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Chains of P-points

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Abstract. It is proved that the Continuum Hypothesis implies that any sequence of rapid P-points of length $< \mathfrak{c}^+$ that is increasing with respect to the Rudin–Keisler ordering is bounded above by a rapid P-point. This is an improvement of a result from B. Kuzeljevic and D. Raghavan. It is also proved that Jensen's diamond principle implies the existence of an unbounded strictly increasing sequence of P-points of length ω_1 in the Rudin–Keisler ordering. This shows that restricting to the class of rapid P-points is essential for the first result.

1 Introduction

The Rudin–Keisler ordering on ultrafilters, introduced in the late sixties [6, 15, 16], turned out to be a very useful tool for studying properties of ultrafilters. A variant of this ordering, the Rudin–Frolík ordering, was used by Frolík [4] to prove in ZFC that the space of non-principal ultrafilters on ω is non-homogeneous. Many combinatorial properties can be characterized in terms of the ordering; *e.g.*, selective (or Ramsey) ultrafilters are precisely those that are minimal in the Rudin–Keisler ordering, Q-points are those that are minimal in the Rudin–Blass ordering, P-points are those below which the Rudin–Keisler and Rudin–Blass orderings coincide.

The first comprehensive study of the Rudin–Keisler (RK) order was done by A. Blass in his thesis [1]. He continued his investigations by considering the lower part of the ordering, *viz.*, the ordering of P-points [2]. He showed that, under suitable assumptions, the ordering can be very rich. Assuming Martin's Axiom (MA), he showed that

- there are $2^{\mathfrak{c}}$ many minimal P-points,
- there are no maximal P-points,
- the ordering of P-points is σ-closed, both downwards and upwards,
- the real line, as well as ω_1, can be embedded into the P-points.

These results were later extended by several authors [8, 12, 14]. The results motivating the research that went into this paper were obtained by B. Kuzeljević and D. Raghavan [7]. They proved the following result.

Theorem 1.1 (Kuzeljević and Raghavan) *Assume* MA. *The ordinal* $\mathfrak{c}^+$ *can be embedded into the ordering of (rapid) P-points.*

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Since any ultrafilter has at most $\mathfrak{c}$-many RK-predecessors, the above is the best possible result as far as embedding of ordinals is concerned. Kuzeljević and Raghavan used the notion of a δ-generic sequence of P-points, which allowed them to carry through an inductive construction of length $\mathfrak{c}^+$ [7].

In this paper we improve upon their results as follows. In Theorem 3.4 we show that, assuming the Continuum Hypothesis (CH), the ordering of *rapid* P-points is, in fact, $\mathfrak{c}^+$-closed.

Theorem 1.2 *Assume CH. Any increasing sequence of rapid P-points of length $< \mathfrak{c}^+$ is bounded above by a rapid P-point.*

Unlike many earlier results, this theorem is more than just an embedding result, for it provides new information about the global structure of the class of rapid P-points under the Rudin–Keisler ordering.

We also show (Theorem 4.10) that the fact that we are looking at *rapid* P-points is crucial. Assuming $\diamond$ (though we suspect that CH is enough), we construct an increasing sequence of P-points of length ω_1 without any P-point upper bound.

The chains of P-points of length $\mathfrak{c}^+$ constructed in [7] enjoy a slightly stronger property than the long chains that can be built using the technique from Section 3 of this paper. The chains of [7] are all increasing in the $\leq^+_{\mathrm{RB}}$ ordering, but our technique is insufficient to ensure this property for any of the chains of length $\mathfrak{c}^+$ here. Thus the existence statement proved in [7] is stronger than the existence result that is derivable from the work in Section 3.

We should also comment on our assumptions. Since S. Shelah [20] showed that P-points need not exist at all, or there might be just one [18, Chapter VI], some assumption guaranteeing that the structure is rich is needed. For simplicity we use CH, though a weaker assumption, *e.g.*, MA, would be sufficient for our results.

2 Preliminaries

In this section we introduce the basic notions and state some standard facts.

Definition 2.1 ([17]) An ultrafilter $\mathcal{U}$ on ω is a *P-point* provided that for any sequence $\langle X_n : n < \omega \rangle$ of elements of $\mathcal{U}$ there is an $X \in \mathcal{U}$ such that $|X \setminus X_n| < \omega$ for each n. The last condition will also be denoted by $X \subseteq^* X_n$.

The following is an alternate characterization that we will often use.

Folklore An ultrafilter $\mathcal{U}$ is a P-point if and only if every function $f : \omega \to \omega$ is either constant or finite-to-one on some set in $\mathcal{U}$.

Definition 2.2 Given a family $\mathcal{P}$ of functions from ω to ω we say that a function $f : \omega \to \omega$ *dominates* $\mathcal{P}$ if $g \leq^* f$ for each $g \in \mathcal{P}$, where

$$g \leq^* f \iff (\forall^\infty n)(g(n) \leq f(n)),$$

and where $\forall^\infty n$ is a shortcut for "for all but finitely many n".

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Definition 2.3 ([11]) An ultrafilter $\mathcal{U}$ on ω is *rapid* if for every $f : \omega \to \omega$ there is an $X \in \mathcal{U}$ such that the function enumerating X in increasing order dominates $\{f\}$. To make notation simpler, we will write $X(n)$ to denote the n-th element of X in its increasing enumeration and $X[n] = X \setminus X(n)$.

Again we will use an alternate characterization.

Fact 2.4 An ultrafilter $\mathcal{U}$ is rapid if and only if for every partition of ω into finite sets $\{K_n : n < \omega\}$ there is $X \in \mathcal{U}$ such that $|X \cap K_n| \leq n$ for all $n < \omega$ if and only if for every infinite $A \subseteq \omega$ there is $X \in \mathcal{U}$ such that $|X \cap n| \leq |A \cap n|^2$ for all $n < \omega$.

Definition 2.5 ([6]) The *Rudin–Keisler ordering* of ultrafilters is defined as follows. Given two ultrafilters $\mathcal{U}, \mathcal{V}$ on ω, we say that $\mathcal{U}$ is *Rudin–Keisler below* (or that it is *Rudin–Keisler reducible* to) $\mathcal{V}$, denoted $\mathcal{U} \leq_{RK} \mathcal{V}$, if there is a function $f : \omega \to \omega$ such that $\mathcal{U} = f_*(\mathcal{V}) = \left\{X \subseteq \omega : f^{-1}[X] \in \mathcal{V}\right\}$. If the function is finite-to-one, we say that $\mathcal{U}$ is *Rudin–Blass* below $\mathcal{V}$, $\mathcal{U} \leq_{\mathrm{RB}} \mathcal{V}$. If the function is both finite-to-one *and nondecreasing*, we write $\mathcal{U} \leq^{+}_{\mathrm{RB}} \mathcal{V}$.

