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MINKOWSKI SYMMETRY SETS OF PLANE CURVES

GRAHAM MARK REEVE AND FARID TARI

Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Avenida Trabalhador São-carlense, 400 – Centro, CEP: 13566-590, São Carlos, São Paulo, Brazil (grahamreeve86@gmail.com; faridtari@icmc.usp.br)

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Abstract We study the Minkowski symmetry set of a closed smooth curve γ in the Minkowski plane. We answer the following question, which is analogous to one concerning curves in the Euclidean plane that was treated by Giblin and O'Shea (1990): given a point p on γ, does there exist a bi-tangent pseudo-circle that is tangent to γ both at p and at some other point q on γ? The answer is yes, but as pseudo-circles with non-zero radii have two branches (connected components) it is possible to refine the above question to the following one: given a point p on γ, does there exist a branch of a pseudo-circle that is tangent to γ both at p and at some other point q on γ? This question is motivated by the earlier quest of Reeve and Tari (2014) to define the Minkowski Blum medial axis, a counterpart of the Blum medial axis of curves in the Euclidean plane.

Keywords: curves; Minkowski plane; symmetry sets; evolutes; caustics; singularities

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Secondary 53B30; 57R45

1. Introduction

The symmetry set of a submanifold M in the Euclidean n-space is the locus of centres of hyperspheres tangent to M in at least two distinct points. The symmetry sets (and some of their subsets) of plane curves and of surfaces of the Euclidean 3-space are well studied and have applications in computer vision and shape recognition (see, for example, [**14**]).

We consider in this paper the Minkowski symmetry set (MSS) of a closed smooth curve γ in the Minkowski plane, which is defined as the locus of centres of pseudo-circles bi-tangent to γ at at least two distinct points. There are three types of pseudo-circles in the Minkowski plane, and points on γ can be space-like, time-like or light-like (see § 2). Also, the pseudo-circles with non-zero radii have two branches. All of these possibilities make the MSS richer than its Euclidean counterpart.

For closed curves in the Euclidean plane, there is a subset of the symmetry set that is of particular interest called the Blum medial axis (see [**3**]). This is defined to be the locus of centres of bi-tangent circles that are completely contained in γ. The Blum medial axis has the property that it can be used to reconstruct the curve γ (see, for example, [**14**]). A Minkowski analogue of the Blum medial axis is defined in [**12**]. The Minkowski Blum

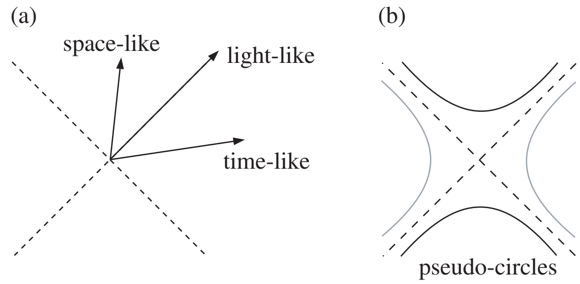

Figure 1. (a) The three types of vectors. (b) Pseudo-circles in $\mathbb{R}^2_1$.

medial axis is the locus of centres of bi-tangent pseudo-circles to γ with the points of tangency lying in just one branch of the pseudo-circles. We call these pseudo-circles 1-branch bi-tangent pseudo-circles.

We answer positively in Theorem 4.1 the following question: given a point p on γ, does there exist a bi-tangent pseudo-circle that is tangent to γ at p and at some other point q? We also consider the same question but restrict ourselves to 1-branch bi-tangent pseudo-circles. We show in Theorem 4.8 that, under some restrictions, given a point p on a space-like or time-like component C of γ, there is a 1-branch bi-tangent pseudo-circle tangent to γ at p and at another point $q \in C$.

Some brief preliminaries are given in § 2. In § 3 the properties of the MSS and of the caustic at finitely determined singularities of the distance-squared function are studied. The results in § 3 are used in § 4 for answering the existence of bi-tangent pseudo-circles questions.

2. Preliminaries

The *Minkowski plane* $(\mathbb{R}^2_1, \langle\cdot,\cdot\rangle)$ is the vector space $\mathbb{R}^2$ endowed with the pseudo-scalar product $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = -u_0v_0 + u_1v_1$ for any $\boldsymbol{u} = (u_0, u_1)$ and $\boldsymbol{v} = (v_0, v_1)$. A vector $\boldsymbol{u} \in \mathbb{R}^2_1$ is called

- *space-like* if $\langle \boldsymbol{u}, \boldsymbol{u}\rangle > 0$,
- *time-like* if $\langle \boldsymbol{u}, \boldsymbol{u}\rangle < 0$,
- *light-like* if $\langle \boldsymbol{u}, \boldsymbol{u}\rangle = 0$.

The norm of $\boldsymbol{u}$ is defined by $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u}\rangle|}$. Throughout the paper, we refer to a pseudo-unit vector $\boldsymbol{u}$, with $\|\boldsymbol{u}\| = 1$, as a unit vector.

The pseudo-circles in $\mathbb{R}^2_1$ with centre $c \in \mathbb{R}^2_1$ and radius $r > 0$ are defined as follows:

$$H^1(c, -r) = \{p \in \mathbb{R}^2_1 \mid \langle p - c, p - c\rangle = -r^2\},$$
$$S^1_1(c, r) = \{p \in \mathbb{R}^2_1 \mid \langle p - c, p - c\rangle = r^2\},$$
$$\mathrm{LC}^*(c) = \{p \in \mathbb{R}^2_1 \setminus 0 \mid \langle p - c, p - c\rangle = 0\}.$$

Observe that $\mathrm{LC}^*(c)$ is the union of the two lines through c with tangent directions $(1, 1)$ and $(1, -1)$, with the point c removed. The pseudo-circle $H^1(c, -r)$ has

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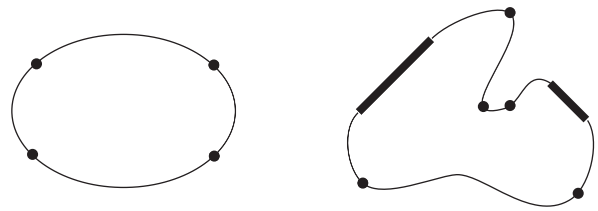

Figure 2. Light-like points as dots or thick lines on a smooth closed curve in $\mathbb{R}^2_1$.

two branches, which can be parametrized by $c + (\pm r\cosh(t), r\sinh(t))$, $t \in \mathbb{R}$. The pseudo-circle $S^1(c,r)$ also has two branches, which can be parametrized by $c + (r\sinh(t), \pm r\cosh(t))$, $t \in \mathbb{R}$. See Figure 1.

Let $\gamma\colon S^1 \to \mathbb{R}^2_1$ be a smooth (C^∞) immersion, where S^1 is the Euclidean unit circle. We call the curve γ the image of the map γ and say that it is a closed smooth curve.

The curve γ at t_0 is said to be space-like if $\gamma'(t_0)$ is space-like and it is said to be time-like if $\gamma'(t_0)$ is time-like. These are open properties, so there is a neighbourhood of t_0 where the curve is either space-like or time-like. If $\gamma'(t_0)$ is light-like, then $\gamma(t_0)$ is said to be a light-like point. It is shown in [**13**, Proposition 2.1] that the set of light-like points of γ is the union of at least four disjoint non-empty and closed subsets of γ (Figure 2). The complement of these sets are disjoint connected space-like or time-like pieces of the curve γ.

We call the restriction of γ to an open interval (λ,μ) of S^1, where λ and μ correspond to light-like points of γ, a *space-like component* if γ is space-like in (λ,μ), and a *time-like component* if γ is time-like in (λ,μ). Thus, (λ,μ) is a maximal interval in which γ is space-like or time-like.

