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The epsilon constant conjecture for higher dimensional unramified twists of $\mathbb{Z}_p^r(1)$

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Abstract. Let N/K be a finite Galois extension of p-adic number fields, and let $\rho^{\mathrm{nr}}: G_K \longrightarrow \mathrm{Gl}_r(\mathbb{Z}_p)$ be an r-dimensional unramified representation of the absolute Galois group G_K, which is the restriction of an unramified representation $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}: G_{\mathbb{Q}_p} \longrightarrow \mathrm{Gl}_r(\mathbb{Z}_p)$. In this paper, we consider the $\mathrm{Gal}(N/K)$-equivariant local ε-conjecture for the p-adic representation $T = \mathbb{Z}_p^r(1)(\rho^{\mathrm{nr}})$. For example, if A is an abelian variety of dimension r defined over $\mathbb{Q}_p$ with good ordinary reduction, then the Tate module $T = T_p\hat{A}$ associated to the formal group $\hat{A}$ of A is a p-adic representation of this form. We prove the conjecture for all tame extensions N/K and a certain family of weakly and wildly ramified extensions N/K. This generalizes previous work of Izychev and Venjakob in the tame case and of the authors in the weakly and wildly ramified case.

1 Introduction

Let p be a prime and N/K a finite Galois extension of p-adic number fields with group $G := \mathrm{Gal}(N/K)$. We write G_K (resp. G_N) for the absolute Galois group of K (resp. N), and for each finite extension $E/\mathbb{Q}_p$, we let F_E denote the arithmetic Frobenius automorphism. Let V denote a p-adic representation of G_K, and let $T \subseteq V$ be a G_K-stable $\mathbb{Z}_p$-sublattice such that $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$.

As in [IV16, BC17], we write $C_{EP}^{na}(N/K, V)$ for the equivariant ε-constant conjecture (see, for example, Conjecture 3.1.1 in [BC17]). For more details and some remarks on the history of the conjecture, we refer the interested reader to the introduction and Section 3.1 of [BC17].

In this manuscript, we will consider $C_{EP}^{na}(N/K, V)$ for higher dimensional unramified twists of $\mathbb{Z}_p^r(1)$ (which should be considered as the Tate module associated with $\mathbb{G}_m^r$). More precisely, by [Cob18, Proposition 1.6], each matrix $U \in \mathrm{Gl}_r(\mathbb{Z}_p)$ gives rise to an unramified representation of G_K by setting $\rho^{\mathrm{nr}}(F_K) := U$. We will be concerned with the module $T = \mathbb{Z}_p^r(1)(\rho^{\mathrm{nr}})$, which by [Cob18, Proposition 1.11] can be considered as the Tate module of an r-dimensional Lubin–Tate formal group.

We recall that for $r = 1$ and representations ρ^{nr} which are restrictions of unramified representations $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}: G_{\mathbb{Q}_p} \longrightarrow \mathbb{Z}_p^\times$, Izychev and Venjakob in [IV16] have proved the validity of $C_{EP}^{na}(N/K, V)$ for tame extensions N/K. The main result of [BC17, Theorem 1] shows that $C_{EP}^{na}(N/K, V)$ holds for certain weakly and wildly ramified finite abelian

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extensions N/K. In this context, we recall that N/K is weakly ramified if the second ramification group in lower numbering is trivial. Generalizing these results, we will show the following theorem.

Theorem 1.1 *Let N/K be a tame extension of p-adic number fields, and let*

$$\rho^{\mathrm{nr}}_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{Gl}_r(\mathbb{Z}_p)$$

be an unramified representation of $G_{\mathbb{Q}_p}$. Let ρ^{nr} denote the restriction of $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$ to G_K. Then, $C^{na}_{EP}(N/K, V)$ is true for N/K and $V = \mathbb{Q}_p^r(1)(\rho^{\mathrm{nr}})$, if $\det(\rho^{\mathrm{nr}}(F_N) - 1) \neq 0$.

Remarks 1.2 (1) The condition $\det(\rho^{\mathrm{nr}}(F_N) - 1) \neq 0$ holds, if and only if $H^2(N, T)$ is finite (see Section 2). It is also equivalent to $\left(\mathbb{Z}_p^r(\rho^{\mathrm{nr}})\right)^{G_N} = 0$.

(2) If $r = 1$, then $\det(\rho^{\mathrm{nr}}(F_N) - 1) = 0$ if and only if $\rho^{\mathrm{nr}}|_{G_N} = 1$. If $r > 1$, then there are "mixed" cases where both $\rho^{\mathrm{nr}}|_{G_N} \neq 1$ and $\det(\rho^{\mathrm{nr}}(F_N) - 1) = 0$ (see, e.g., [Cob18, Example 3.18]).

(3) If $\rho^{\mathrm{nr}}|_{G_N} = 1$, then twisting commutes with taking G_N-cohomology, so that we expect that $C^{na}_{EP}(N/K, V)$ can be proved relying on the fact that the conjecture is known in the untwisted case by [Bre04b]. In the case $r = 1$, this is sketched in [IV16, Appendix A.1]; however, for $r > 1$, we have not checked the details.

In the weakly ramified setting, we will prove the following theorem.

Theorem 1.3 *Let p be an odd prime. Let $K/\mathbb{Q}_p$ be the unramified extension of degree m, and let N/K be a weakly and wildly ramified finite abelian extension with cyclic ramification group. Let d denote the inertia degree of N/K, let $\tilde{d}$ denote the order of $\rho^{\mathrm{nr}}(F_N)$ mod p in $\mathrm{Gl}_r(\mathbb{Z}_p/p\mathbb{Z}_p)$, and assume that m and d are relatively prime. Let*

$$\rho^{\mathrm{nr}}_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{Gl}_r(\mathbb{Z}_p)$$

be an unramified representation of $G_{\mathbb{Q}_p}$, and let ρ^{nr} denote the restriction of $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$ to G_K. Assume that $\det(\rho^{\mathrm{nr}}(F_N) - 1) \neq 0$ and, in addition, that one of the following three conditions holds:

(a) *$\rho^{\mathrm{nr}}(F_N) - 1$ is invertible modulo p;*
(b) *$\rho^{\mathrm{nr}}(F_N) \equiv 1 \pmod p$;*
(c) *$\gcd(\tilde{d}, m) = 1$ and $\det(\rho^{\mathrm{nr}}(F_N)^{\tilde{d}} - 1) \neq 0$.*

Then, $C^{na}_{EP}(N/K, V)$ is true for N/K and $V = \mathbb{Q}_p^r(1)(\rho^{\mathrm{nr}})$.

Remarks 1.4 (a) In the case $r = 1$, we define as in [BC17, equation (14)] a nonnegative integer $\omega = \omega_N := v_p(1 - \rho^{\mathrm{nr}}(F_N))$. Note that the conditions (a) and (b) concerning the reduction of $\rho^{\mathrm{nr}}(F_N)$ modulo p generalize the cases $\omega = 0$ and $\omega > 0$, which were studied separately in [BC17], and which exhaust all the possible cases when $r = 1$. In the higher dimensional setting of the present paper, however, this is not true, even under the assumption $\det(\rho^{\mathrm{nr}}(F_N) - 1) \neq 0$. To deal with the remaining cases, our strategy of proof is to replace the field N by its unramified extension of degree $\tilde{d}$ and to use functoriality with respect to change of fields (see Prop. 7.2). For technical reasons, this forces us to require hypothesis (c).

(b) By [Cob18, Lemma 1.1], we know that $\tilde{d}$ is a divisor of $p^s t$ with $s = (r-1)r/2$ and $t = \prod_{i=1}^r (p^i - 1)$.

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In a more geometrical setting, if $A/\mathbb{Q}_p$ is an abelian variety of dimension r with good ordinary reduction, then by [Cob18, Proposition 1.12] the Tate module of the associated formal group $\hat{A}$ is isomorphic to $\mathbb{Z}_p^r(1)(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$ for an appropriate choice of $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$. Here, it is worth to remark that the converse is not true, i.e., not every module $\mathbb{Z}_p^r(1)(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$ comes from an abelian variety with good ordinary reduction. In this setting, by a result of Mazur [Maz72, Corollary 4.38], we know that $\det(\rho^{\mathrm{nr}}(F_L)-1)\neq 0$ is automatically satisfied for any finite extension $L/\mathbb{Q}_p$ (see Lemma 8.1).

Theorem 1.5 *Let N/K be a tame extension of p-adic number fields, and let $A/\mathbb{Q}_p$ be an r-dimensional abelian variety with good ordinary reduction. Let $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$ be the unramified representation induced by the Tate module $T_p\hat{A}$ of the formal group $\hat{A}$ of A, and let ρ^{nr} be the restriction of $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$ to G_K. Then, $C^{na}_{EP}(N/K,V)$ is true for $V=\mathbb{Q}_p\otimes_{\mathbb{Z}_p}T_p\hat{A}$.*

Theorem 1.6 *Let p be an odd prime, and let $A/\mathbb{Q}_p$ be an r-dimensional abelian variety with good ordinary reduction. Let $K/\mathbb{Q}_p$ be the unramified extension of degree m, and let N/K be a weakly and wildly ramified finite abelian extension with cyclic ramification group. Let $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$ be the unramified representation induced by the Tate module $T_p\hat{A}$ of the formal group $\hat{A}$ of A, and let ρ^{nr} be the restriction of $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$ to G_K. Let d denote the inertia degree of N/K, and let $\tilde{d}$ denote the order of $\rho^{\mathrm{nr}}(F_N)$ mod p in $\mathrm{Gl}_r(\mathbb{Z}_p/p\mathbb{Z}_p)$. Assume that m and d are relatively prime, and, in addition, that one of the following conditions holds:*

(a) $\rho^{\mathrm{nr}}(F_N)-1$ *is invertible modulo p;*
(b) $\rho^{\mathrm{nr}}(F_N)\equiv 1 \pmod{p}$*;*
(c) $(m,\tilde{d})=1$.

Then, $C^{na}_{EP}(N/K,V)$ is true for $V=\mathbb{Q}_p\otimes_{\mathbb{Z}_p}T_p\hat{A}$.

To conclude this introduction, we reference forthcoming work of Nickel [Nic18] and a forthcoming joint paper of Burns and Nickel [BN] where an Iwasawa theoretic approach to $C^{na}_{EP}(N/K,V)$ is developed. In a little more detail, Nickel formulates an Iwasawa theoretic analogue of $C^{na}_{EP}(N/K,\mathbb{Q}_p(1))$, call it $C^{na}_{EP}(N_\infty/K,\mathbb{Q}_p(1))$ for the purpose of this introduction, for the extension N_∞/K where N_∞/N is the unramified $\mathbb{Z}_p$-extension of N. Then, in a second paper, Burns and Nickel show that $C^{na}_{EP}(N_\infty/K,\mathbb{Q}_p(1))$ holds if and only if $C^{na}_{EP}(E/F,\mathbb{Q}_p(1))$ holds for all finite Galois extensions E/F such that $K\subseteq F\subseteq E\subseteq N_\infty$. Furthermore, they prove a certain twist invariance of the conjecture. If $\chi^{\mathrm{nr}}_{\mathbb{Q}_p}$ is a one-dimensional unramified character, they show that $C^{na}_{EP}(N_\infty N'/K,\mathbb{Q}_p(1))$ holds if and only if $C^{na}_{EP}(E/F,\mathbb{Q}_p(1)(\chi^{\mathrm{nr}}))$ holds for all finite Galois extensions E/F such that $K\subseteq F\subseteq E\subseteq N_\infty N'$ where N'/N is a certain unramified extension of degree dividing $p-1$. It will be very interesting to see how this Iwasawa theoretic approach will carry over to the higher dimensional case.

1.1 Notations

We will mostly rely on the notation of [BC17, Cob18]. For a field L, we write L^c for its algebraic closure; for any subfield L of $\mathbb{Q}_p^c$, we let $\bar{L}$ denote the p-adic completion of L. In this paper, N/K will always denote a finite Galois extension of p-adic number fields. We write N^{nr} for the maximal unramified extension, and then set $N_0=\overline{N^{\mathrm{nr}}}$

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and denote by $\widehat{N_0^\times}$ the p-completion of $N_0^\times$. Let N_1 be the maximal unramified subextension of $N/\mathbb{Q}_p$. We will denote by $e_{N/K}$ and $d_{N/K}$ the ramification index and the inertia degree of N/K, $\mathcal{O}_N$ will be the ring of integers of N, and U_N will be its group of units. We also set $\Lambda_N = \prod_r \widehat{N_0^\times}(\rho^{\mathrm{nr}})$, $\Upsilon_N = \prod_r \widehat{U_{N_0}}(\rho^{\mathrm{nr}})$, and $\mathcal{Z} = \mathbb{Z}_p^r(\rho^{\mathrm{nr}})$, and we will mostly use an additive notation for the (twisted) action of the absolute Galois group G_N. The elements fixed by the action of G_N will be denoted by $\Lambda_N^{G_N}$, $\Upsilon_N^{G_N}$, and $\mathcal{Z}^{G_N}$, respectively.

Let φ be the absolute Frobenius automorphism, let F_N be the Frobenius automorphism of N, and let $F = F_K$ be the Frobenius of K.

For an r-dimensional formal group $\mathcal{F}$, we denote by $\mathcal{F}(\mathfrak{p}_N^{(r)})$ the group structure on $\prod_r \mathfrak{p}_N$ induced by $\mathcal{F}$.

For any ring R, we denote by $M_r(R)$ the ring of $r \times r$ matrices with coefficients in R and by $\mathrm{Gl}_r(R)$ the group of invertible matrices. A unity matrix will always be denoted simply by 1. In addition, we write $Z(R)$ for the centre of R.

If Λ and Σ are unital rings and $\Lambda \longrightarrow \Sigma$ a ring homomorphism, then we write $K_0(\Lambda, \Sigma)$ for the relative algebraic K-group defined by Swan [Swa70, p. 215]. If $\Sigma = L[G]$ for a finite group G and a field extension $L/\mathbb{Q}_p$, we write $\mathrm{Nrd}_\Sigma \colon K_1(\Sigma) \longrightarrow Z(\Sigma)^\times$ for the map on K_1 induced by the reduced norm map. We will only be concerned with cases where Nrd_Σ is an isomorphism. In this case, we set $\hat{\partial}^1_{\Lambda,\Sigma} := \partial^1_{\Lambda,\Sigma} \circ \mathrm{Nrd}_\Sigma^{-1} \colon Z(\Sigma)^\times \longrightarrow K_0(\Lambda, \Sigma)$ where $\partial^1_{\Lambda,\Sigma} \colon K_1(\Sigma) \longrightarrow K_0(\Lambda, \Sigma)$ is the canonical map. If there is no danger of confusion, we will often abbreviate $\hat{\partial}^1_{\Lambda,\Sigma}$ to $\hat{\partial}^1$.

For any $\mathbb{Z}_p$-module X and any ring extension $R/\mathbb{Z}_p$, we set $X_R := R \otimes_{\mathbb{Z}_p} X$.

1.2 Plan of the manuscript

We will start recalling some results on the cohomology of $\mathbb{Z}_p^r(1)$ which are proved in [Cob18]. We will also formulate a finiteness hypothesis (F), which we will assume throughout the paper, and we will show some basic consequences of (F). After a short digression on the formal logarithm and exponential function in higher dimension in Section 3, we can start our study of the conjecture $C_{EP}^{na}(N/K, V)$.

As in [BC17], which was motivated by the work in [IV16], we define an element

$$R_{N/K} = C_{N/K} + U_{\mathrm{cris}} + rm\hat{\partial}^1_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}(t) - mU_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p}) - rU_{N/K} + \hat{\partial}^1_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}(\varepsilon_D(N/K, V)) \tag{1.1}$$

in the relative algebraic K-group $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$. The conjecture $C_{EP}^{na}(N/K, V)$ is then equivalent to the vanishing of $R_{N/K}$.

Actually, the element $R_{N/K}$ as defined in (1.1) differs from [BC17, equation (17)] by the term $mU_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$. This new term emerges from the computation of the cohomological term $C_{N/K}$, which was slightly incorrect in [BC17], and has to be compensated in the definition of $R_{N/K}$. For more details on this issue, we refer the reader to Remark 6.6.

We will explicitly compute the terms $C_{N/K}$, U_{cris}, and $\hat{\partial}^1_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}(\varepsilon_D(N/K, V))$ in the definition of $R_{N/K}$ and then use these results to prove $C_{EP}^{na}(N/K, V)$ when N/K is tame (Theorem 1.1) and, under some additional hypotheses, also when N/K

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is weakly and wildly ramified (Theorem 1.3). This generalizes previous work for $r = 1$ of Izychev and Venjakob in [IV16] in the tame case and the authors in [BC17] in the weakly ramified case.

2 The cohomology of $\mathbb{Z}_p^r(1)(\rho^{\mathrm{nr}})$

Let $u \in \mathrm{Gl}_r(\mathbb{Z}_p)$, and let $\rho^{\mathrm{nr}} = \rho_u : G_{\mathbb{Q}_p} \longrightarrow \mathrm{Gl}_r(\mathbb{Z}_p)$ denote the unramified representation attached to u by [Cob18, Proposition 1.6]. By [Haz78, Section 13.3], there is a unique r-dimensional Lubin–Tate formal group $\mathcal{F} = \mathcal{F}_{pu^{-1}}$ attached to the parameter pu^{-1}. As in [Cob18, Proposition 1.10], we can construct an isomorphism $\theta : \mathcal{F} \longrightarrow \mathbb{G}_m^r$ defined over the completion $\overline{\mathbb{Q}_p^{\mathrm{nr}}}$ of $\mathbb{Q}_p^{\mathrm{nr}}$ such that

$$\theta(X) = \varepsilon^{-1}X + \cdots \text{ and } \theta^{\varphi} \circ \theta^{-1} = u^{-1}, \tag{2.1}$$

where $\varepsilon \in \mathrm{Gl}_r(\overline{\mathbb{Z}_p^{\mathrm{nr}}})$ has the defining property $\varphi(\varepsilon^{-1})\varepsilon = u^{-1}$. In the following, we set $T := \mathbb{Z}_p^r(1)(\rho^{\mathrm{nr}})$, and for future reference, we recall that T is isomorphic to the p-adic Tate module $T_p\mathcal{F}$ of $\mathcal{F}$ by [Cob18, Proposition 1.11].

Let $N/\mathbb{Q}_p$ be a finite field extension, and let $N_0 = \overline{N^{\mathrm{nr}}}$ denote the completion of the maximal unramified extension of N. Following [Cob18], we define

$$\Lambda_N := \prod_r \widehat{N_0^{\times}}(\rho^{\mathrm{nr}}), \quad \mathcal{Z} := \mathbb{Z}_p^r(\rho^{\mathrm{nr}}).$$

Then, by [Cob18, Corollary 3.16], we have

$$\begin{aligned} H^1(N,T) &\cong \Lambda_N^{G_N} \cong \mathcal{F}(\mathfrak{p}_N^{(r)}) \times \mathcal{Z}^{G_N}, \\ H^2(N,T) &\cong \mathcal{Z}/(F_N - 1)\mathcal{Z}, \\ H^i(N,T) &= 0 \text{ for } i \neq 1,2. \end{aligned}$$

Remark 2.1 We point out that the above isomorphisms are induced by the explicit representative $C^{\bullet}_{N,\mathcal{F}}$ of $R\Gamma(N,T)$ constructed in [Cob18, Theorem 3.15]. In the formulation of $C_{EP}^{na}(N/K,V)$, however, we will use the identification of the cohomology modules resulting from the use of continuous cochain cohomology. We will address this problem in Section 6.1.

For each finite field extension $N/\mathbb{Q}_p$, we set

$$U_N := \rho_u(F_N) = \rho^{\mathrm{nr}}(F_N) = u^{d_{N/\mathbb{Q}_p}},$$

and in the sequel, always assume the following finiteness hypothesis.

Hypothesis (F): $\det(U_N - 1) \neq 0$.
This hypothesis clearly implies (and, in fact, is equivalent to)

$$\begin{aligned} H^1(N,T) &\cong \mathcal{F}(\mathfrak{p}_N^{(r)}), \\ H^2(N,T) &\cong \mathcal{Z}/(F_N - 1)\mathcal{Z} = \mathbb{Z}_p^r/(U_N - 1)\mathbb{Z}_p^r \text{ is finite.} \end{aligned}$$

The elementary divisor theorem immediately implies

$$\#(\mathcal{Z}/(F_N - 1)\mathcal{Z}) = p^{\omega} \text{ with } \omega := v_p(\det(U_N - 1)),$$

where v_p denotes the normalized p-adic valuation.

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We first study the case when N/K is tame.

Proposition 2.2 *Let N/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(N/K)$. Assume that Hypothesis (F) holds. If N/K is tame, then both $H^1(N,T)$ and $H^2(N,T)$ are G-cohomologically trivial.*

Proof By [Cob18, Theorem 3.3 and Lemma 2.2], it suffices to show that $\mathcal{Z}/(F_N - 1)\mathcal{Z} = \mathbb{Z}_p^r/(U_N-1)\mathbb{Z}_p^r$ is cohomologically trivial.

