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Infinite families of congruences modulo 5 and 7 for the cubic partition function

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In 2010, Hei-Chi Chan introduced the cubic partition function $a(n)$ in connection with Ramanujan's cubic continued fraction. Chen and Lin, and Ahmed, Baruah and Dastidar proved that $a(25n + 22) \equiv 0 \pmod 5$ for $n \geqslant 0$. In this paper, we prove several infinite families of congruences modulo 5 and 7 for $a(n)$. Our results generalize the congruence $a(25n + 22) \equiv 0 \pmod 5$ and four congruences modulo 7 for $a(n)$ due to Chen and Lin. Moreover, we present some non-standard congruences modulo 5 for $a(n)$ by using an identity of Newman. For example, we prove that $a(((15 \times 17^{3\alpha} + 1)/8)) \equiv 3^{\alpha+1} \pmod 5$ for $\alpha \geqslant 0$.

Keywords: cubic partitions; congruences; θ function identities

2010 *Mathematics subject classification:* Primary 05A17; 11P83

1. Introduction

The aim of this paper is to prove infinite families of congruences modulo 5 and 7 for the cubic partition function which originated from the work of Chan [4] in connection with Ramanujan's cubic continued fraction.

Let $a(n)$ denote the number of cubic partitions of n. As usual, $a(0) = 1$. The generating function of $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}, \tag{1.1}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$$

and

$$(a_1, a_2, \ldots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty.$$

In 2010, Chan [4] proved the following elegant identity:

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}, \tag{1.2}$$

which yields

$$a(3n+2) \equiv 0 \pmod 3.$$

Furthermore, Chan [5] discovered congruences modulo any power of 3 for $a(n)$. Using the theory of modular forms, Chen and Lin [6] proved that for $n \geqslant 0$,

$$a(25n+22) \equiv 0 \pmod 5. \tag{1.3}$$

Recently, Ahmed, Baruah and Dastidar [1] proved (1.3) by using θ function identities. Very recently, Chern [7] proved some congruences modulo 5 for the coefficients of $((1/((q;q)_\infty(q^k;q^k)_\infty))$ where $k \in \{7,\ 8,\ 17\}$ by using the theory of modular forms. Moreover, Chen and Lin [6] proved that for $n \geqslant 0$,

$$a(49n+15) \equiv a(49n+29) \equiv a(49n+36) \equiv a(49n+43) \equiv 0 \pmod 7. \tag{1.4}$$

In this paper, we establish several infinite families of congruences modulo 5 and 7 for $a(n)$. Our results generalize (1.3) and (1.4). Furthermore, we prove some non-standard congruences modulo 5 for $a(n)$ by an identity due to Newman [10].

The main results of this paper can be stated as follows.

THEOREM 1.1. *Let p be a prime with $p \equiv 5$ or $7 \pmod 8$. For $n,\ \alpha \geqslant 0$, if $p \nmid n$, then*

$$a\left(5p^{2\alpha+1}n + \frac{15p^{2\alpha+2}+1}{8}\right) \equiv 0 \pmod 5. \tag{1.5}$$

For example, setting $\alpha = 0$ and $p = 13$ in (1.5), we see that for $n \geqslant 0$,

$$a(845n + 65j + 317) \equiv 0 \pmod 5,$$

where $13 \nmid j$.

THEOREM 1.2. *Let p be a prime with $p \equiv 5$ or $7 \pmod 8$. For $n,\ \alpha \geqslant 0$,*

$$a\left(25p^{2\alpha}n + \frac{175p^{2\alpha}+1}{8}\right) \equiv 0 \pmod 5. \tag{1.6}$$

Taking $\alpha = 0$ in (1.6), we get (1.3). Thus, theorem 1.2 is a generalization of (1.3).

THEOREM 1.3. *For all non-negative integers $\alpha,\ \beta,\ \gamma$ and n,*

$$\begin{aligned} &\sum_{k+l=5^\alpha n+((3(5^\alpha-1))/4)} a(5k+1)a(5l+3) \\ &\equiv \sum_{k+l=5^\beta n+((3(5^\beta-1))/4)} a(5k)a(5l+4) \\ &\equiv \sum_{k+l=5^\gamma n+((3(5^\gamma-1))/4)} a(5k+2)a(5l+2) \pmod 5. \end{aligned} \tag{1.7}$$

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In order to state the following theorem, we define

$$\begin{aligned} S_1 &:= \{(0,1),\ (0,4)\}, \\ S_2 &:= \{(1,1),\ (2,4),\ (3,4),\ (4,1)\}, \\ S_3 &:= \{(1,4),\ (2,1),\ (3,1),\ (4,4)\}, \end{aligned}$$

and

$$\sum_{n=0}^{\infty} c(n)q^n := (q;q)_\infty^3 (q^2;q^2)_\infty^3. \tag{1.8}$$

Moreover, assume that $c(((3(p-1))/8)) \equiv r \pmod 5$ and $p^2 \equiv s \pmod 5$ with $0 \leqslant r \leqslant 5$ and $s \in \{1,\ 4\}$, where p is a prime with $p \equiv 1 \pmod 8$. Define

$$\lambda(p) := \begin{cases} 2, & \text{if } (r,s) \in S_1, \\ 3, & \text{if } (r,s) \in S_2, \\ 5, & \text{if } (r,s) \in S_3. \end{cases} \tag{1.9}$$

We deduce the following infinite families of congruences modulo 5 for $a(n)$.

