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ARTICLE

Transversal C_k-factors in subgraphs of the balanced blow-up of C_k[†]

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Abstract

For a subgraph G of the blow-up of a graph F, we let $\delta^*(G)$ be the smallest minimum degree over all of the bipartite subgraphs of G induced by pairs of parts that correspond to edges of F. Johansson proved that if G is a spanning subgraph of the blow-up of C_3 with parts of size n and $\delta^*(G) \geq \frac{2}{3}n + \sqrt{n}$, then G contains n vertex disjoint triangles, and presented the following conjecture of Häggkvist. If G is a spanning subgraph of the blow-up of C_k with parts of size n and $\delta^*(G) \geq \left(1 + \frac{1}{k}\right) \frac{n}{2} + 1$, then G contains n vertex disjoint copies of C_k such that each C_k intersects each of the k parts exactly once. A similar conjecture was also made by Fischer and the case $k = 3$ was proved for large n by Magyar and Martin.

In this paper, we prove the conjecture of Häggkvist asymptotically. We also pose a conjecture which generalises this result by allowing the minimum degree conditions in each bipartite subgraph induced by pairs of parts of G to vary. We support this new conjecture by proving the triangle case. This result generalises Johannson's result asymptotically.

Keywords: extremal graph theory; cycles

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1. Introduction

For a graph F on $[k] := \{1, \ldots, k\}$, we say that B is the *n-blow-up of F* if there exists an ordered partition $(V_1, \ldots, V_k)$ of $V(B)$ such that $|V_1| = \cdots = |V_k| = n$ and we have that $uu' \in E(B)$ if and only if $u \in V_i$ and $u' \in V_j$ for some $ij \in E(F)$. For G a spanning subgraph of B, we call the sequence $V_1, \ldots, V_k$ the *parts* of G and we define

$$\delta^*_F(G) := \min_{ij \in E(F)} \delta\big(G\big[V_i, V_j\big]\big)$$

where $G[V_i, V_j]$ is the bipartite subgraph of G induced by the parts V_i and V_j. We often drop the subscript F when it is clear from the context. For a graph H, we call $\mathcal{T}$ an H-tiling of G if $\mathcal{T}$ consists of vertex disjoint copies of H in G. We say that $\mathcal{T}$ *covers* $V(\mathcal{T}) := \bigcup\{V(H') : H' \in \mathcal{T}\}$ and say that $\mathcal{T}$ is *perfect* or an *H-factor* if it covers every vertex of G. Call a subset of $V(G)$ or a subgraph of G a *transversal* if it intersects each part in exactly one vertex and a *partial transversal* if it intersects each part in at most one vertex. An H-tiling is a *transversal H-tiling* if each copy of H in $\mathcal{T}$ is a transversal. We call a perfect transversal H-tiling a *transversal H-factor*.

Fischer [3] conjectured the following multipartite version of the Hajnal–Szemerédi Theorem: If G is the n-blow-up of K_k, and $\delta^*(G) \geq \left(1 - \frac{1}{k}\right) n$, then G has a K_k-factor. In the same paper, Fischer proved that, when $k \in \{3, 4\}$, such a graph G contains a K_k-tiling of size at least $n - C$,

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where C is a constant that depends only on k. Johansson [4] proved that, for every n, if G is a spanning subgraph of the n-blow-up of K_3 and $\delta^*(G) \geq 2n/3 + \sqrt{n}$, then G contains a K_3-factor, so Johansson proved the triangle case of the conjecture asymptotically. Later, Lo & Märkstrom [8] and, independently, Keevash & Mycroft [5] proved the conjecture asymptotically for every $k \geq 4$. The following theorem, which was proved for $k=3$ by Magyar & Martin [9], for $k=4$ by Martin & Szemerédi [10] and for $k \geq 5$ by Keevash & Mycroft [5], shows that Fischer's original conjecture was nearly true for n sufficiently large. (Keevash & Mycroft actually proved more, see Theorem 1.1 in [5] for details.)

Theorem 1. *For every k there exists $n_0 := n_0(k)$ such that whenever $n \geq n_0$ the following holds for every spanning subgraph G of the n-blow-up of K_k where*

$$\delta^*(G) \geq \left(1 - \frac{1}{k}\right) n.$$

The graph G does not contain a K_k-factor if and only if both n and k are odd, k divides n and G is isomorphic to a specific spanning subgraph $\Gamma_{n,k}$ of the n-blow-up of K_k where $\delta^*(\Gamma_{n,k}) = \left(1 - \frac{1}{k}\right) n$.

The following conjecture of Häggkvist, which appeared in [4], can be seen as a different generalisation of the $k=3$ case of Theorem 1. Independently, Fischer made a similar conjecture in [3].