We set $M := \mathcal{Z}/(F_N-1)\mathcal{Z}$ and write $I = I_{N/K}$ for the inertia group. By [EN18, Lemma 2.3], it suffices to show that $\hat{H}^i(G/I, M^I) = 0$ and $\hat{H}^i(I,M) = 0$ for all $i \in \mathbb{Z}$, where $\hat{H}^i$ denotes Tate cohomology. Because M is a (finite) p-group and $p \nmid \#I$, we get $\hat{H}^i(I,M) = 0$. Hence, it suffices to show that $\hat{H}^i(G/I, M^I) = 0$ for all $i \in \mathbb{Z}$. Because G/I is cyclic and M finite, a standard Herbrand quotient argument shows that it is then enough to prove that $\hat{H}^{-1}(G/I, M^I) = 0$. Note that $M^I = M$. The long exact cohomology sequence attached to the short exact sequence

$$0 \longrightarrow \mathcal{Z} \xrightarrow{F_N-1} \mathcal{Z} \longrightarrow M \longrightarrow 0$$

of G/I-modules yields the exact sequence

$$\hat{H}^{-1}(G/I,\mathcal{Z}) \longrightarrow \hat{H}^{-1}(G/I,M) \longrightarrow \hat{H}^{0}(G/I,\mathcal{Z}).$$

With $F_N = F_K^{d_{N/K}}$, one has $U_N = U_K^{d_{N/K}}$ and

$$1 - U_N = (1 + U_K + \cdots + U_K^{d_{N/K}-1})(1-U_K).$$

Because $U_N - 1$ is invertible, the same is true for $1 - U_K$; hence, $\mathcal{Z}^{G/I} = 0$ and $\hat{H}^0(G/I,\mathcal{Z}) = 0$. To show that $\hat{H}^{-1}(G/I,\mathcal{Z}) = 0$, we note that the above identity also implies that $1 + U_K + \cdots + U_K^{d_{N/K}-1}$ is invertible, and hence, the kernel of the norm map is trivial. Consequently, $\hat{H}^{-1}(G/I,\mathcal{Z}) = 0$. ■

Because of Proposition 2.2, the tame case is much more accessible to proofs of conjecture $C_{EP}^{na}(N/K,V)$ than the wild case. Conversely, the following lemma shows that in the generic wild case, the cohomology modules are not cohomologically trivial.

Lemma 2.3 *Assume that Hypothesis (F) holds. Then, the following are equivalent:*

(i) *$H^2(N,T)$ is trivial.*
(ii) $U_N - 1 \in \mathrm{Gl}_r(\mathbb{Z}_p)$.

If N/K is wildly ramified, then this is also equivalent to:

(iii) *$H^1(N,T)$ is cohomologically trivial.*
(iv) *$H^2(N,T)$ is cohomologically trivial.*

Proof The equivalence of (i) and (ii) is clear. The equivalence of (iii) and (iv) follows from [Cob18, Theorem 3.3 and Lemma 2.2]. To see the equivalence of (i) and (iv) in the wildly ramified case, it suffices to note that I acts trivially on $M := H^2(N,T) = \mathcal{Z}/(F_N-1)\mathcal{Z}$. If P denotes a subgroup of I of order p, then one obviously has

$$H^1(P,M) = \mathrm{Hom}(P,M) = 0 \iff M = 0.$$ ■

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3 Formal logarithm and exponential function in higher dimensions

In this section, we prove some results which are probably well known, but for which we could not find a precise reference in the literature. Throughout this subsection, we let L be a finite extension of $\mathbb{Q}_p$ and ν_L the normalized valuation of L.

The statement and the proof of the following lemma generalize [Frö68, Chapter IV, Section 1, Proposition 1] to a higher dimensional setting. We set $X := (X_1, \dots, X_r)$, and for a homomorphism

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} : \mathcal{F} \to \mathbb{G}_a^r,$$

we write $J_f(X) := \left(\frac{\partial f_i}{\partial X_j}\right)_{1 \le i, j \le r}$ for the Jacobian of f.

Lemma 3.1 *Let $\mathcal{F}$ be an r-dimensional commutative formal group defined over $\mathbb{Z}_p$. Then, there exists a unique isomorphism $\log_{\mathcal{F}} : \mathcal{F} \to \mathbb{G}_a^r$ defined over $\mathbb{Q}_p$, so that the Jacobian $J_{\log_{\mathcal{F}}}(X)$ satisfies $J_{\log_{\mathcal{F}}}(0) = 1$. Furthermore, $J_{\log_{\mathcal{F}}}(X) \in \mathrm{Gl}_r(\mathbb{Z}_p[[X]])$ and $\log_{\mathcal{F}}(x)$ converges for all $x = (x_1, \dots, x_r) \in L^{(r)}$ satisfying* $\min\{\nu_L(x_1), \dots, \nu_L(x_r)\} > 0$.

Proof By [Frö68, Chapter II, Section 2, Theorem 1 and Corollary 1], there exists an isomorphism $g : \mathcal{F} \to \mathbb{G}_a^r$ defined over $\mathbb{Q}_p$. It is then clear that the Jacobian $J_g(0)$ is an invertible matrix. We also note that $J_g(0)^{-1}X$ defines an isomorphism $g_1 : \mathbb{G}_a^r \to \mathbb{G}_a^r$. Thus, the composition $\log_{\mathcal{F}} = g_1 \circ g : \mathcal{F} \to \mathbb{G}_a^r$ is an isomorphism satisfying our normalization $J_{\log_{\mathcal{F}}}(0) = J_{g_1}(g(0))J_g(0) = J_g(0)^{-1}J_g(0) = 1$.

To prove uniqueness, we assume that $f : \mathcal{F} \to \mathbb{G}_a^r$ is another isomorphism with $J_f(0) = 1$. Then,

$$J_{\log_{\mathcal{F}} \circ f^{-1}}(0) = J_{\log_{\mathcal{F}} \circ f^{-1}}(f(0)) = J_{\log_{\mathcal{F}}}(0)J_{f^{-1}}(f(0)) = J_{\log_{\mathcal{F}}}(0)J_f(0)^{-1} = 1.$$

It is easy to see that the isomorphisms $\mathbb{G}_a^r \to \mathbb{G}_a^r$ over $\mathbb{Q}_p$ are in one-to-one correspondence with the matrices in $\mathrm{Gl}_r(\mathbb{Q}_p)$. Hence, we deduce that $\log_{\mathcal{F}} \circ f^{-1}$ is the identity map, i.e., $f = \log_{\mathcal{F}}$.

To show that $J_{\log_{\mathcal{F}}}(X) \in M_r(\mathbb{Z}_p[[X]])$, we write

$$\log_{\mathcal{F}}(\mathcal{F}(X, Y)) = \log_{\mathcal{F}}(X) + \log_{\mathcal{F}}(Y).$$

We view both sides as formal series in the variables Y, calculate the Jacobians, and evaluate at $Y = 0$:

$$J_{\log_{\mathcal{F}}}(\mathcal{F}(X, 0))J_{\mathcal{F}(X,\cdot)}(0) = 0 + J_{\log_{\mathcal{F}}}(0).$$

As a consequence, we obtain

$$J_{\log_{\mathcal{F}}}(X)J_{\mathcal{F}(X,\cdot)}(0) = 1.$$

We let $\mathfrak{a}$ denote the ideal of $\mathbb{Z}_p[[X]]$ which is generated by $X_1, \dots, X_r$ and note that $p\mathbb{Z}_p[[X]] + \mathfrak{a}$ is the maximal ideal of the local ring $\mathbb{Z}_p[[X]]$. By the axioms of formal groups, it follows that $J_{\mathcal{F}(X,\cdot)}(0) = 1 + M$ with a matrix $M \in M_r(\mathbb{Z}_p[[X]])$ with coefficients in $\mathfrak{a}$. Hence, $\det(J_{\mathcal{F}(X,\cdot)}(0)) \equiv 1 \pmod{\mathfrak{a}}$, and we deduce that $\det(J_{\mathcal{F}(X,\cdot)}(0))$

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is a unit in $\mathbb{Z}_p[[X]]$. It follows that $J_{\mathcal{F}(X,\cdot)}(0)$ is invertible in $M_r(\mathbb{Z}_p[[X]])$, so that its inverse $J_{\log_{\mathcal{F}}}(X)$ has integral coefficients and is, in fact, in $\mathrm{Gl}_r(\mathbb{Z}_p[[X]])$.

Hence, a general term of any component $\log_{\mathcal{F},i}$ of $\log_{\mathcal{F}}$ is of the form $\frac{a}{m}\prod_{i=1}^r X_i^{n_i}$, with $m = \gcd(n_1,\ldots,n_r)$ and $a \in \mathbb{Z}_p$. If we set $n = \sum_{i=1}^r n_i$, then

$$v_L\left(\frac{a}{m}\prod_{i=1}^r x_i^{n_i}\right) \geq \sum_{i=1}^r n_i v_L(x_i) - v_L(m)$$
$$\geq n\min\{v_L(x_1),\ldots,v_L(x_r)\} - (\log_p n)v_L(p).$$

This last expression tends to infinity when the total degree n tends to infinity. ∎

As usual, we write $\exp_{\mathcal{F}}$ for the inverse of $\log_{\mathcal{F}}$. To obtain information on the convergence of $\exp_{\mathcal{F}}$, we will need the following lemma whose proof is inspired by the proof of [Sil09, Lemma IV.5.4].

Lemma 3.2 *Let $f,g \in \mathbb{Q}_p[[X]]^r$ be power series without constant term such that $f(g(X)) = X$ for $X = (X_1,\ldots,X_r)$. Assume that $J_g(X) \in M_r(\mathbb{Z}_p[[X]])$ and $J_g(0) = 1$. Then, for all $s \in \mathbb{N}$ and for all $i, n_1,\ldots,n_s \in \{1,\ldots,r\}$, we have*

$$\frac{\partial^s f_i}{\partial X_{n_1}\cdots\partial X_{n_s}}(0) \in \mathbb{Z}_p.$$

Proof In a first step, we prove the following claim.
Claim: For all $s \in \mathbb{N}$ and all $n_1,\ldots,n_s \in \{1,\ldots,r\}$, the expression

$$\sum_{m_1=1}^r \cdots \sum_{m_s=1}^r \frac{\partial^s f_i}{\partial X_{m_1}\cdots\partial X_{m_s}}(g(X))\frac{\partial g_{m_1}}{\partial X_{n_1}}\cdots\frac{\partial g_{m_s}}{\partial X_{n_s}} \tag{3.1}$$

is a polynomial in $\frac{\partial^t f_i}{\partial X_{k_1}\cdots\partial X_{k_t}}(g(X))$ with $1 \leq t \leq s-1$, $k_1,\ldots,k_t \in \{1,\ldots,r\}$ and coefficients in $\mathbb{Z}_p[[X]]$.

Indeed, the chain rule for $\frac{\partial}{\partial X_{n_1}}$ applied to $f_i(g(X)) = X_i$ yields

$$\sum_{m_1=1}^r \frac{\partial f_i}{\partial X_{m_1}}(g(X))\frac{\partial g_{m_1}}{\partial X_{n_1}} = \delta_{i,n_1} \tag{3.2}$$

and thus establishes the claim for $s = 1$.

For the inductive step, we apply $\frac{\partial}{\partial X_{n_{s+1}}}$ to the expression in (3.1), and again, by the chain rule, we obtain

$$\sum_{m_1=1}^r \cdots \sum_{m_s=1}^r \sum_{m_{s+1}=1}^r \frac{\partial^{s+1} f_i}{\partial X_{m_1}\cdots\partial X_{m_s}\partial X_{m_{s+1}}}(g(X))\frac{\partial g_{m_1}}{\partial X_{n_1}}\cdots\frac{\partial g_{m_s}}{\partial X_{n_s}}\frac{\partial g_{m_{s+1}}}{\partial X_{n_{s+1}}}$$
$$= \frac{\partial}{\partial X_{n_{s+1}}}\left(\sum_{m_1=1}^r \cdots \sum_{m_s=1}^r \frac{\partial^s f_i}{\partial X_{m_1}\cdots\partial X_{m_s}}(g(X))\frac{\partial g_{m_1}}{\partial X_{n_1}}\cdots\frac{\partial g_{m_s}}{\partial X_{n_s}}\right)$$
$$- \sum_{m_1=1}^r \cdots \sum_{m_s=1}^r \frac{\partial^s f_i}{\partial X_{m_1}\cdots\partial X_{m_s}}(g(X))\frac{\partial}{\partial X_{n_{s+1}}}\left(\frac{\partial g_{m_1}}{\partial X_{n_1}}\cdots\frac{\partial g_{m_s}}{\partial X_{n_s}}\right).$$

Using the inductive hypothesis for the first term on the right-hand side and the assumption $J_g(X) \in M_r(\mathbb{Z}_p[[X]])$ for the second, one proves the above claim.

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In order to prove the assertion of the lemma, we again proceed by induction on s. For $s = 1$, we specialize (3.2) at $X = 0$ and obtain from $g(0) = 0$

$$\sum_{m_1=1}^{r} \frac{\partial f_i}{\partial X_{m_1}}(0)\frac{\partial g_{m_1}}{\partial X_{n_1}}(0) = \delta_{i,n_1}.$$

Because $J_g(0) = 1$, this implies $\frac{\partial f_i}{\partial X_{n_1}}(0) = \delta_{i,n_1} \in \mathbb{Z}_p$.

For the inductive step, we specialize (3.1) at $X = 0$, and because $J_g(0) = 1$, we simply obtain

$$\frac{\partial^s f_i}{\partial X_{n_1}\cdots\partial X_{n_s}}(0).$$

By the above claim and the inductive hypothesis, this is an element in $\mathbb{Z}_p$. ■

Lemma 3.3 *The isomorphism* $\exp_{\mathcal{F}}$ *converges for all* $x = (x_1, \ldots, x_r) \in L^{(r)}$ *satisfying* $\min\{v_L(x_1), \ldots, v_L(x_r)\} > v_L(p)/(p-1)$.

Proof By Lemmas 3.1 and 3.2, we have

$$\frac{\partial^s \exp_{\mathcal{F},i}}{\partial X_{n_1}\cdots\partial X_{n_s}}(0) \in \mathbb{Z}_p,$$

for any $s \in \mathbb{N}$ and $i, n_1, \ldots, n_s \in \{1, \ldots, r\}$. It follows that each component $\exp_{\mathcal{F},i}$ of $\exp_{\mathcal{F}}$ is of the form

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty} \frac{a_{m_1,\ldots,m_r}}{m_1!\cdots m_r!} X_1^{m_1}\cdots X_r^{m_r},$$

for some $a_{m_1,\ldots,m_r} \in \mathbb{Z}_p$. As in the proof of [Sil09, Lemma IV.6.3(b)], we can show that

$$v_L\left(\frac{a_{m_1,\ldots,m_r}}{m_1!\cdots m_r!}x_1^{m_1}\cdots x_r^{m_r}\right) \geq \sum_{i=0,m_i\neq 0}^{r}\left(v_L(x_i) + (m_i-1)\left(v_L(x_i) - \frac{v_L(p)}{p-1}\right)\right),$$

which under our assumption tends to infinity as the total degree tends to infinity. ■

We summarize our discussion in the next proposition.

Proposition 3.4 *Let L be a finite extension of* $\mathbb{Q}_p$ *with normalized valuation* v_L. *Let* $n > \frac{v_L(p)}{p-1}$ *be an integer. Then, the formal logarithm induces an isomorphism*

$$\log_{\mathcal{F}}: \mathcal{F}((\mathfrak{p}_L^n)^{(r)}) \longrightarrow \mathbb{G}_a^r((\mathfrak{p}_L^n)^{(r)})$$

with inverse induced by $\exp_{\mathcal{F}}$.

Proof Given the results of this section, the proposition follows as in the proof of [Sil09, Theorem IV.6.4]. ■

4 Computation of the term U_{cris}

4.1 Some preliminary results

We will apply the notation introduced and explained in [BB08, Section 1.1]. In particular, B_{cris}, B_{st}, and B_{dR} denote the p-adic period rings constructed by Fontaine.

We recall that the field $B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[1/t]$ is a $\mathbb{Q}_p$-algebra which contains $\mathbb{Q}_p^c$ and carries an action of $G_{\mathbb{Q}_p}$. The uniformizing element $t = \log[\varepsilon]$ depends on the choice of $\varepsilon = (\zeta_{p^n})_{n\geq 0}$ where the primitive p^n-th roots of unity ζ_{p^n} are compatible with respect to $x \mapsto x^p$. We let $\chi_{cyc}: G_{\mathbb{Q}_p} \longrightarrow \mathbb{Z}_p^\times$ denote the cyclotomic character which is uniquely determined by the requirement $\zeta_{p^n}^\sigma = \zeta_{p^n}^{\chi_{cyc}(\sigma)}$ for all $n \geq 0$ and all $\sigma \in G_{\mathbb{Q}_p}$. In particular, we have $\sigma(t) = \chi_{cyc}(\sigma)t$ for all $\sigma \in G_{\mathbb{Q}_p}$.

The subring B_{cris} of B_{dR} contains the element t, and, in addition, there is a Frobenius endomorphism ϕ acting on B_{cris}. In Section 4.2, we will frequently use the formula $\phi(t) = pt$. If V is a p-adic representation of G_K, we put

$$D^K_{\mathrm{dR}}(V) := \left(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}, \quad D^K_{\mathrm{cris}}(V) := \left(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}.$$

The K-vector space $D^K_{\mathrm{dR}}(V)$ is finite dimensional and filtered. The tangent space of V over K is defined by

$$t_V(K) := D^K_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D^K_{\mathrm{dR}}(V).$$

Finally, we write $\exp_V: t_V(K) \longrightarrow H^1(K, V)$ for the exponential map of Bloch and Kato. Note here that $H^1(K, V)$ is defined using continuous cochain cohomology (see Remark 2.1).

For any $\mathbb{Q}_p$-vector space W, we write $W^* = \mathrm{Hom}_{\mathbb{Q}_p}(W, \mathbb{Q}_p)$ for its $\mathbb{Q}_p$-linear dual. For convenience, we usually write $t^*_V(K)$ instead of $t_V(K)^*$.

We fix a matrix $T^{\mathrm{nr}} \in \mathrm{Gl}_r(\overline{\mathbb{Z}}_p^{\mathrm{nr}})$, so that $\varphi(T^{\mathrm{nr}})(T^{\mathrm{nr}})^{-1} = u^{-1}$, which exists by [Cob18, Lemma 1.9].

Lemma 4.1 *Let $v_1^*, \dots, v_r^*$ denote the elements of the canonical $\mathbb{Q}_p$-basis of $V^*(1)$. Then, $e_i^* = \sum_{n=1}^r (T^{\mathrm{nr}})^{-1}_{i,n} \otimes v_n^*$, $i = 1, \dots, r$, constitute a basis of $D^N_{\mathrm{cris}}(V^*(1))$ as an N_1-vector space and of $D^N_{\mathrm{dR}}(V^*(1))$ as an N-vector space. In addition, each element e_i^* is fixed by the action of the Galois group $G_{\mathbb{Q}_p}$.*

Proof The following proof is the r-dimensional generalization of the first part of the proof of [BC17, Lemma 5.2.1].

By definition, we have $T^{\mathrm{nr}} = u\varphi(T^{\mathrm{nr}})$, and by induction, we deduce $F_N(T^{\mathrm{nr}}) = \varphi^{d_N}(T^{\mathrm{nr}}) = u^{-d_N} T^{\mathrm{nr}}$, and hence $u^{d_N} F_N(T^{\mathrm{nr}}) = T^{\mathrm{nr}}$.

First of all, recall that the completion $\overline{\mathbb{Z}}_p^{\mathrm{nr}}$ of $\mathbb{Z}_p^{\mathrm{nr}}$ is contained both in B_{cris} and B_{dR}. We now prove that the elements e_i^* are fixed by the absolute Galois group $G_{\mathbb{Q}_p}$, which will show that the e_i^* are contained in both $D^N_{\mathrm{cris}}(V^*(1))$ and $D^N_{\mathrm{dR}}(V^*(1))$. We note that the inertia group $I_{\mathbb{Q}_p}$ acts trivially on $V^*(1)$, and hence, it remains to prove that e_i^* is fixed by φ. We first need to calculate $\varphi(v_i^*)$. Here, we use the definitions and the fact that the elements v_i constitute the canonical basis of $\mathbb{Q}_p^r(1)(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$:

$$\varphi(v_i^*)(v_j) = v_i^*(u^{-1}v_j) = v_i^*\left(\sum_{k=1}^r (u^{-1})_{k,j} v_k\right) = (u^{-1})_{i,j}.$$

Hence,

$$\varphi(v_i^*) = \sum_{j=1}^r (u^{-1})_{i,j} v_j^*,$$

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and we conclude that

$$\varphi(e_i^*) = \left(\sum_{n=1}^{r}(T^{\mathrm{nr}})^{-1}_{i,n}\otimes v_n^*\right)^{\varphi} = \sum_{n=1}^{r}\sum_{h=1}^{r}(T^{\mathrm{nr}})^{-1}_{i,h}u_{h,n}\otimes\sum_{k=1}^{r}\left(u^{-1}\right)_{n,k}v_k^*$$
$$= \sum_{h=1}^{r}\sum_{k=1}^{r}(T^{\mathrm{nr}})^{-1}_{i,h}\delta_{h,k}\otimes v_k^* = \sum_{k=1}^{r}(T^{\mathrm{nr}})^{-1}_{i,k}\otimes v_k^* = e_i^*.$$

Because $T^{\mathrm{nr}}\in \mathrm{Gl}_r(\overline{\mathbb{Z}}_p^{\mathrm{nr}})\subseteq \mathrm{Gl}_r(B_{\mathrm{cris}})$, the elements $e_1^*,\ldots,e_r^*$ are a B_{cris}-basis of $B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}V^*(1)$. As N_1 is a subfield of B_{cris}, we see that $e_1^*,\ldots,e_r^*$ are linearly independent over N_1. This concludes the proof that the elements e_i^* constitute a basis of $D^N_{\mathrm{cris}}(V^*(1))$, because $\dim_{N_1}D^N_{\mathrm{cris}}(V^*(1))\le\dim_{\mathbb{Q}_p}(V^*(1)) = r$. In particular, this also proves that $V^*(1)$ is cristalline. Then, the elements $e_1^*,\ldots,e_r^*$ must also be a basis of the N-vector space $D^N_{\mathrm{dR}}(V^*(1)) = N\otimes_{N_1}D^N_{\mathrm{cris}}(V^*(1))$. ■

Lemma 4.2 *Let $v_1,\ldots,v_r$ be the elements of the canonical $\mathbb{Q}_p$-basis of V. The elements $e_i=\sum_{n=1}^r t^{-1}T^{\mathrm{nr}}_{n,i}\otimes v_n$, $i=1,\ldots,r$, constitute a basis of $D^N_{\mathrm{cris}}(V)$ as an N_1-vector space and of $D^N_{\mathrm{dR}}(V)$ as an N-vector space. In addition, each element e_i is fixed by the action of the Galois group $G_{\mathbb{Q}_p}$.*

Proof For $\sigma\in I_{\mathbb{Q}_p}$ we compute

$$\sigma(e_i) = \sum_{n=1}^{r}\sigma(t^{-1}T^{\mathrm{nr}}_{n,i})\otimes\sigma(v_n) = \sum_{n=1}^{r}\chi_{cyc}(\sigma^{-1})t^{-1}T^{\mathrm{nr}}_{n,i}\otimes\chi_{cyc}(\sigma)v_n = e_i.$$

Hence, the elements e_i are fixed by the inertia group, and a similar computation as in the proof of Lemma 4.1 shows that $\varphi(e_i)=e_i$. The proof follows as above. ■

Lemma 4.3 *Let $\tilde v_1,\ldots,\tilde v_r$ be the elements of the canonical $\mathbb{Q}_p$-basis of $V(-1)$. The elements $\tilde e_i=\sum_{n=1}^r T^{\mathrm{nr}}_{n,i}\otimes\tilde v_n$ are a basis of $D^N_{\mathrm{cris}}(V(-1))$ as an N_1-vector space and of $D^N_{\mathrm{dR}}(V(-1))$ as an N-vector space. In addition, each element $\tilde e_i$ is fixed by the action of the Galois group $G_{\mathbb{Q}_p}$.*

Proof Similar as above. ■

4.2 Computation of U_{cris}

We recall that $V=\mathbb{Q}_p^r(1)(\rho^{\mathrm{nr}})$ and $V^*(1)=\mathbb{Q}_p^r((\rho^{\mathrm{nr}})^{-1})$ and that we always assume Hypothesis (F). The following lemma (and its proof) is the analogue of [BC17, Lemma 5.1.2].