Theorem 1.4. *Let p be a prime with $p \equiv 1 \pmod 8$. For $n,\ \alpha \geqslant 0$, if $p \nmid (8n+3)$, then*

$$a\left(5p^{\lambda(p)(\alpha+1)-1}n + \frac{15p^{\lambda(p)(\alpha+1)-1}+1}{8}\right) \equiv 0 \pmod 5. \tag{1.10}$$

where $\lambda(p)$ is defined by (1.9).

For example, it is easy to see that $c(6) = -2 \equiv 3 \pmod 5$ and $17^2 \equiv 4 \pmod 5$. Therefore, $\lambda(17) = 3$. If we set $p = 17$ and $\alpha = 0$ in theorem 1.4, we see that for $n \geqslant 0$,

$$a(1445(17n+j)+542) \equiv 0 \pmod 5,$$

where $0 \leqslant j \leqslant 16$ and $j \neq 6$.

We also prove some non-standard congruences modulo 5 for $a(n)$ which can be stated as follows.

Theorem 1.5. *Let p be a prime and $p \equiv 1 \pmod 8$. Then*

$$a\left(\frac{215p+1}{8}\right) \equiv 0 \pmod 5. \tag{1.11}$$

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THEOREM 1.6. *Let p be a prime with $p \equiv 1 \pmod 8$. For $\alpha \geqslant 0$,*

$$a\left(\frac{15p^{\lambda(p)\alpha}+1}{8}\right) \equiv 3V(r,s)^{\alpha} \pmod 5, \tag{1.12}$$

where $\lambda(p)$ is defined by (1.9) and

$$V(r,s) := \begin{cases} -s, & \text{if } (r,s) \in S_1, \\ -rs, & \text{if } (r,s) \in S_2, \\ -r^3 s + 2rs^2, & \text{if } (r,s) \in S_3. \end{cases} \tag{1.13}$$

For example, setting $p = 17$ in theorem 1.6 and using the facts that $c(6) \equiv 3 \pmod 5$ and $17^2 \equiv 4 \pmod 5$, we deduce that

$$a\left(\frac{15 \times 17^{3\alpha}+1}{8}\right) \equiv 3^{\alpha+1} \pmod 5.$$

In order to state congruences modulo 7 for $a(n)$, we introduce the Legendre symbol. Let $p \geqslant 3$ be a prime. The Legendre symbol (a/p) is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\ 0, & \text{if } p \mid a, \\ -1, & \text{if } a \text{ is a non} - \text{quadratic residue modulo } p. \end{cases}$$

THEOREM 1.7. *Let p be a prime with $p \equiv 11, 17, 29, 31, 33, 37, 41, 43, 47, 51, 53, 55$ (mod56).*

(1) *For $n,\ \alpha \geqslant 0$, if $p \nmid n$, then*

$$a\left(7p^{2\alpha+1}n + \frac{7p^{2\alpha+2}+1}{8}\right) \equiv 0 \pmod 7. \tag{1.14}$$

(2) *For $n,\ \alpha \geqslant 0$,*

$$a\left(7p^{2\alpha}(7n+i) + \frac{7p^{2\alpha}+1}{8}\right) \equiv 0 \pmod 7, \tag{1.15}$$

where $i \in \{2,\ 4,\ 5,\ 6\}$.

It should be noted that if we set $\alpha = 0$ in (1.15), we obtain (1.4). Thus, (1.15) is a generalization of (1.4).

This paper is organized as follows. In § 2, we present proofs of Theorems 1.1 and 1.2 by utilizing the theory of quadratic residues. Section 3 is devoted to proving theorem 1.3 by using θ function identities. In § 4, we prove theorem 1.4 by utilizing an identity due to Newman [**10**]. In § 5, we provide proofs of Theorems 1.5 and 1.6. Finally, in § 6, we present a proof of Theorem 1.7 based on a congruence relation due to Lin [**9**].

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2. Proofs of Theorems 1.1 and 1.2

In order to prove theorems 1.1 and 1.2, we first prove the following two lemmas.

LEMMA 2.1. *We have*

$$\sum_{n=0}^{\infty} a(5n+2)q^n \equiv -2(q;q)_\infty^3(q^2;q^2)_\infty^3 \pmod 5. \tag{2.1}$$

Proof. By the binomial theorem,

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod 5. \tag{2.2}$$

From [3, Entry 10(v), p. 262],

$$\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^2} = \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}{(q;q)_\infty(q^{10};q^{10})_\infty} + q\frac{(q^{10};q^{10})_\infty^4}{(q^5;q^5)_\infty^2}. \tag{2.3}$$

In view of (1.1), (2.2) and (2.3),

$$\begin{aligned}\sum_{n=0}^{\infty} a(n)q^n &\equiv \frac{(q;q)_\infty(q^2;q^2)_\infty^4}{(q;q)_\infty^2(q^{10};q^{10})_\infty}\\ &\equiv \frac{(q;q)_\infty}{(q^{10};q^{10})_\infty}\left(\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}{(q;q)_\infty(q^{10};q^{10})_\infty} + q\frac{(q^{10};q^{10})_\infty^4}{(q^5;q^5)_\infty^2}\right)\\ &\equiv \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}{(q^{10};q^{10})_\infty^2} + q\frac{(q;q)_\infty(q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty^2} \pmod 5.\end{aligned} \tag{2.4}$$

Ramanujan [11, p. 212] stated the following identity:

$$(q;q)_\infty = (q^{25};q^{25})_\infty\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right), \tag{2.5}$$

where

$$R(q) = \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}.$$

Hirschhorn [8] gave a simple proof of the above identity by using Jacobi's triple product identity. Substituting (2.5) into (2.4), we get

$$\begin{aligned}\sum_{n=0}^{\infty} a(n)q^n \equiv{}& \frac{(q^5;q^5)_\infty^3(q^{50};q^{50})_\infty}{(q^{10};q^{10})_\infty^2}\left(\frac{1}{R(q^{10})} - q^2 - q^4R(q^{10})\right)\\ &+ q\frac{(q^{10};q^{10})_\infty^3(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^2}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)\end{aligned}$$

