_{Y_0^r} - \mathcal{D}) < 0$. Here, $\mathcal{K}_{Y_n^r}$ stands for the canonical class on Y_n^r, and by [25, Chapter V, Lemma 2.10 and Proposition 3.3], we have that $\mathcal{K}_{Y_n^r} = -2\mathcal{C}_n - (n + 2)\mathcal{F} + \mathcal{E}_1 + \cdots + \mathcal{E}_r$. This implies that $h^2(Y_0^r, \mathcal{D}) = 0$. Consequently, by the Riemann–Roch theorem,

$$h^0(Y_n^r, \mathcal{D}) - h^1(Y_n^r, \mathcal{D}) = 1 + \frac{1}{2}\left(\mathcal{D}^2 - \mathcal{K}_{Y_n^r} \cdot \mathcal{D}\right).$$

In the considered examples, the right-hand side will be a negative integer, and thus the Riemann–Roch theorem cannot conclude that $\mathcal{D}$ is an effective class. However, the decomposition in Theorem 2.1 will imply that our class $\mathcal{D}$ is effective.

Example 3.1 In Case a), the following elements of $\mathrm{NS}(Y_0^r)$ are effective:

(1) The class $\mathcal{D} = \dfrac{r(r+1)(2r+1)}{6}\mathcal{C}_0 - \sum_{j=1}^{r} j^2 \mathcal{E}_j$. In this case,

$$h^0(Y_0^r, \mathcal{D}) - h^1(Y_0^r, \mathcal{D}) = 1 + \frac{1}{2}\sum_{j=1}^{r}(j^2 - j^4),$$

and the number on the right side of the above equation is negative when $r \geq 2$. On the other hand, using the decomposition of this case, we can write

$$\mathcal{D} = \sum_{j=1}^{r-1} j^2\left(\mathcal{C}_0 - \mathcal{E}_j\right) + r^2\left(\mathcal{C}_0 - \mathcal{E}_r\right).$$

(2) The class $\mathcal{D} = (r^2 - 1)\mathcal{C}_0 + \mathcal{F} - \sum_{j=1}^{r} r\mathcal{E}_j$. Here, the right-hand side of the Riemann–Roch theorem is $1 + \frac{1}{2}\left(3r^2 - r^3 - 2\right)$, which is negative once $r \geq 4$.

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In this case, the decomposition of $\mathcal{D}$ is equal to

$$\mathcal{D} = \sum_{j=1}^{r-1} r\left(\mathcal{C}_0 - \mathcal{E}_j\right) + (r-1)\left(\mathcal{C}_0 - \mathcal{E}_r\right) + \left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{j=1}^{r-1} \mathcal{E}_j.$$

(3) The class $\mathcal{D} = \dfrac{r(r+1)}{2}\mathcal{C}_0 + \mathcal{F} - \sum_{j=1}^{r} j\mathcal{E}_j$. The right-hand side in the Riemann–Roch theorem is $1 + \frac{1}{2}\left(2 + \sum_{j=1}^{r}(3j - j^2)\right)$, which is negative when $r \geq 5$. However, it occurs that

$$\mathcal{D} = \sum_{j=1}^{r-1} j\left(\mathcal{C}_0 - \mathcal{E}_j\right) + r\left(\mathcal{C}_0 - \mathcal{E}_r\right) + \left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{j=1}^{r} \mathcal{E}_j.$$

Example 3.2 For Case b), the next elements of $\mathrm{NS}(Y_n^r)$ are effective:

(1) The class $\mathcal{D} = \mathcal{C}_n + \mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j$. It follows that

$$h^0(Y_n^r, \mathcal{D}) - h^1(Y_n^r, \mathcal{D}) = 4 - n - r,$$

and note that $4 - n - r$ is negative when $n + r \geq 5$. However, for Case b.1), we have that

$$\mathcal{D} = \mathcal{C}_n + \left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right),$$

whereas in Case b.2),

$$\mathcal{D} = \left(\mathcal{C}_n - \mathcal{E}_k\right) + \left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + \mathcal{E}_k.$$

