Periodic solutions of a linear differential equation of the second order with periodic coefficients. By Mr E. L. Ince, Trinity College.

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The equation to be considered is of the type

$$\frac{d^2y}{dx^2} + p(x)\,y = 0,$$

where $p(x)$ is continuous for all real values of x, even, and periodic. It is no restriction to suppose that the period is π, and this assumption will be made, so that the equation is virtually Hill's equation.

Of two fundamental solutions the one is even and the other odd. In a previous paper, published by this Society*, it was proved that in a particular case, that of the Mathieu equation, the fundamental solutions cannot both be periodic and of period π or 2π. Other proofs of this theorem have subsequently been discovered, both by the present author and by other writers†. It is now proposed to examine the more general case‡.

By substituting in the equation a formal sine or cosine series-expression for the solution, it may be verified that if the solution is periodic, its period is an integer multiple of π. An even periodic solution will be denoted by $C(x)$, and an odd periodic solution by $S(x)$.

The expression for an even periodic solution, of period $s\pi$, is

$$C(x) = \sum_{k=-\infty}^{\infty} c_k \cos 2\left(\frac{r}{s} + k\right)x,$$

where r is prime to s, and $2r < s$ when $s > 2$.

Now the differential equation itself is unchanged if $\pi - x$ is written for x; if therefore it is satisfied by

$$y = C(x),$$

it is also satisfied by $\quad y = C(\pi - x).$

But

$$C(\pi - x) = \cos\frac{2r\pi}{s}\,C(x) + \sin\frac{2r\pi}{s}\sum_{k=-\infty}^{\infty} c_k \sin 2\left(\frac{r}{s} + k\right)x.$$

When $s > 2$, neither $\cos\frac{2r\pi}{s}$ nor $\sin\frac{2r\pi}{s}$ is zero, and therefore when

* *Proc. Camb. Phil. Soc.* vol. 21 (1922), p. 117.

† E.g. Einar Hille, *Proc. Lond. Math. Soc.* (2), vol. 23 (1923), p. 224.

‡ Another particular case was discussed by the present writer, *Proc. Lond. Math. Soc.* (2), vol. 23 (1923), p. 56.

an even periodic solution exists, there exists also the odd periodic solution

$$S(x) = \sum_{k=-\infty}^{\infty} c_k \sin 2\left(\frac{r}{s} + k\right) x,$$

whose coefficients are precisely the same as those of the even solution.

When $s = 2$, $r = 1$ and the equation is satisfied by $y = C(x)$, then

$$C(\pi - x) = -C(x);$$

if it is satisfied by $y = S(x)$, then

$$S(\pi - x) = S(x).$$

When $s = 1$, $r = 1$ and the equation is satisfied by $y = C(x)$, then

$$C(\pi - x) = C(x);$$

if it is satisfied by $y = S(x)$, then

$$S(\pi - x) = S(x).$$

Now let $s = 2$ and consider the possibility of the coexistence of the even and odd periodic solutions

$$y = C(x), \quad y = S(x).$$

These solutions may be so chosen that

$$C(x)\, S'(x) - S(x)\, C'(x) = 1.$$

If

$$C(x) = \sum_{k=0}^{\infty} a_k \cos(2k+1)x,$$

$$S(x) = \sum_{k=0}^{\infty} b_k \sin(2k+1)x,$$

then

$$\begin{aligned} 1 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (2j+1)\, a_k b_j \cos(2k+1)x \cos(2j+1)x \\ + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (2k+1)\, a_k b_j \sin(2k+1)x \cos(2j+1)x \\ = \sum_{k=0}^{\infty} (2k+1)\, a_k b_k + \sum_{(k>j)} \sum_{j=0}^{\infty} (k-j)(a_j b_k - a_k b_j) \cos 2(j+k+1)x \\ + \sum_{(k>j)} \sum_{j=0}^{\infty} (j+k+1)(a_j b_k + a_k b_j) \cos 2(k-j)x. \end{aligned}$$

This relation is satisfied identically, and therefore

$$\sum_{k=0}^{\infty} (2k+1)\, a_k b_k = 1,$$

$$\sum_{k=0}^{\infty} (2k+2)(a_k b_{k+1} + a_{k+1} b_k) = 0,$$

$$\sum_{k=0}^{\infty} (2k+3)(a_k b_{k+2} + a_{k+2} b_k) + (a_0 b_1 - a_1 b_0) = 0,$$

$$\sum_{k=0}^{\infty} (2k+4)(a_k b_{k+3} + a_{k+3} b_k) + 2(a_0 b_2 - a_2 b_0) = 0,$$

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and so forth. Let the coefficients a_k be regarded as known, then there is an infinite set of linear equations to determine the coefficients b_k. But the formula

$$S(x) = C(x) \int \{C(x)\}^{-2} dx$$

shows that the coefficients b_k are uniquely determined (apart from a constant multiplier) by the knowledge of the coefficients a_k. Hence the system of linear equations has one and only one solution (b_k) such that the series-expression for $S(x)$ converges for all real values of x.

This solution is easily found; it is given by the relations

$$a_j b_k + a_k b_j = 0 \qquad (k - j \text{ odd}),$$
$$a_j b_k - a_k b_j = 0 \qquad (k - j \text{ even}),$$

and these relations are, in turn, satisfied by

$$b_k = (-1)^k a_k.$$

If, therefore, the equation

$$\frac{d^2y}{dx^2} + p(x) y = 0$$

has two periodic solutions, expressible as convergent infinite series, these solutions are of the forms

$$C(x) = \sum_{k=0}^{\infty} a_k \cos(2k+1) x,$$

$$S(x) = \sum_{k=0}^{\infty} (-1)^k a_k \sin(2k+1) x.$$

But now it is an easy matter to verify that $p(x)$ has the period $\frac{1}{2}\pi$, contrary to hypothesis. Consequently when $p(x)$ is of period π, the equation cannot admit of two solutions of period 2π.

If $C(x)$ terminates, and the last term is $a_m \cos(2m+1)x$, the second solution is periodic when, and only when,

$$a_0 = a_1 = \ldots = a_{m-1} = 0,$$

and then $$S(x) = b_m \sin(2m+1) x,$$

and $p(x)$ has the constant value $(2m+1)^2$.

It may also be proved that when $p(x)$ has the period π, two solutions, $C(x)$ and $S(x)$, of period π, cannot coexist except in the trivial case when $p(x)$ has the constant value $4m^2$. In these exceptional cases $p(x)$ is not, strictly speaking, of period π.