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$$\begin{aligned}
&\equiv \frac{(q^5;q^5)_\infty^3 (q^{50};q^{50})_\infty}{(q^{10};q^{10})_\infty^2 R(q^{10})} + q\frac{(q^{10};q^{10})_\infty^3 (q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^2 R(q^5)} \\
&\quad - q^2\frac{(q^5;q^5)_\infty^3 (q^{50};q^{50})_\infty}{(q^{10};q^{10})_\infty^2} - q^2\frac{(q^{10};q^{10})_\infty^3 (q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^2} \\
&\quad - q^3\frac{(q^{10};q^{10})_\infty^3 (q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^2} R(q^5) \\
&\quad - q^4\frac{(q^5;q^5)_\infty^3 (q^{50};q^{50})_\infty}{(q^{10};q^{10})_\infty^2} R(q^{10}) \pmod 5, \qquad (2.6)
\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty} a(5n+2)q^n \equiv -\frac{(q;q)_\infty^3 (q^{10};q^{10})_\infty}{(q^2;q^2)_\infty^2} - \frac{(q^2;q^2)_\infty^3 (q^5;q^5)_\infty}{(q;q)_\infty^2} \pmod 5. \qquad (2.7)$$

Congruence (2.1) follows from (2.2) and (2.7). □

LEMMA 2.2. *For $n \geqslant 0$,*

$$c\left(pn + \frac{3(p^2-1)}{8}\right) = p^2 c(n/p), \qquad (2.8)$$

where p is a prime with $p \equiv 5$ or $7 \pmod 8$ and $c(n)$ is defined by (1.8).

Proof. We have the well-known result of Jacobi [2, p.176] which states that

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{((n(n+1))/2)}. \qquad (2.9)$$

By (1.8) and (2.9),

$$\sum_{n=0}^{\infty} c(n)q^n = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} (-1)^{k+m}(2k+1)(2m+1)q^{((k(k+1))/2)+m(m+1)}, \qquad (2.10)$$

which implies

$$c(n) = \sum_{\substack{\frac{k(k+1)}{2}+m(m+1)=n,\\ k,\ m\geqslant 0}} (-1)^{k+m}(2k+1)(2m+1). \qquad (2.11)$$

We can rewrite (2.11) as

$$c(n) = \sum_{\substack{(2k+1)^2+2(2m+1)^2=8n+3,\\ k,\ m\geqslant 0}} (-1)^{k+m}(2k+1)(2m+1). \qquad (2.12)$$

Therefore,

$$c\left(pn + \frac{3(p^2-1)}{8}\right) = \sum_{\substack{(2k+1)^2+2(2m+1)^2=8pn+3p^2,\\ k,\ m\geqslant 0}} (-1)^{k+m}(2k+1)(2m+1). \qquad (2.13)$$

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Identity $(2k+1)^2 + 2(2m+1)^2 = 8pn + 3p^2$ yields

$$(2k+1)^2 + 2(2m+1)^2 \equiv 0 \pmod{p}.$$

The above congruence implies $p|(2k+1)$ and $p|(2m+1)$ since $p \equiv 5,$ or $7 \pmod{8}$ and $(-2/p) = -1$. Let $2k+1 = p(2k'+1)$ and $2m+1 = p(2m'+1)$. Note that k' and m' are non-negative integers. Thus,

$$\begin{aligned} c\left(pn + \frac{3(p^2-1)}{8}\right) &= \sum_{\substack{p^2(2k'+1)^2+2p^2(2m'+1)^2=8pn+3p^2, \\ k',\ m' \geqslant 0}} (-1)^{pk'+pm'+p-1} \\ &\quad \times p^2(2k'+1)(2m'+1) \\ &= p^2 \sum_{\substack{(2k'+1)^2+2(2m'+1)^2=8n/p+3, \\ k',\ m' \geqslant 0}} (-1)^{k'+m'}(2k'+1)(2m'+1) \\ &= p^2 c(n/p), \qquad \text{by (2.12)} \end{aligned}$$

which is nothing but (2.8). This completes the proof of this lemma. □

Now, we are ready to prove theorem 1.1. Let p be a prime with $p \equiv 5$ or $7 \pmod{8}$. Identity (2.8) implies that for $n \geqslant 0$,

$$c\left(p^2 n + \frac{3(p^2-1)}{8}\right) = p^2 c(n) \tag{2.14}$$

and if $p \nmid n$,

$$c\left(pn + \frac{3(p^2-1)}{8}\right) = 0. \tag{2.15}$$

It follows from (2.14) that for $n,\ \alpha \geqslant 0$,

$$c\left(p^{2\alpha} n + \frac{3(p^{2\alpha}-1)}{8}\right) = p^{2\alpha} c(n). \tag{2.16}$$

Replacing n by $pn + ((3(p^2-1))/8)$ $(p \nmid n)$ in (2.16) and employing (2.15), we see that

$$c\left(p^{2\alpha+1} n + \frac{3(p^{2\alpha+2}-1)}{8}\right) = 0. \tag{2.17}$$

Combining (2.1) and (1.8), we deduce that for $n \geqslant 0$,

$$a(5n+2) \equiv -2c(n) \pmod{5}. \tag{2.18}$$

Congruence (1.5) follows from (2.17) and (2.18). This completes the proof of Theorem 1.1.