More information about the $\leq^{+}_{\mathrm{RB}}$ ordering on the ultrafilters can be found in [9]. A major difference between the $\leq_{\mathrm{RB}}$ and $\leq^{+}_{\mathrm{RB}}$ orderings, which was discovered by Laflamme and Zhu [9], is that $\leq^{+}_{\mathrm{RB}}$ is a tree-like ordering. In other words, for any ultrafilters $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$, if $\mathcal{U} \leq^{+}_{\mathrm{RB}} \mathcal{W}$ and $\mathcal{V} \leq^{+}_{\mathrm{RB}} \mathcal{W}$, then either $\mathcal{U} \leq^{+}_{\mathrm{RB}} \mathcal{V}$ or $\mathcal{V} \leq^{+}_{\mathrm{RB}} \mathcal{U}$. This is very much false for the $\leq_{\mathrm{RB}}$ ordering, even when it is restricted to the class of P-points, as was shown by Blass [2], who constructed a P-point with two incomparable predecessors assuming MA.

It is easy to see that being rapid and being a P-point are preserved when going down in the Rudin–Keisler ordering and that the Rudin–Keisler and Rudin–Blass orderings coincide below every P-point. Also, since Rudin–Keisler reducibility has to be witnessed by some function $f : \omega \to \omega$ and since two RK-inequivalent ultrafilters cannot be witnessed to be below a third by a single function, it immediately follows that every ultrafilter has at most $\mathfrak{c}$-many RK-predecessors.

Another ordering of ultrafilters is the Tukey ordering. It was introduced by Tukey [19] for comparing the cofinal type of arbitrary directed partial orders. Isbell [5] was the first to use the Tukey ordering to compare ultrafilters.

Definition 2.6 ([5]) Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω. We say that $\mathcal{U} \leq_T \mathcal{V}$, *i.e.*, $\mathcal{U}$ is *Tukey reducible* to $\mathcal{V}$ or $\mathcal{U}$ is *Tukey below* $\mathcal{V}$ if there is a map $\phi : \mathcal{V} \to \mathcal{U}$ such that $\forall A, B \in \mathcal{V}[A \subseteq B \implies \phi(A) \subseteq \phi(B)]$ and $\forall A \in \mathcal{U} \exists B \in \mathcal{V}[\phi(B) \subseteq A]$. We say that $\mathcal{U} \equiv_T \mathcal{V}$, *i.e.*, $\mathcal{U}$ is *Tukey equivalent* to $\mathcal{V}$ if $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$.

Recently, interest in this ordering on ultrafilters has been revived [3,10,13].

Finally, to eliminate some extraneous brackets, we will use the convenient standard shorthand $f^{-1}(n)$ to denote the preimage of $\{n\}$ instead of the formally more correct $f^{-1}[\{n\}]$.

3 There Is No Short Unbounded Chain of Rapid P-points

We start with a few simple observations. Below we use Poly to denote the following set of polynomial[1] functions: $\{n^k : k < \omega\}$.

Observation 3.1 If $f : \omega \to \omega$ dominates Poly, then so does $f'(n) = \frac{f(n)}{2n} - n^2$.

It is easy to see that the function n in Fact 2.4 could as well have been replaced by any function tending to infinity.

Observation 3.2 If $s : \omega \to \omega$ is a function tending to infinity (for instance, lim inf $s(n) = \infty$), $\pi : \omega \to \omega$ is a finite-to-one function, and $\mathcal{U}$ is a rapid ultrafilter, then there is $X \in \mathcal{U}$ such that $(\forall n < \omega)(|\pi^{-1}(n) \cap X| \leq s(n))$.

We aim to show that each RK-increasing chain of rapid P-points of length less than ω_2 has a rapid P-point on top. We do this by taking the chain and recursively constructing the future projections (called g in the following proposition) from the top to each ultrafilter in the sequence. If these projections commute with each of the maps witnessing the RK-relations in the chain, then the inverse images of the chain by these projections will generate a P-filter. By a relatively easy argument we can guarantee that it will be an ultrafilter (making sure that at each step we decide one set). To make it rapid, we have to work more. For this purpose, we will also build a tower on the side (the Ts in the following proposition) that will generate a rapid P-point and, moreover, this P-point will be compatible with the final P-filter.

Proposition 3.3 gives a single step of the construction. The set A in the assumption will be later used to make sure that the top filter is both an ultrafilter and rapid. The key property that will keep the induction going will be the fact that the g_αs are finite-to-one, but *not* bounded-to-one in a very strong sense: the size of the preimages of points (we will call this somewhat imprecisely the *growth rate* of g) will dominate a function s which will in turn dominate the set Poly (conditions (i) and (ii)). Moreover the first part of condition (iv) will guarantee that the maps g_α will not be bijections on some large set (otherwise our supposed upper bound would be RK-equivalent to some $\mathcal{U}_\alpha$).

We use the following conventions: we imagine each $\mathcal{U}_\alpha$ lives on a separate copy of ω (the α-the level). We will use the letter m to denote numbers on the first level, *i.e.*, where $\mathcal{U}_0$ lives, the letter n will denote numbers living on some level $0 < \alpha < \delta$, the letter l will denote numbers living on the final level, *i.e.*, where the top ultrafilter we will be constructing lives, and the letter k will denote numbers living on level δ. The letters i and j will be used as unrelated natural numbers. If a function has two ordinal indices $\alpha\beta$, they indicate that it goes from the α-th level down to the β-th level. Finally, the functions g_α go from the final level to the level indicated by their ordinal index.

[1] The fact that they are polynomials is not important. We could as well have chosen all functions of some countable elementary submodel of the universe; all that we need is that each function grows much faster than the previous one.