Lemma 4.4 *We have:*

(1) $t_{V^*(1)}(N)=0$.
(2) $H^1_f(N,V^*(1))=0$.
(3) $H^1_f(N,V)=H^1_e(N,V)=H^1(N,V)$.

Proof Proofs are as for [BC17, Lemma 5.1.2]. For the proof of part (c), we also need that by Lemma 4.7 below the endomorphism $1-\phi$ of $D^N_{\mathrm{cris}}(V)$ is an isomorphism. ■

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By the above lemma, [Cob18, Corollary 3.16], and [BC17, equation (30)], the seven-term exact sequence [BC17, equation (5)] degenerates into the two exact sequences

$$0 \longrightarrow D^N_{\mathrm{cris}}(V) \xrightarrow{1-\phi} D^N_{\mathrm{cris}}(V) \oplus t_V(N) \longrightarrow H^1(N,V) \longrightarrow 0$$

and

$$0 \longrightarrow D^N_{\mathrm{cris}}(V^*(1))^* \xrightarrow{1-\phi^*} D^N_{\mathrm{cris}}(V^*(1))^* \longrightarrow 0.$$

The term $U_{\mathrm{cris}} \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$ is defined by [BC17, equation (26)]. We recall that for a ring R, the abelian group $K_1(R)$ is generated by elements $[P, \alpha]$, where P is a finitely generated projective R-module and α is an automorphism of P. By the computations in loc. cit. (see, in particular, [BC17, equation (32)]), we obtain

(4.1)
$$U_{\mathrm{cris}} = \partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}([D^N_{\mathrm{cris}}(V), 1-\phi]) - \partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}([D^N_{\mathrm{cris}}(V^*(1))^*, 1-\phi^*]).$$

Before computing U_{cris}, we need an easy lemma from linear algebra.

Lemma 4.5 *Let R be a unital commutative ring, and let*

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & B_1 \\ A & 1 & 0 & \cdots & 0 & B_2 \\ 0 & A & 1 & \cdots & 0 & B_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & B_{n-1} \\ 0 & 0 & 0 & \cdots & A & 1+B_n \end{pmatrix} \in M_{nr}(R)$$

be a block matrix, with n^2 square blocks of the same size. Let $\det = \det_R$ *denote the determinant over R. Then,*

$$\det(M) = \det\left(1 + \sum_{i=0}^{n-1} (-A)^i B_{n-i}\right).$$

Proof By Gaussian elimination, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & B_1 \\ 0 & 1 & 0 & \cdots & 0 & B_2 - AB_1 \\ 0 & 0 & 1 & \cdots & 0 & B_3 - AB_2 + A^2 B_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \sum_{i=0}^{n-2} (-A)^i B_{n-1-i} \\ 0 & 0 & 0 & \cdots & 0 & 1 + \sum_{i=0}^{n-1} (-A)^i B_{n-i} \end{pmatrix}.$$

■

The Wedderburn decomposition of $\mathbb{Q}_p[G]$ induces a decomposition of $Z(\mathbb{Q}_p[G])$ as a finite direct sum $\bigoplus_i F_i$ of suitable finite field extensions $F_i/\mathbb{Q}_p$. If $x \in Z(\mathbb{Q}_p[G])$, we let ${}^*x \in Z(\mathbb{Q}_p[G])^\times$ denote the invertible element which is given by $({}^*x_i)$ with ${}^*x_i = 1$ if $x_i = 0$ and ${}^*x_i = x_i$ otherwise.

We now generalize [BC17, Lemmas 5.2.1 and 5.2.2], and in this way, explicitly compute the element U_{cris}. Recall that $F = F_K = \varphi^{d_K}$ is the Frobenius element of K. We write $I = I_{N/K}$ for the inertia subgroup of the Galois extension N/K.

Lemma 4.6 *The endomorphism* $1 - \phi^*$ *of* $D^N_{\mathrm{cris}}(V^*(1))^*$ *is an isomorphism. Furthermore, we have*

$$\partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}([D^N_{\mathrm{cris}}(V^*(1))^*, 1 - \phi^*]) = \hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}({}^*(\det(1 - u^{d_K}F^{-1})e_I))$$

in $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p[G])$.

Proof We have to compute $\phi(e_i^*)$. Using the φ-semilinearity of ϕ, we compute

$$\begin{aligned}\phi(e_i^*) &= \sum_{n=1}^r \varphi(((T^{\mathrm{nr}})^{-1})_{i,n} \otimes v_n^* = \sum_{n=1}^r ((T^{\mathrm{nr}})^{-1}u)_{i,n} \otimes v_n^* \\ &= \sum_{n=1}^r ((T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}}(T^{\mathrm{nr}})^{-1})_{i,n} \otimes v_n^* = \sum_{n=1}^r\sum_{\ell=1}^r ((T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}})_{i,\ell}(T^{\mathrm{nr}})^{-1}_{\ell,n} \otimes v_n^* \\ &= \sum_{\ell=1}^r ((T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}})_{i,\ell}e_\ell^*.\end{aligned}$$

We fix a normal basis element θ of $N_1/\mathbb{Q}_p$. Then, $w_{i,j} := \varphi^{-j}(\theta)e_i^*$, for $i = 1,\dots,r$ and $j = 0,\dots,d_K - 1$, is a $\mathbb{Q}_p[G/I]$-basis of $D^N_{\mathrm{cris}}(V^*(1))$. Let $\psi_{i,j} \in D^N_{\mathrm{cris}}(V^*(1))^*$ for $i = 1,\dots,r$ and $j = 0,\dots,d_K - 1$ be the dual $\mathbb{Q}_p[G/I]$-basis.

For $0 < i, h \le r$, $0 \le j, k < d_K - 1$, and $0 \le n \le d - 1$, we have

$$\begin{aligned}\phi^*(\psi_{i,j})(F^n w_{h,k}) &= \psi_{i,j}(\phi(F^n\varphi^{-k}(\theta)e_h^*)) \\ &= \psi_{i,j}\left(F^n\varphi^{-k+1}(\theta)\sum_{\ell=1}^r ((T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}})_{h,\ell}e_\ell^*\right) \\ &= \sum_{\ell=1}^r ((T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}})_{h,\ell}\psi_{i,j}(F^n\varphi^{-k+1}(\theta)e_\ell^*) \\ &= \sum_{\ell=1}^r ((T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}})_{h,\ell}\psi_{i,j}(F^n w_{\ell,k-1}).\end{aligned}$$

If $n = 0$, $k = j + 1$, and any $0 < h \le r$, this is equal to $((T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}})_{h,i}$; it is 0 otherwise. Hence,

$$\phi^*(\psi_{i,j}) = \sum_{h=1}^r ((T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}})_{h,i}\psi_{h,j+1}.$$

Analogously, for $0 < i \le r$ and $j = d_K - 1$, we have

$$\phi^*(\psi_{i,d_K-1}) = \sum_{h=1}^r ((T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}})_{h,i}F^{-1}\psi_{h,0}.$$

Note here that ϕ^* is indeed defined over $\mathbb{Q}_p[G/I]$, because $(T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}} \in M_r(\mathbb{Q}_p)$.

With respect to the $\mathbb{Q}_p[G/I]$-basis $\psi_{i,j}$ the matrix associated to $1-\phi^*$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -F^{-1}(T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}} \\ -(T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}} & 1 & 0 & \cdots & 0 & 0 \\ 0 & -(T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -(T^{\mathrm{nr}})^{-1}uT^{\mathrm{nr}} & 1 \end{pmatrix}.$$

In this matrix, each entry is an $r\times r$ matrix with coefficients in $\mathbb{Q}_p[G/I]$ (recall that $F=F_K$ generates G/I). In the following, we write det for the determinant over the commutative ring $\mathbb{Q}_p[G/I]$. By Lemma 4.5, the determinant of the above matrix is $\det(1-F^{-1}(T^{\mathrm{nr}})^{-1}u^{d_K}T^{\mathrm{nr}})=\det(1-F^{-1}u^{d_K})$. Because this determinant computes the reduced norm of $1-\phi^*$, the equality in the lemma follows.

To conclude that $1-\phi^*$ is an isomorphism, it is now enough to notice that

$$(1-F^{-1}u^{d_K})(1+F^{-1}u^{d_K}+\cdots+(F^{-1}u^{d_K})^{d_{N/K}-1})=1-U_N,$$

which is invertible, because, by Hypothesis (F), we always have $\det(U_N-1)\neq 0$. ■

Lemma 4.7 *The endomorphism $1-\phi$ of $D^N_{\mathrm{cris}}(V)$ is an isomorphism, and we have*

$$\partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}([D^N_{\mathrm{cris}}(V),1-\phi])=\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}({}^*(\det(1-F(pu)^{-d_K})e_I))$$

in $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p[G])$.

Proof The proof is analogous to the proof of the previous lemma.

To show that $\det(1-F(pu)^{-d_K})$ is invertible in $\mathbb{Q}_p[G/I]$, it is enough to notice that $\sum_{i=0}^{\infty}(F^{-1}(pu)^{d_K})^i$ converges in the matrix ring $M_r(\mathbb{Z}_p[G/I])$ and is the inverse of

$$1-F^{-1}(pu)^{d_K}=-F^{-1}(pu)^{d_K}(1-F(pu)^{-d_K}).$$ ■

Proposition 4.8 *We have*

$$U_{\mathrm{cris}}=\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}({}^*(\det(1-F(pu)^{-d_K})e_I))-\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}({}^*(\det(1-u^{d_K}F^{-1})e_I)).$$

Proof The proof is easily achieved by combining (4.1) and Lemmas 4.6 and 4.7. ■

We conclude this section by proving some functorial properties for the term U_{cris}. To that end, we let L be an intermediate field of N/K and set $H:=\mathrm{Gal}(N/L)$. Then, we let

$$\rho^G_H\colon K_0(\mathbb{Z}_p[G],\mathbb{Q}_p[G])\longrightarrow K_0(\mathbb{Z}_p[H],\mathbb{Q}_p[H]) \tag{4.2}$$

denote the natural restriction of scalars homomorphism, and if H is normal in G, then

$$q^G_{G/H}\colon K_0(\mathbb{Z}_p[G],\mathbb{Q}_p[G])\longrightarrow K_0(\mathbb{Z}_p[G/H],\mathbb{Q}_p[G/H]) \tag{4.3}$$

denotes the homomorphism which is induced by the functor $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z},\cdot)=(\cdot)^H$.

Lemma 4.9 *Let L be an intermediate field of N/K and $H = \mathrm{Gal}(N/L)$. Then:*

(1) $\rho^G_H(U_{\mathrm{cris},N/K}) = U_{\mathrm{cris},N/L}$.
(2) *If H is normal in G, then* $q^G_{G/H}(U_{\mathrm{cris},N/K}) = U_{\mathrm{cris},L/K}$.

Proof Let $u_{\mathrm{cris},N/K} \in Z(\mathbb{Q}_p[G])^\times$ be such that $\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}(u_{\mathrm{cris},N/K}) = U_{\mathrm{cris},N/K}$. We will use an analogous notation for all the other Galois extensions involved in the proof.

Then, for any irreducible character χ of G, we can take

$$(u_{\mathrm{cris},N/K})_\chi = \begin{cases} \frac{\det(1-\chi(F)(pu)^{-d_K})}{\det(1-u^{d_K}\chi(F)^{-1})} & \text{if } \chi|_{I_{N/K}} = 1, \\ 1 & \text{if } \chi|_{I_{N/K}} \neq 1. \end{cases}$$

For two (virtual) characters χ_1 and χ_2 of a finite group J, we write $\langle \chi_1, \chi_2\rangle_J$ for the standard scalar product.

(1) By [BW09, Section 6.1], we have

$$\rho^G_H(u_{\mathrm{cris},N/K}) = \left(\prod_{\chi\in\mathrm{Irr}(G)} (u_{\mathrm{cris},N/K})_\chi^{\langle \chi, \mathrm{Ind}^G_H\psi\rangle_G} \right)_{\psi\in\mathrm{Irr}(H)}.$$

Because $\langle \chi, \mathrm{Ind}^G_H\psi\rangle_G = \langle \chi|_H, \psi\rangle_H$ by Frobenius reciprocity, we obtain

$$(u_{\mathrm{cris},N/K})_\chi^{\langle \chi, \mathrm{Ind}^G_H\psi\rangle_G} = \begin{cases} 1 & \text{if } \chi|_{I_{N/K}} \neq 1, \\ u_{\mathrm{cris},N/K} & \text{if } \chi|_{I_{N/K}} = 1 \text{ and } \chi|_H = \psi, \\ 1 & \text{if } \chi|_{I_{N/K}} = 1 \text{ and } \chi|_H \neq \psi. \end{cases}$$

If $\psi|_{I_{N/L}} \neq 1$, then $\chi|_{I_{N/K}} \neq 1$ whenever $\langle \chi|_H, \psi\rangle_H \neq 0$. Thus, $\rho^G_H(u_{\mathrm{cris},N/K})_\psi = 1$ for those characters ψ.

On the other hand, if $\psi|_{I_{N/L}} = 1$, then ψ is a character of the cyclic group $\overline{H} := H/I_{N/L} = \langle F_L\rangle$. Each character $\chi \in \mathrm{Irr}(G)$ with $(u_{\mathrm{cris},N/K})_\chi \neq 1$ is actually a character of $\overline{G} := G/I_{N/K} = \langle F_K\rangle$. Note that we can naturally identify $\overline{H}$ with a subgroup of $\overline{G}$ and recall that $|\overline{G}/\overline{H}| = d_{L/K}$.

We therefore obtain

$$\prod_{\chi\in\mathrm{Irr}(G)} (u_{\mathrm{cris},N/K})_\chi^{\langle \chi, \mathrm{Ind}^G_H\psi\rangle_G} = \prod_{\chi\in\mathrm{Irr}(\overline{G}),\chi|_{\overline{H}}=\psi} \frac{\det(1-\chi(F_K)(pu)^{-d_K})}{\det(1-u^{d_K}\chi(F_K)^{-1})}.$$

We consider the numerator and the denominator separately and use, in each case, the polynomial identity

$$\prod_{\chi\in\mathrm{Irr}(\overline{G}),\chi|_{\overline{H}}=\psi} (X - \chi(F_K)) = X^{d_{L/K}} - \psi(F_L).$$

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For the numerator, we compute

$$\begin{aligned}\prod_{\chi\in\mathrm{Irr}(\overline{G}),\chi|_{\overline{H}}=\psi} &\det(1-\chi(F_K)(pu)^{-d_K}) \\ &= \det\left(\prod_{\chi\in\mathrm{Irr}(\overline{G}),\chi|_{\overline{H}}=\psi}(pu)^{-d_K}\left((pu)^{d_K}-\chi(F_K)\right)\right) \\ &= \det\left((pu)^{-d_K d_{L/K}}\left((pu)^{d_K d_{L/K}}-\psi(F_L)\right)\right) \\ &= \det\left(1-(pu)^{-d_L}\psi(F_L)\right),\end{aligned}$$

and a similar computation for the denominator shows claim (a).

(2) For any character ψ of G/H, we write $\mathrm{infl}(\psi)$ for the inflated character of G. By [BW09, Section 6.3], $q^G_{G/H}(u_{\mathrm{cris},N/K})_\psi = (u_{\mathrm{cris},N/K})_{\mathrm{infl}(\psi)}$ for any $\psi\in\mathrm{Irr}(G/H)$. This is equal to $(u_{\mathrm{cris},L/K})_\psi$, because $I_{L/K}=I_{N/K}H/H$ and $\mathrm{infl}(\psi)(F_K)=\psi(F_K)$. ■

5 Computation of epsilon constants

As in [IV16, Section 2.3], we define

$$\begin{aligned}\varepsilon_D(N/K,V) &= (\varepsilon(D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V\otimes\rho^*_\chi)),\psi_\xi,\mu_{\mathbb{Q}_p}))_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)}(\mathbb{Q}_p^c)^\times \\ &\cong Z(\mathbb{Q}_p^c[G])^\times.\end{aligned}$$

For all unexplained notation, we refer the reader to [IV16, Section 2.3]. If there is no danger of confusion, we sometimes drop ψ_ξ and $\mu_{\mathbb{Q}_p}$ from our notation. Still following [IV16] (see the proof of Lemma 4.1 of loc. cit.), we obtain

$$D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V\otimes\rho^*_\chi)) \cong D_{\mathrm{pst}}(V)\otimes_{\mathbb{Q}_p^{\mathrm{nr}}} D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*)). \tag{5.1}$$

For an extension L/K of p-adic fields, we write $\mathfrak{D}_{L/K}=\pi_L^{s_{L/K}}\mathcal{O}_L$ for the different of L/K; in the case $K=\mathbb{Q}_p$, we use the notations $\mathfrak{D}_L$ for $\mathfrak{D}_{L/\mathbb{Q}_p}$ and s_L for $s_{L/\mathbb{Q}_p}$. If M/L is a finite abelian extension and η an irreducible character of $\mathrm{Gal}(M/L)$, then we let $\tau_L(\eta)$ denote the abelian local Galois Gauß sum defined, e.g., in [PV13, p. 1184]. For the definition of Galois Gauß sums for a finite Galois extension M/L, we refer the reader to [Frö83, Chapter I, Section 5].

Proposition 5.1 *We have the equality*

$$\begin{aligned}&\hat\partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p^c[G]}(\varepsilon_D(N/K,V)) \\ &\quad= \hat\partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p^c[G]}\left(\left(\det(u)^{-d_K(s_K\chi(1)+m_\chi)}\cdot\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))^{-r}\right)_{\chi\in\mathrm{Irr}(G)}\right),\end{aligned}$$

where we write $\mathfrak{f}(\chi)=\pi_K^{m_\chi}\mathcal{O}_K$ for the Artin conductor of χ.

Proof In the proof, we will use the list of properties in [BB08, Section 2.3]. The field K of loc. cit. corresponds to $\mathbb{Q}_p$ in our situation. Hence, if ψ denotes the standard additive character, we have $n(\psi)=0$ for its conductor.

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Because V is cristalline, the N_1-basis $\{e_i\}$ constructed in Lemma 4.2 is also a $\mathbb{Q}_p^{\mathrm{nr}}$-basis of $D_{\mathrm{pst}}(V)$. A straightforward computation (see the proof of Lemma 4.6 for a similar computation) shows that

$$\phi(e_i) = \sum_{h=1}^{r} p^{-1}((T^{\mathrm{nr}})^{-1}u^{-1}T^{\mathrm{nr}})_{h,i}e_h. \tag{5.2}$$

As in the proof of Lemma 4.2, any element σ of the absolute inertia group acts trivially on the basis elements e_i, whence $D_{\mathrm{pst}}(V)$ is unramified.

Applying (5.1) and [BB08, Section 2.3, proprieté (6)], we obtain

$$\begin{aligned}&\varepsilon(D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V\otimes\rho_\chi^*)),\psi_\xi)\\&\quad=\varepsilon(D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*)),\psi_\xi)^r\cdot\det(D_{\mathrm{pst}}(V))(p)^{m(D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*)))},\end{aligned}$$

where $m(D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*)))$ is the exponent of the Artin conductor.