The space-like and time-like components of γ can be parametrized by arc length. Suppose that $\gamma(s)$, $s \in (\lambda,\mu)$, is an arc-length parametrization of a component of γ. Then $\boldsymbol{t}(s) = \gamma'(s)$ is a unit tangent vector and $\boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s)$, where $\kappa(s)$ is the Minkowski curvature of γ at s and $\boldsymbol{n}$ is the unit Minkowski normal vector at s. The tangent and unit Minkowski normal vectors are pseudo-orthogonal so they are of different types, that is, one is space-like and the other is time-like or vice versa.

The evolute of a space-like or time-like component of γ is the image of the map

$$e(s) = \gamma(s) - \frac{1}{\kappa(s)}\boldsymbol{n}(s).$$

Observe that the evolute goes to infinity at inflection points (i.e. points at which the curvature κ vanishes), so we shall exclude such points when analysing the evolute of a space-like or time-like component of γ.

In general, the curvature tends to infinity as s tends to λ or μ, and the evolute of the curve γ is not defined at the light-like points. However, the caustic of γ is defined everywhere and contains the evolute of γ [**13**]. The caustic can be defined via the *family of distance-squared functions* $f\colon S^1 \times \mathbb{R}^2_1 \to \mathbb{R}$ on γ given by

$$f(t,c) = \langle \gamma(t) - c, \gamma(t) - c\rangle.$$

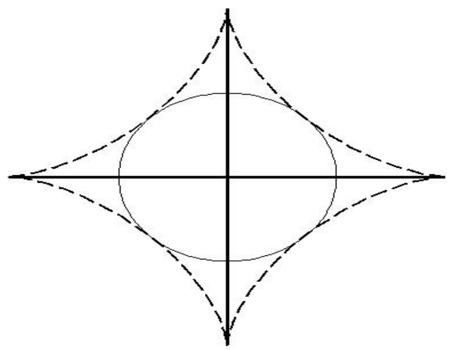

Figure 3. The caustic of an ellipse (dashed) in $\mathbb{R}^2_1$: the points of tangency of the caustic with the ellipse are the light-like points of the ellipse. The thick lines are the Minkowski symmetry set of the ellipse.

Denote by $f_c\colon S^1 \to \mathbb{R}$ the function given by $f_c(t) = f(t, c)$. We say that f_c has an A_k-singularity at t_0 if $f'_c(t_0) = f''_c(t_0) = \cdots = f_c^{(k)}(t_0) = 0$ and $f_c^{(k+1)}(t_0) \neq 0$. This is equivalent to the existence of a local re-parametrization h of γ at t_0 such that $(f_c \circ h)(t) = \pm(t - t_0)^{k+1}$.

Geometrically, f_c has an A_k-singularity if and only if the curve γ has contact of order $k+1$ at $\gamma(t_0)$ with the pseudo-circle of centre c and radius $r = \|\gamma(t_0) - c\|$. Denote this pseudo-circle by $C(c, r)$. Thus, the curve γ has order of contact 1 with a pseudo-circle at t_0 if it intersects transversally the pseudo-circle at $\gamma(t_0)$. The order of contact is 2 if the circle and the curve have ordinary tangency at $\gamma(t_0)$.

The caustic of γ is the local component $\mathcal{B}_1$ of the bifurcation set of the family f, given by

$$\mathcal{B}_1 = \{c \in \mathbb{R}^2_1 \mid \exists t \in S^1 \text{ such that } f'_c(t) = f''_c(t) = 0\}.$$

This is the set of points $c \in \mathbb{R}^2_1$ such that the germ f_c has a degenerate singularity at some point t. The caustic of γ is defined at all points on γ including its light-like points. The caustic of a generic curve γ is a smooth curve at the light-like points of γ and has ordinary tangency with γ at such points (see [**13**] and Figure 3).

3. The Minkowski symmetry set

The multi-local component of the bifurcation set of the family f is defined as

$$\mathcal{B}_2 = \{c \in \mathbb{R}^2_1 \mid \exists t_1,\ t_2 \text{ such that } t_1 \neq t_2,\ f_c(t_1) = f_c(t_2),\ f'_c(t_1) = f'_c(t_2) = 0\}.$$

The full-bifurcation set of f is defined as

$$\operatorname{Bif}(f) = \mathcal{B}_1 \cup \mathcal{B}_2.$$

Definition 3.1. The MMS of γ is the locus of centres of pseudo-circles that are

(1) bi-tangent to γ at at least two distinct points p and q, then the pair p, q is called a bi-tangent pair; or

(2) tangent to γ at a single point with contact of order at least 4.

Table 1. *The geometric characterization of the local singularity of f_c at a non-light-like point.*

the singularity of f_c	the geometric conditions
A_0	γ transversally intersects $C(c,r)$ at $\gamma(t_0)$
A_1	c is on the normal line to γ at $\gamma(t_0)$ but not on its evolute
A_2	c is on the evolute of γ and $\kappa'(t_0) \neq 0$
A_3	c is on the evolute of γ, $\kappa'(t_0) = 0$, $\kappa''(t_0) \neq 0$

It is clear that the MSS is precisely the closure of the multi-local component $\mathcal{B}_2$ of the bifurcation set of the family of distance-squared function f on γ. (In Definition 3.1, we allow limit points of the MSS at which the bi-tangent points coincide to belong to the MSS.)

It follows from Thom's transversality theorem (see, for example, [**4**, **9**]) that for an open and dense set of immersions $\gamma\colon S^1 \to \mathbb{R}^2_1$ the function f_c has only local singularities of type A_1, A_2, A_3 and multi-local singularities of type A_1^2, A_1A_2, A_1^3. The geometric characterization of the local singularity of f_c at a non-light-like point is as in Table 1. At a light-like point, the function f_c has generically an A_1-singularity [**13**] and the curve and the caustic are on different sides of their common tangent line (see Figure 3).

An application of Thom's transversality theorem (see, for example, [**5**, Theorem 1]) also asserts that, for generic curves, the family f is a versal unfolding of the generic singularities of f_c (see, for example, [**5**, Appendix] for more details on versal unfoldings). In fact, we have the following result (the proof is technical and is included for completeness; the importance of the result is given in Corollary 3.3).

Theorem 3.2. *The family f is always a versal unfolding of the generic singularities A_1, A_2, A_3, A_1^2, A_1A_2 and A_1^3 of f_{c_0}.*

Proof. The A_1-singularity is stable and so is automatically versally unfolded by f. For the local A_k-singularity, $k = 2, 3$, of f_{c_0}, which we shall assume to be at $t = 0$, we use the following criterion for showing that f is a versal unfolding of f_{c_0} (see [**4**, 6.10p]). Write $c_0 = (a_0, b_0)$, $\gamma(t) = (x(t), y(t))$ and

$$j^{k-1}\frac{\partial f}{\partial a}(0, c_0) = \alpha_{1,1}t + \alpha_{1,2}t^2 + \cdots + \alpha_{1,k-1}t^{k-1},$$
$$j^{k-1}\frac{\partial f}{\partial b}(0, c_0) = \alpha_{2,1}t + \alpha_{2,2}t^2 + \cdots + \alpha_{2,k-1}t^{k-1},$$

where $j^l g$ denotes the Taylor polynomial of g of degree l, at $t = 0$, without the constant term. Then f is a versal unfolding of the A_k-singularity of f_{c_0} if and only if the matrix

$$J_{A_k} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,k-1} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,k-1} \end{pmatrix}$$

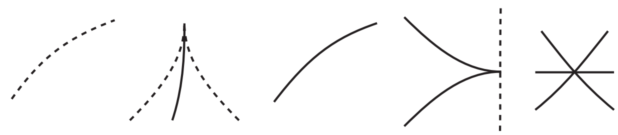

Figure 4. Generic models of the evolute (dashed line) and of the MSS (solid line).

has rank $k-1$. We have

$$J_{A_2} = \begin{pmatrix} -2x'(0) \\ 2y'(0) \end{pmatrix},$$

which has rank 1 as the curve γ is regular, and

$$J_{A_3} = \begin{pmatrix} -2x'(0) & -x''(0) \\ 2y'(0) & y''(0) \end{pmatrix},$$

which has rank 2 as the curvature of γ is assumed to be non-zero. (As observed above, the function f_{c_0} has generically only an A_1-singularity at a light-like point, so the A_3-singularities occur generically on a time-like or a space-like component of γ.)