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Proposition 3.3 *Assume $\delta < \omega_1$ and $\langle \mathcal{U}_\alpha : \alpha \le \delta \rangle$ is an RK-increasing sequence of rapid P-points as witnessed by finite-to-one maps $\Pi = \{\pi_{\alpha\beta} : \beta \le \alpha \le \delta\}$ with $\pi_{\alpha\alpha} = \mathrm{Id}$ for each $\alpha \le \delta$. Also, let $\bar{s} = \langle s_\alpha : \alpha < \delta \rangle$ be a sequence of maps, each dominating* Poly, *let $\bar{g} = \langle g_\alpha : \alpha < \delta \rangle$ be a sequence of finite-to-one maps, let $\overline{T} = \langle T_\alpha : \alpha < \delta \rangle$ be a $\subseteq^*$-decreasing sequence of subsets of ω, and let $A \subseteq \omega$. Suppose, moreover, that the following conditions are satisfied.*

(i) *The growth rate of g_α dominates $s_\alpha \circ \pi_{\alpha 0}$,* i.e.,

$$(\forall \alpha)(\forall^\infty n)(|g_\alpha^{-1}[\{n\}]| \ge s_\alpha(\pi_{\alpha 0}(n))).$$

(ii) *The sequence $\bar{s}$ is $<^*$ decreasing in the following (stronger) sense:*

$$(\forall \alpha < \beta < \delta)(\forall^\infty m)\Big(s_\alpha(m) \ge \frac{s_\alpha(m)}{2m} - m^2 \ge s_\beta(m)\Big).$$

(iii) *$\Pi \cup \{g_\alpha : \alpha < \delta\}$ commute,* i.e., *for all $\beta \le \alpha < \delta$, the following diagram commutes on a $\mathcal{U}_\alpha$-large set:*

$$\begin{tikzcd} \omega \arrow[d, "\pi_{\alpha\beta}"'] & \omega \arrow[l, "g_\alpha"'] \arrow[dl, "g_\beta"] \\ \omega & \end{tikzcd}.$$

Formally, there is $X \in \mathcal{U}_\alpha$ such that $g_\beta(l) = \pi_{\alpha\beta}(g_\alpha(l))$ for each $l \in g_\alpha^{-1}[X]$.

(iv) *For each $\alpha < \delta$, there is $X \in \mathcal{U}_\alpha$ such that $\lim_{n \in X} |g_\alpha^{-1}(n) \cap T_\alpha| = \infty$, while also $|g_\alpha^{-1}(n) \cap T_\alpha| \le \min\big(g_\alpha^{-1}(n) \cup \{\pi_{\alpha 0}(n)\}\big)$ for each $n \in X$.*

Then we can extend the sequences $\bar{g}, \bar{s}, \overline{T}$ by constructing the maps g_δ and s_δ and a set T_δ so that (corresponding modifications of) (i–iv) *are still satisfied and, moreover, $(\forall i)(|T_\delta \cap i| \le |A \cap i|^2)$ and T_δ decides A,* i.e., *$T_\delta \subseteq A$ or $T_\delta \cap A = \varnothing$.*

Proof We first introduce some notation. Fix $D \subseteq \delta$ a cofinal subset of δ of order type ω such that $0 \in D$. In our construction we will only deal with $\alpha \in D$. Given $\alpha \in D$, we write $\alpha^+ = \min(\{\beta \in D : \alpha < \beta\})$ for the successor of α in D. We also let $\#\alpha = |D \cap \alpha|$, *i.e.*, α is the $\#\alpha$-th element of D. Next we use D to enumerate Poly in an increasing sequence: Poly $= \{p_\alpha : \alpha \in D\}$, where $p_\alpha \le p_{\alpha^+}$.

Given $\alpha \in D$, let $c_n^\alpha = \pi_{\delta\alpha}^{-1}(n)$. Since $\mathcal{U}_\delta$ is rapid, we can use Fact 2.4 to find $X_\alpha \in \mathcal{U}_\delta$ such that $|X_\alpha \cap c_n^\alpha| \le n$ for each $n < \omega$. We can also assume[2] that

$$|g_\alpha^{-1}(n)| \ge s_\alpha(\pi_{\alpha 0}(n)) \tag{3.1}$$

for all n such that $X_\alpha \cap c_n^\alpha \ne \varnothing$ and that if $n \in \pi_{\delta\alpha}[X_\alpha]$, then

$$|g_\alpha^{-1}(n) \cap T_\alpha| \le \min\big(g_\alpha^{-1}(n) \cup \{\pi_{\alpha 0}(n)\}\big). \tag{3.2}$$

Next, choose $Y_\alpha \in \mathcal{U}_\delta$ such that

$$\big\{g_\beta : \beta \in D \;\&\; \beta \le \alpha^+\big\} \cup \{\pi_{\gamma\beta} : \gamma \le \beta \le \alpha^+, \gamma, \beta \in D\big\}$$

[2]Otherwise, throw finitely many elements of X_α away to get the first requirement, and for the second, intersect it with the set $\pi_{\delta\alpha}^{-1}[X]$, where X is the set guaranteed to exist by condition (iv) above.

commute on Y_α; more precisely, for every $\beta \le \gamma \le \alpha^+$ all in D and every $k \in Y_\alpha$ and any $l \in g_\gamma^{-1}[\pi_{\delta\gamma}(k)]$ we have

$$\pi_{\gamma\beta}(\pi_{\delta\gamma}(k)) = \pi_{\delta\beta}(k) = g_\beta(l). \tag{3.3}$$

Finally, since g_α and T_α satisfy the first part of (iv), we can use Observation 3.2 to find $Z_\alpha \in \mathcal{U}_\delta$ such that

$$(\forall n)\left(|\pi_{\delta\alpha}^{-1}(n) \cap Z_\alpha| \le \frac{|g_\alpha^{-1}(n) \cap T_\alpha|}{\#\alpha}\right), \tag{3.4}$$

i.e., Z_α is $\#\alpha$-times more sparse then T_α. (Just apply Observation 3.2 to $\pi = \pi_{\delta\alpha}$, $\mathcal{U} = \mathcal{U}_\delta$, and $s(n) = |g_\alpha^{-1}(n) \cap T_\alpha|/\#\alpha$.)

Since $\mathcal{U}_\delta$ is a P-point, we can find $X \in \mathcal{U}_\delta$ that is a pseudo-intersection of $\{X_\alpha, Y_\alpha, Z_\alpha : \alpha \in D\}$. Recursively construct a partition $\{K_\alpha : \alpha \in D\}$ of X into finite sets and $s_\delta : \omega \to \omega$ such that $K_\alpha \subseteq \bigcap_{\beta\in D\cap\alpha^+} X_\beta \cap Y_\beta \cap Z_\beta$, and

(iv) $s_\alpha(m) \ge s_\alpha(m)/m - m^2 \ge s_\delta(m) \ge p_\alpha(m)$ whenever $m = \pi_{\delta 0}(n)$ and $n \in K_\alpha$;
(v) $\pi_{\delta\alpha}^{-1}[\pi_{\delta\alpha}[K_\alpha]] \cap X \subseteq K_\alpha$;
(vi) $\#\alpha \le \left|A \cap \min\left(\pi_{\delta 0}[K_\alpha]\right) \cap \min\left(g_\alpha^{-1}[\pi_{\delta\alpha}[K_\alpha]]\right)\right|$.