Conjecture 2. *For every $k \geq 3$, if G is a spanning subgraph of the n-blow-up of C_k and*

$$\delta^*(G) \geq \left(1 + \frac{1}{k}\right) \frac{n}{2} + 1, \tag{1}$$

then G has a transversal C_k-factor.

Our first result establishes an asymptotic version of Conjecture 2.

Theorem 3. *For every $\varepsilon > 0$ and positive integer $k \geq 4$ there exists $n_0 := n_0(k, \varepsilon)$ such that for every $n \geq n_0$ the following holds. If G is a spanning subgraph of the n-blowup of C_k and*

$$\delta^*(G) \geq \left(1 + \frac{1}{k} + \varepsilon\right) \frac{n}{2}, \tag{2}$$

then G has a transversal C_k-factor.

Note that Theorem 1 shows that Conjecture 2 is tight when $k=3$. The following example from [4] shows that, for $k \geq 4$, the minimum degree condition (1) in Conjecture 2 cannot be decreased by more than 1. Call $Z \subseteq V(G)$ a *transversal C_k-cover* if every transversal C_k in G intersects Z and let the *transversal C_k-cover number* of G be the order of a smallest transversal C_k-cover. This example relies on the observation that, because every transversal C_k has at least one vertex in a transversal C_k-cover, the maximum size of a transversal C_k-tiling is bounded above by the transversal C_k-cover number. (Note that we always view arithmetic on elements of $[k] := \{1, \ldots, k\}$ modulo k.)

Example 4. For $k \geq 3$ and $m \geq 1$, let $n := 2km$ and $V_1, \ldots, V_k$ be disjoint sets each of size n. For $i \in [k-1]$, let $\{U_i, W_i, Z_i\}$ be a partition of V_i such that $|U_i| = (k-1)m$, $|W_i| = (k-1)m$ and $|Z_i| = 2m$, and let $\{U_k, W_k, Z_k\}$ be a partition of V_k such that $|U_k| = (k-1)m$, $|W_k| = (k-1)m + 1$ and $|Z_k| = 2m - 1$. Let G be the spanning subgraph of the n-blow-up of C_k with parts $V_1, \ldots, V_k$ where $E(G)$ consists of the union of the edges in the following graphs:

- the complete bipartite graphs with parts Z_i, V_{i-1} and Z_i, V_{i+1} for each $i \in [k]$,
- the complete bipartite graphs with parts U_i, U_{i+1} and W_i, W_{i+1} for each $i \in [k-1]$,
- the complete bipartite graphs with parts U_k, W_1 and W_k, U_1.

Note that $\delta^*(G) = (k+1)m - 1 = \left(1 + \frac{1}{k}\right)\frac{n}{2} - 1$, and that every transversal C_k has at least one vertex in $Z := Z_1 \cup \cdots \cup Z_k$, that is., Z is a transversal C_k-cover of G. The fact that $|Z| = 2mk - 1 < n$ then implies that G does not contain a transversal C_k-factor.

We make the following conjecture which, if true, would be a strengthening of Theorem 3.

Conjecture 5. *For every $k \geq 3$ and $\varepsilon > 0$, there exists $n_0 := n_0(k, \varepsilon)$ such that for every $n \geq n_0$ the following holds. Let G be a spanning subgraph of the n-blow-up of C_k with parts $V_1, \ldots, V_k$. If there exist $\delta_1, \delta_2, \ldots, \delta_k \geq n/2$ such that $\delta(G[V_i, V_{i+1}]) \geq \delta_i$, for every $i \in [k]$, and*

$$\frac{1}{k}\sum_{i\in[k]} \delta_i \geq \left(1 + \frac{1}{k} + \varepsilon\right)\frac{n}{2}, \tag{3}$$

then G has a transversal C_k-factor.

Note that Theorem 3 is a special, uniform case of Conjecture 5, namely the case when $\delta_1 = \delta_2 = \ldots = \delta_k$. Also, note that the condition $\delta_1, \ldots, \delta_k \geq n/2$ is necessary because a transversal C_k-factor in G defines a perfect matching in $G[V_i, V_{i+1}]$ for every $i \in [k]$ and $n/2$ is the smallest minimum degree condition necessary to guarantee a perfect matching in a bipartite graph with parts of size n.

Our second result shows that Conjecture 5 holds for $k = 3$. Note that this result can also be seen as a strengthening of an asymptotic version of the $k = 3$ case of Theorem 1.