For the multi-local singularities, we use the criterion in [**5**, Appendix]. For the A_1^2-singularity, say at t_1 and t_2, the family f is a versal unfolding of f_{c_0} if and only if

$$\begin{pmatrix} x(t_2) - x(t_1) \\ y(t_2) - y(t_1) \end{pmatrix}$$

has maximal rank, which it has as the two points $\gamma(t_1)$ and $\gamma(t_2)$ are distinct. For the A_1A_2-singularity, we require

$$\begin{pmatrix} x(t_2) - x(t_1) & x'(t_2) \\ y(t_2) - y(t_1) & y'(t_2) \end{pmatrix}$$

to have maximal rank. This happens if and only if the tangent direction to γ at the A_2-singularity t_2 is not parallel to $\gamma(t_1) - \gamma(t_2)$, or, alternatively, when the tangent line to γ at t_2 does not pass through $\gamma(t_1)$. But this is always true as $\gamma(t_1)$ and $\gamma(t_2)$ also belong to a pseudo-circle. Therefore, f is always a versal unfolding of the A_1A_2-singularity of f_{c_0}.

For the A_1^3-singularity, say at t_1, t_2 and t_3, f is a versal unfolding if and only if

$$\begin{pmatrix} x(t_2) - x(t_1) & x(t_3) - x(t_1) \\ y(t_2) - y(t_1) & x(t_3) - x(t_1) \end{pmatrix}$$

has maximal rank. This is always the case as the three points $\gamma(t_1)$, $\gamma(t_2)$ and $\gamma(t_3)$ are also on a pseudo-circle, and so cannot be collinear. □

Corollary 3.3. *The evolute and the MSS of a generic curve γ in the Minkowski plane are diffeomorphic to one of the models in Figure 4.*

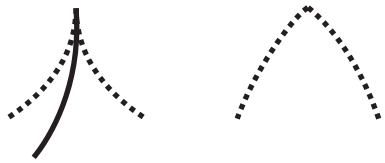

Figure 5. Models of the evolute (dashed line) and MSS (solid line) at higher vertices.

Proof. The proof follows from the fact that two versal families of the same singularity type and with the same number of parameters have diffeomorphic bifurcation sets. The bifurcation sets in Figure 4 are those of the model versal families of the singularities in Theorem 3.2. □

The function f_c, with c the intersection of two light-like tangent lines of γ at $\gamma(t_1)$ and $\gamma(t_2)$, has generically an A_1^2-singularity at these points. The MSS is a smooth curve at c and has tangent direction along $\gamma(t_1) + \gamma(t_2)$ at that point [**13**].

We consider now the structure of the caustic and of the MSS at a non-generic singularity of f_c. We start with the case in which the singular point is not light-like, so the caustic coincides with the evolute.

Theorem 3.4. *Let $p = \gamma(t_0)$ be a non-light-like point and suppose that f_c has an A_k-singularity at t_0, with $k \geqslant 3$.*

(i) *The evolute can be parametrized locally in a suitable system of coordinates in the form $(t^{k-1}, t^k\eta(t))$, with $\eta(0) \neq 0$. See Figure 5.*

(ii) *The MSS is locally empty if k is even, and consists of a smooth branch ending at the singular point of the evolute if k is odd. When k is odd, the branch of the MSS can be parametrized in a suitable system of coordinates in the form $(t^{k-1}, t^{k+1}\xi(t))$, with $\xi(0) \neq 0$. See Figure 5.*

Proof. (i) Take γ parametrized by arc length so that its evolute can be parametrized by

$$e(t) = \gamma(t) - \frac{1}{\kappa(t)}\boldsymbol{n}(t),$$

where $\boldsymbol{n}(t)$ is the unit normal vector such that $(\boldsymbol{t}(t), \boldsymbol{n}(t))$ form a positive basis. One can show by induction that the function $f_{e(t_0)}$ has an A_k-singularity at t_0, with $k \geqslant 3$, if and only if $\kappa'(t_0) = \kappa''(t_0) = \cdots = \kappa^{(k-2)}(t_0) = 0$ and $\kappa^{(k-1)}(t_0) \neq 0$ (we call such a point a $(k-2)$-vertex). Then the result follows from the fact that $e(t)$ can be written, in the coordinate system with origin the $(k-2)$-vertex $e(t_0)$ and basis $(\boldsymbol{t}(t_0), \boldsymbol{n}(t_0))$, in the form

$$\left(\frac{\kappa^{(k-1)}(t_0)}{\kappa(t_0)}(t-t_0)^k + \cdots, \frac{\kappa^{(k-1)}(t_0)}{\kappa(t_0)^2}(t-t_0)^{k-1} + \cdots\right).$$

In particular, the singularity of the defining equation of the evolute is of type A_2 (i.e. it is equivalent by smooth changes of coordinates in the source to $x^2 + y^3 = 0$) if $k = 3$,

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of type E_6 (i.e. equivalent to $x^3+y^4=0$) if $k=4$, and has a non-simple singularity if $k>4$ (see [**1**] for terminology).

(ii) Suppose that γ is time-like (the space-like case follows in a similar way). We make a Lorentz transformation in the target and a reparametrization in the source so that γ is written locally at $t_0=0$ in the form $\gamma(t)=(t,\beta(t))$, with

$$\beta(t)=\beta_2t^2+\beta_3t^3+\cdots+\beta_kt^k+\text{h.o.t.},$$

where 'h.o.t.' indicates higher-order terms.

Now, $\beta_2\neq 0$ as $\kappa(t_0)\neq 0$. The centre of the pseudo-circle with degenerate contact with γ at the origin is $(0,-1/(2\beta_2))$. We take the nearby centres in the form $c=(a,b-1/(2\beta_2))$, with a, b near zero, and consider the germ of the family of distance-squared functions $f\colon\mathbb{R}\times\mathbb{R}^2,(0,0)\to\mathbb{R}$ given by

$$\begin{aligned}f(t,(a,b))&=-(t-a)^2+\left(\beta(t)-\left(b-\frac{1}{2\beta_2}\right)\right)^2\\&=-a^2+\left(b-\frac{1}{2\beta_2}\right)^2+2at-2b\beta(t)+\phi(t)\end{aligned}$$

with $\phi(t)=-t^2+\beta(t)^2+\beta(t)/\beta_2$. We have $\phi(t)=f_0(t)-1/(4\beta_2^2)$, so

$$\phi(t)=\phi_{k+1}t^{k+1}+\text{h.o.t.},$$

with $\phi_{k+1}\neq 0$ as f_0 is assumed to have an A_k-singularity at $t=0$.