This is not hard to do: first find an increasing sequence $\{k_\alpha : \alpha \in D\}$ of natural numbers such that $X \setminus k_\alpha \subseteq \bigcap_{\beta\in D\cap\alpha^+} X_\beta \cap Y_\beta \cap Z_\beta$, and

$$s_\alpha(m) \ge \frac{s_\alpha(m)}{m} - m^2 \ge p_{\alpha^+}(m) \quad \text{and} \quad \#\alpha \le |A \cap m \cap i|$$

for all $m = \pi_{\delta 0}(n)$ and $i \in g_\alpha^{-1}(\pi_{\delta\alpha}(n))$ with $n \in X \setminus k_\alpha$. Then let

$$K_\alpha = \pi_{\delta\alpha}^{-1}[\pi_{\delta\alpha}[X \cap [k_\alpha, k_{\alpha^+})]] \quad \text{and} \quad s_\delta \restriction \pi_{\delta 0}[K_\alpha] = p_\alpha.$$

(Formally, this will not be a partition, since it will not cover $X \cap [0, k_0)$; we can just throw these finitely many elements out of X).

Let $J_\alpha = \pi_{\delta\alpha}[K_\alpha]$ and $L_\alpha = g_\alpha^{-1}[J_\alpha]$. Notice that since $K_\alpha \subseteq Y_\alpha$, we can use (3.3) and (v) to conclude that $L_\alpha \cap L_\beta = \varnothing$ for distinct $\alpha \ne \beta \in D$. This allows us to define g_δ separately on each L_α (see Figure 1). For $n \in J_\alpha$ let $b_n^\alpha = g_\alpha^{-1}(n)$.

Fix $\alpha \in D$ and $n \in J_\alpha$ and let $m = \pi_{\alpha 0}(n)$. Then, since $K_\alpha \subseteq X_\alpha$, by (3.1) and (iv) we have $|b_n^\alpha| \ge m s_\delta(m) + m^3$. Moreover, since $K_\alpha \subseteq Z_\alpha$, by (3.4) and (3.2) we have

$$|K_\alpha \cap \pi_{\delta\alpha}^{-1}(n)| \le m,$$
$$|K_\alpha \cap \pi_{\delta\alpha}^{-1}(n)| \le \frac{|b_n^\alpha \cap T_\alpha|}{\#a} \le \frac{\min(b_n^\alpha \cup \{m\})}{\#a}.$$

It follows that we can partition b_n^α into pieces $\{e_k^n : k \in K_\alpha \cap \pi_{\delta\alpha}^{-1}(n)\}$, each of size $\ge s_\delta(m) + m^2$, that, moreover, satisfy $\#\alpha \le |e_k^n \cap T_\alpha| \le \min(b_n^\alpha \cup \{m\})$.

Due to (vi) we can shrink e_k^n to a smaller set d_k^n (throwing away at most m-many elements of $e_k^n \cap T_\alpha$) such that

$$\#\alpha \le |d_k^n \cap T_\alpha| \le |A \cap m \cap \min b_n^\alpha|. \tag{3.5}$$

Since we threw away at most m elements from each e_k^n, we still have $|d_k^n| \ge s_\delta(m)$. Now let $g_\delta[d_k^n] = \{k\}$ and extend g_δ to all of ω arbitrarily so that the new values are outside of X and that the requirements on g_δ are satisfied, *i.e.*, that it is finite-to-one,

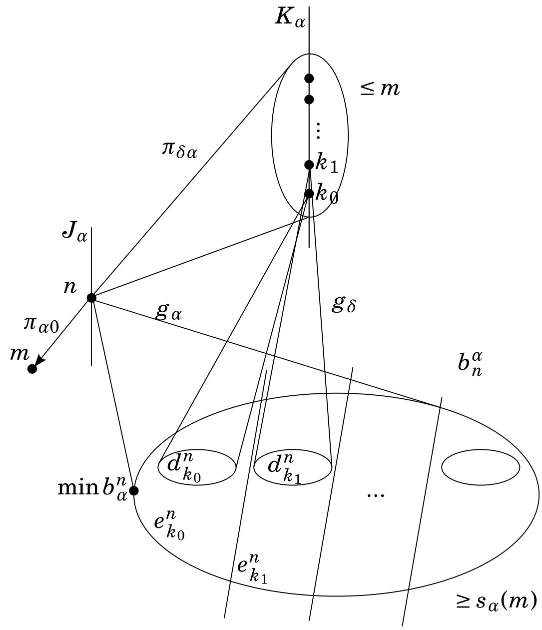

Figure 1: Constructing g_δ.

its growth rate is bounded below by s_δ, etc. This finishes the construction of g_δ. For future reference, let us note that $g_\delta \restriction (T_\alpha \cap l)$ is at most $|A \cap l|$-to-one for any $l < \omega$.

Notice that if $\beta < \alpha \in D$, $k \in K_\alpha$, and $g_\delta(l) = k$, then $g_\beta(l) = \pi_{\alpha\beta}(g_\alpha(l))$, and so $\pi_{\delta\beta}(k) = \pi_{\alpha\beta}(\pi_{\delta\alpha}(k)) = g_\beta(l)$; so Proposition 3.3(iii) is satisfied for g_δ. That condition (ii) for s_δ is satisfied follows from (iv). That condition (i) is satisfied for s_δ and g_δ follows from the construction ($|d^n_k| \geq s_\delta(m)$). (Formally, we have only checked the conditions for $\beta, \alpha \in D$, but this is clearly enough, since D is cofinal in δ.)