(2) The class $\mathcal{D} = \mathcal{C}_n + r\mathcal{F} - \sum_{j=1}^{r} j\mathcal{E}_j$. In fact, for Case b.1), it occurs that

$$\mathcal{D} = \mathcal{C}_n + r\left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{j=1}^{r} (r - j)\,\mathcal{E}_j,$$

whereas for Case b.2),

$$\mathcal{D} = \left(\mathcal{C}_n - \mathcal{E}_k\right) + r\left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{\substack{j=1 \\ j\neq k}}^{r} (r - j)\,\mathcal{E}_j + (r + 1 - k)\,\mathcal{E}_k.$$

In both cases, the right-hand side of the Riemann–Roch theorem is $2 + 2r - n - \frac{1}{2}\sum_{j=1}^{r}\left(j + j^2\right)$, which is negative when $r \geq 2$.

(3) The class $\mathcal{D} = \mathcal{C}_n + r^2\mathcal{F} - \sum_{j=1}^{r} j^2\mathcal{E}_j$. Note that the right-hand side of the Riemann–Roch theorem is $2 + 2r^2 - n - \frac{1}{2}\sum_{j=1}^{r}\left(j^2 + j^4\right)$, which is negative if $r \geq 2$. Moreover, for Case b.1), we have that

$$\mathcal{D} = \mathcal{C}_n + r^2\left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{j=1}^{r} \left(r^2 - j^2\right)\mathcal{E}_j,$$

whereas in Case b.2),

$$\mathcal{D} = (\mathcal{C}_n - \mathcal{E}_k) + r^2\left(\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{\substack{j=1\\ j\neq k}}^{r} \left(r^2 - j^2\right)\mathcal{E}_j + \left(r^2 + 1 - k^2\right)\mathcal{E}_k.$$

Example 3.3 In Case c), the following elements of $\mathrm{NS}(Y_n^r)$ are effective:

(1) The class $\mathcal{D} = \mathcal{C}_n + (r-1)\mathcal{F} - \sum_{j=1}^{r} \mathcal{E}_j$. In this case, it follows that

$$h^0(Y_n^r, \mathcal{D}) - h^1(Y_n^r, \mathcal{D}) = r - n,$$

and the right-hand side of this equation is negative when $n \geq r+1$. However, using the corresponding decomposition, we have

$$\mathcal{D} = \left(\mathcal{C}_n - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{j=1}^{r-1} \left(\mathcal{F} - \mathcal{E}_j\right) + \sum_{j=1}^{r-1} \mathcal{E}_j.$$

(2) The class $\mathcal{D} = \mathcal{C}_n + \left(r^2 - 1\right)\mathcal{F} - \sum_{j=1}^{r} r\mathcal{E}_j$. Indeed,

$$\mathcal{D} = \left(\mathcal{C}_n - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{j=1}^{r-1} r\left(\mathcal{F} - \mathcal{E}_j\right) + (r-1)\left(\mathcal{F} - \mathcal{E}_r\right) + \sum_{j=1}^{r-1} \mathcal{E}_j.$$

From the right-hand side of the Riemann–Roch theorem, we obtain the quantity $1 + \frac{1}{2}\left(3r^2 - r^3 - 2n - 2\right)$, which is negative whenever $r \geq 4$.

(3) The class $\mathcal{D} = \mathcal{C}_n + 3r\mathcal{F} - \sum_{j=1}^{r-1} \mathcal{E}_j - r\mathcal{E}_r$. For this class, we have that

$$\mathcal{D} = \left(\mathcal{C}_n - \sum_{j=1}^{r} \mathcal{E}_j\right) + \sum_{j=1}^{r-1} \left(\mathcal{F} - \mathcal{E}_j\right) + (2r+1)\left(\mathcal{F} - \mathcal{E}_r\right) + \sum_{j=1}^{r-1} \mathcal{E}_j + (r+2)\mathcal{E}_r,$$

whereas the right-hand side of the Riemann–Roch theorem $1 + \frac{1}{2}(9r + 4 - r^2 - 2n)$ is negative for $r \geq 10$.