We consider the first factor and note that

$$D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*))\cong\mathbb{Q}_p^{\mathrm{nr}}\otimes_{\mathbb{Q}_p}\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*),$$

so that we deduce from [BC17, Proposition 6.1.3] that

$$\varepsilon(D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*)),\psi_\xi)=\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*)).$$

As a consequence of [Mar77, Proposition II.4.1(ii)], we get

$$\begin{aligned}&\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*))\\&\quad=\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))^{-1}\cdot p^{m(D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*)))}\cdot(\det((\mathrm{Ind}_{K/\mathbb{Q}_p}\chi)(-1))).\end{aligned}$$

For the second factor, we first note that

$$\det(D_{\mathrm{pst}}(V))(p)=\det(\varphi^{-1},D_{\mathrm{pst}}(V)).$$

By [BB08, p. 625], the action of the Weil group $W_{\mathbb{Q}_p}$ on $D_{\mathrm{pst}}(V)$ is defined, so that the action of the geometric Frobenius φ^{-1} coincides with the usual action of $\varphi^{-1}\phi$ on $D_{\mathrm{pst}}(V)$. We recall from (5.2) that with respect to the basis $\{e_i\}$ of $D_{\mathrm{pst}}(V)$ the element $\varphi^{-1}\phi$ acts as $\varphi^{-1}(p^{-1}(T^{\mathrm{nr}})^{-1}u^{-1}T^{\mathrm{nr}})$ on $D_{\mathrm{pst}}(V)$, so that we derive $\det(\varphi^{-1},D_{\mathrm{pst}}(V))=p^{-r}\det(u^{-1})$.

Finally, by [Neu92, Chapter VII, Theorem 11.7], we get

$$\mathfrak{f}(\mathbb{Q}_p^{\mathrm{nr}}\otimes_{\mathbb{Q}_p}\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*))=\mathfrak{f}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*))=\mathfrak{d}_K^{\chi(1)}N_{K/\mathbb{Q}_p}(\mathfrak{f}(\chi^*))=p^{d_K(s_K\chi(1)+m_\chi)},$$

so that

$$m(D_{\mathrm{pst}}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*)))=m(\mathbb{Q}_p^{\mathrm{nr}}\otimes_{\mathbb{Q}_p}\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi^*))=d_K(s_K\chi(1)+m_\chi).$$

We conclude that

$$\begin{aligned}\varepsilon_D(N/K,V)_\chi=\big(\det(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))(-1)\big)^r\cdot\det(u)^{-d_K(s_K\chi(1)+m_\chi)}\\\cdot\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))^{-r}.\end{aligned}$$

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The proposition is now immediate from [BC17, Lemma 6.2.2], which shows that

$$\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p[G]}\left(\left(\det(\mathrm{Ind}_{K/\mathbb{Q}_p}\chi)(-1)\right)_{\chi\in\mathrm{Irr}(G)}\right) = 0. \qquad \blacksquare$$

Concerning functoriality with respect to change of fields, we have the following lemma.

Lemma 5.2 *Let L be an intermediate field of N/K and $H = \mathrm{Gal}(N/L)$.*

(1) $\rho^G_H(\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p^c[G]}(\varepsilon_D(N/K,V))) = \hat{\partial}^1_{\mathbb{Z}_p[H],\mathbb{Q}_p^c[H]}(\varepsilon_D(N/L,V))$.

(2) *If H is normal in G, then*

$$q^G_{G/H}(\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p^c[G]}(\varepsilon_D(N/K,V))) = \hat{\partial}^1_{\mathbb{Z}_p[G/H],\mathbb{Q}_p^c[G/H]}(\varepsilon_D(L/K,V)).$$

Proof By Proposition 5.1, we have

$$\begin{aligned}&\rho^G_H(\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p^c[G]}(\varepsilon_D(N/K,V)))\\ &\quad= \rho^G_H\left(\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p^c[G]}\left(\left(\det(u)^{-d_K(s_K\chi(1)+m_\chi)}\cdot\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))^{-r}\right)_{\chi\in\mathrm{Irr}(G)}\right)\right).\end{aligned}$$

By [Bre04b, Lemma 2.3], we have

$$\begin{aligned}&\rho^G_H(\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p^c[G]}\left(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))^{-r}\right)_{\chi\in\mathrm{Irr}(G)})\\ &\quad= \hat{\partial}^1_{\mathbb{Z}_p[H],\mathbb{Q}_p^c[H]}\left(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{L/\mathbb{Q}_p}(\psi))^{-r}\right)_{\psi\in\mathrm{Irr}(H)},\end{aligned}$$

whereas [BW09, Section 6.1] implies

$$\rho^G_H\left(\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p^c[G]}\left(\det(u)^{-d_K(s_K\chi(1)+m_\chi)}\right)\right) = \hat{\partial}^1_{\mathbb{Z}_p[H],\mathbb{Q}_p^c[H]}\left((\alpha_\psi)_{\psi\in\mathrm{Irr}(H)}\right)$$

with

$$\alpha_\psi = \prod_{\chi\in\mathrm{Irr}(G)}\det(u)^{-d_K(s_K\chi(1)+m_\chi)\langle\chi,\mathrm{Ind}^G_H\psi\rangle_G}.$$

From [Neu92, Theorem VII.11.7] and the obvious relation

$$\mathrm{Ind}^G_H\psi = \sum_{\chi\in\mathrm{Irr}(G)}\langle\chi,\mathrm{Ind}^G_H\psi\rangle_G\chi,$$

we derive

$$\prod_{\chi\in\mathrm{Irr}(G)}\mathfrak{f}(N/K,\chi)^{\langle\chi,\mathrm{Ind}^G_H\psi\rangle_G} = \mathfrak{f}(N/K,\mathrm{Ind}^G_H\psi) = \mathfrak{d}^{\psi(1)}_{L/K}N_{L/K}(\mathfrak{f}(N/L,\psi)),$$

where $\mathfrak{d}_{L/K}$ denotes the discriminant of L/K. This implies

$$\sum_{\chi\in\mathrm{Irr}(G)} m_\chi\langle\chi,\mathrm{Ind}^G_H\psi\rangle_G = d_{L/K}s_{L/K}\psi(1) + d_{L/K}m_\psi.$$

Furthermore, we note

$$\sum_{\chi\in\mathrm{Irr}(G)}\langle\chi,\mathrm{Ind}^G_H\psi\rangle_G\cdot\chi(1) = (\mathrm{Ind}^G_H\psi)(1) = [G:H]\psi(1) = e_{L/K}d_{L/K}\psi(1).$$

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Hence, we deduce from $d_L = d_K d_{L/K}$

$$\begin{aligned}\sum_{\chi\in\mathrm{Irr}(G)} & d_K(s_K\chi(1)+m_\chi)\langle\chi,\mathrm{ind}_H^G\psi\rangle_G\\ &= d_K(s_K e_{L/K} d_{L/K}\psi(1) + d_{L/K}s_{L/K}\psi(1) + d_{L/K}m_\psi)\\ &= d_L s_K e_{L/K}\psi(1) + d_L s_{L/K}\psi(1) + d_L m_\psi\\ &= d_L\psi(1)(s_K e_{L/K} + s_{L/K}) + d_L m_\psi\\ &= d_L\psi(1)s_L + d_L m_\psi,\end{aligned}$$

where the last equality follows from the multiplicativity of differents (see [Neu92, Theorem III.2.2(i)]). The first functoriality property is now obvious.

The second functoriality property follows easily from [Bre04b, Lemma 2.3] and [Neu92, Lemma VII.11.7(ii)]. ■

6 Computation of the cohomological term

6.1 Identifying cohomology

In the following, we will take the opportunity to clarify some of the constructions of [BC17, Section 7.1, p. 356]. This is necessary, because, in the definition of $C_{N/K}$, we use the identification of $H^1(N,T)$ with $\mathcal{F}(\mathfrak{p}_N^{(r)})$ coming from continuous cochain cohomology combined with Kummer theory, whereas in the computations in [BC17, Section 7.1], we use the identification coming from [BC17, Theorem 4.3.1] combined with [BC17, Lemma 4.1.1]. In this manuscript, we work in the r-dimensional setting based on the results of [Cob18, Section 3], where the special case $r = 1$ covers the situation of [BC17].

Let

$$C^\bullet_{N,\rho^{\mathrm{nr}}} := [\mathcal{I}_{N/K}(\chi^{\mathrm{nr}}) \longrightarrow \mathcal{I}_{N/K}(\chi^{\mathrm{nr}}) \longrightarrow \mathbb{Q}_p^r(\rho^{\mathrm{nr}})/(F_N-1)\cdot\mathbb{Q}_p^r(\rho^{\mathrm{nr}})],$$

with nontrivial modules in degrees 0, 1, and 2, be the complex of [Cob18, Theorem 3.12], and let

$$C^\bullet_{N,T} := [\mathcal{I}_{N/K}(\chi^{\mathrm{nr}}) \longrightarrow \mathcal{I}_{N/K}(\chi^{\mathrm{nr}})],$$

with nontrivial modules in degrees 1 and 2, be the complex of [Cob18, Theorem 3.15]. We also deduce from [Cob18, Theorem 3.3] combined with [Cob18, Lemma 2.1] the short exact sequence

$$(6.1)\qquad 0\longrightarrow \mathcal{F}(\mathfrak{p}_N^{(r)}) \xrightarrow{f_{\mathcal{F},N}} \mathcal{I}_{N/K}(\chi^{\mathrm{nr}}) \longrightarrow \mathcal{I}_{N/K}(\chi^{\mathrm{nr}}) \longrightarrow \mathbb{Z}/(F_N-1)\mathbb{Z}\longrightarrow 0.$$

In the sequel, we will use dotted arrows for morphisms in the derived category and solid arrows for those which are actual morphisms of complexes.

In the proof of [Cob18, Theorem 3.12], we construct an isomorphism

$$\tau\colon C^\bullet_{N,\rho^{\mathrm{nr}}} \dashrightarrow R\Gamma(N,\mathcal{F})$$

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in the derived category which induces the identity on H^0. In a second step (see [Cob18, Corollary 3.13]), we produce quasi-isomorphisms

$$\eta\colon P^\bullet \xrightarrow{\sim} C^\bullet_{N,\rho^{\mathrm{nr}}} \text{ and } \tilde{\eta}\colon \tilde{P}^\bullet \xrightarrow{\sim} C^\bullet_{N,T}[1],$$

where

$$\begin{aligned} P^\bullet &:= [P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \mathbb{Q}_p^r/(F_N - 1)\cdot \mathbb{Q}_p^r(\rho^{\mathrm{nr}})], \\ \tilde{P}^\bullet &:= [P^{-1} \longrightarrow P^0 \longrightarrow P^1]. \end{aligned}$$

Here, the $\mathbb{Z}_p[G]$-modules P^{-1}, P^0, P^1 are finitely generated and projective, and the uniquely divisible module $\mathbb{Q}_p^r(\rho^{\mathrm{nr}})/(F_N - 1)\cdot \mathbb{Q}_p^r(\rho^{\mathrm{nr}})$ is G-cohomologically trivial. Composing η and τ, we obtain an isomorphism $P^\bullet \dashrightarrow R\Gamma(N, \mathcal{F})$ in the derived category. Passing to the projective limit in [Cob18, Lemma 3.14], we obtain another quasi-isomorphism $\varphi\colon \tilde{P}^\bullet \xrightarrow{\sim} R\Gamma(N,T)[1]$ and thus obtain the following commutative diagram in the derived category:

(6.2)
$$\begin{array}{ccccc} & C^\bullet_{N,\rho^{\mathrm{nr}}} & \xleftarrow{\quad\iota\quad} & C^\bullet_{N,T}[1] & \\ & \sim\uparrow\eta & & \sim\uparrow\tilde{\eta} & \\ \tau & P^\bullet & \xleftarrow{\quad\iota\quad} & \tilde{P}^\bullet & \xi \\ & \sim\downarrow\tau\circ\eta & & \sim\downarrow\varphi & \\ & R\Gamma(N,\mathcal{F}) & \xleftarrow{\tau\circ\eta\circ\varphi^{-1}} & R\Gamma(N,T)[1] & \end{array}$$

On H^0, we therefore obtain the commutative diagram:

(6.3)
$$\begin{array}{ccccc} \mathcal{F}(\mathfrak{p}_N^{(r)}) & \xrightarrow{f_{\mathcal{F},N}} & H^0(C^\bullet_{N,\rho^{\mathrm{nr}}}) & \xrightarrow{\mathrm{id}} & H^1(C^\bullet_{N,T}) \\ \downarrow \mathrm{id} & & \downarrow H^0(\tau) & & \downarrow H^0(\xi) \\ \mathcal{F}(\mathfrak{p}_N^{(r)}) & \xrightarrow{\mathrm{id}} & H^0(N,\mathcal{F}) & \xrightarrow{H^0(\varphi\circ\eta^{-1}\circ\tau^{-1})} & H^1(N,T) \end{array}$$

By the proof of [BC17, Theorem 4.3.1] (which is used also for [Cob18, Theorem 3.15]), we know that the composite

$$H^0(N,\mathcal{F}) \xrightarrow{H^0(\varphi\circ\eta^{-1})} H^1(N,T) \longrightarrow H^1(N,\mathcal{F}[p^n])$$

is the Kummer map $\partial_{Ku,n}$ resulting from the distinguished triangle

$$R\Gamma(N,\mathcal{F}) \xrightarrow{p^n} R\Gamma(N,\mathcal{F}) \longrightarrow R\Gamma(N,\mathcal{F}[p^n])[1] \longrightarrow .$$

By the universal property of projective limits, we obtain

$$H^0(\varphi\circ\eta^{-1}) = \partial_{Ku}, \text{ respectively, } H^0(\xi) = \partial_{Ku}.$$

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6.2 Definition of the twist invariant

In this subsection, we define an invariant $U_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$ in the relative algebraic K-group $K_0(\mathbb{Z}_p[G],\overline{\mathbb{Q}_p^{\mathrm{nr}}}[G])$. We recall that $T^{\mathrm{nr}} \in \mathrm{Gl}_r(\overline{\mathbb{Z}_p^{\mathrm{nr}}})$ satisfies the matrix equality

$$\varphi(T^{\mathrm{nr}}) = u^{-1}T^{\mathrm{nr}}. \tag{6.4}$$

This equality determines T^{nr} up to right multiplication by a matrix $S \in \mathrm{Gl}_r(\mathbb{Z}_p)$; explicitly, if $\tilde{T}^{\mathrm{nr}}$ is a second matrix satisfying (6.4), then $T^{\mathrm{nr}} = \tilde{T}^{\mathrm{nr}}S$. It is thus immediate that the element

$$U_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p}) := \hat{\partial}^1_{\mathbb{Z}_p[G],\overline{\mathbb{Q}_p^{\mathrm{nr}}}[G]}(\det(T^{\mathrm{nr}})) \tag{6.5}$$

does not depend on the specific choice of T^{nr} satisfying (6.4).

Remark 6.1 The element $U_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$ clearly becomes trivial under the canonical map $K_0(\mathbb{Z}_p[G],\overline{\mathbb{Q}_p^{\mathrm{nr}}}[G]) \longrightarrow K_0(\overline{\mathbb{Z}_p^{\mathrm{nr}}}[G],\overline{\mathbb{Q}_p^{\mathrm{nr}}}[G])$.

6.3 The cohomological term $C_{N/K}$

In this subsection, we clarify and correct the computation of the cohomological term $C_{N/K}$ of [BC17, Section 7.1]. In particular, we produce a detailed proof of [BC17, Lemma 7.1.2], which in loc. cit. was quoted from [IV16, Lemma 6.1]. It is this part of the computation where the new term $U_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$ emerges.

We recall that we throughout assume Hypothesis (F), in particular, $\rho^{\mathrm{nr}}|_{G_N} \neq 1$. Then, by [BC17, equations (15) and (16)], the cohomological term $C_{N/K}$ is defined by

$$C_{N/K} = -\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet, \exp_V \circ \mathrm{comp}_V^{-1}), \tag{6.6}$$

where

$$M^\bullet = R\Gamma(N,T) \oplus \mathrm{Ind}_{N/\mathbb{Q}_p} T[0]. \tag{6.7}$$

We fix a $\mathbb{Z}_p[G]$-projective sublattice $\mathcal{L} \subseteq \mathcal{O}_N$ such that the exponential map $\exp_{\mathcal{F}}$ of Lemma 3.3 converges on $\mathcal{L}^{(r)}$. We set $X(\mathcal{L}) := \exp_{\mathcal{F}}(\mathcal{L}^{(r)})$ and note that $X(\mathcal{L}) \subseteq \mathcal{F}(\mathfrak{p}_N^{(r)})$.

The embedding $X(\mathcal{L}) \hookrightarrow f_{\mathcal{F},\mathcal{N}}H^1(C^\bullet_{N,T})$, where $f_{\mathcal{F},\mathcal{N}}$ is the first map in the exact sequence (6.1), induces an injective map of complexes $X(\mathcal{L})[-1] \longrightarrow C^\bullet_{N,T}$. We set

$$\begin{aligned} K^\bullet(\mathcal{L}) &:= \mathrm{Ind}_{N/\mathbb{Q}_p}(T)[0] \oplus X(\mathcal{L})[-1], \\ M^\bullet(\mathcal{L}) &:= [\mathcal{I}_{N/K}(\rho^{\mathrm{nr}})/f_{\mathcal{F},\mathcal{N}}(X(\mathcal{L})) \longrightarrow \mathcal{I}_{N/K}(\rho^{\mathrm{nr}})], \end{aligned} \tag{6.8}$$

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with modules in degrees 1 and 2, and have thus constructed an exact sequence of complexes

$$0 \longrightarrow K^\bullet(\mathcal{L}) \longrightarrow C^\bullet_{N,T} \oplus \mathrm{Ind}_{N/\mathbb{Q}_p}(T)[0] \longrightarrow M^\bullet(\mathcal{L}) \longrightarrow 0.$$

We first rewrite $C_{N/K}$ in terms of the middle complex and obtain

$$C_{N/K} = -\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(C^\bullet_{N,T} \oplus \mathrm{Ind}_{N/\mathbb{Q}_p}(T)[0], \exp_V \circ \mathrm{comp}_V^{-1} \circ H^0(\xi))$$

with ξ as in (6.2). We then use additivity of refined Euler characteristics in distinguished triangles and derive

$$C_{N/K} = [X(\mathcal{L}), \lambda, \mathrm{Ind}_{N/\mathbb{Q}_p}(T)] - \chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}), 0), \tag{6.9}$$

where λ is the following composite map:

$$X(\mathcal{L})_{B_{\mathrm{dR}}} = \mathcal{F}(\mathfrak{p}_N^{(r)})_{B_{\mathrm{dR}}} \xrightarrow{f_{\mathcal{F},N}} H^1(C^\bullet_{N,T})_{B_{\mathrm{dR}}}$$
$$\xrightarrow{H^0(\xi)} H^1(N,T)_{B_{\mathrm{dR}}} \xrightarrow{comp_V \circ \exp_V^{-1}} (\mathrm{Ind}_{N/\mathbb{Q}_p}(T))_{B_{\mathrm{dR}}}.$$

Then, the term $\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}), 0)$ is precisely the term which is computed in [BC17, Section 7.2] in the one-dimensional weakly ramified case. We will compute this term in arbitrary dimension $r \geq 1$ in Section 6.4 in the tame case and in Section 6.5 in the weakly ramified case.

For the first term, we obtain

$$[X(\mathcal{L}), \lambda, \mathrm{Ind}_{N/\mathbb{Q}_p}(T)] = \left[X(\mathcal{L}), \lambda_2, \bigoplus_{i=1}^r \mathcal{L}e_i\right] + \left[\bigoplus_{i=1}^r \mathcal{L}e_i, \mathrm{comp}_V, \mathrm{Ind}_{N/\mathbb{Q}_p} T\right], \tag{6.10}$$

where the elements $e_1, \dots, e_r$ are defined in Lemma 4.2 and λ_2 is the composite map

$$X(\mathcal{L})_{B_{\mathrm{dR}}[G]} \xrightarrow{H^0(\xi)\circ f_{\mathcal{F},N}} H^1(N,T)_{B_{\mathrm{dR}}[G]} \xrightarrow{\exp_V^{-1}} t_V(N)_{B_{\mathrm{dR}}[G]}. \tag{6.11}$$

Note that $e_1, \dots, e_r$ constitute an N-basis of $D^N_{\mathrm{dR}}(V) = t_V(N)$.

In the next three lemmas, we will compute the summands in (6.10).

Lemma 6.2 *With λ_2 denoting the composite map defined in (6.11), we have*

$$\left[X(\mathcal{L}), \lambda_2, \bigoplus_{i=1}^r \mathcal{L}e_i\right] = 0.$$

Proof This proof is an expanded version of the arguments of [IV16, p. 509].

Recall the isomorphism $\theta : \mathcal{F} \to \mathbb{G}_m^r$ from (2.1), which satisfies $\theta(x) \equiv \varepsilon^{-1}x \pmod{\deg \geq 2}$. Let $\pi_{B_{\mathrm{dR}}} : {B_{\mathrm{dR}}}^+ \to \mathbb{C}_p$ be the natural projection to the residue field.