Finally we must construct T_δ. Without loss of generality we may assume that $T_\alpha \subseteq T_\beta$ for $\beta < \alpha \in D$. (Otherwise we could have carried out the construction for some finite modifications of T_αs and the resulting T_δ would still work for the original T_αs). Let $T' = \bigcup_{\alpha \in D} g_\delta^{-1}[K_\alpha] \cap T_\alpha$. Then T' is a pseudo-intersection of $\{T_\alpha : \alpha \in D\}$. Moreover, g_δ was constructed (see (3.5)) so that $\#\alpha \leq |g_\delta^{-1}(k) \cap T_\alpha| \leq \min\{m\} \cup g_\delta^{-1}(k)$ for each $k \in K_\alpha$ and $m = \pi_{\delta 0}(k)$. It follows that T' and g_δ satisfy (the corresponding modification of) Proposition 3.3(iv) and, in particular, that $g_\delta[T'] \in \mathcal{U}_\delta$.

Claim *There is an $X' \in \mathcal{U}_\delta$, $X' \subseteq X \cap g_\delta[T']$ such that $|X' \cap g_\delta[l]| \leq |A \cap l|$,*

Proof of Claim Since $|g_\delta[l]| \leq l$, we can find a bijection $\pi : \omega \to \omega$ such that $\pi[g_\delta[l]] \subseteq l$. Since π is a bijection, the ultrafilter $\pi^*(\mathcal{U}_\delta)$ is rapid, so there is $Y \in \pi^*(\mathcal{U}_\delta)$ such that $|Y \cap l| \leq |A \cap l|$. Let $X' = \pi^{-1}[Y] \cap X \cap g_\delta[T'] \in \mathcal{U}_\delta$. Then we have

$$X' \cap g_\delta[l] \subseteq X' \cap \pi^{-1}[l] \subseteq \pi^{-1}[Y] \cap \pi^{-1}[l] = \pi^{-1}[Y \cap l].$$

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Using this, the fact that π is a bijection, and the choice of Y, we get

$$|X' \cap g_\delta[l]| \leq |\pi^{-1}[Y \cap l]| = |Y \cap l| \leq |A \cap l|,$$

which finishes the proof of the claim. ■

Let $T'' = g_\delta^{-1}[X'] \subseteq T'$. Now, since $g_\delta \upharpoonright T' \cap l$ is at most $|A \cap l|$-to-1, we have

$$|T'' \cap l| \leq |A \cap l| \cdot |X' \cap g_\delta[l]| \leq |A \cap l| \cdot |A \cap l| \leq |A \cap l|^2.$$

Finally notice that

$$g_\delta[T'' \cap A] \cup g_\delta[T'' \setminus A] = g_\delta[T''] = X' \in \mathcal{U}_\delta.$$

So, since $\mathcal{U}_\delta$ is an ultrafilter, we can choose $T_\delta \subseteq T''$ that decides A and still satisfies Proposition 3.3(iv). This finishes the proof. ■

Theorem 3.4 *Assume CH. Every RK-increasing chain of rapid P-points of length ω_1 has an upper bound that is also a rapid P-point.*

Proof Let $\langle \mathcal{U}_\alpha : \alpha < \omega_1 \rangle$ be an RK-increasing chain of rapid P-points as witnessed by finite-to-one maps $\Pi = \{\pi_{\alpha\beta} : \beta \leq \alpha < \omega_1\}$. Without loss of generality we can assume that the maps commute in the sense of Proposition 3.3(iii). Enumerate $[\omega]^\omega$ as $\{A_\alpha : \alpha < \omega_1\}$. Recursively build a sequence of finite-to-one maps $\langle g_\alpha : \alpha < \omega_1 \rangle$ and a decreasing tower $\langle T_\alpha : \alpha < \omega_1 \rangle$ so that T_α decides A_α and $|T_\alpha \cap n| \leq |A_\alpha \cap n|^2$. This can be done by repeatedly applying Proposition 3.3 at each step. In the end $\langle T_\alpha : \alpha < \omega_1 \rangle$ generates a rapid P-point and the map g_α witnesses that this P-point is above $\mathcal{U}_\alpha$. ■

We do not know the optimal hypothesis needed to carry out the above proof. We leave it as a question for further research.

Question What is the optimal hypothesis needed to get the conclusion of Theorem 3.4 with ω_1 replaced by $\mathfrak{c}$? In particular, does this hold if we replace CH by $\mathfrak{b} = \mathfrak{c}$? Or even $\mathfrak{d} = \mathfrak{c}$?

4 A Short Unbounded Chain of P-points

In this section we show, assuming $\diamondsuit$, that there is an RK-chain of P-points of length ω_1 that has no P-point RK-above. We assume $\diamondsuit$ only for simplicity; a more involved argument using the Devlin–Shelah weak diamond can be used to construct the chain, *e.g.*, under CH.

Definition 4.1 Let $\overline{\mathcal{U}} = \langle \mathcal{U}_\alpha : \alpha < \delta \rangle$ be a sequence of ultrafilters and $\Pi = \langle \pi_{\alpha\beta} : \beta \leq \alpha \leq \delta \rangle$ a family of maps from ω to ω. We say that Π *commutes with respect to* $\overline{\mathcal{U}}$, if for $\beta \leq \alpha \leq \gamma \leq \delta$ there is $X \in \mathcal{U}_\gamma$ such that $\pi_{\alpha\beta}(\pi_{\gamma\alpha}(i)) = \pi_{\gamma\beta}(i)$ for all $i \in X$. When the sequence $\overline{\mathcal{U}}$ is clear from the context, we just say that Π commutes.

Notation 4.2 Given two families $\Pi_i = \langle \pi^i_{\delta\alpha} : \alpha < \delta \rangle$, $i < 2$ of maps and a sequence of ultrafilters $\overline{\mathcal{U}}$ as above, we write $f : \Pi_0 \to_{\overline{\mathcal{U}}} \Pi_1$ to indicate that f is a map from ω to ω and for each $\alpha < \delta$ there is an $X \in (\pi^0_{\delta\alpha})^{-1}[\mathcal{U}_\alpha]$ such that $\pi^0_{\delta\alpha}(n) = \pi^1_{\delta\alpha}(f(n))$ for all $n \in X$.