4 The minimal generating set of the nef and semi-ample monoids

Now, we study the nef classes on the surface Y_n^r. First, we determine the generators of the nef monoid of Y_n^r and we present an explicit decomposition for every nef class. Later on, we prove that Y_n^r satisfies the anticanonical orthogonal property (see Definition 1.2). As a consequence, we conclude that Y_n^r is a Harbourne–Hirschowitz surface (see Definition 1.3) and that the Cox ring of Y_n^r is finitely generated. Finally, we prove that the complete linear system associated with any nef divisor is base-point-free. It is worth noting that we consider the distinct possibilities that can occur for a configuration of collinear points, i.e., we consider Cases a), b), and c) stated on page 2.

Now, we prove Theorem 1.2.

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Theorem 4.1 *With notation as above, the nef monoid* $\mathrm{Nef}(Y_n^r)$ *is finitely generated and its minimal generating set* $\mathcal{N}$ *is given by the following:*

Case a)
1. $\mathcal{C}_0$,
2. $\mathcal{F}$,
3. $\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_i$ *for all* $i = 1, \dots, r$,
4. $2\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_i - \mathcal{E}_j$ *for all* $1 \le i < j \le r$,
5. $3\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_i - \mathcal{E}_j - \mathcal{E}_\ell$ *for all* $1 \le i < j < \ell \le r$,

$\vdots$

$r + 2$. $r\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_1 - \cdots - \mathcal{E}_r$.

Case b.1)
1. $\mathcal{C}_n + n\mathcal{F}$,
2. $\mathcal{F}$,
3. $\mathcal{C}_n + n\mathcal{F} - \mathcal{E}_i$ *for all* $i = 1, \dots, r$.

Case b.2)
1. $\mathcal{C}_n + n\mathcal{F}$,
2. $\mathcal{F}$,
3. $\mathcal{C}_n + n\mathcal{F} - \mathcal{E}_i$ *for all* $i = 1, \dots, r$ *with* $i \neq k$,
4. $\mathcal{C}_n + (n+1)\mathcal{F} - \mathcal{E}_k$.

Case c)
1. $\mathcal{C}_n + n\mathcal{F}$,
2. $\mathcal{F}$,
3. $\mathcal{C}_n + (n+1)\mathcal{F} - \mathcal{E}_i$ *for all* $i = 1, \dots, r$,
4. $\mathcal{C}_n + (n+2)\mathcal{F} - \mathcal{E}_i - \mathcal{E}_j$ *for all* $1 \le i < j \le r$,
5. $\mathcal{C}_n + (n+3)\mathcal{F} - \mathcal{E}_i - \mathcal{E}_j - \mathcal{E}_\ell$ *for all* $1 \le i < j < \ell \le r$,

$\vdots$

$r + 2$. $\mathcal{C}_n + (n+r)\mathcal{F} - \mathcal{E}_1 - \cdots - \mathcal{E}_r$.

Proof Let $\mathcal{N}$ be the set described in Theorem 4.1. For each case, it is clear that $\mathcal{N}$ is contained in $\mathrm{Nef}(Y_n^r)$. On the other hand, we consider an element $\mathcal{D}$ in $\mathrm{Nef}(Y_n^r)$. Then there exist integer numbers $a, b, c_1, \dots, c_r$ such that $\mathcal{D} = a\mathcal{C}_n + b\mathcal{F} - c_1\mathcal{E}_1 - \cdots - c_r\mathcal{E}_r$. Without loss of generality, we assume that $\mathcal{D}$ is irreducible and different from the elements of $\mathcal{N}$ in each case.