We analyse the existence of bi-tangent pairs (t_1,t_2) near $(0,0)$. The pair (t_1,t_2) is a bi-tangent pair if there exist (a,b) such that

$$\left.\begin{array}{c}\dfrac{\partial f}{\partial t}(t_1,(a,b))=0,\qquad \dfrac{\partial f}{\partial t}(t_2,(a,b))=0,\\ f(t_1,(a,b))=f(t_2,(a,b)).\end{array}\right\}\tag{3.1}$$

Equivalently,

$$\begin{aligned}a-b\beta'(t_1)+\tfrac{1}{2}\phi'(t_1)&=0,\\a-b\beta'(t_2)+\tfrac{1}{2}\phi'(t_2)&=0,\\a-b\frac{\beta(t_1)-\beta(t_2)}{t_1-t_2}+\frac{\phi(t_1)-\phi(t_2)}{2(t_1-t_2)}&=0.\end{aligned}$$

Solving the first two equations in a and b gives

$$\begin{aligned}a&=\frac{\beta'(t_2)\phi'(t_1)-\beta'(t_1)\phi'(t_2)}{2(\beta'(t_1)-\beta'(t_2))},\\b&=\frac{\phi'(t_1)-\phi'(t_2)}{2(\beta'(t_1)-\beta'(t_2))}.\end{aligned}$$

Substituting into the third equation gives an expression of the form

$$\frac{g(t_1,t_2)}{2(t_1-t_2)(\beta'(t_1)-\beta'(t_2))}=0, \tag{3.2}$$

with

$$g(t_1,t_2)=(t_1-t_2)(\beta'(t_2)\phi'(t_1)-\beta'(t_1)\phi'(t_2))-(\beta(t_1)-\beta(t_2))(\phi'(t_1)-\phi'(t_2)) \\ +(\beta'(t_1)-\beta'(t_2))(\phi(t_1)-\phi(t_2)).$$

It can be shown, by computing the successive partial derivatives of g, that

$$g(t_1,t_1)=\frac{\partial g}{\partial t_1}(t_1,t_1)=\frac{\partial^2 g}{\partial t_1^2}(t_1,t_1)=\frac{\partial^3 g}{\partial t_1^3}(t_1,t_1)=0$$

and

$$\frac{\partial^4 g}{\partial t_1^4}(t_1,t_1)=4k(k^2-1)(k-2)\beta_2\phi_{k+1}t_1^{k-2}+\text{h.o.t.}$$

Therefore, we can write $g(t_1,t_2)=(t_1-t_2)^4\tilde{g}(t_1,t_2)$ for some germ of a smooth function $\tilde{g}$ that does not vanish at $t_1=t_2$ when $t_1\neq 0$. Similarly, the denominator in (3.2) can be written in the form $(t_1-t_2)^2\tilde{\beta}(t_1,t_2)$, with $\tilde{\beta}$ non-vanishing at $t_1=t_2$ when $t_1\neq 0$ as $\beta_2\neq 0$.

The points on the diagonal $t_1=t_2$ do not give bi-tangent pairs, so the bi-tangent pairs are given by the zero set of $\tilde{g}$. We now determine the structure of the zero set of $\tilde{g}$.

Write the $(k+2)$-jet of g in the form

$$j^{k+2}g(t_1,t_2)=\beta_2\phi_{k+1}(t_1-t_2)P_k(t_1,t_2),$$

with P_k the homogeneous polynomial

$$P_k(t_1,t_2)=(1-k)t_1^{k+1}+(1+k)t_1^k t_2-(1+k)t_1t_2^k-(1-k)t_2^{k+1}.$$

We claim that $t_1=t_2$ is the only repeated factor of P_k. Indeed, suppose that $t_1=\alpha t_2$ is a repeated factor of P_k. Then $P_k(\alpha t_2,t_2)=\partial P_k/\partial t_1(\alpha t_2,t_2)=0$, that is,

$$(1-k)\alpha^{k+1}+(1+k)\alpha^k-(1+k)\alpha-(1-k)=0, \tag{3.3}$$

$$(1-k)\alpha^k+k\alpha^{k-1}-1=0. \tag{3.4}$$

The linear combination $[-(k+(1-k)\alpha)](3.3)+[(1-k)\alpha^2+(1+k)\alpha](3.4)$ of (3.3) and (3.4) gives

$$k(1-k)(\alpha-1)^2=0,$$

so $\alpha=1$.

Therefore, $j^{k-2}\tilde{g}$ is a homogeneous polynomial with no repeated (complex or real) factors. It follows that the ideal generated by the partial derivatives of $\tilde{g}$ is Newton non-degenerate (see [**2**, **10**] for terminology), and so has finite codimension. This implies that there exists a germ of a diffeomorphism in the source such that $\tilde{g}\circ h=j^l\tilde{g}$ for some

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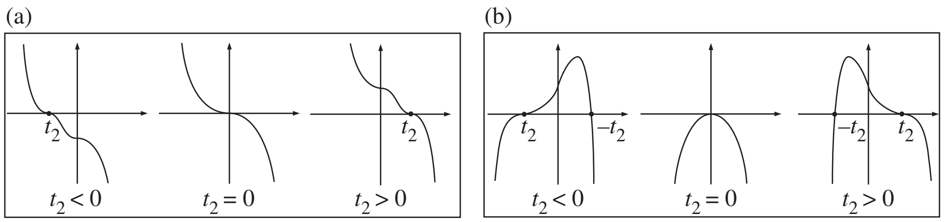

Figure 6. Graphs of the function $f(t_1) = P_k(t_1, t_2)$ for t_2 fixed when (a) k is even and (b) k is odd.

$l \geqslant k-2$. Thus, the zero set of $\tilde{g}$ is diffeomorphic to that of $j^l\tilde{g}$ (which is an algebraic curve). The number of branches, at the origin, of $j^l\tilde{g} = 0$ is determined by its tangent cone, i.e. by the number of solutions of $j^{k-2}\tilde{g} = 0$, and as these are all simple, the branches are all smooth curves. We now determine the number of these curves.

It is clear that $t_1 + t_2$ is a factor of P_k if and only if k is odd, so using the fact that $(t_1 - t_2)^4$ is a factor of g, we have

$$P_k(t_1, t_2) = \begin{cases} (t_1 - t_2)^3 (t_1 + t_2) J_k(t_1, t_2) & \text{if } k \text{ is odd}, \\ (t_1 - t_2)^3 L_k(t_1, t_2) & \text{if } k \text{ is even}, \end{cases}$$

where $J_k(t_1, t_2)$ and $L_k(t_1, t_2)$ are homogeneous polynomials with no repeated factors.

To show that L_k and J_k have no real factors, we consider the function $f(t_1) = P_k(t_1, t_2)$ with t_2 fixed and draw its graph. Elementary calculations show that these graphs are as in Figure 6, where the inflections at $t_1 = t_2$ are ordinary ones. Therefore, L_k and J_k vanish only at $(0, 0)$, i.e. they have no real factors. Therefore, there exists one smooth branch of $\tilde{g} = 0$ if k is odd, and none if k is even. When k is odd, the solution of $\tilde{g} = 0$ has initial term $t_2 = -t_1 + \text{h.o.t.}$, so the MSS has a parametrization of the form

$$(a, b) = \left(-\frac{(k+2)\phi_{k+2}}{2} t_1^{k+1} + \text{h.o.t.}, \frac{(k+1)\phi_{k+1}}{4\beta_2} t_1^{k-1} + \text{h.o.t.}\right).$$

□

Remark 3.5. When $k = 2l$, the polynomial L_k in the proof of Theorem 3.4 is in fact the following self-reciprocal polynomial of degree $2l - 2$:

$$L_{2l} = -(a_0 t_1^{2l-2} + \cdots + a_{l-2} t_1^l t_2^{l-2} + a_{l-1} t_1^{l-1} t_2^{l-1} + a_{l-2} t_1^{l-2} t_2^{l+1} + \cdots + a_0 t_2^{2l-2})$$

with

$$a_i = l^2 - (l - 1 - i)^2, \quad i = 0, \ldots, l-1.$$

When $k = 2l + 1$, the polynomial J_{2l+1} in the proof of Theorem 3.4 is also a self-reciprocal polynomial of degree $2l - 2$, given by

$$J_{2l+1} = b_0 t_1^{2l-2} + \cdots + b_{l-2} t_1^l t_2^{l-2} + b_{l-1} t_1^{l-1} t_2^{l-1} + b_{l-2} t_1^{l-2} t_2^{l+1} + \cdots + b_0 t_2^{2l-2}$$

with

$$b_0 = l, \quad b_1 = l-1, \quad b_2 = 2(l-1), \quad b_3 = 2(l-2),$$
$$b_i = 2b_{i-1} - 2b_{i-2} + b_{i-3}, \quad i = 4, \dots, l-1.$$

The proof of Theorem 3.4 shows that L_{2l} and J_{2l+1} have no real roots (see, for example, [**11**] on the zeros of self-reciprocal polynomials). In fact, the polynomials L_{2l} and J_{2l+1} are self-reciprocal because $j^{k+2}g$ is a self-reciprocal polynomial. Indeed, if (t_1, t_2) is a bi-tangent pair, so is (t_2, t_1).