Similarly to [BK90, p. 360], we can construct a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \xrightarrow{\theta^{-1}} & \varprojlim \mathcal{F}(\mathfrak{p}^{(r)}_{\mathcal{O}_{\mathbb{C}_p}}) & \longrightarrow & \mathcal{F}(\mathfrak{p}^{(r)}_{\mathcal{O}_{\mathbb{C}_p}}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \varepsilon\circ\theta} & & \\
0 & \longrightarrow & T & \longrightarrow & \varprojlim (\mathcal{O}^{\times}_{\mathbb{C}_p})^r(\rho^{\mathrm{nr}}) = (R^{\times})^r(\rho^{\mathrm{nr}}) & \xrightarrow{x\mapsto \varepsilon x_0} & (\mathcal{O}^{\times}_{\mathbb{C}_p})^r & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow & & \downarrow{\scriptstyle \log_p} & & \\
0 & \longrightarrow & V & \longrightarrow & (B_{\mathrm{cris}}^{\varphi=p} \cap B_{\mathrm{dR}}{}^{+})^r(\rho^{\mathrm{nr}}) & \xrightarrow{\varepsilon\circ\pi_{B_{\mathrm{dR}}}} & \mathbb{C}_p^r & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle v\mapsto t^{-1}T^{\mathrm{nr}}v} & & \\
0 & \longrightarrow & V & \longrightarrow & B_{\mathrm{cris}}^{\varphi=1} \otimes V & \longrightarrow & (B_{\mathrm{dR}}/B_{\mathrm{dR}}{}^{+}) \otimes V & \longrightarrow & 0
\end{array}$$

Note that (differently from [BK90]) some of the objects are twisted by ρ^{nr} in order to make all maps G_N-invariant; ε always denotes multiplication by the element $\varepsilon \in \overline{\mathbb{Q}_p^{\mathrm{nr}}}$ which occurs in (2.1). We observe that by Lemma 3.1 we have the equality $\log_p \circ \varepsilon \circ \theta = \log_{\mathcal{F}}$.

Taking G_N-fixed elements and cohomology, we obtain

$$\begin{array}{ccc}
(\mathcal{F}(\mathfrak{p}^{(r)}_{\mathcal{O}_{\mathbb{C}_p}}))^{G_N} = \mathcal{F}(\mathfrak{p}_N^{(r)}) & \xrightarrow{\partial_{Ku}} & H^1(N, T) \\
\downarrow{\scriptstyle \log_{\mathcal{F}}} & & \downarrow{\scriptstyle \mathrm{incl}} \\
(\mathbb{C}_p^r)^{G_N} = N^r & & H^1(N, V) \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle =} \\
((B_{\mathrm{dR}}/B_{\mathrm{dR}}{}^{+}) \otimes V)^{G_N} = t_V(N) = \bigoplus_{i=1}^r Ne_i & \xrightarrow{\exp_V} & H^1(N, V),
\end{array}$$

where the map s is such that $s(v_i) = e_i$ for all i. Note that this diagram is the higher dimensional version of [IV16, equation (3.4)]. It makes the identification s of N^r and the tangent space $t_V(N)$ explicit.

We rewrite λ_2 in terms of the maps in the last diagram and get

$$\begin{array}{ccccc}
X(\mathcal{L})_{B_{\mathrm{dR}}[G]} = \mathcal{F}(\mathfrak{p}_N^{(r)})_{B_{\mathrm{dR}}[G]} & \xrightarrow{H^0(\xi)\circ f_{\mathcal{F},\mathcal{N}}} & H^1(N, T)_{B_{\mathrm{dR}}[G]} & \xrightarrow{\partial_{Ku}^{-1}} & \mathcal{F}(\mathfrak{p}_N^{(r)})_{B_{\mathrm{dR}}[G]} \\
 & & & & \downarrow{\scriptstyle \log_{\mathcal{F}}} \\
 & \searrow{\scriptstyle \lambda_2} & & & B_{\mathrm{dR}}{}^{(r)} \\
 & & & & \downarrow{\scriptstyle s} \\
 & & & & t_V(N)_{B_{\mathrm{dR}}[G]}
\end{array}$$

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By diagram (6.3), we see that $(\partial_{Ku}^{-1} \circ H^0(\xi) \circ f_{\mathcal{F},\mathcal{N}})(X(\mathcal{L})) = X(\mathcal{L})$, so that it remains to show

$$[X(\mathcal{L}), s \circ \log_{\mathcal{F}}, \bigoplus_{i=1}^{r} \mathcal{L} e_i] = 0,$$

which is immediate from $\log_{\mathcal{F}}(X(\mathcal{L})) = \mathcal{L}^{(r)}$ and $s(v_i) = e_i$ for $i = 1, \dots, r$. ■

Lemma 6.3 *With notation as in (6.10) and $m := [K : \mathbb{Q}_p]$, we have*

$$\left[\bigoplus_{i=1}^{r} \mathcal{L} e_i, \mathrm{comp}_V, \mathrm{Ind}_{N/\mathbb{Q}_p} T\right] = -rm\hat{\partial}^1_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(t) + \left[\bigoplus_{i=1}^{r} \mathcal{L}\tilde{e}_i, \mathrm{comp}_{V(-1)}, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)\right],$$

where the elements $\tilde{e}_i$ are defined in Lemma 4.3.

Proof It is easy to see that in the r-dimensional setting, we also have a diagram as in [IV16, equation (6.1)]. If we denote the vertical maps in this diagram by f_1 and f_2, then

$$\left[\bigoplus_{i=1}^{r} \mathcal{L} e_i, \mathrm{comp}_V, \mathrm{Ind}_{N/\mathbb{Q}_p} T\right] + [\mathrm{Ind}_{N/\mathbb{Q}_p} T, f_2, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)] = \left[\bigoplus_{i=1}^{r} \mathcal{L} e_i, f_1, \bigoplus_{i=1}^{r} \mathcal{L}\tilde{e}_i\right] + \left[\bigoplus_{i=1}^{r} \mathcal{L}\tilde{e}_i, \mathrm{comp}_{V(-1)}, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)\right].$$

Because f_1 sends each basis element e_i to $\tilde{e}_i$, the first summand on the right-hand side is trivial. Both $\mathrm{Ind}_{N/\mathbb{Q}_p} T$ and $\mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)$ are isomorphic to $\mathbb{Z}_p[G]^{rm}$ as $\mathbb{Z}_p[G]$-modules. Via these isomorphisms, the map f_2 corresponds to multiplication by t, and so we obtain

$$[\mathrm{Ind}_{N/\mathbb{Q}_p} T, f_2, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)] = \left[\mathbb{Z}_p[G]^{rm}, t, \mathbb{Z}_p[G]^{rm}\right] = rm\hat{\partial}^1_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(t).$$

■

Let $\beta \in N$ be a normal basis element of N/K, i.e., $N = K[G]\beta$. Let

$$\rho_\beta = (\rho_{\beta,\chi})_{\chi\in\mathrm{Irr}(G)} \in Z(\mathbb{Q}_p^c[G])^\times = \prod_{\chi\in\mathrm{Irr}(G)} (\mathbb{Q}_p^c)^\times$$

be defined by

$$\rho_{\beta,\chi} = \mathfrak{d}_K^{\chi(1)} \mathcal{N}_{K/\mathbb{Q}_p}(\beta|\chi),$$

where $\mathfrak{d}_K$ denotes the discriminant of $K/\mathbb{Q}_p$ and $\mathcal{N}_{K/\mathbb{Q}_p}(\beta|\chi)$ the usual norm resolvent (see, e.g., [PV13, Section 2.2]).

We also recall the definition of the twist invariant $U_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$ in Section 6.2. The next lemma corrects an error in [BC17, Lemma 7.1.2] where we just quoted the proof of [IV16, Lemma 6.1]. However, whereas we work in the relative group $K_0(\mathbb{Z}_p[G], B_{\mathrm{dR}}[G])$, the authors of loc. cit. work in $K_0(\overline{\mathbb{Z}_p^{\mathrm{nr}}}[G], B_{\mathrm{dR}}[G])$ where $U_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$ vanishes by Remark 6.1.

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Lemma 6.4 *With $m = [K : \mathbb{Q}_p]$ and $\hat{\partial}^1 = \hat{\partial}^1_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}$, we have*

$$\left[\bigoplus_{i=1}^{r} \mathcal{L}\tilde{e}_i, \mathrm{comp}_{V(-1)}, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)\right] = r\left[\mathcal{L}, \mathrm{id}, \mathcal{O}_K[G]\beta\right] + mU_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p}) + r\hat{\partial}^1(\rho_\beta).$$

Proof We let $T_{\mathrm{triv}} = \mathbb{Z}_p^{(r)}$ and $V_{\mathrm{triv}} = \mathbb{Q}_p^{(r)}$ denote the trivial representations. Let $z_1, \ldots, z_r$ denote the canonical $\mathbb{Z}_p$-basis of T_{triv}.

In the following, we choose to use for each G_N-representation W

$$\mathrm{Ind}_{N/\mathbb{Q}_p}(W) = \{x : G_{\mathbb{Q}_p} \longrightarrow W \mid x(\tau\sigma) = \tau x(\sigma) \text{ for all } \tau \in G_N, \sigma \in G_{\mathbb{Q}_p}\}$$

as the definition for the induction. Note that if $L/\mathbb{Q}_p$ is any field extension (e.g., $L = B_{\mathrm{dR}}$) which carries an action of $G_{\mathbb{Q}_p}$ and W is an L-space, then $\mathrm{Ind}_{N/\mathbb{Q}_p}(W)$ is also an L-space with $(\alpha x)(\sigma) = \sigma(\alpha)x(\sigma)$ for all $\alpha \in L$ and $\sigma \in G_{\mathbb{Q}_p}$. We also note that

$$\mathrm{Ind}_{N/\mathbb{Q}_p}(W) \longrightarrow L[G_{\mathbb{Q}_p}] \otimes_{L[G_N]} W, \quad x \mapsto \sum_{\sigma \in G_{\mathbb{Q}_p}/G_N} \sigma \otimes x(\sigma^{-1}),$$

is a well-defined isomorphism of $L[G]$-modules. For the comparison isomorphism comp_W, we then obtain the following simple description:

$$\begin{aligned} \mathrm{comp}_W : L \otimes_{\mathbb{Q}_p} \left(L \otimes_{\mathbb{Q}_p} W\right)^{G_N} &\longrightarrow \mathrm{Ind}_{N/\mathbb{Q}_p}(L \otimes_{\mathbb{Q}_p} W), \\ l \otimes z &\mapsto l y_z, \end{aligned}$$

where $l \in L$, $z \in \left(L \otimes_{\mathbb{Q}_p} W\right)^{G_N}$ and $y_z(\sigma) := z$ for all $\sigma \in G_{\mathbb{Q}_p}$ (and hence, $(l y_z)(\sigma) = \sigma(l)z$).

We define a G-equivariant isomorphism

$$\tilde{h} : \left(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_{\mathrm{triv}}\right)^{G_N} \longrightarrow \left(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-1)\right)^{G_N}, \quad 1 \otimes z_i \mapsto \tilde{e}_i,$$

and

$$h : \mathrm{Ind}_{N/\mathbb{Q}_p}\left(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_{\mathrm{triv}}\right) \longrightarrow \mathrm{Ind}_{N/\mathbb{Q}_p}\left(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-1)\right), \quad x \mapsto \tilde{h} \circ x.$$

Then, similar as in the proof of [IV16, Lemma 6.1], we obtain a commutative diagram:

$$\begin{array}{ccc} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_{\mathrm{triv}})^{G_N} & \xrightarrow{\mathrm{comp}_{V_{\mathrm{triv}}}} & \mathrm{Ind}_{N/\mathbb{Q}_p}(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_{\mathrm{triv}}) \\ \big\downarrow {\scriptstyle B_{\mathrm{dR}} \otimes \tilde{h}} & & \big\downarrow {\scriptstyle h} \\ B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-1))^{G_N} & \xrightarrow{\mathrm{comp}_{V(-1)}} & \mathrm{Ind}_{N/\mathbb{Q}_p}(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-1)). \end{array}$$

As a consequence, we derive

$$\begin{aligned}
&\left[\bigoplus_{i=1}^{r}\mathcal{L}\tilde{e}_i, \mathrm{comp}_{V(-1)}, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)\right]\\
&\quad = \left[\bigoplus_{i=1}^{r}\mathcal{L}z_i, B_{\mathrm{dR}}\otimes\tilde{h}, \bigoplus_{i=1}^{r}\mathcal{L}\tilde{e}_i\right] + \left[\bigoplus_{i=1}^{r}\mathcal{L}z_i, \mathrm{comp}_{V_{\mathrm{triv}}}, \mathrm{Ind}_{N/\mathbb{Q}_p} T_{\mathrm{triv}}\right]\\
&\qquad\qquad + \left[\mathrm{Ind}_{N/\mathbb{Q}_p} T_{\mathrm{triv}}, h, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)\right]\\
&\quad = \left[\bigoplus_{i=1}^{r}\mathcal{L}z_i, \mathrm{comp}_{V_{\mathrm{triv}}}, \mathrm{Ind}_{N/\mathbb{Q}_p} T_{\mathrm{triv}}\right] + \left[\mathrm{Ind}_{N/\mathbb{Q}_p} T_{\mathrm{triv}}, h, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)\right]\\
&\quad = r\left[\mathcal{L}, \mathrm{id}, \mathcal{O}_K[G]\beta\right] + \left[\bigoplus_{i=1}^{r}(\mathcal{O}_K[G]\beta z_i), \mathrm{comp}_{V_{\mathrm{triv}}}, \mathrm{Ind}_{N/\mathbb{Q}_p} T_{\mathrm{triv}}\right]\\
&\qquad\qquad + \left[\mathrm{Ind}_{N/\mathbb{Q}_p} T_{\mathrm{triv}}, h, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)\right]
\end{aligned}$$

The computations in [IV16, pp. 512–513] show that

$$\left[\bigoplus_{i=1}^{r}\mathcal{O}_K[G]\beta z_i, \mathrm{comp}_{V_{\mathrm{triv}}}, \mathrm{Ind}_{N/\mathbb{Q}_p} T_{\mathrm{triv}}\right] = r\hat{\partial}^1_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(\rho_\beta).$$

It finally remains to prove that

$$\left[\mathrm{Ind}_{N/\mathbb{Q}_p} T_{\mathrm{triv}}, h, \mathrm{Ind}_{N/\mathbb{Q}_p} T(-1)\right] = mU_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p}).$$

To that end, we write

$$G_{\mathbb{Q}_p} = \bigcup_{\bar{\rho}\in G}\ \bigcup_{\sigma_i\in G_K\backslash G_{\mathbb{Q}_p}} G_N\rho\sigma_i$$

and define elements $x_{ij}\in\mathrm{Ind}_{N/\mathbb{Q}_p}(T_{\mathrm{triv}})$ and $y_{ij}\in\mathrm{Ind}_{N/\mathbb{Q}_p}(T(-1))$, for $i=1,\dots,m$ and $j=1,\dots,r$, by

$$x_{ij}(\rho\sigma_k) = \begin{cases} z_j, & \text{if } i=k \text{ and } \bar{\rho}=1,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$y_{ij}(\rho\sigma_k) = \begin{cases} \tilde{v}_j, & \text{if } i=k \text{ and } \bar{\rho}=1,\\ 0, & \text{otherwise.}\end{cases}$$

Without loss of generality, we assume $\rho=1$ for $\bar{\rho}=1$. Then, the x_{ij}, respectively, the y_{ij}, constitute a $\mathbb{Z}_p[G]$-basis of $\mathrm{Ind}_{N/\mathbb{Q}_p}(T_{\mathrm{triv}})$, respectively, $\mathrm{Ind}_{N/\mathbb{Q}_p}(T(-1))$.

For fixed i and j and for $1\le k\le m$, we compute

$$\left(h(x_{ij})\right)(\rho\sigma_k) = \begin{cases} \tilde{e}_j, & \text{if } i=k \text{ and } \bar{\rho}=1,\\ 0, & \text{otherwise.}\end{cases}$$

For $1\le s\le m$ and $1\le t\le r$, let ξ_{st} be indeterminates (with values in B_{dR}). Then,

$$\left(\sum_{s,t}\xi_{st}y_{st}\right)(\rho\sigma_k) = \begin{cases} \sum_t(\rho\sigma_k)(\xi_{kt})\tilde{v}_t, & \text{if } \bar{\rho}=1,\\ 0, & \text{otherwise.}\end{cases}$$

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Because $\tilde{e}_j = \sum_t T^{\mathrm{nr}}_{tj} \otimes \tilde{v}_t$, we obtain

$$\sum_t (\rho\sigma_k)(\xi_{kt})\tilde{v}_t = \begin{cases} \sum_t T^{\mathrm{nr}}_{tj} \otimes \tilde{v}_t, & \text{if } i = k, \\ 0, & \text{if } i \neq k, \end{cases}$$

and hence, for all $1 \le t \le r$,

$$\xi_{it} = \sigma_i^{-1}(T^{\mathrm{nr}}_{tj}), \quad \xi_{kt} = 0 \text{ for } k \neq i.$$

We conclude that with respect to the chosen basis, the map h is represented by a block matrix of the form

$$C = \begin{pmatrix} \sigma_1^{-1}(T^{\mathrm{nr}}) & & \\ & \ddots & \\ & & \sigma_m^{-1}(T^{\mathrm{nr}}) \end{pmatrix}.$$

We recall that $\varphi(T^{\mathrm{nr}}) = u^{-1}T^{\mathrm{nr}}$ and fix $\alpha_i \in \hat{\mathcal{F}}$ such that $\sigma_i|_{\mathbb{Q}_p^{\mathrm{nr}}} = \varphi^{\alpha_i}$. Note that for all $n \in \mathcal{F}$, $\varphi^{-n}(T^{\mathrm{nr}}) \cdot (T^{\mathrm{nr}})^{-1} = u^n$, which coincides with $\rho^{\mathrm{nr}}(\varphi^n)$. By a continuity argument, we have $\varphi^{-\alpha_i}(T^{\mathrm{nr}}) \cdot (T^{\mathrm{nr}})^{-1} = u^{\alpha_i}$, where u^{α_i} is well defined by [Cob18, Lemma 1.5]. Then,

$$\det(C) = \prod_{i=1}^{m} \det\left(u^{\alpha_i} T^{\mathrm{nr}}\right) = \det(u)^{\alpha} \det(T^{\mathrm{nr}})^m$$

with $\alpha = \sum_i \alpha_i$. Because $\det(u)^{\alpha} \in \mathbb{Z}_p^{\times} \subseteq \mathbb{Z}_p[G]^{\times}$, the result follows from the definition of $U_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$. ∎

We summarize the results of the previous lemmas in the following proposition.

Proposition 6.5 *With $\hat{\partial}^1 = \hat{\partial}^1_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}$ and $\chi = \chi_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}$, we have*

$$C_{N/K} = -rm\hat{\partial}^1(t) + r[\mathcal{L}, \mathrm{id}, \mathcal{O}_K[G]\beta] + r\hat{\partial}^1(\rho_\beta) + mU_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p}) - \chi(M^\bullet(\mathcal{L}), 0).$$

Remark 6.6 To compare Proposition 6.5 and [BC17, equation (55)], we first note that in loc. cit., we have $\mathcal{L} = \mathcal{O}_K[G]\beta$. The additional new term $U_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$ emerges from the computations in Lemma 6.4. The error does not affect the validity of any of the arguments in [BC17]; it just forces us to adapt our definition of $R_{N/K}$ and $\tilde{R}_{N/K}$.

To finish the proof of the conjecture, it is necessary to compute explicitly the term $\chi_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}), 0)$. For this, we will consider the tame and the weakly ramified case separately.

6.4 The tame case

In this subsection, we let N/K be tame und compute the term $\chi_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}), 0)$ from (6.9). In the tame case, by results of Ullom, we can and will use $\mathcal{L} = \mathfrak{p}_N^{\nu}$ for a large enough positive integer ν and we also fix $\beta \in \mathcal{O}_N$ such that $\mathcal{O}_N = \mathcal{O}_K[G]\beta$.

Proposition 6.7 *We have*

$$\chi_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}), 0) = r[\mathfrak{p}_N^{\nu}, \mathrm{id}, \mathfrak{p}_N] - \hat{\partial}^1_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}({}^*(\det(1 - u^{d_K}F^{-1})e_I)).$$

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Proof The key point in the proof is that by Proposition 2.2 the cohomology modules of $M^\bullet(\mathcal{L})$ are perfect, so that we can compute the refined Euler characteristic of $M^\bullet(\mathcal{L})$ in terms of cohomology without explicitly using the complex. In a little more detail, we note that the mapping cone of

$$\mathcal{F}(\mathfrak{p}_N^{(r)})/X(\mathcal{L})[1] \to M^\bullet(\mathcal{L}),$$

where the map in degree 1 is induced by $f_{\mathcal{F},\mathcal{N}}$, is isomorphic to $H^2(N,T)[2]$. We also recall from Section 2 that we identify $H^2(N,T)$ with $\mathcal{Z}/(F_N-1)\mathcal{Z}$. Hence, we conclude from [BB05, Theorem 5.7] that

$$\begin{aligned}&\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}),0)\\ &\quad= \chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}\left(\mathcal{F}(\mathfrak{p}_N^{(r)})/X(\mathcal{L})[1],0\right) + \chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(\mathcal{Z}/(F_N-1)\mathcal{Z},0).\end{aligned}$$

To compute the first summand, we observe that by Proposition 3.4 we have $X(\mathcal{L}) = \mathcal{F}\left((\mathfrak{p}_N^{\nu})^{(r)}\right)$. Because, for each integer $i \ge 0$, the identity map induces isomorphisms

$$\mathcal{F}\left((\mathfrak{p}_N^{i})^{(r)}\right)/\mathcal{F}\left((\mathfrak{p}_N^{i+1})^{(r)}\right) \cong (\mathfrak{p}_N^{i})^{(r)}/(\mathfrak{p}_N^{i+1})^{(r)}, \tag{6.12}$$

a standard argument shows that

$$\begin{aligned}\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}\left(\mathcal{F}(\mathfrak{p}_N^{(r)})/X(\mathcal{L})[1],0\right) &= \chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}\left(\mathfrak{p}_N^{(r)}/(\mathfrak{p}_N^{\nu})^{(r)}[1],0\right)\\ &= r[\mathfrak{p}_N^{\nu},\mathrm{id},\mathfrak{p}_N].\end{aligned}$$

For the computation of the second term, we consider the short exact sequence of G-modules

$$0 \to \mathbb{Z}_p^r[G/I] \xrightarrow{F^{-1}u^{d_K}-1} \mathbb{Z}_p^r[G/I] \xrightarrow{\pi} \mathcal{Z}/(F_N-1)\mathcal{Z} \to 0,$$

where $\pi(z(\bar{g})) = g\cdot(z+(F_N-1)\mathcal{Z})$ for all $z\in\mathbb{Z}_p^r$ and $g\in G$ and where G acts on $\mathcal{Z}/(F_N-1)\mathcal{Z}$ through any lift of its elements to G_K (which is well defined, because elements of G_N act trivially).