Case a) The hypothesis on $\mathcal{D}$ implies that $\mathcal{D} \cdot (\mathcal{F} - \sum_{j=1}^r \mathcal{E}_j) \ge 0$, i.e., $a - \sum_{j=1}^r c_i \ge 0$. Also, we know that $\mathcal{D} \cdot (\mathcal{C}_0 - \mathcal{E}_i) = b - c_i \ge 0$ for all $i = 1, \dots, r$. Without loss of generality, we can make the assumption $c_1 \ge \cdots \ge c_r$, and thus $c_i - c_{i+1} \ge 0$ for all $i = 1, \dots, r-1$. There are two possibilities that can occur:

Case I: $b \ge \sum_{j=1}^r c_j$. For this possibility, we consider the following decomposition:

$$\left(a - \sum_{j=1}^r c_j\right)\mathcal{C}_0 + \left(b - \sum_{j=1}^r c_j\right)\mathcal{F} + c_1(\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_1) + \cdots + c_r(\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_r) = \mathcal{D}.$$

Case II: $b < \sum_{j=1}^r c_j$. In this case, the following decomposition recovers $\mathcal{D}$:

$$\left(a - \sum_{j=1}^r c_j\right)\mathcal{C}_0 + (b - c_1)\mathcal{F} + (c_1 - c_2)(\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_1)$$

$$+ (c_2 - c_3)(2\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_1 - \mathcal{E}_2) + (c_3 - c_4)(3\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3) + \cdots$$

$$+(c_{r-1}-c_r)\left((r-1)\mathcal{C}_0+\mathcal{F}-\sum_{j=1}^{r-1}\mathcal{E}_j\right)+c_r\left(r\mathcal{C}_0+\mathcal{F}-\sum_{j=1}^{r}\mathcal{E}_j\right)=\mathcal{D}.$$

Case b.1) Since $\mathcal{D}$ is a nef class, then $\mathcal{D}\cdot(\mathcal{F}-\sum_{i=1}^r\mathcal{E}_j)\geq 0$, that is, $a-\sum_{j=1}^r c_j\geq 0$. Since $\mathcal{D}\cdot\mathcal{C}_n\geq 0$, then we have that $b\geq na$. Now, note that

$$c_1(\mathcal{C}_n+n\mathcal{F}-\mathcal{E}_1)+\cdots+c_r(\mathcal{C}_n+n\mathcal{F}-\mathcal{E}_r)+\left(a-\sum_{j=1}^r c_j\right)(\mathcal{C}_n+n\mathcal{F})+(b-na)\mathcal{F}=\mathcal{D}.$$

Case b.2) Using the fact that $\mathcal{D}$ is nef, we have that $\mathcal{D}\cdot(\mathcal{F}-\sum_{j=1}^r\mathcal{E}_j)=a-\sum_{j=1}^r c_j\geq 0$. Also, note that $\mathcal{D}\cdot(\mathcal{C}_n-\mathcal{E}_k)=b-na-c_k\geq 0$. Thus, we have the equality

$$c_1(\mathcal{C}_n+n\mathcal{F}-\mathcal{E}_1)+\cdots+c_k(\widehat{\mathcal{C}_n+n\mathcal{F}}-\mathcal{E}_k)+\cdots+c_r(\mathcal{C}_n+n\mathcal{F}-\mathcal{E}_r)$$

$$+c_k(\mathcal{C}_n+(n+1)\mathcal{F}-\mathcal{E}_k)+\left(a-\sum_{j=1}^r c_j\right)(\mathcal{C}_n+n\mathcal{F})+(b-na-c_k)\mathcal{F}=\mathcal{D};$$

here, the term $c_k(\widehat{\mathcal{C}_n+n\mathcal{F}}-\mathcal{E}_k)$ is omitted.