Theorem 6. *For every $\varepsilon > 0$ there exists $n_0 := n_0(\varepsilon)$ such that for every $n \geq n_0$ the following holds. Let G be a spanning subgraph of the n-blow-up of a triangle with parts V_1, V_2, V_3. If there exist $\delta_1, \delta_2, \delta_3 \geq n/2$ such that $\delta(G[V_i, V_{i+1}]) \geq \delta_i$, for every $i \in [3]$, and*

$$\frac{\delta_1 + \delta_2 + \delta_3}{3} \geq \left(1 + \frac{1}{3} + \varepsilon\right)\frac{n}{2} = \frac{2n}{3} + \frac{\varepsilon n}{2},$$

then G has a triangle factor.

Because of Example 4, the condition on the average of the minimum degrees in Conjecture 5 is asymptotically sharp. However, it might be possible to weaken the degree condition by only placing a lower bound on the average of some proper subset of the minimum degrees. For example, in the triangle case, we do not have an example of a graph without a triangle factor in which all of the minimum degrees are at least $n/2$ and the average of only the two largest minimum degrees is at least $2n/3$. Often one tries to find such examples that either have an independent set which is larger than n or have a triangle cover of size less than n, since either one of these two conditions imply that the graph cannot contain n vertex disjoint triangles. It is a straightforward exercise to show that, under these conditions, the independence number must be n. The following theorem proves that the triangle cover number must be n as well.

Theorem 7. *For every $n \in \mathbb{N}$, the following holds for every spanning subgraph G of the n-blow-up of C_3 with parts V_1, V_2, V_3. If $\delta_1 \geq \delta_2 \geq \delta_3 \geq n/2$, $\delta(G[V_i, V_{i+1}]) \geq \delta_i$ for $i \in [3]$, and*

$$\frac{\delta_1 + \delta_2}{2} \geq \frac{2n}{3},$$

then the triangle cover number of G is n.

Moreover, for every rational $\gamma \in \left(\frac{3}{4}, \frac{7}{9}\right] \cup \left\{\frac{2}{3}\right\}$ there are infinitely many $n \in \mathbb{N}$ such that when $\beta = 4/3 - \gamma$ there exists a spanning subgraph G of the n-blow-up of C_3 with parts A, B and C such that $\delta(G[A, B]) \geq \gamma n - 1$, $\delta(G[A, C]) \geq \beta n$ and $\delta(G[B, C]) \geq n/2$ that has a triangle cover of order less than n.

Suppose that for every sufficiently small $\varepsilon > 0$ there exists n_0 such that for every $n \geq n_0$ there exists a subgraph of the n-blow-up of C_3 with parts V_1, V_2, V_3 that meets the stronger degree

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conditions $\delta_1 \geq \delta_2 \geq \delta_3 \geq (1+\varepsilon)n/2$ and $(\delta_1+\delta_2)/2 \geq (2/3+\varepsilon)n$ yet does not have a triangle factor. In Section 2 (Lemma 13 and Proposition 15), we will show that, under these conditions, we can apply the absorbing method. This would therefore mean that, for some $\sigma > 0$ and for every sufficiently large n, there would exist a subgraph of the n-blow-up of C_3 that meets the degree conditions of first part of Theorem 7 in which every triangle factor has order at most $(1-\sigma)n$, but has triangle cover number n and independence number n.[1]

1.1 Additional observations and remarks related to Conjecture 5

Let $k, \delta_1, \ldots, \delta_k$, n and G be as in Conjecture 5.

Observation 8. In the case of $\delta_i \leq (1+\varepsilon)\frac{n}{2}$ for some $i \in [k]$, the problem in Conjecture 5 for k can be reduced to $k-1$. To see this, assume $i = k$ (for convenience) and first note that

$$\sum_{i\in[k-1]} \delta_i \geq k\left(1+\frac{1}{k}+\varepsilon\right)\frac{n}{2} - (1+\varepsilon)\frac{n}{2} = (k-1)\left(1+\frac{1}{k-1}+\varepsilon\right)\frac{n}{2}. \tag{4}$$

Because $\delta(G[V_1, V_k]) \geq \frac{n}{2}$, Hall's Theorem implies that we can match every $v \in V_1$ to a unique $f_v \in V_k$ that is adjacent to v. Let G' be the graph derived from G by collapsing each edge vf_v into v for each $v \in V_1$, that is, G' is $G - V_k$ with an edge between $v \in V_1$ and $u \in V_{k-1}$ if and only if f_v is adjacent to u in G. It is easy to see that G' is a spanning subgraph of the n-blow-up of C_{k-1} with parts $V_1, \ldots, V_{k-1}$ such that $\delta(G[V_i, V_{i+1}]) \geq \delta_i$ for each $i \in [k-1]$, and that any transversal C_{k-1}-factor in G' can be extended to a transversal C_k-factor in G. So, by (4), if the $k-1$ case of Conjecture 5 holds, then G has a C_k-factor.