Consider now the case in which the distance-squared function has a degenerate singularity at a light-like point.

Theorem 3.6. *Suppose that $p = \gamma(t_0)$ is a light-like point and that f_{c_0} has an A_k-singularity at p.*

(i) *If $k \geqslant 2$, the caustic of γ is the union of the light-like line tangent to γ at p and a smooth curve that has $(k+1)$-order of contact with γ at p.*

(ii) *The MSS is empty if k is odd, and it consists of two smooth curves tangent to γ at p if k is even.*

Proof. (i) Take p to be the origin and the tangent line to γ at p to be parallel to the direction $(1,1)$; the case in which the tangent line is parallel to $(1,-1)$ follows similarly. The curve can then be parametrized locally in the form $\gamma(t) = (t, t+\beta(t))$, where β is a germ at 0 of a smooth function with $\beta(0) = \beta'(0) = 0$. We write $c = (a,b)$ so that $f(t,c) = -(t-a)^2 + (t+\beta(t)-b)^2$. By hypothesis, the origin is an A_k-singularity of some distance-squared function f_{c_0} with $c_0 = (a_0, b_0)$ and $b_0 = a_0$. Thus, $\beta(t) = \beta_{k+1}t^{k+1} + \text{h.o.t.}$, with $\beta_{k+1} \neq 0$.

The centre c belongs to the caustic of γ if and only if there exists t such that $(\partial f/\partial t)(t,c) = (\partial^2 f/\partial t^2)(t,c) = 0$. Solving this system for a and b gives, for $t \neq 0$,

$$b = b(t) = (2\beta'(t) + t\beta''(t) + \beta''(t)\beta(t) + \beta'(t)^2)/\beta''(t) = (2/k)t + h(t),$$
$$a = a(t) = b(t) + b(t)\beta'(t) - \beta(t) - t\beta'(t) - \beta'(t)\beta(t)$$

for some germ of a smooth function h with $h(0) = h'(0) = 0$.

Therefore, the caustic contains the germ, at the light-like point p, of the smooth curve C parametrized by $c(t) = (a(t), b(t))$. If $\beta''(0) \neq 0$, the smooth curve C is the entire caustic. When $\beta''(0) = 0$, the light-like line $a - b = 0$ is a solution of the system of equations giving the caustic for $t = 0$, and so is also part of the caustic. (The determination of the order of contact between γ and its caustic follows by standard calculations.)

(ii) We proceed as in the proof of Theorem 3.4. The pair (t_1, t_2), with $t_1 \neq t_2$, is a bi-tangent pair if there exists $c = (a,b)$ for which the system of three equations (3.1) is satisfied, where f is as in case (i). We make a change of variables in the parameters plane

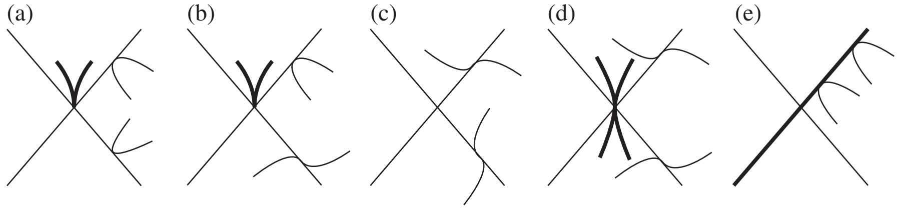

Figure 7. The MSS (thick line), the two pieces of γ (thin line) and the two light-like tangent lines: (a), (b) k and l are not both even; (c), (d) k and l are both even; (e) the bi-tangency is with one branch of the lightcone.

and set $u = a - b$ and $v = b$, so system (3.1) becomes

$$u - v\beta'(t_1) + \beta(t_1) + t_1\beta'(t_1) + \beta'(t_1)\beta(t_1) = 0, \tag{3.5}$$

$$u - v\beta'(t_2) + \beta(t_2) + t_2\beta'(t_2) + \beta'(t_2)\beta(t_2) = 0, \tag{3.6}$$

$$u(t_2 - t_1) - v(\beta(t_2) - \beta(t_1)) + t_2\beta(t_2) - t_1\beta(t_1) + \beta^2(t_2) - \beta^2(t_1) = 0. \tag{3.7}$$

Equations (3.5) and (3.6) give

$$v = v(t_1, t_2) = \frac{\beta(t_2) - \beta(t_1) + t_2\beta'(t_2) - t_1\beta'(t_1) + \beta'(t_2)\beta(t_2) - \beta'(t_1)\beta(t_1)}{\beta'(t_2) - \beta'(t_1)}. \tag{3.8}$$

Denote the numerator of $v(t_1, t_2)$ in (3.8) by $\xi(t_1, t_2)$ and its denominator by $\eta(t_1, t_2)$. Substituting $v(t_1, t_2)$ for v in (3.5) gives

$$u = u(t_1, t_2) = \frac{\xi(t_1, t_2)}{\eta(t_1, t_2)}\beta'(t_1) - \beta(t_1) - t_1\beta'(t_1) - \beta'(t_1)\beta(t_1). \tag{3.9}$$

Substituting $v(t_1, t_2)$ for v and $u(t_1, t_2)$ for u in (3.7) and eliminating the denominator gives an equation of the form $g(t_1, t_2) = 0$. The rest of the proof follows in a similar way to that of Theorem 3.4. We show that $g(t_1, t_2) = (t_1 - t_2)^4\tilde{g}(t_1, t_2)$ for some germ of a smooth function $\tilde{g}$ with $\tilde{g}(t_1, t_1)$ not vanishing identically. We then analyse the $2(k + 1)$-jet of g, which is its first non-zero jet, and deduce from that the number of solutions of $\tilde{g}(t_1, t_2) = 0$. □

We consider now the multi-local singularities of f_c at two light-like points.

Theorem 3.7. *Suppose that f_{c_0} has an A_kA_l-singularity at two light-like points $p = \gamma(t_1)$ and $q = \gamma(t_2)$, with c_0 distinct from p and q.*

(i) *Suppose that p and q are on different lines of the lightcone centred at c_0. If k and l are not both even, then the MSS is a curve with limiting tangent direction parallel to $p + q$. If k and l are both even, then the MSS is either an isolated point or a pair of curves with common limiting tangent directions parallel to $p + q$.*

(ii) *If p and q are on the same light-like line L of the lightcone centred at c_0, then the MSS consists of the light-like line L.*

See Figure 7.

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Proof. (i) We parametrize one piece of the curve by $\gamma_1(t) = (x_1 + t, -x_1 - t + \beta_1(t))$ and the second by $\gamma_2(s) = (x_2 + s, x_2 + s + \beta_2(s))$, with s, t varying independently near zero, $p = \gamma_1(0) = (x_1, -x_1)$, $q = \gamma_2(0) = (x_2, x_2)$ and $c_0 = (0, 0)$. Suppose here that $x_1 \neq 0$ and $x_2 \neq 0$, that is, p and q are distinct from c_0.