Next, we turn to prove theorem 1.2. By (2.9), it is trivial to check that

$$(q;q)_\infty^3 \equiv (q^{10},q^{15},q^{25};q^{25})_\infty - 3q(q^5,q^{20},q^{25};q^{25})_\infty \pmod 5. \tag{2.19}$$

Substituting (2.19) into (1.8), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c(n)q^n &\equiv \big((q^{10},q^{15},q^{25};q^{25})_\infty - 3q(q^5,q^{20},q^{25};q^{25})_\infty\big)\\ &\qquad \times \big((q^{20},q^{30},q^{50};q^{50})_\infty - 3q^2(q^{10},q^{40},q^{50};q^{50})_\infty\big)\\ &\equiv (q^{10},q^{15},q^{25};q^{25})_\infty(q^{20},q^{30},q^{50};q^{50})_\infty\\ &\quad - 3q(q^5,q^{20},q^{25};q^{25})_\infty(q^{20},q^{30},q^{50};q^{50})_\infty\\ &\quad - 3q^2(q^{10},q^{15},q^{25};q^{25})_\infty(q^{10},q^{40},q^{50};q^{50})_\infty\\ &\quad + 4q^3(q^5,q^{20},q^{25};q^{25})_\infty(q^{10},q^{40},q^{50};q^{50})_\infty \pmod 5,\end{aligned} \tag{2.20}$$

which yields

$$c(5n+4) \equiv 0 \pmod 5. \tag{2.21}$$

Replacing n by $5n+4$ in (2.16) and employing (2.21), we see that

$$c\left(p^{2\alpha}(5n+4) + \frac{3(p^{2\alpha}-1)}{8}\right) \equiv 0 \pmod 5, \tag{2.22}$$

where p is a prime with $p \equiv 5$ or $7 \pmod 8$. Replacing n by $p^{2\alpha}(5n+4) + ((3(p^{2\alpha}-1))/8)$ in (2.18) and utilizing (2.22), we arrive at (1.6). The proof of Theorem 1.2 is complete.

3. Proof of Theorem 1.3

In this section, we present a proof of Theorem 1.3.

It follows from (2.2) and (2.6) that

$$\sum_{n=0}^{\infty} a(5n)q^n \equiv \frac{(q;q)_\infty^3(q^2;q^2)_\infty^3}{R(q^2)} \pmod 5, \tag{3.1}$$

$$\sum_{n=0}^{\infty} a(5n+1)q^n \equiv \frac{(q;q)_\infty^3(q^2;q^2)_\infty^3}{R(q)} \pmod 5, \tag{3.2}$$

$$\sum_{n=0}^{\infty} a(5n+3)q^n \equiv -(q;q)_\infty^3(q^2;q^2)_\infty^3 R(q) \pmod 5, \tag{3.3}$$

$$\sum_{n=0}^{\infty} a(5n+4)q^n \equiv -(q;q)_\infty^3(q^2;q^2)_\infty^3 R(q^2) \pmod 5. \tag{3.4}$$

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It follows from (2.1) and (3.1)–(3.4) that

$$\sum_{n=0}^{\infty}\sum_{k+l=n} a(5k)a(5l+4)q^n \equiv \sum_{n=0}^{\infty}\sum_{k+l=n} a(5k+1)a(5l+3)q^n \equiv \sum_{n=0}^{\infty}\sum_{k+l=n} a(5k+2)a(5l+2)q^n \equiv -(q;q)_\infty^6(q^2;q^2)_\infty^6 \pmod 5. \tag{3.5}$$

The above congruence implies that for $n \geqslant 0$,

$$\sum_{k+l=n} a(5k)a(5l+4) \equiv \sum_{k+l=n} a(5k+1)a(5l+3) \equiv \sum_{k+l=n} a(5k+2)a(5l+2) \equiv d(n) \pmod 5, \tag{3.6}$$

where $d(n)$ is defined by

$$\sum_{n=0}^{\infty} d(n)q^n := -(q;q)_\infty^6(q^2;q^2)_\infty^6. \tag{3.7}$$

It follows from (2.2), (2.5) and (3.7) that

$$\begin{aligned}\sum_{n=0}^{\infty} d(n)q^n &\equiv -(q;q)_\infty(q^2;q^2)_\infty(q^5;q^5)_\infty(q^{10};q^{10})_\infty\\ &\equiv -(q^5;q^5)_\infty(q^{10};q^{10})_\infty(q^{25};q^{25})_\infty(q^{50};q^{50})_\infty\\ &\quad\times\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right)\left(\frac{1}{R(q^{10})}-q^2-q^4R(q^{10})\right)\\ &\equiv -(q^5;q^5)_\infty^6(q^{10};q^{10})_\infty^6\left(\frac{1}{R(q^5)R(q^{10})}-\frac{q}{R(q^{10})}-\frac{q^2}{R(q^5)}-\frac{q^2R(q^5)}{R(q^{10})}\right.\\ &\quad\left.+q^3-q^4\frac{R(q^{10})}{R(q^5)}+q^4R(q^5)+q^5R(q^{10})+q^6R(q^5)R(q^{10})\right) \pmod 5,\end{aligned} \tag{3.8}$$

which yields

$$\sum_{n=0}^{\infty} d(5n+3)q^n \equiv -(q;q)_\infty^6(q^2;q^2)_\infty^6 \pmod 5. \tag{3.9}$$

Because of (3.7) and (3.9), we see that for $n \geqslant 0$,

$$d(5n+3) \equiv d(n) \pmod 5. \tag{3.10}$$

By (3.10) and mathematical induction, we find that for $\alpha \geqslant 0$,

$$d\left(5^\alpha n+\frac{3(5^\alpha-1)}{4}\right) \equiv d(n) \pmod 5. \tag{3.11}$$

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Congruence (1.7) follows from (3.6) and (3.11). This completes the proof.