Let $x = \sum_{i=0}^{d-1}\alpha_i F^{-i} \in \mathbb{Z}_p^r[G/I]$ be an element in the kernel of the map on the left. Then, $u^{d_K}\alpha_{i-1}-\alpha_i = 0$ for all i. Hence, $(u^{dd_K}-1)\alpha_i = 0$, and by the assumption $\mathcal{Z}^{G_N} = 1$, it follows that $\alpha_i = 0$ for all i. Hence, the map on the left is injective.

Next, we see that $\pi((F^{-1}u^{d_K}-1)e_i) = (\rho^{\mathrm{nr}}(F^{-1})u^{d_K}-1)e_i = 0$.

Conversely, let $x = \sum_{i=0}^{d-1}\alpha_i F^{-i} \in \mathbb{Z}_p^r[G/I]$ be such that $\pi(x) = 0$. Modulo the image of $F^{-1}u^{d_K}-1$, x has a representative $y\in\mathbb{Z}_p^r$. We must show that $y\in\mathrm{im}(F^{-1}u^{d_K}-1)$. Because $\pi(y) = 0$, there exists $z\in\mathbb{Z}_p^r$ such that $y = (u^{dd_K}-1)z = ((F^{-1}u^{d_K})^d-1)z$, which is in the image of $F^{-1}u^{d_K}-1$. Hence, we have exactness in the middle term.

To prove the exactness of the sequence, it remains to check the surjectivity of the map on the right, which is obvious.

Because we are considering the case of a tame extension, $\mathbb{Z}_p^r[G/I]$ is a projective $\mathbb{Z}_p[G]$-module and we have:

$$\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(H^2(N,T)[2],0) = -[\mathbb{Z}_p^r[G/I],F^{-1}u^{d_K}-1,\mathbb{Z}_p^r[G/I]].$$

The results follows. ∎

6.5 The weakly ramified case

In this subsection, we let p be an odd prime. Let $K/\mathbb{Q}_p$ be the unramified extension of degree m. We let N/K be a weakly and wildly ramified finite abelian extension with cyclic ramification group. We let $d = d_{N/K}$ be the inertia degree of N/K and assume that m and d are relatively prime.

The aim of this subsection is to compute the term $\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\lambda), 0)$ from (6.9) in this weakly and wildly ramified situation. For that purpose, we aim to generalize the methods of [BC17]; however, this forces us to introduce a further technical condition which might be either

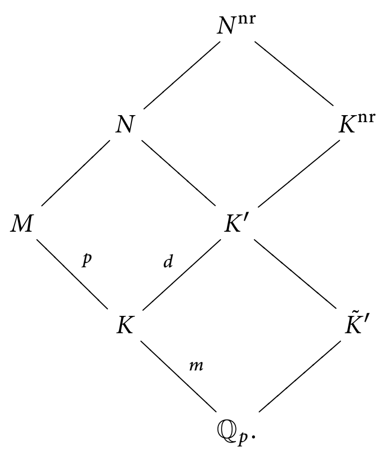
Hypothesis (T): $U_N \equiv 1 \pmod{p}$
or
Hypothesis (I): $U_N - 1$ is invertible modulo p.

Here, (T) stands for trivial reduction modulo p and (I) for invertible modulo p. If we set $\omega := v_p(\det(U_N - 1))$, then we have the following equivalences:

$$(I) \text{ holds} \iff U_N - 1 \in \mathrm{Gl}_r(\mathbb{Z}_p) \iff \omega = 0.$$

Note also that Hypothesis (T) immediately implies $\omega > 0$. However, in the higher dimensional setting, there are mixed cases, where none of our hypotheses holds.

As in [BC17], we have a diagram of fields as follows:



Here, K'/K is the maximal unramified subextension of N/K, M/K is a weakly and wildly ramified cyclic extension of degree p, and $N = MK'$. Because $\gcd(m, d) = 1$, there exists $\tilde{K}'/\mathbb{Q}_p$ of degree d such that $K' = K\tilde{K}'$.

The following lemma generalizes [BC17, Lemma 7.2.1].

Lemma 6.8 *For $n \geq 2$, one has*

$$\mathcal{F}((\mathfrak{p}_N^n)^{(r)}) \textit{ is } \mathbb{Z}_p[G]\textit{-projective} \iff n \equiv 1 \pmod{p}.$$

Moreover,

$$\mathcal{F}(\mathfrak{p}_N^{(r)}) \textit{ is } \mathbb{Z}_p[G]\textit{-projective} \iff \textit{Hypothesis (I) holds}.$$

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Proof For $n \geq 2$, the formal logarithm induces an isomorphism $\mathcal{F}((\mathfrak{p}_N^n)^{(r)}) \cong (\mathfrak{p}_N^n)^{(r)}$ of $\mathbb{Z}_p[G]$-modules by Proposition 3.4. Hence, the first assertion follows from [Köc04, Theorem 1.1 and Proposition 1.3].

We henceforth assume $n = 1$. By Lemma 2.3, we know that $\mathcal{F}(\mathfrak{p}_N^{(r)})$ is cohomologically trivial, if and only if Hypothesis (I) holds. Hence, it suffices to prove that $\mathcal{F}(\mathfrak{p}_N^{(r)})$ is torsion-free. By [Cob18, Lemma 2.1], the module $\mathcal{F}(\mathfrak{p}_N^{(r)})$ is isomorphic to $\left(\prod_r \widehat{N_0^\times}(\rho^{\mathrm{nr}})\right)^{G_N}$ which is torsion-free. Indeed, any tuple $(\zeta_1, \dots, \zeta_r)$ of pth roots of unity $\zeta_1, \dots, \zeta_r \in \widehat{N_0^\times} \cong \mathbb{Z}_p \times U_{N_0}^{(1)}$ must be contained in N (because N_0/N is unramified). Hence, $(\zeta_1, \dots, \zeta_r)$ is fixed by G_N, if and only if it is fixed by F_N, if and only if $\zeta_1 = \cdots = \zeta_r = 1$ (using Hypothesis (I)). ∎

By Lemma 6.8, we can and will take $\mathcal{L} = \mathfrak{p}_N^{p+1}$ and thus obtain

$$X(\mathcal{L}) = \mathcal{F}\left((\mathfrak{p}_N^{p+1})^{(r)}\right).$$

We recall some of the notations from [BC17]. We put $q = p^m$, $b = F^{-1}$ and consider an element $a \in \mathrm{Gal}(N^{\mathrm{nr}}/K)$ such that $\mathrm{Gal}(M/K) = \langle a|_M \rangle$, $a|_{K^{\mathrm{nr}}} = 1$. Because there will be no ambiguity, we will denote by the same letters a, b their restrictions to N. Then, $\mathrm{Gal}(N/K) = \langle a, b \rangle$ and $\mathrm{ord}(a) = p, \mathrm{ord}(b) = d$. We also define $\mathcal{T}_a := \sum_{i=0}^{p-1} a^i$.

Let $\theta_1 \in M$ be such that $\mathcal{T}_{M/K}\theta_1 = p$, where $\mathcal{T}_{M/K}$ denotes the trace map from M to K, and $\mathcal{O}_K[\mathrm{Gal}(M/K)]\theta_1 = \mathfrak{p}_M$. Let θ_2 (resp. A) be a normal integral basis generator of trace one for the extension $\tilde{K}'/\mathbb{Q}_p$ (resp. $K/\mathbb{Q}_p$). Let $\alpha_1 \in \mathcal{O}_K^\times$ be such that $\theta_1^{a-1} \equiv 1 - \alpha_1\theta_1 \pmod{\mathfrak{p}_M^2}$. If we set

$$\alpha_1, \alpha_2 = \alpha_1 A, \alpha_3 = \alpha_1 A^{\varphi}, \dots, \alpha_m = \alpha_1 A^{\varphi^{m-2}}, \tag{6.13}$$

then these elements form a $\mathbb{Z}_p$-basis of $\mathcal{O}_K$ (see [BC17, equation (60)]).

Furthermore, we use [Cob18, Lemma 2.4] to find for $i = 1, \dots, r$ an element $\gamma_i \in \prod_r U_{N_0}^{(1)}$ such that

$$(F_N - 1) \cdot \gamma_i = (\theta_1)_i, \tag{6.14}$$

where

$$(\theta_1)_i := (1, \dots, 1, \theta_1^{a-1}, 1, \dots, 1)$$

with the nontrivial entry is the ith component.

Let

$$W' = \mathbb{Z}_p[G]^r z_1 \oplus \mathbb{Z}_p[G]^r z_2,$$

$$W_{\geq n} = \bigoplus_{j=n}^{p-1} \bigoplus_{k=1}^{m} \mathbb{Z}_p[G]^r v_{k,j} \cong \bigoplus_{j=n}^{p-1} \bigoplus_{k=1}^{m} \mathbb{Z}_p[G]^r \alpha_k w_j = \bigoplus_{j=n}^{p-1} \mathcal{O}_K[G]^r w_j$$

and put

$$W = W_{\geq 0}.$$

If we write $e_1, \dots, e_r$ for the standard $\mathbb{Z}_p$-basis of $\mathbb{Z}_p{}^{(r)}$, then a general element of W is of the form

$$\sum_{i=1}^{r}\sum_{j=1}^{p-1}\sum_{k=1}^{m}\lambda_{i,j,k}e_i v_{k,j} = \sum_{i=1}^{r}\sum_{j=1}^{p-1}\mu_{i,j,k}e_i w_j$$

with $\lambda_{i,j,k} \in \mathbb{Z}_p[G]$ or $\mu_{i,j} \in \mathcal{O}_K[G]$. We will apply this convention analogously for the modules W' and $W_{\geq n}$.

We define a matrix $E \in M_r(\mathcal{O}_{K'})$ by

$$E = \begin{cases} 1 & \text{under Hypothesis (I)} \\ \sum_{i=0}^{dm-1}(A\theta_2)^{\varphi^i}u^{-i} & \text{under Hypothesis (T)} \end{cases} \tag{6.15}$$

and a matrix

$$\tilde{u} = \begin{cases} 1 & \text{under Hypothesis (I)} \\ u & \text{under Hypothesis (T).} \end{cases} \tag{6.16}$$

We also recall that in [Cob18, Lemma 1.9], we constructed an element $\varepsilon \in \mathrm{Gl}_r(\overline{\mathbb{Z}_p^{\mathrm{nr}}})$ such that $u = \varepsilon^{-1} \cdot \varphi(\varepsilon)$.

Lemma 6.9 *The following assertions hold:*

(1) $E \in \mathrm{Gl}_r(\mathcal{O}_{K'})$.
(2) $\varphi(E) \equiv \tilde{u}E \equiv E\tilde{u} \pmod{p\mathcal{O}_{K'}}$.
(3) $\varphi(\varepsilon^{-1}E) \equiv u^{-1}\tilde{u}\varepsilon^{-1}E \pmod{p}$.

Proof Let $H = \mathrm{Gal}(K'/\mathbb{Q}_p)$ and let $f\colon H \to K'$ be defined by $f(\sigma) = (A\theta_2)^{\sigma^{-1}}$. Then, applying [Was97, Lemma 5.26(a)], we obtain

$$\det((A\theta_2)^{\tau\sigma^{-1}})_{\sigma,\tau\in H} = \prod_{\chi\in\hat{H}}\sum_{\sigma\in H}(A\theta_2)^{\sigma^{-1}}\chi(\sigma).$$

Because $A\theta_2$ is an integral normal basis generator and $K'/\mathbb{Q}_p$ is unramified, the left-hand side $\det(\tau\sigma^{-1}(A\theta_2))_{\sigma,\tau\in H}$ is a unit (whose square is the discriminant of $K'/\mathbb{Q}_p$). Therefore, for each character $\chi \in \hat{H}$, the factor

$$\sum_{\sigma\in H}(A\theta_2)^{\sigma^{-1}}\chi(\sigma) = \sum_{i=0}^{dm-1}(A\theta_2)^{\varphi^i}\chi(\varphi^{-i})$$

is a unit, and hence, $\sum_{i=0}^{dm-1}(A\theta_2)^{\varphi^i}\varphi^{-i}$ is a unit in the maximal order $\mathcal{M}$ of $K'[H]$. Because $\sum_{i=0}^{dm-1}(A\theta_2)^{\varphi^i}\varphi^{-i} \in \mathcal{O}_{K'}[H]$, we deduce from the well-known fact

$$\mathcal{M}^\times \cap \mathcal{O}_{K'}[H] = \mathcal{O}_{K'}[H]^\times$$

that $\sum_{i=0}^{dm-1}(A\theta_2)^{\varphi^i}\varphi^{-i} \in \mathcal{O}_{K'}[H]^\times$. We now apply the character $\rho_{\mathbb{Q}_p}^{\mathrm{nr}}$ and easily derive (a).

For the proof of (b), we can assume Hypothesis (T) and we compute

$$\varphi(E) = \sum_{i=0}^{dm-1}(A\theta_2)^{\varphi^{i+1}}u^{-(i+1)}u \equiv Eu \equiv uE \pmod{p\mathcal{O}_{K'}},$$

where the congruences hold, because we have $u^{md} \equiv 1 \pmod{p}$ by hypothesis (T).

If Hypothesis (T) holds, then part (c) is an immediate consequence of [Cob18, Lemma 1.9]. Under Hypothesis (I), it follows from the definitions. ∎

Generalizing the approach of [BC17, Section 6.3], we define a $\mathbb{Z}_p[G]$-module homomorphism

$$\tilde{f}_{3,W}\colon W \longrightarrow \mathcal{F}(\mathfrak{p}_N^{(r)})$$

by

$$\tilde{f}_{3,W}(e_i v_{k,j}) = E\alpha_k(a-1)^j\theta e_i = E\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_k(a-1)^j\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

for all i, j, k, where $\theta = \theta_1\theta_2$. We denote by $f_{3,W}$ the composition of $\tilde{f}_{3,W}$ with the projection to $\mathcal{F}(\mathfrak{p}_N^{(r)})/\mathcal{F}((\mathfrak{p}_N^{p+1})^{(r)})$.

In order to generalize [BC17, Lemma 7.2.4], we first need a higher dimensional version of [BC16, Lemma 4.1.7].

Lemma 6.10 *For $v \in \mathcal{O}_K[G]$, $i = 1, \ldots, r$ and $j = 0, \ldots, p-1$, we have*

$$f_{3,W}(ve_iw_j) \in \mathfrak{p}_N^{j+1}e_i \text{ and } f_{3,W}(ve_iw_j) \equiv vE(a-1)^j\theta e_i \pmod{\mathfrak{p}_N^{j+2}}.$$

Proof We write $v = \sum_{h,l,k} v_{h,l,k}a^hb^l\alpha_k$ for some $v_{h,l,k} \in \mathbb{Z}_p$. Then,

$$f_{3,W}\left(\sum_{h,l,k} v_{h,l,k}a^hb^le_i\alpha_kw_j\right) = \sum_{h,l,k} v_{h,l,k}a^hb^l \cdot E\alpha_k(a-1)^j\theta e_i \in \mathfrak{p}_N^{j+1}e_i$$

by [BC16, Lemma 3.2.5]. Using the isomorphism in (6.12), this is congruent to

$$\sum_{h,l,k} v_{h,l,k}a^hb^lE\alpha_k(a-1)^j\theta e_i \pmod{\mathfrak{p}_N^{j+2}}.$$ ∎

Lemma 6.11 *The map $f_{3,W}$ is surjective. More precisely, for $j \geq 0$,*

$$f_{3,W}(W_{\geq j}) = \mathcal{F}((\mathfrak{p}_N^{j+1})^{(r)})/\mathcal{F}((\mathfrak{p}_N^{p+1})^{(r)}).$$

Proof For $j = p$, $W_{\geq p} = \{0\}$ and $\mathcal{F}((\mathfrak{p}_N^{p+1})^{(r)})/\mathcal{F}((\mathfrak{p}_N^{p+1})^{(r)}) = \{0\}$, so the result is trivial.

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We assume the result for $j+1$ and proceed by descending induction. Let $x \in \mathcal{F}((\mathfrak{p}_N^{j+1})^{(r)})$. As in the proof of [BC17, Lemma 7.2.4], there exist $v_{h,\ell,i,k} \in \mathbb{Z}_p$ such that

$$\begin{aligned} x &\equiv E \sum_{h,\ell,i,k} v_{h,\ell,i,k} a^h b^\ell \alpha_k (a-1)^j \theta e_i \pmod{\mathfrak{p}_N^{j+2}} \\ &\equiv \sum_{h,\ell,i,k} \tilde{u}^{m\ell} E^{\varphi^{-m\ell}} v_{h,\ell,i,k} a^h b^\ell \alpha_k (a-1)^j \theta e_i \pmod{\mathfrak{p}_N^{j+2}} \\ &\equiv \sum_{h,\ell,i,k} \tilde{u}^{m\ell} v_{h,\ell,i,k} a^h b^\ell E \alpha_k (a-1)^j \theta e_i \pmod{\mathfrak{p}_N^{j+2}} \\ &\equiv \tilde{f}_{3,W}\left(\sum_{h,\ell,i,k} \tilde{u}^{ml} v_{h,\ell,i,k} a^h b^\ell e_i \alpha_k w_j \right) \pmod{\mathfrak{p}_N^{j+2}}. \end{aligned}$$

This means that the class $\pi(x)$ of x in $\mathcal{F}(\mathfrak{p}_N^{(r)})/\mathcal{F}((\mathfrak{p}_N^{p+1})^{(r)})$ is the sum of an element in the image of $W_{\geq j}$ and an element in $\mathcal{F}((\mathfrak{p}_N^{j+2})^{(r)})/\mathcal{F}((\mathfrak{p}_N^{p+1})^{(r)})$, which is by assumption in the image of $W_{\geq j+1} \subseteq W_{\geq j}$. ∎

Lemma 6.12 *Let $1 \leq i \leq r$, $0 \leq j \leq p-1$, $1 \leq k \leq m$. Then there exists $\mu_{i,j,k} \in W_{\geq j+2}$ such that the element*

$$s_{i,j,k} = \alpha_k (a-1) e_i w_j - \alpha_k e_i w_{j+1} + \mu_{i,j,k}$$

is in the kernel of $f_{3,W}$. Here, w_p should be interpreted as 0.

Proof As in the proof of [BC17, Lemma 7.2.5], we see that the formal subtraction of X and Y takes the form

$$X -_{\mathcal{F}} Y = X - Y + (X^t A_h Y)_h - (Y^t A_h Y)_h + (\deg \geq 3),$$

with $A_h \in M_r(\mathbb{Z}_p)$ for $h = 1, \ldots, r$. In analogy to the proof of [BC17, Lemma 7.2.5], we set

$$x := E\alpha_k (a-1)^j a \theta e_i, \quad y := E\alpha_k (a-1)^j \theta e_i, \quad z := x - y = E\alpha_k (a-1)^{j+1} \theta e_i,$$

and we obtain that $x -_{\mathcal{F}} y -_{\mathcal{F}} z \equiv 0 \pmod{\mathcal{F}((\mathfrak{p}_N^{j+3})^{(r)})}$. Therefore,

$$\tilde{f}_{3,W}(\alpha_k (a-1) e_i w_j - \alpha_k e_i w_{j+1}) \equiv 0 \pmod{\mathcal{F}((\mathfrak{p}_N^{j+3})^{(r)})},$$

and we conclude the proof of the lemma using Lemma 6.11. ∎

Lemma 6.13 *The elements*

$$r_{i,1} = \mathcal{T}_a e_i \alpha_1 w_0 + \varepsilon u^{-1} \tilde{u}^{1-m\tilde{m}} \varepsilon^{-1} b^{-\tilde{m}} e_i \alpha_1 w_{p-1} - e_i \alpha_1 w_{p-1},$$

$$r_{i,k} = \mathcal{T}_a e_i \alpha_k w_0 + \varepsilon u^{-1} \tilde{u}^{1-m\tilde{m}} \varepsilon^{-1} b^{-\tilde{m}} e_i \alpha_{k+1} w_{p-1} - e_i \alpha_k w_{p-1},$$

$$r_{i,m} = \mathcal{T}_a e_i \alpha_m w_0 + \varepsilon u^{-1} \tilde{u}^{1-m\tilde{m}} \varepsilon^{-1} b^{-\tilde{m}} e_i \left(\alpha_1 - \sum_{i=2}^{m} \alpha_i \right) w_{p-1} - e_i \alpha_m w_{p-1},$$

for $1 \leq i \leq r$ and $1 < k < m$, are in the kernel of $f_{3,W}$. Note that $\varepsilon u^{-1} \tilde{u}^{1-m\tilde{m}} \varepsilon^{-1}$ has coefficients in $\mathbb{Z}_p^{\mathrm{nr}}$ and is fixed by φ; hence, it has coefficients in $\mathbb{Z}_p$.