Definition 4.3 Given two families of maps $\Pi_i = \langle \pi^i_{\delta\alpha} : \alpha < \delta \rangle$, $i < 2$, we say that $\Pi_1 \prec \Pi_0$ if

$$(\forall \alpha < \delta)(\forall^\infty n < \omega)\big(|(\pi^1_{\delta\alpha})^{-1}(n)| > n \cdot |(\pi^0_{\delta\alpha})^{-1}(n)|\big).$$

Moreover, if π_0, π_1 are two maps, $\mathcal{U}$ is an ultrafilter, and $X \in [\omega]^\omega$, we write

$$\pi_1 \prec_{X,\mathcal{U}} \pi_0$$

if there is a $Y \in \mathcal{U}$ and $s : \omega \to \omega$ tending to infinity such that

$$(\forall n \in Y)\big(|\pi_1^{-1}(n) \cap X| > s(n) \cdot |\pi_0^{-1}(n) \cap X|\big).$$

Observation 4.4 Assume $\overline{\mathcal{U}} = \langle U_\alpha : \alpha < \delta \rangle$ is an RK-increasing chain of P-points of length $\delta < \omega_1$ as witnessed by a family of finite-to-one maps $\Pi = \langle \pi_{\alpha\beta} : \beta \leq \alpha < \delta \rangle$. Suppose, moreover, that we are given a family of finite-to-one maps $\Pi_0 = \langle \pi^0_{\delta\alpha} : \alpha < \delta \rangle$ such that $\Pi \cup \Pi_0$ commute. Then there is a family $\Pi_1 = \langle \pi^1_{\delta\alpha} : \alpha < \delta \rangle$ such that $\Pi_1 \prec \Pi_0$ and $\Pi \cup \Pi_1$ still commute.

Proof Fix an arbitrary finite-to-one π such that $|\pi^{-1}(n)| \geq n$ and let $\pi^1_{\delta\alpha}(n) = \pi^0_{\delta\alpha}(\pi(n))$. ∎

Definition 4.5 Given an RK-increasing chain of P-points $\overline{\mathcal{U}}$ of length δ for some limit $\delta < \omega_1$, a family of finite-to-one maps $\Pi = \langle \pi_{\alpha\beta} : \beta \leq \alpha < \delta \rangle$ witnessing that the chain is RK-increasing, and two families of finite-to-one maps $\Pi_i = \langle \pi^i_{\delta\alpha} : \alpha < \delta \rangle$, $i < 2$, such that $\Pi_1 \prec \Pi_0$ and $\Pi \cup \Pi_i$ commutes with respect to $\overline{\mathcal{U}}$ for $i < 2$, we define the forcing

$$\mathbb{P}(\overline{\mathcal{U}}, \Pi, \Pi_0, \Pi_1) = \big(\big\{X \in [\omega]^\omega : (\forall \alpha < \delta)(\pi^1_{\delta\alpha} \prec_{X,\mathcal{U}_\alpha} \pi^0_{\delta\alpha})\big\}, \subseteq^*\big).$$

For the following observation and propositions, fix δ, $\overline{\mathcal{U}}$, Π, Π_0, and Π_1 as in the definition.

Observation 4.6 The forcing $\mathbb{P}(\overline{\mathcal{U}}, \Pi, \Pi_0, \Pi_1)$ contains ω.

Proposition 4.7 *The forcing $\mathbb{P}(\overline{\mathcal{U}}, \Pi, \Pi_0, \Pi_1)$ is σ-closed.*

Proof Let $\langle X_n : n < \omega \rangle$ be a descending sequence of conditions and, without loss of generality, assume $X_{n+1} \subseteq X_n$ for all $n < \omega$. Fix a cofinal subset $D \subseteq \delta$ of order type ω and, as before, write $\alpha^+ = \min\big(D \setminus (\alpha+1)\big)$ and $\#\alpha = |D \cap \alpha|$. For each $n < \omega$ and $\alpha \in D$, fix $Y^\alpha_n \in \mathcal{U}_\alpha$ and s^α_n witnessing $\pi^1_{\delta\alpha} \prec_{X_n,\mathcal{U}_\alpha} \pi^0_{\delta\alpha}$. Then for each $\alpha \in D$, let $Y^\alpha \in \mathcal{U}_\alpha$ be a pseudo-intersection of $\{Y^\alpha_n : n < \omega\} \subseteq \mathcal{U}_\alpha$ and fix a function $s : \omega \to \omega$, tending to infinity, such that $s \leq^* s^\alpha_n$ for all $\alpha \in D$, $n < \omega$. For each $\alpha \in D$, fix $m_\alpha < \omega$ such that

$$|(\pi^1_{\delta\beta})^{-1}(m) \cap X_{\#\alpha}| \geq s(m) \cdot |(\pi^0_{\delta\beta})^{-1}(m) \cap X_{\#\alpha}|$$

for each $\beta \in D \cap \alpha^+$ and $m \in Y^\alpha \setminus m_\alpha$. We also choose each m_α large enough to make sure that

$$\max\big((\pi^i_{\delta\beta'})^{-1}[m']\big) < \min\big((\pi^j_{\delta\beta})^{-1}(m)\big) \tag{4.1}$$

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for each $\beta, \beta' \in D \cap \alpha^+$, $i, j < 2$ and $m' < m_\alpha < m_{\alpha+} < m$. For $\alpha \in D$ let

$$X^\alpha = \bigcup_{\beta \in D \cap \alpha^+} (\pi^1_{\delta\beta})^{-1}[m_\alpha, m_{\alpha^+}) \cap X_{\#\alpha}.$$

Next let D_0 and D_1 be the set of even and odd elements of D, respectively. Choose a cofinal $D' \subseteq D$ and $i < 2$ so that $Z^\alpha = \bigcup_{\beta \in D_i} [m_\beta, m_{\beta^+}) \cap Y^\alpha \in \mathcal{U}_\alpha$. Finally, define $X = \bigcup_{\alpha \in D_i} X^\alpha$. By (4.1) it is clear that

$$X \cap (\pi^j_{\delta\beta})^{-1}[m_\alpha, m_{\alpha^+}) = X_{\#\alpha} \cap (\pi^j_{\delta\beta})^{-1}[m_\alpha, m_{\alpha^+}) \tag{4.2}$$

for each $\alpha \in D_i$, $\beta < \alpha$, and $j < 2$. It is clear that X is a pseudo-intersection of the X_ns. We need to show that $X \in \mathbb{P}(\overline{\mathcal{U}}, \Pi, \Pi_0, \Pi_1)$, *i.e.*, that for each $\alpha < \delta$ we have

$$\pi^1_{\delta\alpha} \prec_{X, \mathcal{U}_\alpha} \pi^0_{\delta\alpha}. \tag{4.3}$$

First assume $\alpha \in D'$. We show that $Z^\alpha \setminus m_\alpha$ witnesses (4.3). Let $m \in Z^\alpha \setminus m_\alpha$ be arbitrary. Find $\alpha' \in D_i$ so that $m \in [m_{\alpha'}, m_{\alpha'^+})$. Then $\alpha < \alpha'$ so, in particular, we have

$$|(\pi^1_{\delta\alpha})^{-1}(m) \cap X_{\#\alpha'}| \geq s(m) \cdot |(\pi^0_{\delta\alpha})^{-1}(m) \cap X_{\#\alpha'}|.$$