4. Proof of Theorem 1.4

In order to prove theorem 1.4, we first prove the following two lemmas.

LEMMA 4.1. *Let* $(r, s) \in S_1 \bigcup S_2 \bigcup S_3$ *and define*

$$S_{r,s} := \Big\{ p | p \text{ is a prime}, \; p \equiv 1 \pmod 8, \\ c\left(\frac{3(p-1)}{8}\right) \equiv r \pmod 5 \text{ and } p^2 \equiv s \pmod 5 \Big\}.$$

If $p \in S_{r,s}$*, then for* $n, \; \alpha \geqslant 0$,

$$c\left(p^{\alpha}n + \frac{3(p^{\alpha}-1)}{8}\right) \equiv A_{r,s}(\alpha) c\left(pn + \frac{3(p-1)}{8}\right) + B_{r,s}(\alpha)c(n) \pmod 5, \quad (4.1)$$

where $c(n)$ *is defined by* (1.8) *and* $A_{r,s}(\alpha)$ *and* $B_{r,s}(\alpha)$ *are defined by*

$$A_{r,s}(\alpha+2) = rA_{r,s}(\alpha+1) - sA_{r,s}(\alpha), \quad (4.2)$$

$$B_{r,s}(\alpha+2) = rB_{r,s}(\alpha+1) - sB_{r,s}(\alpha), \quad (4.3)$$

with $B_{r,s}(0) = A_{r,s}(1) = 1$ *and* $B_{r,s}(1) = A_{r,s}(0) = 0$.

Proof. We prove (4.1) by induction on α. It is routine to check that (4.1) holds when $\alpha = 0$ and $\alpha = 1$ since $A_{r,s}(1) = B_{r,s}(0) = 1$ and $A_{r,s}(0) = B_{r,s}(1) = 0$ for $(r, s) \in S_1 \bigcup S_2 \bigcup S_3$. Suppose that (4.1) holds when $\alpha = m$ and $\alpha = m + 1$ ($m \geqslant 0$), that is,

$$c\left(p^{m}n + \frac{3(p^{m}-1)}{8}\right) \equiv A_{r,s}(m) c\left(pn + \frac{3(p-1)}{8}\right) + B_{r,s}(m)c(n) \pmod 5, \quad (4.4)$$

and

$$c\left(p^{m+1}n + \frac{3(p^{m+1}-1)}{8}\right) \equiv A_{r,s}(m+1) c\left(pn + \frac{3(p-1)}{8}\right) \\ + B_{r,s}(m+1)c(n) \pmod 5, \quad (4.5)$$

where $p \in S_{r,s}$. From the definition of $S_{r,s}$,

$$c\left(\frac{3(p-1)}{8}\right) \equiv r \pmod 5 \quad \text{and} \quad p^2 \equiv s \pmod 5. \quad (4.6)$$

Newman [10] proved that

$$c\left(pn + \frac{3(p-1)}{8}\right) = \nu(p)c(n) - p^2 c\left(\frac{n}{p} - \frac{3(p-1)}{8p}\right), \quad (4.7)$$

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where p is a prime with $p \equiv 1 \pmod 8$ and $\nu(p)$ is a function in p. Setting $n = 0$ in (4.7) and using the fact that $c(0) = 1$, we get

$$\nu(p) = c\left(\frac{3(p-1)}{8}\right). \tag{4.8}$$

Replacing n by $pn + ((3(p-1))/8)$ in (4.7) and utilizing (4.8), we get

$$c\left(p^2 n + \frac{3(p^2-1)}{8}\right) = c\left(\frac{3(p-1)}{8}\right) c\left(pn + \frac{3(p-1)}{8}\right) - p^2 c(n). \tag{4.9}$$

Thanks to (4.6) and (4.9),

$$c\left(p^2 n + \frac{3(p^2-1)}{8}\right) \equiv rc\left(pn + \frac{3(p-1)}{8}\right) - sc(n) \pmod 5, \tag{4.10}$$

where $p \in S_{r,s}$. Replacing n by $p^m n + ((3(p^m-1))/8)$ in (4.10) and utilizing (4.2)–(4.5) yields

$$\begin{aligned} & c\left(p^{m+2} n + \frac{3(p^{m+2}-1)}{8}\right) \\ & \quad \equiv rc\left(p^{m+1} n + \frac{3(p^{m+1}-1)}{8}\right) - sc\left(p^m n + \frac{3(p^m-1)}{8}\right) \\ & \quad \equiv rA_{r,s}(m+1)c\left(pn + \frac{3(p-1)}{8}\right) + rB_{r,s}(m+1)c(n) \\ & \qquad - sA_{r,s}(m)c\left(pn + \frac{3(p-1)}{8}\right) - sB_{r,s}(m)c(n) \\ & \quad \equiv (rA_{r,s}(m+1) - sA_{r,s}(m))c\left(pn + \frac{3(p-1)}{8}\right) \\ & \qquad + (rB_{r,s}(m+1) - sB_{r,s}(m))c(n) \\ & \quad \equiv A_{r,s}(m+2)c\left(pn + \frac{3(p-1)}{8}\right) + B_{r,s}(m+2)c(n) \pmod 5, \end{aligned}$$

which implies that (4.1) is true when $\alpha = m + 2$. Congruence (4.1) is proved by induction and this completes the proof of Lemma 4.1. □

Lemma 4.2. *If $(r, s) \in S_1 \bigcup S_2 \bigcup S_3$ and $p \in S_{r,s}$, then for $\alpha \geqslant 0$,*

$$rA_{r,s}(\lambda(p)(\alpha+1) - 1) + B_{r,s}(\lambda(p)(\alpha+1) - 1) \equiv 0 \pmod 5, \tag{4.11}$$

where $\lambda(p)$, $A_{r,s}(\alpha)$ and $B_{r,s}(\alpha)$ are defined by (1.9), (4.2) and (4.3), respectively.