Proof We denote by v_j the jth component of a vector v. Using [BC16, Lemma 3.2.2] and Lemma 6.9(c), we calculate

$$\begin{aligned}(\mathcal{N}_{N_0/K_0}(\varepsilon^{-1}E\alpha_k\theta e_i))_j &= \mathcal{N}_{N_0/K_0}(\theta_1)(\varepsilon^{-1}E\alpha_k\theta_2 e_i)_j{}^p \\ &\equiv -\alpha_k^p\theta_2^p\alpha_1^{1-p}p(\varepsilon^{-1}Ee_i)_j{}^p \pmod{\mathfrak{p}_{N_0}^{p+1}} \\ &\equiv -\alpha_k^p\theta_2^p\alpha_1^{1-p}p(\varphi(\varepsilon^{-1}E)e_i)_j \pmod{\mathfrak{p}_{N_0}^{p+1}} \\ &\equiv -\alpha_k^p\theta_2^p\alpha_1^{1-p}p\left(u^{-1}\tilde{u}\varepsilon^{-1}Ee_i\right)_j \pmod{\mathfrak{p}_{N_0}^{p+1}}.\end{aligned}$$

We also compute

$$\mathcal{T}_{N_0/K_0}(\varepsilon^{-1}E\alpha_k\theta e_i) = \mathcal{T}_{N_0/K_0}(\theta_1)\varepsilon^{-1}E\alpha_k\theta_2 e_i \equiv p\alpha_k\theta_2\varepsilon^{-1}Ee_i \pmod{\mathfrak{p}_{N_0}^{p+1}}.$$

With this in mind, we can do analogous calculations to those in [BC16, Lemma 4.2.6] and [BC17, Lemma 7.2.6] and obtain

$$f_{3,W}(\mathcal{T}_a e_i\alpha_k w_0) \equiv \varepsilon\left(\alpha_k\theta_2 - \alpha_1\left(\frac{\alpha_k}{\alpha_1}\right)^p\theta_2^{b^{-\tilde{m}}}u^{-1}\tilde{u}\right)p\varepsilon^{-1}Ee_i \pmod{\mathfrak{p}_{N_0}^{p+1}}.$$

Furthermore,

$$\begin{aligned}&f_{3,W}(\varepsilon u^{-1}\tilde{u}^{1-m\tilde{m}}\varepsilon^{-1}b^{-\tilde{m}}e_i\alpha_{k+1}w_{p-1}) \\ &\quad\equiv \varepsilon u^{-1}\tilde{u}^{1-m\tilde{m}}\varepsilon^{-1}E^{\varphi^{m\tilde{m}}}\alpha_{k+1}p\theta_2^{b^{-\tilde{m}}}e_i \pmod{\mathfrak{p}_{N_0}^{p+1}} \\ &\quad\equiv \varepsilon u^{-1}\tilde{u}^{1-m\tilde{m}}u^{m\tilde{m}}(\varepsilon^{-1}E)^{\varphi^{m\tilde{m}}}\alpha_{k+1}p\theta_2^{b^{-\tilde{m}}}e_i \pmod{\mathfrak{p}_{N_0}^{p+1}} \\ &\quad\equiv \varepsilon u^{-1}\tilde{u}\varepsilon^{-1}E\alpha_{k+1}p\theta_2^{b^{-\tilde{m}}}e_i \pmod{\mathfrak{p}_{N_0}^{p+1}}.\end{aligned}$$

It is straightforward to adapt the remaining calculations from [BC17, Lemma 7.2.6] and conclude that $r_{i,k} \in \ker f_{3,W}$ for $1 \le i \le r$ and $1 < k < m$. The proof that $r_{i,m} \in \ker f_{3,W}$ is analogous. ■

From now on, we have to distinguish the cases of Hypotheses (T) and (I).
We start assuming Hypothesis (I).
Following the computations in [BC17, Section 7.3], we have

$$\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}),0) = [\mathcal{F}((\mathfrak{p}_N^{p+1})^{(r)}),\mathrm{id},\mathcal{F}(\mathfrak{p}_N^{(r)})] = [\ker(f_{3,W}),\mathrm{id},W].$$

Lemma 6.14 *The pmr elements $r_{i,k}, s_{i,j,k}$, for $1 \le i \le r$, $0 \le j \le p-2$, $1 \le k \le m$, constitute a $\mathbb{Z}_p[G]$-basis of* $\ker f_{3,W}$.

Proof We adapt the proof of [BC17, Lemmas 7.3.1]. We write the coefficients of the $e_i\alpha_k w_{p-1}$-components, $i = 1,\dots,r$, $k = 1,\dots,m$, of the elements $r_{i,j}$, $j = 1,\dots,m$, into

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the columns of an $mr \times mr$ matrix, which we call $\mathcal{M}$, and whose entries are $r \times r$ blocks,

$$\mathcal{M} = \begin{pmatrix} \varepsilon u^{-1}\varepsilon^{-1}b^{-\tilde{m}}-1 & 0 & \cdots & 0 & 0 & \varepsilon u^{-1}\varepsilon^{-1}b^{-\tilde{m}} \\ 0 & -1 & \cdots & 0 & 0 & -\varepsilon u^{-1}\varepsilon^{-1}b^{-\tilde{m}} \\ 0 & \varepsilon u^{-1}\varepsilon^{-1}b^{-\tilde{m}} & \cdots & 0 & 0 & -\varepsilon u^{-1}\varepsilon^{-1}b^{-\tilde{m}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \varepsilon u^{-1}\varepsilon^{-1}b^{-\tilde{m}} & -1 & -\varepsilon u^{-1}\varepsilon^{-1}b^{-\tilde{m}} \\ 0 & 0 & \cdots & 0 & \varepsilon u^{-1}\varepsilon^{-1}b^{-\tilde{m}} & -1-\varepsilon u^{-1}\varepsilon^{-1}b^{-\tilde{m}} \end{pmatrix}.$$

By Lemma 4.5 and analogous computations as in [BC17], we obtain

$$\det \mathcal{M} = (-1)^{r(m-1)} \det(u^{-m}b^{-1}-1).$$

The rest of the proof works exactly as in [BC17]; note that Hypothesis (I) plays the role of the assumption $\omega = 0$ in the one-dimensional setting. ■

We recall that $G = \mathrm{Gal}(N/K) = \langle a \rangle \times \langle b \rangle$. Any irreducible character ψ of G decomposes as $\psi = \chi\phi$, where χ is an irreducible character of $\langle a \rangle$ and ϕ an irreducible character of $\langle b \rangle$. We will denote by χ_0 the trivial character.

Proposition 6.15 *Assume Hypothesis (I). For $\mathcal{L} = \mathfrak{p}_N^{p+1}$, the element*

$$\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}),0) \in K_0(\mathbb{Z}_p[G],B_{\mathrm{dR}}[G])$$

is contained in $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p[G])$ and represented by $\varepsilon \in \mathbb{Q}_p[G]^\times$ where

$$\kappa_{\chi\phi} = \begin{cases} p^{mr} & \text{if } \chi = \chi_0 \\ (-1)^{r(m-1)} \det(u^{-m}\phi(b)^{-1}-1)(\chi(a)-1)^{mr(p-1)} & \text{if } \chi \neq \chi_0. \end{cases}$$

Proof We choose the elements $r_{i,k}$ and $s_{i,j,k}$ of Lemma 6.14 as a $\mathbb{Z}_p[G]$-basis of $\ker(f_{3,W})$ and fix the canonical $\mathbb{Z}_p[G]$-basis of W. Then,

$$\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}),0) = [\ker(f_{3,W}),\mathrm{id},W]$$

is represented by the determinant of

$$\mathfrak{M} = \begin{pmatrix} \mathcal{T}_a I & (a-1)I & 0 & \cdots & 0 & 0 \\ 0 & -I & (a-1)I & \cdots & 0 & 0 \\ 0 & * & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & * & \cdots & -I & (a-1)I \\ \mathcal{M} & * & * & \cdots & * & -I \end{pmatrix},$$

where in the above $pmr \times pmr$ all the entries are $mr \times mr$ blocks. Recalling that p is odd, we get:

$$\begin{aligned} \det(\chi\phi(\mathfrak{M})) &= \begin{cases} p^{mr}(-1)^{mr(p-1)} & \text{if } \chi = \chi_0 \\ (-1)^{m^2r^2(p-1)} \det(\chi\phi(\mathcal{M}))(\chi(a)-1)^{mr(p-1)} & \text{if } \chi \neq \chi_0 \end{cases} \\ &= \begin{cases} p^{mr} & \text{if } \chi = \chi_0 \\ (-1)^{r(m-1)} \det(u^{-m}\phi(b)^{-1}-1)(\chi(a)-1)^{mr(p-1)} & \text{if } \chi \neq \chi_0. \end{cases} \end{aligned}$$

■

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It remains to consider the case of Hypothesis (T). Here, we follow the strategy of [BC17, Section 7.4].

Lemma 6.16 *There is a commutative diagram of $\mathbb{Z}_p[G]$-modules with exact rows*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X(2)\oplus W & \longrightarrow & W'\oplus W & \xrightarrow{\delta_2} & \mathbb{Z}_p[G]^r z_0 & \xrightarrow{\pi} & \mathbb{Z}/(F_N-1)\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde f_{3,W}} & & \downarrow{\scriptstyle \tilde f_3} & & \downarrow{\scriptstyle \tilde f_2} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & \mathcal{F}(\mathfrak{p}_N^{(r)}) & \xrightarrow{f_{\mathcal{F},\mathcal{N}}} & \mathcal{I}_{N/K}(\rho^{\mathrm{nr}}) & \xrightarrow{(F-1)\times 1} & \mathcal{I}_{N/K}(\rho^{\mathrm{nr}}) & \longrightarrow & \mathbb{Z}/(F_N-1)\mathbb{Z} & \longrightarrow & 0
\end{array}$$

where

$$\begin{aligned}
\delta_2(e_i z_1) &= (u^m b-1)e_i z_0,\\
\delta_2(e_i z_2) &= (a-1)e_i z_0,\\
\delta_2(e_i v_{j,k}) &= 0,\\
\tilde f_2(e_i z_0) &= [(\theta_1)_i, 1,\dots,1],\\
\tilde f_3(e_i z_1) &= [(\theta_1)_i, 1,\dots,1],\\
\tilde f_3(e_i z_2) &= [\gamma_i,\dots,\gamma_i],\\
\tilde f_3(e_i v_{k,j}) &= f_{\mathcal{F},\mathcal{N}}(\tilde f_{3,W}(e_i v_{k,j})),\\
\pi(e_i z_0) &= e_i,
\end{aligned}$$

for all i, j, and k (recall that γ_i was defined in (6.14)). Furthermore, $X(2)=\ker(\delta_2|_{W'})$ and $\tilde f_{3,W}$ is the restriction of $\tilde f_3$ to $X(2)\oplus W$.

Proof We recall from [Cob18, Section 2] that the action of $\mathrm{Gal}(K_0/K)\times G$ on $\mathcal{I}_{N/K}(\rho^{\mathrm{nr}})$ is characterized by

$$\begin{aligned}
(F\times 1)[x_1,\dots,x_d] &= [U_N x_d^{F_N}, x_2,\dots,x_{d-1}],\\
(F^{-n}\times\sigma)[x_1,\dots,x_d] &= [\rho^{\mathrm{nr}}(\tilde\sigma)x_1^{\tilde\sigma},\dots,\rho^{\mathrm{nr}}(\tilde\sigma)x_d^{\tilde\sigma}],
\end{aligned}$$

where $x_i\in\prod_r \widehat{N_0^\times}$, the elements F^{-n} and $\sigma\in G$ have the same restriction to $N\cap K_0$, and $\tilde\sigma\in\mathrm{Gal}(N_0/K)$ is uniquely defined by $\tilde\sigma|_{K_0}=F^{-n}$ and $\tilde\sigma|_N=\sigma$. Furthermore, we remark that the action of G on $\mathbb{Z}/(U_N-1)\mathbb{Z}$ is induced by

$$a\cdot e_i = e_i,\quad b\cdot e_i = \rho^{\mathrm{nr}}(F^{-1})e_i = U_K^{-1}e_i.$$

The bottom sequence is exact by [Cob18, Theorem 3.3 and Lemma 2.1]. The proofs of the exactness of the top sequence as well as the proof of commutativity follow along the lines of proof in the one-dimensional case (see [BC17, Lemma 7.4.1]). For example, if we denote the coefficients of $U_K=u^m$ by u_{ij}, then

$$\begin{aligned}
\tilde f_2\circ\delta_2(e_i z_1) = \tilde f_2((u^m b-1)e_i) &= \tilde f_2\left(\sum_{j=1}^r b u_{j,i}e_j - e_i\right)\\
&= \sum_{j=1}^r (1\times b)u_{j,i}\cdot[(\theta_1)_j,1,\dots,1] - [(\theta_1)_i,1,\dots,1]
\end{aligned}$$

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$$\begin{aligned}
&= (1 \times b) \cdot [(\theta_1^{u_{1,i}}, \theta_1^{u_{2,i}}, \dots, \theta_1^{u_{r,i}}), 1, \dots, 1] - [(\theta_1)_i, 1, \dots, 1] \\
&= (1 \times b)[u(\theta_1)_i, 1, \dots, 1] - [(\theta_1)_i, 1, \dots, 1] \\
&= (F \times 1)(F^{-1} \times b)[u(\theta_1)_i, 1, \dots, 1] - [(\theta_1)_i, 1, \dots, 1] \\
&= (F \times 1)[u^{-1}u(\theta_1)_i, 1, \dots, 1] - [(\theta_1)_i, 1, \dots, 1] \\
&= ((F-1) \times 1) \circ \tilde{f}_3(e_i z_1). \qquad \blacksquare
\end{aligned}$$

We will need an explicit description of $X(2)$, which generalizes the one given in [BC17, Lemma 7.4.3].

Lemma 6.17 *We have*

$$X(2) = \langle (a-1)e_i z_1 - (u^m b - 1)e_i z_2, \mathcal{T}_a e_i z_2 \rangle_{\mathbb{Z}_p[G]}.$$

Proof The proof is just the r-dimensional analogue of that of [BC17, Lemma 7.4.3]. ■

We let

$$f_{3,W} \colon X(2) \oplus W \longrightarrow \mathcal{F}(\mathfrak{p}_N^{(r)})/\mathcal{F}((\mathfrak{p}_N^{p+1})^{(r)})$$

denote the composite of $\tilde{f}_{3,W}$ with the canonical projection. By Lemma 6.11, the homomorphism $f_{3,W}$ is surjective.

In the next proposition, which is the analogue of [BC17, Lemma 7.4.2], we will obtain an explicit representative for the complex $M^\bullet(\mathcal{L})$, which we will use to compute the Euler characteristic $\chi_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}), 0)$.

Proposition 6.18 *The complex*

$$F^\bullet := [\ker f_{3,W} \longrightarrow W' \oplus W \longrightarrow \mathbb{Z}_p[G]^r z_0]$$

with modules in degrees $0, 1$*, and* 2 *is a representative of* $M^\bullet(\mathcal{L})$ *for* $\mathcal{L} = \mathfrak{p}_N^{p+1}$.

Proof If we recall the definition of $M^\bullet(\mathcal{L})$ from (6.8), then it follows readily from Lemmas 6.11 and 6.16 that we have a quasi-isomorphism of complexes

$$F^\bullet \longrightarrow M^\bullet(\mathcal{L}). \qquad \blacksquare$$

For the computation of the Euler characteristic of $M^\bullet(\mathcal{L})$, we continue to closely follow the approach of [BC17, Section 7.4]. On these grounds, we will only sketch the proofs, pointing out the parts which are specific for the higher dimensional setting.

The next result is an analogue of [BC17, Lemma 7.4.5] and [BC17, Lemma 7.4.6]. Recall that by assumption $(m, d) = 1$, and let $\tilde{m}$ denote an integer such that

$$m\tilde{m} \equiv 1 \pmod{d}.$$

Lemma 6.19 *There exist* $y_{i,1} \in W_{\geq 1}$ *such that*

$$\begin{aligned}
t_{i,1} := &(a-1)e_i z_1 - (u^m b - 1)e_i z_2 + \\
&+ \left(\sum_{j=2}^{m} \alpha_j (u^m b)^{1-(j-2)\tilde{m}} + \left(\alpha_1 - \sum_{j=2}^{m} \alpha_j \right) (u^m b)^{\tilde{m}} \right) e_i w_0 + y_{i,1}
\end{aligned}$$

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and

$$t_{i,2} := \begin{cases} \mathcal{T}_a e_i z_2 - \alpha_1 e_i w_{p-1} & \text{if } m = 1 \\ \mathcal{T}_a e_i z_2 - \alpha_2 e_i w_{p-1} & \text{if } m > 1 \end{cases}$$

are in the kernel of $f_{3,W}$.

Proof Let $x_2 \in \mathcal{O}_{K^{\mathrm{nr}}}$ be such that $x_2/\alpha_1 \pmod{\mathfrak{p}_{K'}}$ is a root of $X^p - X + A\theta_2$, and let $(1 + x_2\theta_1)_i$ be the element in $\prod_r U_{N_0}^{(1)}$ whose ith component is $1 + x_2\theta_1$ and all the other components are 1. By the same proof as in [BC17, Lemma 7.2.2], we obtain

$$(1 + x_2\theta_1)_i^{\rho^{\mathrm{nr}}(F_N)F_N - 1} \equiv (1 - \alpha_1\theta_1)_i \pmod{\mathfrak{p}_{N_0}^2}.$$

Therefore, by [Cob18, Lemma 2.4], we may assume $\gamma_i \equiv (1 + x_2\theta_1)_i \pmod{\mathfrak{p}_{N_0}^2}$.

With this in mind, the proof of the lemma is the same as in [BC17, Lemma 7.4.5] and [BC17, Lemma 7.4.6]. ∎

In the case of Hypothesis (T), we need to redefine the elements $r_{i,1}$ in a different way:

Lemma 6.20 *The elements*

$$r_{i,1} = \mathcal{T}_a t_{i,1} + (u^m b - 1)t_{i,2}$$

belong to $\ker f_{3,W} \cap W$, *and their* $\alpha_1 w_0$*-components are* $(u^m b)^{\tilde{m}} \mathcal{T}_a e_i$.

Proof Straightforward by the same calculations as in [BC16, Lemma 4.2.5]. ∎

We also redefine the matrix $\mathcal{M}$ considered in the case of Hypothesis (I) using the elements $t_{i,2}$ instead of the elements $r_{i,1}$. For $m > 1$, we obtain:

$$\mathcal{M} = \left(\begin{array}{c|ccccc} \overbrace{}^{\mathcal{M}_1} & & & \overbrace{}^{\tilde{\mathcal{M}}} & & \\ 0 & 0 & \cdots & 0 & 0 & \varepsilon u^{-m\tilde{m}}\varepsilon^{-1}b^{-\tilde{m}} \\ -1 & -1 & \cdots & 0 & 0 & -\varepsilon u^{-m\tilde{m}}\varepsilon^{-1}b^{-\tilde{m}} \\ 0 & \varepsilon u^{-m\tilde{m}}\varepsilon^{-1}b^{-\tilde{m}} & \cdots & 0 & 0 & -\varepsilon u^{-m\tilde{m}}\varepsilon^{-1}b^{-\tilde{m}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \varepsilon u^{-m\tilde{m}}\varepsilon^{-1}b^{-\tilde{m}} & -1 & -\varepsilon u^{-m\tilde{m}}\varepsilon^{-1}b^{-\tilde{m}} \\ 0 & 0 & \cdots & 0 & \varepsilon u^{-m\tilde{m}}\varepsilon^{-1}b^{-\tilde{m}} & -1-\varepsilon u^{-m\tilde{m}}\varepsilon^{-1}b^{-\tilde{m}} \end{array} \right).$$

We call $\mathcal{M}_1$ the matrix consisting of the first r columns of $\mathcal{M}$, and $\tilde{\mathcal{M}}$ the matrix of the remaining columns.

For $m = 1$, the matrix $\mathcal{M}$ is determined only by the $t_{i,2}$, and, recalling their definition from Lemma 6.19, we get minus the identity matrix.

By an easy calculation, $\det(\mathcal{M}) = (-1)^{mr}(\det(u)^m b)^{-\tilde{m}(m-1)}$.

Lemma 6.21 *The* $r(pm+1)$ *elements* $t_{1,i}, t_{2,i}$, *for* $i = 1, \ldots, r$, $r_{i,k}$, *for* $i = 1, \ldots, r$, $k = 2, \ldots, m$, *and* $s_{i,j,k}$, *for* $i = 1, \ldots, r$, $j = 0, \ldots, p-2$, $k = 1, \ldots, m$, *constitute a* $\mathbb{Z}_p[G]$*-basis of* $\ker(f_{3,W})$.

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Proof It is enough to follow the proof of [BC17, Lemma 7.4.9]. Note that we can construct the matrix $\mathcal{M}$ as in [BC17] and that it also takes exactly the same shape up to the fact that all the entries are $r \times r$ blocks. ∎

By the same proof as in [BC17], we can now reduce the computation of the term $\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}),0)$ to the determinant of a matrix $(w,\mathfrak{M})$, which looks exactly as in [BC17], with the convention the elements in $\mathbb{Z}_p[G]$ must be thought as diagonal $r \times r$ matrices. So, in particular, $m \times m$ blocks become $mr \times mr$ blocks and so on. Of course, for the block $\mathcal{M}$, we have to take the one defined above and not that of [BC17]. With this in mind, the matrix looks as follows:

$$\begin{pmatrix}
\frac{\sum_{i=0}^{d-1}(u^m b)^i}{u^{dm}-1} & (a-1)I_r & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1-u^m b & \mathcal{T}_a I_r & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & v & 0 & \mathcal{T}_a \tilde{I} & (a-1)I_{rm} & 0 & \cdots & 0 & 0 \\
0 & * & 0 & 0 & -I_{rm} & (a-1)I_{rm} & \cdots & 0 & 0 \\
0 & * & 0 & 0 & * & -I_{rm} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & * & 0 & 0 & * & * & \cdots & -I_{rm} & (a-1)I_{rm} \\
0 & * & \mathcal{M}_1 & \tilde{\mathcal{M}} & * & * & \cdots & * & -I_{rm}
\end{pmatrix}.$$

Here, I_{rm} (resp. I_r) is the $rm \times rm$ (resp. $r \times r$) identity matrix, $\tilde{I}$ is obtained by I_{rm} by removing the first r columns, v is an $r \times rm$ matrix, whose first r rows coincide with the matrix $(u^m b)^{\tilde{m}}$, and $\frac{1}{u^{dm}-1}$ is the inverse of the matrix $u^{dm}-1$.