This, together with (4.2), shows (4.3). Finally notice that if $\alpha < \beta < \alpha'$,

$$\pi^1_{\delta\alpha} \prec_{X, \mathcal{U}_\alpha} \pi^0_{\delta\alpha}, \quad \text{and} \quad \pi^1_{\delta\alpha'} \prec_{X, \mathcal{U}_{\alpha'}} \pi^0_{\delta\alpha'},$$

then also $\pi^1_{\delta\beta} \prec_{X, \mathcal{U}_\beta} \pi^0_{\delta\beta}$. Since D' was cofinal in δ, this finishes the proof of (4.3) for all $\alpha < \delta$. ∎

Proposition 4.8 *If $A \subseteq \omega$, then the set*

$$D_A = \{X \in \mathbb{P}(\overline{\mathcal{U}}, \Pi, \Pi_0, \Pi_1) : X \subseteq A \vee X \subseteq \omega \setminus A\}$$

is dense.

Proof Notice that if $\pi^1_{\delta\alpha} \prec_{X, \mathcal{U}_\alpha} \pi^0_{\delta\alpha}$, then either

$$\pi^1_{\delta\alpha} \prec_{X \cap A, \mathcal{U}_\alpha} \pi^0_{\delta\alpha} \quad \text{or} \quad \pi^1_{\delta\alpha} \prec_{X \setminus A, \mathcal{U}_\alpha} \pi^0_{\delta\alpha}.$$

This follows from the fact that either

$$|(\pi^1_{\delta\alpha})^{-1}(m) \cap X \cap A| \geq \frac{|(\pi^1_{\delta\alpha})^{-1}(m) \cap X|}{2}$$

or

$$|(\pi^1_{\delta\alpha})^{-1}(m) \cap X \setminus A| \geq \frac{|(\pi^1_{\delta\alpha})^{-1}(m) \cap X|}{2}$$

for $\mathcal{U}_\alpha$-many ms and that if s tends to infinity then so does $s/2$. The result then immediately follows because one of the two cases must happen for cofinally many $\alpha < \delta$. ∎

Proposition 4.9 *If $f : \Pi_0 \to_{\overline{\mathcal{U}}} \Pi_1$ is finite-to-one, then the set*

$$D_f = \{X \in \mathbb{P}(\overline{\mathcal{U}}, \Pi, \Pi_0, \Pi_1) : (\exists Y \in \mathcal{U}_0)(X \cap f[(\pi^0_{\delta 0})^{-1}[Y]] = \emptyset\}$$

is dense.

Proof Let $X \in \mathbb{P}(\overline{\mathcal{U}}, \Pi, \Pi_0, \Pi_1)$. For each $\alpha < \delta$, choose $Y_\alpha \in \mathcal{U}_\alpha$ and $s_\alpha : \omega \to \omega$ tending to infinity witnessing $\pi^1_{\delta\alpha} \prec_{X, \mathcal{U}_\alpha} \pi^0_{\delta\alpha}$. We can also assume that

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(1) $\pi^1_{\delta\alpha}(f(k)) = \pi^0_{\delta\alpha}(k)$ for all $k \in (\pi^0_{\delta\alpha})^{-1}[Y_\alpha]$,
(2) $\pi_{\alpha 0}(\pi^1_{\delta\alpha}(k)) = \pi^1_{\delta 0}(k)$ for each $k \in (\pi^1_{\delta\alpha})^{-1}[Y_\alpha]$ and $\alpha < \delta$, and
(3) $Y_\alpha \subseteq (\pi_{\alpha 0})^{-1}[Y_0]$.

Since $\mathcal{U}_0$ is a P-point, there is a $Y \in \mathcal{U}_0$ that is a pseudo-intersection of $\pi_{\alpha 0}[Y_\alpha]$ and let $n_\alpha < \omega$ be such that $\pi_{\alpha 0}[Y_\alpha \setminus n_\alpha] \subseteq Y$. Also write $Z = (\pi^0_{\delta 0})^{-1}[Y]$. Let $X' = \big((\pi^1_{\delta 0})^{-1}[Y] \cap X\big) \setminus f[Z \cap X]$. Notice that for each $n \in Y_\alpha \setminus n_\alpha$, we have $f^{-1}\big[(\pi^1_{\delta\alpha})^{-1}(n)\big] \cap X \subseteq (\pi^0_{\delta\alpha})^{-1}(n) \cap X$ (since $f, \pi^0_{\delta\alpha}, \pi^1_{\delta\alpha}$ commute on $Y_\alpha \setminus n_\alpha$), so that

$$\big|(\pi^1_{\delta\alpha})^{-1}(n) \cap X \cap f[Z \cap X]\big| \le \big|(\pi^0_{\delta\alpha})^{-1}(n) \cap X\big|.$$

By the choice of Y_α we also have $|(\pi^1_{\delta\alpha})^{-1}(n) \cap X| \ge s_\alpha(n) \cdot |(\pi^0_{\delta\alpha})^{-1}(n) \cap X|$. Putting this together gives

$$|(\pi^1_{\delta\alpha})^{-1}(n) \cap X'| \ge (s_\alpha(n) - 1) \cdot |(\pi^0_{\delta\alpha})^{-1}(n) \cap X'|.$$

Since s_α tends to infinity, so does $s_\alpha - 1$. This shows that $s_\alpha - 1$ and $Y_\alpha \setminus n_\alpha$ witness the fact that $X' \in \mathbb{P}(\overline{\mathcal{U}}, \Pi, \Pi_0, \Pi_1)$. ∎

We now put Propositions 4.8 and 4.9 together and prove the following.

Theorem 4.10 *Assume $\diamondsuit$. There is a sequence of ultrafilters of length ω_1 that is strictly increasing in the RK-order and has no upper bound that would be a P-point.*

Proof Let $\langle \Pi^\alpha : \alpha < \omega_1 \rangle$, where $\Pi^\alpha = \langle p^\alpha_{\omega_1\gamma} : \gamma < \alpha \rangle$, be a diamond sequence guessing sequences of functions from ω to ω, *i.e.*, such that, for every sequence of such functions $\Pi = \langle \pi_{\omega_1\alpha} : \alpha < \omega_1 \rangle$, the set $\{\alpha \in \mathrm{Lim}(\omega_1) : \Pi \upharpoonright \alpha = \Pi^\alpha\}$ is stationary. We recursively construct an RK-increasing sequence $\langle \mathcal{U}_\alpha : \alpha < \omega_1 \rangle$ of P-points and witnessing maps $\Pi^\alpha_1 = \langle \pi^1_{\alpha\beta} : \beta < \alpha \rangle$ as follows.