Proof. We also prove (4.11) by induction on α. It is easy to verify that (4.11) holds for all $(r, s) \in S_1 \bigcup S_2 \bigcup S_3$ when $\alpha = 0$. Assume that (4.11) is true when $\alpha = m$

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$(m \geqslant 0)$, namely,

$$rA_{r,s}(\lambda(p)m+\lambda(p)-1)+B_{r,s}(\lambda(p)m+\lambda(p)-1)\equiv 0 \pmod 5, \tag{4.12}$$

where $(r,s)\in S_1\bigcup S_2\bigcup S_3$ and $p\in S_{r,s}$. Based on (4.2) and (4.3),

$$\begin{aligned} &rA_{r,s}(\lambda(p)m+2\lambda(p)-1)+B_{r,s}(\lambda(p)m+2\lambda(p)-1)\\ &\quad = U(r,s)(rA_{r,s}(\lambda(p)m+\lambda(p))+B_{r,s}(\lambda(p)m+\lambda(p)))\\ &\qquad + V(r,s)(rA_{r,s}(\lambda(p)m+\lambda(p)-1)+B_{r,s}(\lambda(p)m+\lambda(p)-1)), \end{aligned} \tag{4.13}$$

where

$$U(r,s):=\begin{cases} r=0, & \text{if } (r,s)\in S_1,\\ r^2-s, & \text{if } (r,s)\in S_2,\\ r^4-3r^2s+s^2, & \text{if } (r,s)\in S_3, \end{cases} \tag{4.14}$$

and $V(r,s)$ is defined by (1.13). It is easy to check that for any pair $(r,s)\in S_1\bigcup S_2\bigcup S_3$,

$$U(r,s)\equiv 0 \pmod 5. \tag{4.15}$$

Combining (4.12), (4.13) and (4.15), we see that for $(r,s)\in S_1\bigcup S_2\bigcup S_3$,

$$rA_{r,s}(\lambda(p)m+2\lambda(p)-1)+B_{r,s}(\lambda(p)m+2\lambda(p)-1)\equiv 0 \pmod 5,$$

which implies that (4.11) is true when $\alpha=m+1$ and (4.11) is proved by induction. This completes the proof of this lemma. □

Now, we turn to prove theorem 1.1.

Let p be a prime with $p\equiv 1 \pmod 8$. Assume that $c(((3(p-1))/8))\equiv r \pmod 5$ and $p^2\equiv s \pmod 5$ with $0\leqslant r\leqslant 5$ and $s\in\{1,\ 4\}$. Thus, for any prime $p\equiv 1 \pmod 8$, there exists a pair $(r,s)\in S_1\bigcup S_2\bigcup S_3$ such that $p\in S_{r,s}$, where $S_{r,s}$ is defined in lemma 4.1. In view of (4.1), (4.6), (4.7) and (4.8),

$$\begin{aligned} c\left(p^\alpha n+\frac{3(p^\alpha-1)}{8}\right) &\equiv A_{r,s}(\alpha)\left(c\left(\frac{3(p-1)}{8}\right)c(n)-p^2c\left(\frac{n-3(p-1)/8}{p}\right)\right)\\ &\quad + B_{r,s}(\alpha)c(n)\\ &\equiv (rA_{r,s}(\alpha)+B_{r,s}(\alpha))c(n)\\ &\quad - sA_{r,s}(\alpha)c\left(\frac{n-3(p-1)/8}{p}\right) \pmod 5. \end{aligned} \tag{4.16}$$

Replacing α by $\lambda(p)(\alpha+1)-1$ in (4.16) and utilizing (4.11), we obtain

$$\begin{aligned} &c\left(p^{\lambda(p)(\alpha+1)-1}n+\frac{3(p^{\lambda(p)(\alpha+1)-1}-1)}{8}\right)\\ &\equiv -sA_{r,s}(\lambda(p)(\alpha+1)-1)c\left(\frac{n-3(p-1)/8}{p}\right) \pmod 5. \end{aligned} \tag{4.17}$$

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Note that if $p \nmid (8n+3)$, then $((n-3(p-1)/8)/p)$ is not an integer and

$$c\left(\frac{n-3(p-1)/8}{p}\right)=0. \tag{4.18}$$

Combining (4.17) and (4.18), we deduce that if $p \nmid (8n+3)$, then for $\alpha \geqslant 0$,

$$c\left(p^{\lambda(p)(\alpha+1)-1}n+\frac{3(p^{\lambda(p)(\alpha+1)-1}-1)}{8}\right)\equiv 0 \pmod 5. \tag{4.19}$$

Replacing n by $p^{\lambda(p)(\alpha+1)-1}n+((3(p^{\lambda(p)(\alpha+1)-1}-1))/8)$ in (2.18) and utilizing (4.19), we arrive at (1.10). The proof is completed.

5. Proofs of Theorems 1.5 and 1.6

In this section, we always let p be a prime with $p \equiv 1 \pmod 8$. In order to prove theorems 1.5 and 1.6, we first prove the following lemma.