Repeated applications of Observation 8 and Theorem 6 imply the following.

Remark 9. For every $k \geq 3$ and sufficiently large n, Conjecture 5 holds if

$$\frac{\delta_i+\delta_j+\delta_\ell}{3} \geq \left(1+\frac{1}{3}+\varepsilon\right)\frac{n}{2} \qquad \text{for three distinct } i, j, \ell \in [k],$$

that is, Conjecture 5 holds in the case when all the excess values of δ_i compared to $n/2$ are concentrated in at most 3 members of $\delta_1, \delta_2, \ldots \delta_k$.

To further support Conjecture 5 we now mention a natural extension of Conjecture 5 for the case when $k = 2$ which is easily proved with Hall's Theorem. To motivate this extension, first consider a spanning subgraph G of the n-blow-up of a triangle with parts V_1, V_2, V_3 that meets the conditions of the $k = 3$ case of the conjecture. Since $G[V_1, V_3] \geq n/2$, Hall's Theorem implies that we can match every $v \in V_1$ to some $f_v \in V_3$. Similarly to Observation 8, we note that a perfect matching in the bipartite graph $G[V_1, V_2]$ that is simultaneously a perfect matching in the bipartite graph H with parts V_1 and V_2 in which $v \in V_1$ is adjacent to $u \in V_2$ in H if and only if u is adjacent to f_v in $G[V_2, V_3]$ corresponds to a triangle factor of G.

This leads to the aforementioned extension of Conjecture 5 for $k = 2$: Suppose that H and H' are two balanced bipartite graphs both with the same partite sets V_1 and V_2 where $|V_1| = |V_2| = n$ and that $\frac{1}{2}\big(\delta(H)+\delta(H')\big) \geq \frac{3n}{4}$. Then there exists $M \subseteq E(H) \cap E(H')$ that is simultaneously a perfect matching of both H and H'. Indeed, every vertex in $v \in V_1 \cup V_2$ is incident to at least $d_H(v) + d_{H'}(v) - n \geq n/2$ edges in $E(H) \cap E(H')$, so Hall's Theorem implies the desired matching exists.

[1]It is well known that this assumption would also imply that there would exist a family of examples that meet the degree conditions of first part of Theorem 7 and do not have a perfect factional triangle tiling. By the duality theorem from linear programming, such a family of examples then must have a fractional triangle cover of size less than the size of the parts.

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This with Observation 8, implies the following.

Remark 10. For every $k \geq 3$ and n, Conjecture 5 holds whenever $\frac{\delta_i+\delta_j}{2} \geq \left(1+\frac{1}{2}\right)\frac{n}{2} = \frac{3n}{4}$ for distinct $i, j \in [k]$.

Using Remark 10, Observation 8, and a straightforward application of the absorbing method of Rödl, Ruciński and Szemerédi we will show (see Lemma 16 of Section 2) that one only needs to prove the following weaker conjecture to establish Conjecture 5. We use this reduction in our proof of Theorem 6.

Conjecture 11. *For every $k \geq 3$, $\varepsilon > 0$ and $\sigma > 0$, there exists $n_0 := n_0(k, \varepsilon, \sigma)$ such that for every $n \geq n_0$ the following holds. Let G be a spanning subgraph of the n-blow-up of C_k with parts $V_1, \ldots, V_k$. If there exist $\delta_1, \delta_2, \ldots, \delta_k \geq (1+\varepsilon)n/2$ such that $\delta(G[V_i, V_{i+1}]) \geq \delta_i$, for every $i \in [k]$, and*

$$\frac{1}{k}\sum_{i\in[k]} \delta_i \geq \left(1+\frac{1}{k}+\varepsilon\right)\frac{n}{2},$$

then G has a transversal C_k-tiling of size at least $(1-\sigma)n$.

1.2 Notation

For a graph G, $e(G)$ denotes the number of edges in G. For $S, T \subseteq V(G)$, we let $N_G(S, T) := T \cap \left(\bigcap\{N_G(v) : v \in S\}\right)$ be the *common neighbourhood of S in T* and we let $d_G(S, T) := |N_G(S, T)|$. For $v \in V(G)$, we define $N_G(v, T) := N_G(\{v\}, T)$ and $d_G(v, T) = d_G(\{v\}, T)$. We typically drop the subscript from this notation when it is clear from the context. For a tiling $\mathcal{T}$, we let $U(\mathcal{T}) := V(G) \setminus V(\mathcal{T})$ be the vertices *uncovered* by $\mathcal{T}$ and if $v \in U(\mathcal{T})$ we say that *v is uncovered by $\mathcal{T}$*. Similarly, if $e \in E(G)$, and both endpoints of e are uncovered by $\mathcal{T}$, we say that *e is uncovered by $\mathcal{T}$*.