The family of distance-squared functions is given by the bi-germ

$$f_1(t, c) = -(x_1 + t - a)^2 + (x_1 + t - \beta_1(t) + b)^2,$$
$$f_2(s, c) = -(x_2 + s - a)^2 + (x_2 + s + \beta_2(s) - b)^2,$$

with $c = (a, b)$ varying near $(0, 0)$. The centre c is on the MSS if and only if there exist s, t such that

$$\frac{\partial f_1}{\partial t}(t, c) = 0, \qquad \frac{\partial f_2}{\partial t}(s, c) = 0,$$
$$f_1(t, c) = f_2(s, c);$$

equivalently,

$$a + b - b\beta_1'(t) - x_1\beta_1'(t) - \beta_1(t) - t\beta_1'(t) + \beta_1'(t)\beta_1(t) = 0, \tag{3.10}$$
$$a - b - b\beta_2'(s) + x_2\beta_2'(s) + \beta_2(s) + s\beta_2'(s) + \beta_2'(s)\beta_2(s) = 0, \tag{3.11}$$
$$-(x_1 + t - a)^2 + (x_1 + t - \beta_1(t) + b)^2 = -(x_2 + s - a)^2 + (x_2 + s + \beta_2(s) - b)^2. \tag{3.12}$$

Subtracting (3.11) from (3.10) we get

$$b = \frac{x_2\beta_2'(s) + x_1\beta_1'(t) + \beta_2(s) + \beta_1(t) + s\beta_2'(s) + t\beta_1'(t) + \beta_2'(s)\beta_2(s) - \beta_1'(t)\beta_1(t)}{2 + \beta_2'(s) - \beta_1'(t)},$$

and substituting into (3.10) (or into (3.11)) yields a as a function of s, t.

We write $\beta_1(t) = \beta_{k+1}t^{k+1} + \text{h.o.t.}$ and $\beta_2(s) = \beta_{l+1}t^{l+1} + \text{h.o.t.}$ and assign the weights weight$(s) = k$ and weight$(t) = l$ to s and t. Then

$$a = \tfrac{1}{2}(x_1(k+1)\beta_{k+1}t^k - x_2(l+1)\beta_{l+1}s^l) + \text{h.o.w.},$$
$$b = \tfrac{1}{2}(x_1(k+1)\beta_{k+1}t^k + x_2(l+1)\beta_{l+1}s^l) + \text{h.o.w.},$$

where h.o.w. is short for 'terms of higher-order weights'.

Substituting into (3.12), the expressions for a and b as functions of (s, t) give an equation of the form

$$g(s, t) = 0,$$

with

$$g(s, t) = x_1^2(k+1)\beta_{k+1}t^k + x_2^2(l+1)\beta_{l+1}s^l + \text{h.o.w.}$$

The function g is Newton non-degenerate, so, following the same arguments in the proof of Theorem 3.6, its zero set consists of a finite number of branches. The number of branches and their limiting tangent directions are determined by the solutions of the initial part: $x_1^2(k+1)\beta_{k+1}t^k + x_2^2(l+1)\beta_{l+1}s^l = 0$.

Suppose that k and l are not both even. Then $g(s,t)=0$ consists of one branch that can be parametrized, for instance, when k is not even, by

$$(s,t)=(\omega^k+\text{h.o.w.},\alpha\omega^l+\text{h.o.w.}), \tag{3.13}$$

with $\alpha^k=-x_2^2(l+1)\beta_{l+1}/(x_1^2(k+1)\beta_{k+1})$. (In particular, the curve is smooth if and only if $k=1$ or $l=1$; compare with [**7**, **13**].) Substituting this into a and b gives

$$a=\frac{x_2(l+1)\beta_{l+1}}{x_1}(x_2+x_1)\omega^{kl}+\text{h.o.w.},$$
$$b=\frac{x_2(l+1)\beta_{l+1}}{x_1}(x_2-x_1)\omega^{kl}+\text{h.o.w.},$$

which defines a curve in the (a,b)-plane with limiting tangent direction at the origin parallel to $(x_2+x_1,x_2-x_1)=p_2+p_1$. (Observe that the curve is smooth if and only if $k=l=1$; compare with [**7**, **13**].)

If k and l are both even, $g(s,t)=0$ is an isolated point if $\beta_{k+1}\beta_{l+1}>0$, and a pair of curves with the same limiting tangent directions if $\beta_{k+1}\beta_{l+1}<0$. The result follows by similar calculations to those for the case in which k and l are not both even.

(ii) The proof is similar to that of (i). We parametrize one piece of the curve by $\gamma_1(t)=(x_1+t,x_1+t+\beta_1(t))$ and the second by $\gamma_2(s)=(x_2+s,x_2+s+\beta_2(s))$, with s, t varying independently near zero, $p=\gamma_1(0)=(x_1,x_1)$, $q=\gamma_2(0)=(x_2,x_2)$ and $c_0=(0,0)$. We suppose here that $x_1\neq 0$, $x_2\neq 0$ and $x_1\neq x_2$.

The family of distance-squared functions is given by the bi-germ

$$f_1(t,c)=-(x_1+t-a)^2+(x_1+t+\beta_1(t)-b)^2,$$
$$f_2(s,c)=-(x_2+s-a)^2+(x_2+s+\beta_2(s)-b)^2,$$

with $c=(a,b)$ varying near $(0,0)$. The centre c is on the MSS if and only if there exist s, t such that

$$a-b-b\beta_1'(t)+x_1\beta_1'(t)+\beta_1(t)+t\beta_1'(t)+\beta_1'(t)\beta_1(t)=0, \tag{3.14}$$
$$a-b-b\beta_2'(s)+x_2\beta_2'(s)+\beta_2(s)+s\beta_2'(s)+\beta_2'(s)\beta_2(s)=0, \tag{3.15}$$
$$-(x_1+t-a)^2+(x_1+t+\beta_1(t)-b)^2=-(x_2+s-a)^2+(x_2+s+\beta_2(s)-b)^2. \tag{3.16}$$

It is clear that when $s=t=0$, the whole line $a=b$ is a solution of the above system. Now, subtracting (3.15) from (3.14) gives

$$b=b(s,t)=\frac{x_1\beta_1'(t)-x_2\beta_2'(s)+\beta_1(t)-\beta_2(s)t\beta_1'(t)-s\beta_2'(s)+\beta_1'(t)\beta_1(t)-\beta_2'(s)\beta_2(s)}{\beta_1'(t)-\beta_2'(s)}.$$

Substituting $b(s,t)$ for b in (3.14) gives an expression for a as a function in (s,t), say $a(s,t)$. Substituting in (3.16) $a(s,t)$ and $b(s,t)$ for a and b gives an equation in s, t. Substituting the solution of this equation (when it exists) in $b(s,t)$ gives $b(0,0)\neq 0$. Therefore, the system of equations (3.14)–(3.16) gives a unique branch of the MSS that is the light-like line tangent to the two pieces of curves at p and q. □

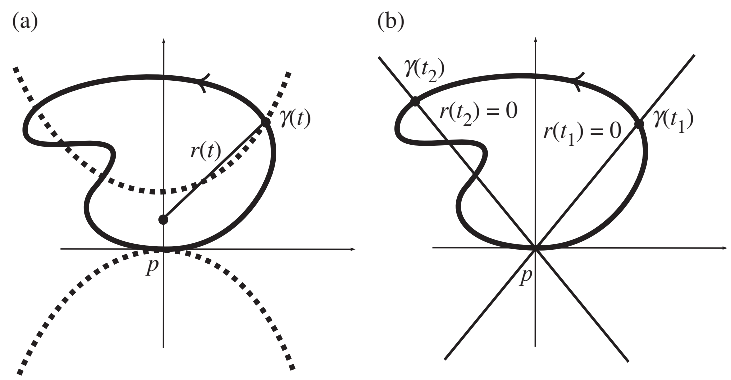

Figure 8. Bi-tangent pairs.