LEMMA 5.1. *Let $(r,s)\in S_1 \bigcup S_2 \bigcup S_3$ and let $S_{r,s}$ be defined in lemma 4.1. If $p \in S_{r,s}$, then for $\alpha \geqslant 0$,*

$$A_{r,s}(\lambda(p)\alpha)\equiv 0 \pmod 5. \tag{5.1}$$

and

$$B_{r,s}(\lambda(p)\alpha)\equiv V(r,s)^{\alpha} \pmod 5, \tag{5.2}$$

where $V(r,s)$ is defined by (1.13).

Proof. We prove (5.1) and (5.2) by induction on α. It is easy to see that (5.1) and (5.2) hold when $\alpha=0$ since $A_{r,s}(0)=0$ and $B_{r,s}(0)=1$. Suppose that (5.1) and (5.2) are true when $\alpha=m$, namely,

$$A_{r,s}(\lambda(p)m)\equiv 0 \pmod 5 \tag{5.3}$$

and

$$B_{r,s}(\lambda(p)m)\equiv V(r,s)^{m} \pmod 5. \tag{5.4}$$

Thanks to (4.2) and (4.3),

$$A_{r,s}(\lambda(p)m+\lambda(p))=U(r,s)A_{r,s}(\lambda(p)m+1)+V(r,s)A_{r,s}(\lambda(p)m) \tag{5.5}$$

and

$$B_{r,s}(\lambda(p)m+\lambda(p))=U(r,s)B_{r,s}(\lambda(p)m+1)+V(r,s)B_{r,s}(\lambda(p)m), \tag{5.6}$$

where $U(r,s)$ and $V(r,s)$ are defined by (4.14) and (1.13), respectively. Because of (4.15), (5.3) and (5.5), we see that (5.1) is true when $\alpha=m+1$. Combining (4.15) and (5.6) yields

$$B_{r,s}(\lambda(p)m+\lambda(p))\equiv V(r,s)B_{r,s}(\lambda(p)m) \pmod 5. \tag{5.7}$$

It follows from (5.4) and (5.7) that (5.2) is true when $\alpha=m+1$. Therefore, lemma 5.1 is proved by induction. This completes the proof of this lemma. □

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Now, we are ready to prove theorem 1.5.

By (4.7) and (4.8), we see that if $p \nmid (8n+3)$, then $(n/p) - ((3(p-1))/(8p))$ is not an integer and

$$c\left(pn + \frac{3(p-1)}{8}\right) = c\left(\frac{3(p-1)}{8}\right) c(n). \tag{5.8}$$

Setting $n = 5$ in (5.8) and using the facts that $c(5) \equiv 0 \pmod 5$ and $p \nmid (8 \times 5 + 3)$, we get

$$c\left(\frac{43p-3}{8}\right) \equiv 0 \pmod 5. \tag{5.9}$$

Theorem 1.5 follows from (2.18) and (5.9).

Now, we turn to prove theorem 1.6.

Setting $n = 0$ in (4.1) and using the fact that $c(0) = 1$, we get

$$c\left(\frac{3(p^\alpha - 1)}{8}\right) \equiv A_{r,s}(\alpha) c\left(\frac{3(p-1)}{8}\right) + B_{r,s}(\alpha) \pmod 5. \tag{5.10}$$

Replacing α by $\lambda(p)\alpha$ in (5.10) and employing (5.1) and (5.2) yields,

$$c\left(\frac{3(p^{\lambda(p)\alpha} - 1)}{8}\right) \equiv V(r,s)^\alpha \pmod 5. \tag{5.11}$$

Replacing n by $\frac{3(p^{\lambda(p)\alpha}-1)}{8}$ in (2.18) and using (5.11), we arrive at (1.12). This completes the proof.

6. Proof of Theorem 1.7

In order to prove theorem 1.7, we first prove the following lemma.

LEMMA 6.1. *Define*

$$\sum_{n=0}^{\infty} e(n) q^n := (q;q)_\infty^3 \frac{(q^7;q^7)_\infty^2}{(q^{14};q^{14})_\infty} + q(q^2;q^2)_\infty^3 \frac{(q^{14};q^{14})_\infty^2}{(q^7;q^7)_\infty}. \tag{6.1}$$

If $p \geqslant 3$ *is a prime with* $p \equiv 11, 17, 29, 31, 33, 37, 41, 43, 47, 51, 53, 55 \pmod{56}$, *then*

$$e\left(pn + \frac{p^2-1}{8}\right) = (-1)^{((p-1)/2)} p e(n/p). \tag{6.2}$$

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Proof. The following identities follow from Jacobi's triple product identity:

$$\sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \tag{6.3}$$

and

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}. \tag{6.4}$$

By (2.9), (6.1), (6.3) and (6.4),

$$\begin{aligned}\sum_{n=0}^{\infty} e(n)q^n =& \sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty} (-1)^{k+m}(2k+1)q^{((k(k+1))/2)+7m^2} \\ &+ \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^r(2r+1)q^{r(r+1)+7((s(s+1))/2)+1},\end{aligned}$$

which yields

$$e(n) = \sum_{\substack{((k(k+1))/2)+7m^2=n,\\ k\geqslant 0, -\infty<m<+\infty}} (-1)^{k+m}(2k+1) + \sum_{\substack{r(r+1)+7((s(s+1))/2)=n-1,\\ r,\ s\geqslant 0}} (-1)^r(2r+1). \tag{6.5}$$

We can rewrite (6.5) as

$$e(n) = \sum_{\substack{(2k+1)^2+56m^2=8n+1,\\ k\geqslant 0, -\infty<m<+\infty}} (-1)^{k+m}(2k+1) + \sum_{\substack{2(2r+1)^2+7(2s+1)^2=8n+1,\\ r,\ s\geqslant 0}} (-1)^r(2r+1). \tag{6.6}$$