2. The absorbing method

We use a straightforward application of the absorbing method of Rödl, Ruciński and Szemerédi [11]. Propositions 14 and 15 are essentially all that is necessary to derive appropriate absorbing lemmas in this setting.

Definition 12. For $k \geq 3$, let G be a subgraph of the n-blow-up of C_k with parts $V_1, \ldots, V_k$. For vertices v, v' in the same part, we call a t-tuple of distinct vertices $(v_1, \ldots v_t)$ a (v, v', t)-linking sequence if both $G[\{v, v_1, \ldots v_t\}]$ and $G[\{v', v_1, \ldots, v_t\}]$ have a transversal C_k-factor. We allow $v = v'$ in this definition. We say that G is (η, t)-linked if, for every $i \in [k]$ and $v, v' \in V_i$, the number of (v, v', t)-linking sequences is at least ηn^t.

The proof of the following lemma is standard (e.g., it is very similar to Lemma 1.1 in [7]), but we include a proof in the appendix for completeness.

Lemma 13. *The Absorbing Lemma For $k \geq 3$, $t \geq k-1$, $\eta > 0$ and $0 < \sigma \leq \frac{0.1\eta^{k+1}}{(k(t+1))^2+1}$, there exists $n_0(k, t, \eta, \sigma)$ such that for every $n \geq n_0$ the following holds. Suppose that G is a subgraph of the n-blow-up of C_k with parts $V_1, \ldots, V_k$ that is $(2\eta, t)$-linked. For some $z \leq \sigma n$, there exists $A \subseteq V(G)$ where $|A \cap V_i| = z$ for every $i \in [k]$ such that if $G - A$ has a transversal C_k-tiling of size at least $n - z - \sigma^2 n$, then G has a transversal C_k-factor.*

Note that the degree condition in the following proposition is weaker than the degree condition in Conjecture 2.

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Proposition 14. *For $k \geq 4$ and $\varepsilon > 0$, if G is a subgraph of the n-blow-up of C_k and $\delta^*(G) \geq (1+\varepsilon)n/2$, then G is $(\varepsilon^3/2^k, k-1)$-linked.*

Proof. Let $V_1, \ldots, V_k$ be the parts of G. Without loss of generality we can assume that $v, v' \in V_1$. We can construct $(v_2, \ldots, v_k)$ a (v, v')-linking sequence by first selecting $v_2 \in N(\{v, v'\}, V_2)$ and then $v_k \in N(\{v, v'\}, V_k)$ each in at least $2\delta^*(G) - n \geq \varepsilon n$ ways. Iteratively, for i from 3 to $k-2$ we can select $v_i \in N(v_{i-1}, V_i)$ in at least $\delta^*(G) \geq n/2$ ways. Finally, we can select $v_{k-1} \in N(\{v_{k-2}, v_k\}, V_{k-1})$ in at least $2\delta^*(G) - n \geq \varepsilon n$ ways. □

Proposition 15. *For every $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for every $n \geq n_0$ the following holds. Let G be a subgraph of the n-blow-up of a triangle with parts V_1, V_2, V_3. If $\delta_1 \geq \delta_2 \geq \delta_3 \geq (1+\varepsilon)n/2$ are such that $\delta(G[V_i, V_{i+1}]) \geq \delta_i$ for every $i \in [3]$, and*

$$\frac{\delta_1 + \delta_2}{2} \geq \left(\frac{2}{3} + \varepsilon\right) n,$$

then G is $(\varepsilon^3/100, 5)$-linked.

Proof. There are at least $n \cdot \delta_3 \cdot (\delta_1 + \delta_2 - n) \geq n^3/6$ triangles in G, because we can pick any $w_3 \in V_3$, then any $w_1 \in N(w_3, V_1)$ and then any $w_2 \in N(w_1) \cap N(w_3) \cap V_2$ to form a triangle. We will also need the following fact:

$$\forall u_1, u_1' \in V_1 \text{ there are at least } 6\varepsilon^2 n^2 \text{ edges } u_2u_3 \text{ s.t. } \backslash\ u_1u_2u_3 \text{ and } u_1'u_2u_3 \text{ are triangles.} \quad (5)$$

To see (5), note that there are at least $2\delta_3 - n \geq 2\varepsilon n$ ways to pick a vertex $u_3 \in V_3$ adjacent to both u_1 and u_1' and that then there are at least $2\delta_1 + \delta_2 - 2n \geq 3(\delta_1 + \delta_2)/2 - 2n \geq 3\varepsilon n$ ways to select a vertex $u_2 \in V_2$ that is adjacent to u_1, u_1' and u_3.