4. Existence of bi-tangent pairs

We consider in this section the problem of existence of bi-tangent pseudo-circles in $\mathbb{R}^2_1$ (for curves and surfaces in the Euclidean case see [**8**] and for general hypersurfaces in the Euclidean n-space see [**6**]).

Theorem 4.1. *Let γ be a closed simple smooth curve in $\mathbb{R}^2_1$ and let p be a point on γ. Then there exists a pseudo-circle or a line tangent to γ at p and at some other point q on γ distinct from p.*

Proof. We consider separately the cases in which p is space-like, time-like or light-like.

Suppose that p is a light-like point with tangent direction $(\pm 1, 1)$. It follows from the proof of Proposition 1.2 in [**13**] that there is a point q on γ (in fact there are at least two such points) with tangent directions $(\mp 1, 1)$. The tangent lines to γ at p and q intersect at some point c. The lightcone $\mathrm{LC}^*(c)$ is therefore bi-tangent to γ at p and q.

Suppose that γ is time-like at p, which we take to be the origin. We follow the same setting as in [**8**]. By a Lorentz transformation, we write $\gamma = (x(t), y(t))$ with $\gamma(0) = p$ and $\gamma'(0) = (1, 0)$. Let $\gamma(t)$ be a point on the curve γ that is not on the x-axis (so $y(t) \neq 0$) and let $S^1_1(c(t), r(t))$ be the unique pseudo-circle that is tangent to γ at p and passes through $\gamma(t)$ (see Figure 8 (a)).

We have $c(t) = (0, r(t))$ so that

$$\langle (x(t), y(t) - r(t)), (x(t), y(t) - r(t)) \rangle = r(t)^2$$

gives

$$r(t) = \frac{-x(t)^2 + y(t)^2}{2y(t)}.$$

Note that the case in which $y(t) = 0$ corresponds to the ‘pseudo-circle’ being a straight line. We have $r'(t) = (-2x(t)y(t)x'(t) + (x(t)^2 + y(t)^2)y'(t))/(2y(t)^2)$, which vanishes if and only

$$\langle (x'(t), y'(t)), (2x(t)y(t), x(t)^2 + y(t)^2) \rangle = 0.$$

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The above condition implies that the tangent line to the curve is pseudo-orthogonal to the Minkowski normal of the above pseudo-circle. Therefore, $r'(t) = 0$ if and only if the $S^1_1(c(t), r(t))$ is bi-tangent to γ at the origin and at $\gamma(t)$.

We consider the lightcone $\mathrm{LC}^*(p)$ given by the two lines $(\pm s, s)$, $s \in \mathbb{R} \setminus 0$. Suppose, without loss of generality, that the interior of γ contains points in the semi-plane $y > 0$. Let $\gamma(t_1)$ and $\gamma(t_2)$ be the last points of intersection of γ with the semi-lines (s, s) and $(-s, s)$ with $s > 0$, respectively (see Figure 8 (b)). Here, last point of intersection means that there are no points of intersection of the curve with the semi-lines $\gamma(t_1) + (s, s)$ and $\gamma(t_2) + (-s, s)$, $s > 0$.

We orient γ so that $0 < t_1 < t_2$, and, as it is a simple curve, we have $\gamma(t) \neq p$ for all $t \in [t_1, t_2]$. The function $r(t)$ is well defined at t_1 and t_2 and is smooth in the interval $[t_1, t_2]$. As $r(t_1) = r(t_2) = 0$, there exists $t_3 \in (t_1, t_2)$ for which $r'(t_3) = 0$. This implies that there exists a bi-tangent pseudo-circle to γ at $p = \gamma(0)$ and $q = \gamma(t_3)$, with $q \neq p$.

The case in which γ is space-like at p follows in a similar way to the time-like case. □

Remark 4.2. When $p \in \gamma$ is time-like or space-like, the proof of Theorem 4.1 shows that there exists a bi-tangent pair (p, q) with p and q belonging to different time-like or space-like components of γ. Indeed, from the setting in the proof (for the time-like case), the tangent directions to γ at p and q are parallel to $(1, 0)$, so there must be points on γ between p and q and between q and p where the tangent line is parallel to $(\pm 1, 1)$. That is, p and q are on different components of γ.

We now consider bi-tangent pairs belonging to the same space-like or time-like component of γ.

Proposition 4.3. *Bi-tangent pairs on the same time-like (or space-like) component of γ can only occur on one branch of a pseudo-circle $S^1_1(c, r)$ (or $H^1(c, r)$).*

Proof. Consider a time-like component C of γ; the space-like case follows similarly. Suppose that C is tangent to one branch of a pseudo-circle $S^1_1(c, r)$ at $\gamma(t_1)$, and to the other branch of $S^1_1(c, r)$ at $\gamma(t_2)$. Then the normal directions $\gamma(t_1) - c$ to γ at t_1 and $\gamma(t_2) - c$ to γ at t_2 are in two different space-like components of $\mathbb{R}^2_1$. Therefore, there must exist $t_3 \in (t_1, t_2)$ where the normal to C at t_3 is a light-like direction. This is a contradiction as C is a time-like component of γ. □

We consider the question of the existence of bi-tangent pairs on the same space-like or time-like component of γ. If the component of γ is not strictly convex, then the following example shows that there may not exist bi-tangent pairs.

Proposition 4.4. *There are no bi-tangent pairs on the time-like or space-like components of the curve (t, t^3) or (t^3, t), with $t \in I = (-1/\sqrt{3}, 1/\sqrt{3})$.*

Proof. Consider the case in which $\gamma(t) = (t, t^3)$ (the other case follows similarly). The curve γ is time-like for $t \in I$ and $\gamma(\pm 1/\sqrt{3})$ are light-like points. In I, γ can only be tangent to the pseudo-circles $S^1_1(c, r)$.

There exists a bi-tangent pair (t_1, t_2) on γ in I if and only if there exist $c = (a, b)$ and $r > 0$ such that the function

$$d(t) = -(t-a)^2 + (t^3 - b)^2 - r^2 = t^6 - 2bt^3 - t^2 + 2at - a^2 + b^2 - r^2$$

has a double root at t_1 and t_2. That is,

$$d(t) = (t - t_1)^2 (t - t_2)^2 (t^2 + \lambda t + \mu)$$

for some λ and μ in $\mathbb{R}$. Equating the coefficients of t^i in both expressions of $d(t)$ yields

$$\begin{aligned}
&\text{coefficient of } t^5: && -2t_1 - 2t_2 + \lambda = 0,\\
&\text{coefficient of } t^4: && t_1^2 + 4t_1t_2 + t_2^2 - 2\lambda(t_1 + t_2) + \mu = 0,\\
&\text{coefficient of } t^2: && -7t_1^2t_2^2 - 4t_1^3t_2 - 4t_1t_2^3 + \mu(t_1^2 + 4t_1t_2 + t_2^2) = -1.
\end{aligned}$$

The first two equations give λ and μ as functions of (t_1, t_2), and substituting these into the third equation gives

$$3(t_1^2 + t_1t_2 + t_2^2)(t_1^2 + 3t_1t_2 + t_2^2) = -1. \tag{4.1}$$

Consider the function $g_{t_2}(t_1) = 3(t_1^2 + t_1t_2 + t_2^2)(t_1^2 + 3t_1t_2 + t_2^2)$, with t_2 fixed in the interval I. We have $g'_{t_2}(t_1) = 6(t_1 + 2t_2)(2t_1^2 + 2t_1t_2 + t_2^2)$, so g_{t_2} has a minimum at $t_1 = -2t_2$ with $g_{t_2}(-2t_2) = -9t_2^4$. As $t_1, t_2 \in I$, $g_{t_2}(t_1) \geqslant g_{t_2}(-2t_2) > -1$. Therefore, (4.1) has no solutions for $t_1, t_2 \in I$. □

In view of Proposition 4.4, we shall restrict ourselves to time-like/space-like components of curves γ with nowhere vanishing curvature. Such components are strictly convex curves. At light-like points we say that γ is strictly convex if it has A_1 contact with its tangent line.