At a successor step $\alpha + 1$, we just construct an arbitrary P-point $\mathcal{U}_{\alpha+1}$ above $\mathcal{U}_\alpha$ and let $\Pi^{\alpha+1}_1$ be the appropriate witnessing maps.

At a limit step α, let $\Pi = \langle \pi^1_{\beta\gamma} : \gamma \le \beta < \alpha \rangle$, $\overline{\mathcal{U}} = \langle \mathcal{U}_\beta : \beta < \alpha \rangle$, and write $\Pi^\alpha = \Pi^\alpha_0 = \{\pi^0_{\alpha\beta} : \beta < \alpha\}$, *i.e.*, $\pi^0_{\alpha\beta} = p^\alpha_{\omega_1\beta}$. If $\Pi \cup \Pi^\alpha_0$ do not commute with respect to $\overline{\mathcal{U}}$, we construct $\mathcal{U}_\alpha$ to be an arbitrary P-point above $\overline{\mathcal{U}}$ and let Π^α_1 be appropriate finite-to-one witnessing maps. Otherwise, we use Theorem 4.4 to construct $\Pi^\alpha_1 = \{\pi^1_{\alpha\beta} : \beta < \alpha\}$ satisfying

(1) $\Pi^\alpha_1 \prec \Pi^\alpha_0$.

Then we recursively construct a P-filter $\mathcal{U}_\alpha$ on $\mathbb{P}(\overline{\mathcal{U}}, \Pi, \Pi^\alpha_0, \Pi^\alpha_1)$ so that

(2) for all finite-to-one $f : \Pi^\alpha_0 \to_{\overline{\mathcal{U}}} \Pi^\alpha_1$ there is $X \in \mathcal{U}_\alpha$ and $Y \in \mathcal{U}_0$ such that

$$\varnothing = f\big[(\pi^0_{\alpha 0})^{-1}[Y]\big] \cap X.$$

To guarantee (1) we just need to ensure that it hits each of the ω_1-many dense sets $\{D_f : f : \Pi^\alpha_0 \to_{\overline{\mathcal{U}}} \Pi^\alpha_1\}$; we also make sure that it hits the dense sets $\{D_A : A \subseteq \omega\}$, so that it is an ultrafilter. This can be done since the forcing is σ-closed by Theorem 4.7. This finishes the recursive construction.

Finally notice that the chain of P-points thus constructed cannot have a P-point on top. Otherwise suppose $\mathcal{U}$ is RK-above the chain as witnessed by finite-to-one maps $\Pi^{\omega_1} = \{\pi_{\omega_1\alpha} : \alpha < \omega_1\}$ that commute with $\bigcup_{\alpha<\omega_1} \Pi^\alpha_1$. Since the Π^αs formed a diamond sequence, there is a limit $\alpha < \omega_1$ such that $\Pi^{\omega_1} \upharpoonright \alpha = \Pi^{\alpha}$. Then $\Pi^{\alpha} = \Pi_0^{\alpha}$ commutes with Π, so we can apply (2) to $f = \pi_{\omega_1\alpha}$ and conclude that there are $X \in \mathcal{U}_\alpha$ and $Y \in \mathcal{U}_0$ such that

$$\varnothing = \pi_{\omega_1\alpha}\big[(\pi^0_{\alpha 0})^{-1}[Y]\big] \cap X = \pi_{\omega_1\alpha}\big[(p^{\alpha}_{\omega_1 0})^{-1}[Y]\big] \cap X = \pi_{\omega_1\alpha}\big[\pi^{-1}_{\omega_1 0}[Y]\big] \cap X,$$

contradicting the fact that $\pi_{\omega_1\alpha}$ witnesses that $\mathcal{U}$ is above $\mathcal{U}_\alpha$. ∎

5 Concluding Remarks

It was proved in Section 3 that, given a Rudin–Keisler increasing chain of rapid P-points $\langle \mathcal{U}_\alpha : \alpha < \omega_1 \rangle$ together with a commuting sequence of finite-to-one witnessing maps $\langle \pi_{\beta,\alpha} : \alpha \leq \beta < \omega_1 \rangle$, it is possible to find a sequence of finite-to-one maps $\langle g_\alpha : \alpha < \omega_1 \rangle$ together with a rapid P-point $\mathcal{V}$ such that g_α is a witness to $\mathcal{U}_\alpha \leq_{\mathrm{RK}} \mathcal{V}$. However, the argument in Section 3 does not guarantee that any of the g_α will be nondecreasing, even when it is given that each of the maps $\pi_{\beta\alpha}$ is nondecreasing. In other words, the rapid P-point $\mathcal{V}$ may not be an $\leq^+_{\mathrm{RB}}$ upper bound of the sequence $\langle \mathcal{U}_\alpha : \alpha < \omega_1 \rangle$ even if that sequence itself is assumed to be $\leq^+_{\mathrm{RB}}$-increasing.

It appears that one must fall back on the construction given in [7] if one wants a chain of P-points of length $\mathfrak{c}^+$ that is increasing in the $\leq^+_{\mathrm{RB}}$ ordering. Nevertheless, the ideas from Section 3 can be combined with the work in [7] to show that CH implies that the rapid P-points are $\mathfrak{c}^+$-closed with respect to $\leq^+_{\mathrm{RB}}$. More precisely, the following theorem will appear in a forthcoming paper of Kuzeljević, Raghavan, and Verner. Assume the Continuum Hypothesis. Suppose $\delta < \mathfrak{c}^+$. If $\langle \mathcal{U}_\alpha : \alpha < \delta \rangle$ is any sequence of rapid P-points that is increasing with respect to $\leq^+_{\mathrm{RB}}$, then there exists a rapid P-point $\mathcal{V}$ such that $\forall \alpha < \delta [\mathcal{U}_\alpha \leq^+_{\mathrm{RB}} \mathcal{V}]$. Therefore every strictly increasing sequence of rapid P-points of length $< \mathfrak{c}^+$ can be extended to one of length $\mathfrak{c}^+$ with respect to the $\leq^+_{\mathrm{RB}}$ ordering.

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