The fact that there are at least $n^3/6$ triangles and (5) immediately implies that, for every $v, v' \in V_1$, the number of $(v, v', 5)$-linking sequences is at least $\frac{\varepsilon^3 n^5}{100}$, because the sequence $(u_2, u_3, w_1, w_2, w_3)$ is a $(v, v', 5)$-linking sequence whenever vu_2u_3 and $v'u_2u_3$ are both triangles and $w_1w_2w_3$ is a triangle disjoint from $\{v, v', u_2, u_3\}$.

So we are left to consider the case when $v, v' \in V_i$ for $i \in \{2, 3\}$. Let $j \in \{2, 3\}$ so $i \neq j$. We can pick $u_j \in N(v) \cap N(v') \cap V_j$ in at least $2\delta_2 - n \geq 2\varepsilon n$ ways. Then we can pick $u_1 \in N(v) \cap N(u_j) \cap V_1$ in at least $\delta_1 + \delta_3 - n \geq n/6$ ways. Similarly, we can now pick $u_1' \in N(v') \cap N(u_j) \cap V_1$ distinct from u_1 in at least $\delta_1 + \delta_3 - n - 1 \geq n/6$ ways. Observe that vu_ju_1 and $v'u_ju_1'$ are both triangles. By (5), there are at least $\frac{1}{2} \cdot 6\varepsilon^2 n^2$ ways to now pick u_2 and u_3 such that $u_1u_2u_3$ and $u_1'u_2u_3$ are both triangles and such that u_2 and u_3 are disjoint from $\{v, v', u_j\}$. All together there are at least

$$2\varepsilon \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot 3\varepsilon^2 \cdot n^5 \geq \frac{\varepsilon^3 n^5}{100}$$

ways to make these selection. To complete the proof, we observe that every such selection $(u_j, u_1, u_2, u_3, u_1')$ is a $(v, v', 5)$-linking sequence, because vu_ju_1 and $u_1'u_2u_3$ are both triangles and $v'u_ju_1'$ and $u_1u_2u_3$ are both triangles. □

Lemma 16. *Let $k \geq 3$. If Conjecture 11 holds for k, then Conjecture 5 holds for k.*

Proof. We can assume σ is small enough and n is large enough so that the following holds:

- $\sigma^{1/2} < \varepsilon$ and, because Observation 8 implies that if Conjecture 11 holds for k, then Conjecture 11 holds for every ℓ less than k, we can assume that for every $n' \geq (1 - \sigma^{1/2})\, n$ and for every $3 \leq \ell \leq k$, we can apply Conjecture 11 with ℓ, σ, n' and $\varepsilon - \sigma^{1/2}$ playing the roles of k, σ, n and ε, respectively;

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- for every $4 \leq \ell \leq k$, we can apply Lemma 13 with ℓ, $\ell - 1$, $\varepsilon^3/2^\ell$ and $\sigma^{1/2}$ playing the roles of k, t, η and σ, respectively; and
- we can apply Lemma 13 with 3, 5, $\varepsilon^3/100$ and $\sigma^{1/2}$ playing the roles of k, t, η and σ, respectively.

Let G and $\delta_1, \ldots, \delta_k$ be as in the statement of Conjecture 5. Let $I := \left\{i \in [k] : \delta_i < (1+\varepsilon)\frac{n}{2}\right\}$, let $\ell = k - |I|$ and let $i_1 < \cdots < i_{|I|}$ be an ordering of the elements of I. In the manner described after the statement of Theorem 6, iteratively, for j from 1 to $|I|$, we can match every $v \in V_{i_j}$ to a unique $f_v \in V_{i_j+1}$ and then collapse the edge vf_v into f_v. Let G' be the resulting graph, so G' will be a subgraph of the n-blow-up of C_ℓ such that a transversal C_ℓ factor of G' corresponds to a transversal C_k factor of G. For convenience, we relabel the parts of G' as $V'_1, \ldots, V'_\ell$ so that, for $i \in [\ell]$, we have $G'[V'_i, V'_{i+1}] \geq \delta'_i$. Note that $\delta'_i \geq (1+\varepsilon)\frac{n}{2}$ for $i \in [\ell]$ and

$$\sum_{i=1}^{\ell} \delta'_i = \sum_{i=1}^{k} \delta_i - \sum_{j=1}^{k-\ell} \delta_{i_j} > k\left(1 + \frac{1}{k} + \varepsilon\right)\frac{n}{2} - (k-\ell)(1+\varepsilon)\frac{n}{2} = \ell\left(1 + \frac{1}{\ell} + \varepsilon\right)\frac{n}{2}. \quad (6)$$