Theorem 4.5. *Let C be a space-like or time-like component of γ defined in an open interval I and suppose that it is strictly convex on $\bar{I}$. Suppose that the derivative κ' of the curvature of C is not flat at any point in I. Then C has bi-tangent pairs.*

Proof. As C is strictly convex, its curvature κ does not vanish on I, and so has constant sign on I, say positive. The curvature tends to $+\infty$ as t tends to the boundaries of I (see [**13**]), so it must have a global minimum on I, say at t_0. As the curvature of C is not flat, we take k to be the least integer such that $\kappa^{(k-1)}(t_0) \neq 0$. As t_0 is a minimum of κ, k is an odd integer, and the result follows by Theorem 3.4 (ii). □

The result in Theorem 4.5 asserts the existence of bi-tangent pairs. However, we would like to know, given any point p on the component C of γ, whether or not there exists another point q on C such that (p, q) is a bi-tangent pair. One idea is to keep track of the bi-tangent pairs born at the global minimum of the curvature t_0 in the proof of Theorem 4.5 and hope that they will reach the boundary points of I. The problem is that at an A_1A_2-singularity of the distance-squared function one of the points of the bi-tangent pair turns back (see Figure 9). Thus, Theorem 4.5 cannot guarantee that given any point p on C there exists another q on C such that (p, q) is a bi-tangent pair.

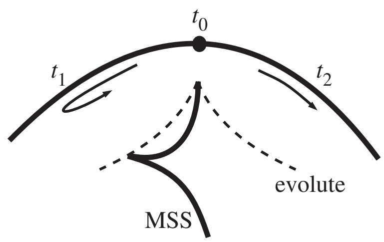

Figure 9. Turning points of bi-tangent pairs.

We need the following concept. A germ g of a smooth function is said to be finitely determined if there exists an integer k such that any other germ f with the same k-jet as g can be written as $f = g \circ h$, where h is a germ of a diffeomorphism.

Definition 4.6. We say that a space-like or a time-like curve γ (not necessarily connected) satisfies the property (d) if its pre-MSS is locally at any of its points the zero set of a finitely determined germ of a smooth function.

Remark 4.7. The pre-MSS is a smooth curve at the multi-local singularities of the distance-squared function on a generic curve (see [**7**, **13**]). This is also true at the A_3-singularities of the distance squared function. Thus, property (d) is a generic property of plane curves. Suppose that the distance-squared function f_{c_0} has a more degenerate multi-local singularity of type A_kA_l at p and q, with say $l \geqslant k$. Take p to be the origin and parametrize the curve there by $(t, \alpha_2 t^2 + \text{h.o.t.})$. Parametrize the curve at $q = (x_0, y_0)$ by $(x_0 + s, y_0 + \beta_1 s + \beta_2 s^2 + \text{h.o.t.})$, $\beta_1 \neq 0$. The distance-squared functions at p and q have the form $f_1(t, c_0) = \phi_{k+1} t^{k+1} + \text{h.o.t.}$ and $f_2(s, c_0) = \psi_{l+1} s^{l+1} + \text{h.o.t.}$, respectively, with $\phi_{k+1}\psi_{l+1} \neq 0$. Let $g(t, s)$ be the function whose zero set is the pre-MSS. Following the same calculations as in the proof of Theorem 3.7, and setting weight$(t) = l$ and weight$(s) = k+1$, we show that the principal part of $g(t, s)$ (i.e. the part that determines its Newton polyhedron; see [**2**, **10**]) is given by

$$-k\phi_{k+1} t^{k+1} + \frac{\beta_2}{\beta_1} x_0 \phi_{k+1} s t^k + \frac{1}{2\beta_1} y_0 (l+1) \psi_l s^l. \tag{4.2}$$

If $l \neq k+1$ or if $l = k+1$ but (4.2) does not have a repeated root, the principal part of g is non-degenerate. It follows then that g is finitely determined, so γ satisfies property (d) at the A_kA_l-singularity of f_c.

Theorem 4.8. *Let C be a space-like or time-like component of γ defined in the open interval $I = (\lambda, \mu)$ and satisfying property (d). Let c_0 be the centre of the lightcone bi-tangent to γ at λ and μ, and suppose that the distance-squared function f_{c_0} has an A_kA_l-singularity, with k and l both odd (i.e. γ is convex at λ and μ). Then given any point p on C, there exists a branch of a pseudo-circle that is bi-tangent to C at p and at another point q on C.*

Proof. Consider the *pre-MSS* that consists of bi-tangent pairs (t_1, t_2) in $A = \bar{I} \times \bar{I}$, where $\bar{I} = [\lambda, \mu]$. From the parametrization (3.13) of the *pre-MSS*, when k and l are both

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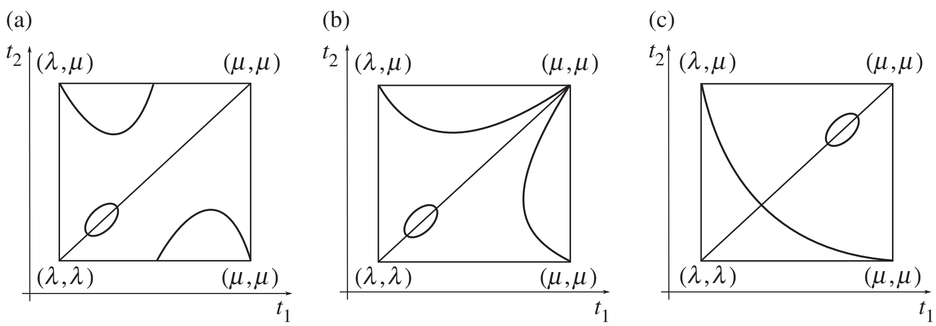

Figure 10. (a), (b) Impossible and (c) possible configurations of the pre-MSS of a time-like or space-like component of γ. The pre-MSS can have other components such as the loop depicted in each figure.

odd, the pre-MSS has a single branch S starting at the point (λ,μ) with the following property: for any t_1 near λ with $t_1 > \lambda$, there exists t_2 near μ with $t_2 < \mu$ such that $(t_1,t_2) \in S$. (Here the germs of curves at $\gamma(\lambda)$ and $\gamma(\mu)$ come from the same convex component C of γ, so β_{k+1} and β_{l+1} in the proof of Theorem 3.7 have the same sign, which implies that $\alpha < 0$ in (3.13).) Therefore, there is one smooth segment of the branch S in the interior of A near the corner (λ,μ). We call (λ,μ) the starting point of S.

Suppose that the branch S returns to the line segment $t_2 = \mu$ in A at a point (t_1,μ). The $\lambda < t_1 < \mu$ case is not possible as the tangent line to γ at t_1 is not light-like (Figure 10 (a)). The $t_1 = \lambda$ case is also not possible because there is only one branch of S in the interior of A near (λ,μ) (see (3.13)). The $t_1 = \mu$ case is not possible either for the following reason. As the pre-MSS is symmetric with respect to the diagonal, $t_1 = \mu$ means that the MSS would have two branches at the point $\gamma(t_2)$ (see Figure 10 (b)). By Theorem 3.6 this is not possible as l is odd (the MSS should be locally empty at $\gamma(\mu)$).

It follows from the above considerations that the branch S does not return to the line segment $t_2 = \mu$. As the pre-MSS is a compact set, S must end somewhere in A. Because C satisfies property (d) the end point cannot be an interior point of A, so S must cross the diagonal in A at a point that corresponds to a vertex of C and, by symmetry, ends at (μ,λ), the diagonally opposite corner in A of the starting point of S (Figure 10 (c)). This means that for any point $p \in C$, there exists $q \in C$ such that p, q is a bi-tangent pair. □

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