We are ready to record the result of the computation of the refined Euler characteristic of $M^\bullet(\mathcal{L})$.

Proposition 6.22 *We assume the hypotheses (F) and (T), and let $\mathcal{L} = \mathfrak{p}_N^{p+1}$. Then, the element $\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}),0)$ is contained in $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p[G])$ and represented by $\varepsilon \in \mathbb{Q}_p[G]^\times$ where*

$$\kappa_{\chi\phi} = \begin{cases} (-1)^r \frac{\det(u)^{m\tilde{m}}\phi(b)^{r\tilde{m}}p^{rm}}{\det(u^m\phi(b)-1)} & \text{if } \chi = \chi_0 \\ (-1)^{(m-1)r}(\det(u)^m\phi(b)^r)^{-\tilde{m}(m-1)}(\chi(a)-1)^{rm(p-1)} & \text{if } \chi \neq \chi_0 \end{cases}$$

and $\chi\phi$ is the decomposition explained before Proposition 6.15.

Proof The proof is a straightforward adaption of the computations in the proof of [BC17, Proposition 7.4.10]. ∎

7 Rationality and functoriality

From now on, we set $\hat{\partial}^1 = \hat{\partial}^1_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}$. As in [BC17], we consider the term

$$R_{N/K} = C_{N/K} + U_{\mathrm{cris}} + rm\hat{\partial}^1(t) - mU_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p}) - rU_{N/K} + \hat{\partial}^1(\varepsilon_D(N/K,V)),$$

which a priori lives in $K_0(\mathbb{Z}_p[G],B_{\mathrm{dR}}[G])$. The unramified term $U_{N/K}$ is defined by Breuning in [Bre04b, Proposition 2.12]. Recall that $R_{N/K}$ differs from [BC17, equation (17)], because we have to adapt the definition of $R_{N/K}$ as explained in Remark 6.6.

Proposition 7.1 *The element $R_{N/K}$ is rational, i.e., $R_{N/K} \in K_0(\mathbb{Z}_p[G],\mathbb{Q}_p[G])$.*

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Proof For elements $x, y \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p^c[G])$, we use the notation $x \equiv y$ when $x - y \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$. By Propositions 4.8, 5.1, and 6.5, we get

$$R_{N/K} \equiv r\hat{\partial}^1(\rho) - rU_{N/K} - rT_{N/K},$$

where $T_{N/K} := \hat{\partial}\left(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))_{\chi\in\mathrm{Irr}(G)}\right)$ is precisely the element defined by Breuning in [Bre04b, Section 2.3]. The result now follows, because this is exactly r times the element obtained in [BC17, Proposition 7.1.3]. ∎

Proposition 7.2 *Let L be an intermediate field of N/K and $H = \mathrm{Gal}(N/L)$. Let ρ_H^G and $q_{G/H}^G$ be the restriction and quotient maps from (4.2) and (4.3), respectively. Then,*

(1) $\rho_H^G(R_{N/K}) = R_{N/L}$.
(2) *If H is normal in G, then* $q_{G/H}^G(R_{N/K}) = R_{L/K}$.

Proof By definition of the cohomological term, we have

$$\begin{aligned} R_{N/K} = &-\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet, \exp_{V,N}\circ\mathrm{comp}_{V,N}^{-1}) + rm\hat{\partial}^1(t) - mU_{tw}(\rho_{\mathbb{Q}_p}^{\mathrm{nr}}) \\ &+ U_{\mathrm{cris},N/K} - rU_{N/K} + \hat{\partial}^1_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(\varepsilon_D(N/K,V)) \end{aligned}$$

with $M^\bullet = R\Gamma(N,T) \oplus \mathrm{Ind}_{N/\mathbb{Q}_p}T[0]$. The functoriality properties of $rm\hat{\partial}^1(t)$ and $mU_{tw}(\rho_{\mathbb{Q}_p}^{\mathrm{nr}})$ follow easily using the general formulas in [BW09, Sections 6.1 and 6.3]. Recalling also Lemmas 4.9 and 5.2 and [Bre04a, Lemma 4.5], it remains to show

(7.1)
$$\rho_H^G\left(\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet, \exp_{V,N}\circ\mathrm{comp}_{V,N}^{-1})\right) = \chi_{\mathbb{Z}_p[H],B_{\mathrm{dR}}[H]}(M^\bullet, \exp_{V,N}\circ\mathrm{comp}_{V,N}^{-1})$$

and

$$\begin{aligned} &q_{G/H}^G\left(\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet, \exp_{V,N}\circ\mathrm{comp}_{V,N}^{-1})\right) \\ &\quad = \chi_{\mathbb{Z}_p[G/H],B_{\mathrm{dR}}[G/H]}(M^\bullet, \exp_{V,L}\circ\mathrm{comp}_{V,L}^{-1}). \end{aligned} \tag{7.2}$$

The proof of (7.1) follows along the same line of argument as the proof of [Bre04a, Lemma 4.14(1)]. We just have to replace $A = \mu_{p^n}$ in loc. cit. by $\mathcal{F}[p^n]$.

Because (7.2) is essentially proved in the same way as part (2) of [Bre04b, Lemma 4.14], we only give a brief sketch. As in (17) of loc. cit., we obtain a canonical isomorphism

$$R\Gamma(L,T) \cong R\mathrm{Hom}_{\mathbb{Z}_p[H]}(\mathbb{Z}_p, R\Gamma(N,T)) \tag{7.3}$$

in the derived category. For each i, the induced map from $H^i(\mathbb{Q}_p^c \otimes R\Gamma(L,T)) \cong \mathbb{Q}_p^c \otimes H^i(L,T)$ to $H^i(\mathbb{Q}_p^c \otimes R\mathrm{Hom}_{\mathbb{Z}_p[H]}(\mathbb{Z}_p, R\Gamma(N,T))) \cong \mathbb{Q}_p^c \otimes H^i(N,T)^H$ is the map induced by the restriction $H^i(L,T) \longrightarrow H^i(N,T)$.

For $i = 1$, this restriction map is clearly given by the inclusion $\mathcal{F}(\mathfrak{p}_L^{(r)}) \subseteq \mathcal{F}(\mathfrak{p}_N^{(r)})$, and for $i = 2$, cohomology vanishes after tensoring with $\mathbb{Q}_p^c$ by our hypothesis (F).

We further note that we have a canonical isomorphism

$$\left(\mathrm{Ind}_{N/\mathbb{Q}_p}T\right)^H \cong \mathrm{Ind}_{L/\mathbb{Q}_p}T, \tag{7.4}$$

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which is given by $x \mapsto x$, if we identify $\mathrm{Ind}_{N/\mathbb{Q}_p} T$ with the set of maps $x\colon G_{\mathbb{Q}_p} \longrightarrow T$ satisfying $x(\tau\sigma) = \tau x(\sigma)$ for all $\tau \in G_N$ and $\sigma \in G_{\mathbb{Q}_p}$.

From (7.3) and (7.4), we derive a canonical isomorphism $M_L^\bullet \cong R\mathrm{Hom}_{\mathbb{Z}_p[H]}(\mathbb{Z}_p, M_N^\bullet)$, and it finally remains to show that the induced maps (drawn as dotted arrows below) in the following diagram coincide with $\mathrm{comp}_{V,L}^{-1}$ and $\exp_{V,L}$, respectively:

$$\begin{array}{ccccc}
\left(\mathrm{Ind}_{N/\mathbb{Q}_p}(V)_{B_{\mathrm{dR}}}\right)^H & \xrightarrow{\mathrm{comp}_{V,N}^{-1}} & \left(D_{\mathrm{dR}}^N(V)_{B_{\mathrm{dR}}}\right)^H & \xrightarrow{\exp_{V,N}} & \left(H^1(N,V)_{B_{\mathrm{dR}}}\right)^H = \mathcal{F}(\mathfrak{p}_N^{(r)})_{B_{\mathrm{dR}}}^H \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\mathrm{Ind}_{L/\mathbb{Q}_p}(V)_{B_{\mathrm{dR}}} & \overset{\mathrm{comp}_{V,L}^{-1}}{\dashrightarrow} & D_{\mathrm{dR}}^L(V)_{B_{\mathrm{dR}}} & \overset{\exp_{V,L}}{\dashrightarrow} & H^1(L,V)_{B_{\mathrm{dR}}} = \mathcal{F}(\mathfrak{p}_L^{(r)})_{B_{\mathrm{dR}}}
\end{array}$$

For $\exp_V$, this is immediate, because it is defined as a connecting homomorphism (see, for example, [BB08, p. 612]) and thus compatible with restriction. In addition, for comp_V, it follows from the definitions (see [BB08, p. 625]). ■

8 Proof of the main results

As in [BC17], we also define

$$\tilde{R}_{N/K} = C_{N/K} + U_{\mathrm{cris}} + rm\hat{\partial}^1(t) - mU_{tw}(\rho_{\mathbb{Q}_p}^{\mathrm{nr}}) + \hat{\partial}^1(\varepsilon_D(N/K,V)),$$

so that $R_{N/K} = \tilde{R}_{N/K} - rU_{N/K}$.

We now argue as in the proof of [BC17, Proposition 3.2.6] (see page 359 of loc. cit.). By Taylor's fixed point theorem together with [Bre04b, Proposition 2.12], it can be shown that $R_{N/K} = 0$ in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$ if and only if $\tilde{R}_{N/K} = 0$ in $K_0(\overline{\mathbb{Z}_p^{\mathrm{nr}}}[G], \overline{\mathbb{Q}_p^{\mathrm{nr}}}[G])$.

From Propositions 4.8, 5.1, and 6.5, we conclude that

$$\begin{aligned}
\tilde{R}_{N/K} =& r\hat{\partial}^1(\rho_\beta) + r[\mathcal{L}, \mathrm{id}, \mathcal{O}_K[G]\cdot\beta] - \chi(M^\bullet(\mathcal{L}), 0) \\
&+ \hat{\partial}^1({}^*(\det(1 - Fp^{-d_K}u^{-d_K})e_I)) - \hat{\partial}^1({}^*(\det(1 - u^{d_K}F^{-1})e_I)) \\
&+ \hat{\partial}^1\left(\det(u)^{-d_K(s_K\chi(1)+m_\chi)}\right)_{\chi\in\mathrm{Irr}(G)} - r\hat{\partial}^1\left(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))\right)_{\chi\in\mathrm{Irr}(G)}.
\end{aligned} \tag{8.1}$$

As in the one-dimensional case (see [BC17, Proposition 3.2.6]), the following three statements are equivalent in our present setting:

(a) $C_{EP}^{na}(N/K, V)$ is valid.
(b) $R_{N/K} = 0$ in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$.
(c) $\tilde{R}_{N/K} = 0$ in $K_0(\overline{\mathbb{Z}_p^{\mathrm{nr}}}[G], \overline{\mathbb{Q}_p^{\mathrm{nr}}}[G])$.

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Proof of Theorem 1.1 It suffices to show that $\tilde{R}_{N/K} = 0$. From (8.1) together with Proposition 6.7, we obtain

$$\begin{aligned}\tilde{R}_{N/K} &= r\hat{\partial}^1(\rho_\beta) + r[\mathfrak{p}_N, \mathrm{id}, \mathcal{O}_N] + \hat{\partial}^1({}^*(\det(1 - Fp^{-d_K}u^{-d_K})e_I)) \\ &\quad + \hat{\partial}^1\left(\det(u)^{-d_K(s_K\chi(1)+m_\chi)}\right)_{\chi\in\mathrm{Irr}(G)} - r\hat{\partial}^1\left(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))\right)_{\chi\in\mathrm{Irr}(G)}.\end{aligned}$$

The term

$$\hat{\partial}^1\left(\left(\det(u)^{-d_K(s_K\chi(1)+m_\chi)}\right)_\chi\right)$$

vanishes by [IV16, Lemma 6.2]. Because N/K is tame, the group ring idempotent e_I is contained in $\mathbb{Z}_p[G]$, and it is then easy to show that

$${}^*(\det(p^{d_K} - Fu^{-d_K})e_I) \in \mathbb{Z}_p[G]^\times.$$

We conclude that

$$\begin{aligned}\hat{\partial}^1({}^*(\det(1 - Fp^{-d_K}u^{-d_K})e_I)) &= -r\hat{\partial}^1({}^*(p^{d_K}e_I)) \\ &= -r[\mathfrak{p}_N, \mathrm{id}, \mathcal{O}_N],\end{aligned}$$

where the last equality follows from [IV16, equation (6.7)]. In conclusion, we have shown that

$$\tilde{R}_{N/K} = r\left(\hat{\partial}^1(\rho_\beta) - \hat{\partial}^1\left(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))\right)_{\chi\in\mathrm{Irr}(G)}\right).$$

We finally conclude as in the proof of [Bre04b, Theorem 3.6] or as in [IV16, p. 517] to show that

$$\hat{\partial}^1(\rho_\beta) - \hat{\partial}^1\left(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))\right)_{\chi\in\mathrm{Irr}(G)} = 0.$$

We recall that these proofs crucially use a fundamental result of M. Taylor (see [Frö83, Theorem 31]) which computes the quotient of norm resolvents and Galois Gauss sums. We also note that the so-called nonramified characteristic which occurs in these results is an integral unit itself. ∎

Proof of Theorem 1.3 The crucial input in our proof is a result of Picket and Vinatier in [PV13]. In [BC16, Section 5.1], we used this result to construct an integral normal basis generator $\beta = p^2\alpha_M\theta_2$ of $\mathcal{L} = \mathfrak{p}_N^{p+1}$. Hence, the expression in (8.1) simplifies to

$$\begin{aligned}\tilde{R}_{N/K} =& r\hat{\partial}^1(\rho_\beta) - \chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(M^\bullet(\mathcal{L}), 0) \\ &+ \hat{\partial}^1({}^*(\det(1 - Fp^{-d_K}u^{-d_K})e_I)) - \hat{\partial}^1({}^*(\det(1 - u^{d_K}F^{-1})e_I)) \\ &+ \hat{\partial}^1\left(\det(u)^{-d_K(s_K\chi(1)+m_\chi)}\right)_{\chi\in\mathrm{Irr}(G)} - r\hat{\partial}^1\left(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))\right)_{\chi\in\mathrm{Irr}(G)}.\end{aligned} \tag{8.2}$$

By [BC16, Proposition 5.2.1], the element $\hat{\partial}^1(\rho_\beta) - \hat{\partial}^1\left(\tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\chi))\right)_{\chi\in\mathrm{Irr}(G)}$ is represented by $\hat{\partial}^1(\eta^{-1})$ with η as in [BC16, Proposition. 5.2.1]. Furthermore, recalling that $m = d_K$ and $s_K = 0$,

$$\hat{\partial}^1\left(\det(u)^{-d_K(s_K\chi(1)+m_\chi)}\right)_{\chi\in\mathrm{Irr}(G)} = \hat{\partial}^1\left(\det(u)^{-mm_\chi}\right)_{\chi\in\mathrm{Irr}(G)}.$$

From now on, we have to distinguish the three conditions in the statement of the Theorem. Let us start with (a), i.e., Hypothesis (I). In this case, $\tilde{R}_{N/K}$ is represented by

$$\tilde{r}_{\chi\phi} = \begin{cases} \frac{\mathfrak{d}_K^{r/2}\mathcal{N}_{K/\mathbb{Q}_p}(\theta_2|\phi)^r p^{2rm}\det(1-p^{-m}u^{-m}\phi(b^{-1}))}{p^{rm}\det(1-u^m\phi(b))} & \text{if } \chi = \chi_0 \\ \frac{\mathfrak{d}_K^{r/2}\mathcal{N}_{K/\mathbb{Q}_p}(\theta_2|\phi)^r p^{rm}\chi(4)^r\phi(b)^{-2r}\det(u)^{-2m}}{(-1)^{r(m-1)}(u^{-m}\phi(b)^{-1}-1)(\chi(a)-1)^{rm(p-1)}} & \text{if } \chi \neq \chi_0 \end{cases}$$

$$= \begin{cases} (-1)^r\frac{\mathfrak{d}_K^{r/2}\mathcal{N}_{K/\mathbb{Q}_p}(\theta_2|\phi)^r}{\det(1-u^m\phi(b))}\det(u)^{-m}\phi(b)^{-r}\det(1-p^mu^m\phi(b)) & \text{if } \chi = \chi_0 \\ (-1)^r\frac{\mathfrak{d}_K^{r/2}\mathcal{N}_{K/\mathbb{Q}_p}(\theta_2|\phi)^r}{\det(1-u^m\phi(b))}\det(u)^{-m}\phi(b)^{-r}\left(\frac{-p}{(\chi(a)-1)^{p-1}}\right)^{rm}\chi(4)^r & \text{if } \chi \neq \chi_0. \end{cases}$$

Let $W_{\theta_2} \in \mathcal{O}_p^t[G] \subset \overline{\mathbb{Z}_p^{\mathrm{nr}}}[G]^\times$ be defined by

$$\chi\phi(W_{\theta_2}) = (-1)^r\mathfrak{d}_K^{r/2}\mathcal{N}_{K/\mathbb{Q}_p}(\theta_2|\phi)^r.$$

So

$$\tilde{r} = \frac{W_{\theta_2}\det(u)^{-m}b^{-r}}{\det(1-u^mb)}\left(\det(1-p^mu^mb)e_a + (-\tilde{u})^m\sigma_4(1-e_a)\right),$$

where $\tilde{u} \in \mathbb{Z}_p[a]$ is a unit whose augmentation is congruent to 1 modulo p, as in [BC17, Section 8]. Then, the proof works as in [BC17].

Let us now consider case (b), i.e., Hypothesis (T). In this case, $\tilde{R}_{N/K}$ is represented by

$$\tilde{r}_{\chi\phi} = \begin{cases} (-1)^r\frac{\mathfrak{d}_K^{r/2}\mathcal{N}_{K/\mathbb{Q}_p}(\theta_2|\phi)^r p^{2rm}\det(u^m\phi(b)-1)\det(1-p^{-m}u^{-m}\phi(b^{-1}))}{\det(u)^{m\tilde{m}}\phi(b)^{r\tilde{m}}p^{rm}\det(1-u^m\phi(b))} & \text{if } \chi = \chi_0 \\ (-1)^{r(m+1)}\frac{\mathfrak{d}_K^{r/2}\mathcal{N}_{K/\mathbb{Q}_p}(\theta_2|\phi)^r p^{rm}\chi(4)^r\phi(b)^{-2r}\det(u)^{-2m}}{(\det(u)^m\phi(b)^r)^{-\tilde{m}(m-1)}(\chi(a)-1)^{rm(p-1)}} & \text{if } \chi \neq \chi_0 \end{cases}$$

$$= \begin{cases} \chi\phi(W_{\theta_2})\phi(b)^{-r-r\tilde{m}}\det(u)^{-m-m\tilde{m}}\det(1-p^mu^m\phi(b)) & \text{if } \chi = \chi_0 \\ \chi\phi(W_{\theta_2})\phi(b)^{-r-r\tilde{m}}\det(u)^{-m-m\tilde{m}}\left(\frac{-p}{(\chi(a)-1)^{p-1}}\right)^{rm}\chi(4)\det(u)^{m^2\tilde{m}-m} & \text{if } \chi \neq \chi_0. \end{cases}$$

So

$$\tilde{r} = W_{\theta_2}b^{r-r\tilde{m}}\det(u)^{-m-m\tilde{m}}\left(\det(1-p^mu^mb)e_a + (-\tilde{u})^m\sigma_4\det(u)^{m^2\tilde{m}-m}(1-e_a)\right).$$

As in [BC17], Hypothesis (T) implies that $\det(u)^{m^2\tilde{m}-m} \equiv 1 \pmod{1-\zeta_p}$. Then, the proof works as in [BC17], using the computations of the case of Hypothesis (I).

It remains to consider the case (c). Let E/K' be the unramified extension of degree $\tilde{d}$. By the functoriality result of Proposition 7.2 (a), it is enough to show $R_{NE/K} = 0$, which is true, because, for NE/K, we can apply part (b) of the theorem. ∎

We finally prove Theorems 1.5 and 1.6. For the definition of the twist matrix of an abelian variety A with good ordinary reduction, we refer the reader to [LR78, p. 237]. We recall the following lemma.

Lemma 8.1 *Let A be an r-dimensional abelian variety defined over the p-adic number field L with good ordinary reduction, and let U be the twist matrix of A. Then,* $\det(U-1) \neq 0$.

Proof This is shown in [Maz72, Corollary 4.38] or in the proof of [LR78, Theorem 2]. ∎

Proof of Theorems 1.5 and 1.6 The results are immediate from Theorems 1.1 and 1.3. Note that Hypothesis (F) and the condition $\det(\rho^{\mathrm{nr}}(F_N)^{\tilde{d}} - 1) \neq 0$ are automatically satisfied by Lemma 8.1. ∎

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