Clearly (6) implies $\ell \geq 2$ and that we can assume $\ell \geq 3$ by Remark 10. If $\ell = 3$, then Proposition 15 implies that G is $\left(\varepsilon^3/100, 5\right)$-linked and if $\ell \geq 4$, Proposition 14 implies that G is $(\varepsilon^3/2^\ell, \ell - 1)$-linked. So by the selection of σ and n, we can apply Lemma 13 with G' and $\sigma^{1/2}$ playing the roles of G and σ to find a set $A \subseteq V(G')$ with $z = |V'_i \cap A| \leq \sigma^{1/2}$ for $i \in [\ell]$ guaranteed by Lemma 13. Conjecture 11 then implies that $G' - A$ has a transversal C_ℓ-tiling of size at least $n - z - \sigma n$ which implies that G' has a transversal C_ℓ-factor. This in turn implies that G has a transversal C_k-factor. □

3. Proof of Theorem 3

Informally the proof of Theorem 3 proceeds as follows: Given a spanning subgraph of the n-blow-up of C_k with parts $V_1, \ldots, V_k$ that satisfies the degree condition (2), we independently select, for every $i \in k$ and for large $T := T(k, \varepsilon)$, a partition of almost all of V_i into $T+1$ parts $U_{i,1}, W_{i,1}, W_{i,2}, \ldots, W_{i,T}$ each of size mk. The Chernoff and union bounds imply that, if n is sufficiently large, there exists an outcome where, for every $i \in [k]$ and every $v \in V_{i-1} \cup V_{i+1}$, the vertex v has at least $(1 + 1/k + \varepsilon/2)mk/2$ neighbours in each of the $T+1$ parts of V_i. Therefore, for t from 1 to T, we can iteratively apply the following lemma (Lemma 17) to find a transversal C_k-tiling $\mathcal{T}_t$ of size mk contained in $\bigcup_{i \in k}\left(U_{i,t} \cup W_{i,t}\right)$ so that, if, for $i \in [k]$, we let $U_{i,t+1}$ be the vertices in $U_{i,t} \cup W_{i,t}$ uncovered by $\mathcal{T}_t$, we can continue with the next iteration. In this way, we can cover almost all of the vertices, so with absorbing (i.e., Proposition 14 and Lemma 13) we can find a transversal C_k-factor.

Lemma 17. *For $\varepsilon > 0$ and integer $k \geq 3$, there exists $m_0 := m_0(k, \varepsilon)$ such that for every $m \geq m_0$ the following holds for $n \geq 2mk$. Suppose that G is a subgraph of the n-blow-up of C_k with parts $V_1, \ldots, V_k$, and that, for every $i \in [k]$, there exist disjoint $U_i, W_i \subseteq V_i$ where $|U_i| = |W_i| = mk$ and the following conditions hold for every $v \in V_{i-1} \cup V_{i+1}$:*

(A) $d(v, U_i) \geq (1+\sigma)mk/2$, and

(B) $d(v, W_i) \geq \left(1 + 1/k + \sigma\right)mk/2$.

Then G contains a transversal C_k-tiling $\mathcal{T}$ of size mk contained in $\bigcup_{i \in [k]} U_i \cup W_i$ such that for every $i \in [k]$ and every $v \in V_{i-1} \cup V_{i+1}$ with $U'_i := (U_i \cup W_i) \setminus V(\mathcal{T})$ we have

$$d\left(v, U'_i\right) \geq (1+\sigma)mk/2.$$

Proof. For every $i \in [k]$, independently and uniformly at random select a partition of U_i into parts $U_{i,1}, \ldots, U_{i,k}$ each of size m. For every $i, j \in [k]$ and every $v \in V_{i-1} \cup V_{i+1}$, the random variable

$d(v, U_{i,j})$ is hypergeometrically distributed with expected value $d(v, U_i)\frac{|U_{i,j}|}{|U_i|} = \frac{d(v,U_i)}{k}$. Therefore, by (C1) and the Chernoff and union bounds, there exists an outcome such that for every $i, j \in [k]$ and every $v \in V_{i-1} \cup V_{i+1}$ we have

$$d(v, U_{i,j}) \geq m/2. \tag{7}$$

This implies that for every $i, j \in [k]$, the bipartite graph $G[U_{i,j}, U_{i+1,j}]$ is balanced with parts of size m and minimum degree at least $m/2$, so, by Hall's Theorem, it contains a perfect matching $M_{i,j}$. For $j \in [k]$, let H_j be the graph with vertex set $U_{1,j} \cup \cdots \cup U_{k,j}$ such that

$$E(H_j) := \bigcup_{i=1}^{k} M_{i,j} \setminus (M_{j-1,j} \cup M_{j,j}).$$

Note that H_j consists of a collection $\mathcal{P}_j$ of m vertex disjoint paths each on $k-1$ vertices such that

- $V(\mathcal{P}_j) = V(H_j) \setminus U_{j,j}$;
- every $P \in \mathcal{P}_j$ has exactly one vertex in each of the sets $U_{1,j}, \ldots, U_{k,j}$ except $U_{j,j}$; and
- every $P \in \mathcal{P}_j$ has one end-vertex in $U_{j-1,j}$ and the other end-vertex in $U_{j+1,j}$.