Therefore,

$$\begin{aligned}e\left(pn+\frac{p^2-1}{8}\right) =& \sum_{\substack{(2k+1)^2+56m^2=8pn+p^2,\\ k\geqslant 0, -\infty<m<+\infty}} (-1)^{k+m}(2k+1) \\ &+ \sum_{\substack{2(2r+1)^2+7(2s+1)^2=8pn+p^2,\\ r,\ s\geqslant 0}} (-1)^r(2r+1),\end{aligned}$$

where p is a prime with $p \equiv 11, 17, 29, 31, 33, 37, 41, 43, 47, 51, 53, 55 \pmod{56}$. It is easy to check that $(-14/p) = -1$. Identities $(2k+1)^2 + 56m^2 = 8pn + p^2$ and $2(2r+1)^2 + 7(2s+1)^2 = 8pn + p^2$ imply

$$(2k+1)^2 + 56m^2 \equiv 0 \pmod{p}$$

and

$$2(2r+1)^2 + 7(2s+1)^2 \equiv 0 \pmod{p}.$$

The above two congruences yield $p|(2k+1)$, $p|m$, $p|(2r+1)$ and $p|(2s+1)$ since $(-14/p) = -1$. Let $2k+1 = p(2k'+1)$, $m = pm'$, $2r+1 = p(2r'+1)$ and

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$2s+1=p(2s'+1)$. Note that k', r' and s' are non-negative integers and m' is an integer. Hence,

$$\begin{aligned} e\left(pn+\frac{p^2-1}{8}\right) &= \sum_{\substack{p^2(2k'+1)^2+56p^2m'^2=8pn+p^2,\\ k'\geqslant 0, -\infty<m'<+\infty}} (-1)^{pk'+((p-1)/2)+pm'}p(2k'+1) \\ &\quad + \sum_{\substack{2p^2(2r'+1)^2+7p^2(2s'+1)^2=8pn+p^2,\\ r',\ s'\geqslant 0}} (-1)^{pr'+((p-1)/2)}p(2r'+1) \\ &= (-1)^{((p-1)/2)}p\left(\sum_{\substack{(2k'+1)^2+56m'^2=8n/p+1,\\ k'\geqslant 0, -\infty<m'<+\infty}} (-1)^{pk'+pm'}(2k'+1)\right. \\ &\quad \left. + \sum_{\substack{2(2r'+1)^2+7(2s'+1)^2=8n/p+1,\\ r',\ s'\geqslant 0}} (-1)^{pr'}(2r'+1)\right) \\ &= (-1)^{((p-1)/2)}p\left(\sum_{\substack{(2k'+1)^2+56m'^2=8n/p+1,\\ k'\geqslant 0, -\infty<m'<+\infty}} (-1)^{k'+m'}(2k'+1)\right. \\ &\quad \left. + \sum_{\substack{2(2r'+1)^2+7(2s'+1)^2=8n/p+1,\\ r',\ s'\geqslant 0}} (-1)^{r'}(2r'+1)\right) \\ &= (-1)^{((p-1)/2)}pe(n/p), \quad \text{by (6.6)} \end{aligned}$$

which is nothing but (6.2). The proof of this lemma is complete. □

To conclude this section, we provide a proof of Theorem 1.7. Let p be a prime with $p\equiv 11, 17, 29, 31, 33, 37, 41, 43, 47, 51, 53, 55 \pmod{56}$. Identity (6.2) implies that if $p\nmid n$, then

$$e\left(pn+\frac{p^2-1}{8}\right)=0. \tag{6.7}$$

Replacing n by pn in (6.2), we get

$$e\left(p^2n+\frac{p^2-1}{8}\right)=(-1)^{((p-1)/2)}pe(n). \tag{6.8}$$

By (6.8) and mathematical induction, we deduce that for $n,\ \alpha\geqslant 0$,

$$e\left(p^{2\alpha}n+\frac{p^{2\alpha}-1}{8}\right)=(-1)^{((\alpha(p-1))/2)}p^{\alpha}e(n). \tag{6.9}$$

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Replacing n by $pn + ((p^2 - 1)/8)$ $(p \nmid n)$ in (6.9) and using (6.7), we deduce that if $p \nmid n$, then

$$e\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2} - 1}{8}\right) = 0. \tag{6.10}$$

In his thesis [9], Lin proved that

$$\sum_{n=0}^{\infty} a(7n+1)q^n \equiv (q;q)_\infty^3 \frac{(q^7;q^7)_\infty^2}{(q^{14};q^{14})_\infty} + q(q^2;q^2)_\infty^3 \frac{(q^{14};q^{14})_\infty^2}{(q^7;q^7)_\infty} \pmod 7. \tag{6.11}$$

Combining (6.1) and (6.11), we obtain

$$a(7n+1) \equiv e(n) \pmod 7. \tag{6.12}$$

Replacing n by $p^{2\alpha+1}n + ((p^{2\alpha+2} - 1)/8)$ $(p \nmid n)$ in (6.12) and employing (6.10), we obtain (1.14). Thanks to (6.9) and (6.12),

$$a\left(7p^{2\alpha}n + \frac{7p^{2\alpha} + 1}{8}\right) \equiv (-1)^{((\alpha(p-1))/2)} p^\alpha a(7n+1) \pmod 7. \tag{6.13}$$

Replacing n by $7n + i$ $(i \in \{2,\ 4,\ 5,\ 6\})$ in (6.13) and employing (1.4), we reach (1.15). This completes the proof.

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