Case c) The condition $\mathcal{D}$ nef implies that $\mathcal{D}\cdot(\mathcal{C}_n-\sum_{j=1}^r\mathcal{E}_j)=b-na-\sum_{j=1}^r c_i\geq 0$. In addition, we have that $\mathcal{D}\cdot(\mathcal{F}-\mathcal{E}_i)=a-c_i\geq 0$ for all $i=1,\ldots,r$. We distinguish two cases:

Case I: $a\geq\sum_{j=1}^r c_j$. Here, we have that

$$\left(a-\sum_{j=1}^r c_j\right)(\mathcal{C}_n+n\mathcal{F})+\left(b-na-\sum_{j=1}^r c_j\right)\mathcal{F}+c_1(\mathcal{C}_n+(n+1)\mathcal{F}-\mathcal{E}_1)+\cdots$$
$$+c_r(\mathcal{C}_n+(n+1)\mathcal{F}-\mathcal{E}_r)=\mathcal{D}.$$

Case II. $a<\sum_{j=1}^r c_j$. Without loss of generality, we assume that $c_1\geq\cdots\geq c_r$. Hence, we have that $c_i-c_{i+1}\geq 0$ for all $i=1,\ldots,r-1$. In this case, we consider the following decomposition:

$$\left(b-na-\sum_{j=1}^r c_j\right)\mathcal{F}+(a-c_1)(\mathcal{C}_n+n\mathcal{F})+(c_1-c_2)(\mathcal{C}_n+(n+1)\mathcal{F}-\mathcal{E}_1)$$
$$+(c_2-c_3)(\mathcal{C}_n+(n+2)\mathcal{F}-\mathcal{E}_1-\mathcal{E}_2)+(c_3-c_4)(\mathcal{C}_n+(n+3)\mathcal{F}-\mathcal{E}_1-\mathcal{E}_2-\mathcal{E}_3)$$
$$+\cdots+(c_{r-1}-c_r)\left(\mathcal{C}_n+(n+(r-1))\mathcal{F}-\sum_{j=1}^{r-1}\mathcal{E}_j\right)+c_r\left(\mathcal{C}_n+(n+r)\mathcal{F}-\sum_{j=1}^{r}\mathcal{E}_j\right)$$
$$=\mathcal{D}.$$

This completes the proof. ∎

In order to prove the fulfillment of the anticanonical orthogonal property, we determine a decomposition for the anticanonical class of Y_n^r in each of the possible cases that can occur.

Proposition 4.2 *With the above notation, the surface Y_n^r is anticanonical. Moreover, we can write the anticanonical class $-\mathcal{K}_{Y_n^r}$ using $r+2$ classes of smooth rational curves:*

- *For Case a),*

$$-\mathcal{K}_{Y_0^r} = 2\left(\mathcal{C}_0 - \mathcal{E}_1\right) + 2\left(\mathcal{F} - \sum_{j=1}^{r}\mathcal{E}_j\right) + 3\mathcal{E}_1 + \sum_{j=2}^{r}\mathcal{E}_j.$$

- *For Case b.1),*

$$-\mathcal{K}_{Y_n^r} = 2\mathcal{C}_n + (n+2)\left(\mathcal{F} - \sum_{j=1}^{r}\mathcal{E}_j\right) + (n+1)\sum_{j=1}^{r}\mathcal{E}_j.$$

- *For Case b.2),*

$$-\mathcal{K}_{Y_n^r} = 2\left(\mathcal{C}_n - \mathcal{E}_k\right) + (n+2)\left(\mathcal{F} - \sum_{j=1}^{r}\mathcal{E}_j\right) + (n+3)\mathcal{E}_k + (n+1)\sum_{\substack{j=1\\ j\neq k}}^{r}\mathcal{E}_j.$$

- *For Case c),*

$$-\mathcal{K}_{Y_n^r} = 2\left(\mathcal{C}_n - \sum_{j=1}^{r}\mathcal{E}_j\right) + (n+2)\left(\mathcal{F} - \mathcal{E}_1\right) + (n+3)\mathcal{E}_1 + \sum_{j=2}^{r}\mathcal{E}_j.$$

Proof In each case, the elements that appear in the right side of the equalities are the classes of the exceptional divisors, the class of the strict transform $\widetilde{C_n}$ of the curve C_n (that is, $\mathcal{C}_0 - \mathcal{E}_1$ in Case a), $\mathcal{C}_n$ in Case b.1), $\mathcal{C}_n - \mathcal{E}_k$ in Case b.2), and $\mathcal{C}_n - \sum_{j=1}^{r}\mathcal{E}_j$ in Case c)), and the class of the strict transform $\tilde{F}$ of an adequate fiber in the respective case: in the Cases a), b.1), and b.2), the fiber containing all the points (that is, the class of $\tilde{F}$ is equal to $\mathcal{F} - \sum_{j=1}^{r}\mathcal{E}_j$), and in the Case c), the fiber containing the point p_1 (that is, the class of $\tilde{F}$ is equal to $\mathcal{F} - \mathcal{E}_1$). So, we have that such classes are effective. We conclude the result by [25, Chapter V, Corollary 2.11 and Proposition 3.3]. ∎

Now, we prove that the surface Y_n^r satisfies the anticanonical orthogonal property and some consequences of this fact.

Theorem 4.3 *With the above notation, the surface Y_n^r satisfies the anticanonical orthogonal property.*

Proof Let H be a nef divisor on Y_n^r, and denote its class in $\mathrm{NS}(Y_n^r)$ by $\mathcal{H}$. Then there are nonnegative integers $a, b, c_1, \ldots, c_r$ such that $\mathcal{H} = a\mathcal{C}_n + b\mathcal{F} - \sum_{i=1}^{r} c_i\mathcal{E}_i$. Now, assume that $-\mathcal{K}_{Y_n^r}\cdot\mathcal{H} = 0$. Using this hypothesis, the corresponding decomposition of the anticanonical class of Proposition 4.2 in each case, and the fact that the intersection number of $\mathcal{H}$ with each component of $-\mathcal{K}_{Y_n^r}$ is equal to zero (since $\mathcal{H}$ is nef), we conclude that the class $\mathcal{H}$ is equal to zero. Indeed,

Case a) In this case, the equation $-\mathcal{K}_{Y_0^r}\cdot\mathcal{H} = 0$ implies the conditions $(\mathcal{C}_0 - \mathcal{E}_1)\cdot\mathcal{H} = 0$, $(\mathcal{F} - \sum_{j=1}^{r}\mathcal{E}_j)\cdot\mathcal{H} = 0$, and $\mathcal{E}_i\cdot\mathcal{H} = 0$ for every $i = 1, \ldots, r$. So, $b - c_1 = 0$, $a -$

$\sum_{j=1}^r c_j = 0$, and $c_i = 0$ for every $i = 1, \ldots, r$. Thus, the integers $a, b, c_1, \ldots, c_r$ are zero. Therefore, $\mathcal{H} = 0$.

Case b.1) Here, we have the conditions $\mathcal{C}_n \cdot \mathcal{H} = 0$, $(\mathcal{F} - \sum_{j=1}^r \mathcal{E}_j) \cdot \mathcal{H} = 0$, and $\mathcal{E}_i \cdot \mathcal{H} = 0$ for every $i = 1, \ldots, r$. These imply that $b = na$, $a - \sum_{j=1}^r c_j = 0$, and $c_i = 0$ for every $i = 1, \ldots, r$. Consequently, $\mathcal{H}$ is equal to zero.

Case b.2) In this case, the conditions $(\mathcal{C}_n - \mathcal{E}_k) \cdot \mathcal{H} = 0$, $(\mathcal{F} - \sum_{j=1}^r \mathcal{E}_j) \cdot \mathcal{H} = 0$, and $\mathcal{E}_i \cdot \mathcal{H} = 0$ for every $i = 1, \ldots, r$ hold. Hence, $b = na + c_k$, $a - \sum_{j=1}^r c_j = 0$, and $c_i = 0$ for every $i = 1, \ldots, r$. So, $\mathcal{H} = 0$.

Case c) For the last case, we know that $(\mathcal{C}_n - \sum_{j=1}^r \mathcal{E}_j) \cdot \mathcal{H} = 0$, $(\mathcal{F} - \mathcal{E}_1) \cdot \mathcal{H} = 0$, and $\mathcal{E}_i \cdot \mathcal{H} = 0$ for every $i = 1, \ldots, r$. Hence, $b = na + \sum_{j=1}^r c_j$, $a - c_1 = 0$ and $c_i = 0$ for every $i = 1, \ldots, r$. Therefore, $\mathcal{H}$ is the zero class.

In all cases, we obtain that $\mathcal{H} = 0$; therefore, we conclude that $H = 0$. ■

Corollary 4.4 *With the above notation, the surface Y_n^r is Harbourne–Hirschowitz.*

Proof The surface Y_n^r satisfies the anticanonical orthogonal property by the above theorem. We conclude the statement by [13, Theorem 2.5]. ■

Corollary 4.5 *With the above notation, the Cox ring of Y_n^r is finitely generated.*

Proof The surface Y_n^r has a finitely generated effective monoid (see Theorem 2.1) and satisfies the anticanonical orthogonal property (see Theorem 4.3). So, we are done using [7, Theorem 1]. ■

Theorem 4.6 *With the above notation, if H is a nef divisor on Y_n^r, then the complete linear system $|H|$ is base-point-free.*

Proof Let H be a nonzero nef divisor on Y_n^r. By Theorem 4.3, the integer $-K_{Y_n^r} \cdot H$ is greater than zero. There exist nonnegative integers $a, b, c_1, \ldots, c_r$ such that $\mathcal{H} = a\mathcal{C}_n + b\mathcal{F} - \sum_{i=1}^r c_i \mathcal{E}_i$, where $\mathcal{H}$ is the class of H in $\mathrm{NS}(Y_n^r)$. By [24, Theorem III.1(a)], it is sufficient to ensure that the intersection number of $-\mathcal{K}_{Y_n^r}$ and $\mathcal{H}$ is greater than or equal to 2. So, we proceed by contradiction: suppose that $-\mathcal{K}_{Y_n^r} \cdot \mathcal{H} = 1$. Again, we use the decomposition of the anticanonical class given in Proposition 4.2 and the fact that the intersection number of $\mathcal{H}$ with only one of the components of $-\mathcal{K}_{Y_n^r}$ is equal to one, whereas the other intersection numbers are equal to zero (since $\mathcal{H}$ is nef).

Case a) Here, we have the existence of $i = 2, \ldots, r$ such that $\mathcal{E}_i \cdot \mathcal{H} = 1$ and $\mathcal{E}_j \cdot \mathcal{H} = 0$ for $j = 1, \ldots, r$ with $j \neq i$, and also we have that $(\mathcal{C}_0 - \mathcal{E}_1) \cdot \mathcal{H} = 0$ and $(\mathcal{F} - \sum_{j=1}^r \mathcal{E}_j) \cdot \mathcal{H} = 0$. From these equalities, it follows that $\mathcal{H} = \mathcal{C}_0 - \mathcal{E}_i$, but such class is not nef. Indeed, the self-intersection of $\mathcal{C}_0 - \mathcal{E}_i$ is negative.

Case b) The condition $-\mathcal{K}_{Y_n^r} \cdot \mathcal{H} = 1$ is impossible because the coefficients in the decomposition of the anticanonical class in both Cases b.1) and b.2) are larger than one.

Case c) The only possibility that may occur is $\mathcal{E}_i \cdot \mathcal{H} = 1$ for some $i = 2, \ldots, r$, $\mathcal{E}_j \cdot \mathcal{H} = 0$ for every $j = 1, \ldots, r$ with $j \neq i$, $(\mathcal{C}_n - \sum_{j=1}^r \mathcal{E}_j) \cdot \mathcal{H} = 0$, and $(\mathcal{F} - \mathcal{E}_1) \cdot \mathcal{H} = 0$.

Thus, $\mathcal{H}$ would be equal to $\mathcal{F} - \mathcal{E}_i$, but the latter is not nef. Indeed, the self-intersection of $\mathcal{F} - \mathcal{E}_i$ is negative.

In all cases, we obtain a contradiction. Therefore, $-\mathcal{K}_{Y_n^r} \cdot \mathcal{H}$ is at least equal to 2. ∎

Corollary 4.7 *With the above notation, the semi-ample and nef monoids of Y_n^r are equal.*

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