By (C2), the number of common neighbours in W_j of the endpoints of every path in $\mathcal{P}_j$ is at least

$$2(1 + 1/k + \sigma)mk/2 - mk > m = |\mathcal{P}_j|.$$

Therefore, we can greedily select such a common neighbour for every path in $\mathcal{P}_j$ to form $\mathcal{T}_j$, a transversal C_k-tiling of size m. The union $\mathcal{T} := \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_k$ is a transversal C_k-tiling of G of size mk. For every $i \in [k]$, let $U'_i := (U_i \cup W_i) \setminus V(\mathcal{T}) = U_{i,i} \cup (W_i \setminus V(\mathcal{T}))$, so

$$|U'_i| = |U_{i,i}| + |W_i| - m = mk.$$

With (C2) and (7), for every $v \in V_{i-1} \cup V_{i+1}$, we have that

$$d(v, U'_i) \geq m/2 + (1 + 1/k + \sigma)mk/2 - m = (1 + \sigma)mk/2.$$

□

Proof of Theorem 3. Define $\eta := k^{-3} \cdot 2^{-k}$ and $\sigma := \min\{\varepsilon/4, 0.1\eta^{k+1}/(k^4 + 1)\}$. By Proposition 14, G is $(\eta, k-1)$-linked. Let A be the set guaranteed by Lemma 13, so there exists $z \leq \sigma n$ such that $|A \cap V_i| = z$ for every $i \in [k]$. Let $m := \left\lfloor \frac{\sigma^2 n}{2k} \right\rfloor$ and let $T := \left\lfloor \frac{n-z}{mk} \right\rfloor - 1$ and note that $(T+1)mk \leq n - z \leq (T+2)mk$ and that T is bounded above by a constant that depends only on k and ε. For every $i \in [k]$, let $V'_i \subseteq V_i \setminus A$ where $|V'_i| = (T+1)mk$. We will construct T disjoint transversal C_k-tilings each of size mk that each avoid A. Because

$$mkT = (T+2)mk - 2mk \geq n - z - \sigma^2 n,$$

this will imply the theorem by the properties of A from Lemma 13.

Note that, by (2), for every $i \in [k]$ and $v \in V_{i-1} \cup V_{i+1}$, we have

$$d(v, V'_i) \geq \delta^*(G) - (n - |V'_i|) \geq (1 + 1/k + 2\sigma)\, n/2 \geq (1 + 1/k + 2\sigma)\, |V'_i|/2. \tag{8}$$

For every $i \in [k]$, independently and uniformly at random select a partition of V'_i into $T+1$ parts $W_{i,0}, \ldots, W_{i,T}$ each of size mk. For every $i \in [k]$, every $0 \leq t \leq T$ and every $v \in V_{i-1} \cup V_{i+1}$, the random variable $d(v, W_{i,t})$ is hypergeometrically distributed with expected value $d(v, V'_i)\, \frac{|W_{i,t}|}{|V'_i|}$. Therefore, by (8) and the Chernoff and union bounds, there exists an outcome such that for every $i \in [k]$, $0 \leq t \leq T$ and $v \in V_{i-1} \cup V_{i+1}$ we have

$$d(v, W_{i,t}) \geq (1 + 1/k + \sigma)mk/2. \tag{9}$$

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We will now show by induction on t from 1 to $T+1$ that there exist $t-1$ disjoint transversal C_k-tilings $\mathcal{T}_1, \ldots, \mathcal{T}_{t-1}$ each of size mk that are contained in $\bigcup_{i\in k}\bigcup_{s=0}^{t-1} W_{i,s}$, and that, for every $i \in [k]$, if we let $U_{i,t} := \left(\bigcup_{s=0}^{t-1} W_{i,s}\right) \setminus \left(\bigcup_{s=1}^{t-1} V(\mathcal{T}_s)\right)$, then the following holds:

$$d(v, U_{i,t}) \geq (1+\sigma)mk/2 \qquad \text{for every } v \in V_{i-1} \cup V_{i+1}. \tag{10}$$

This will prove the theorem.

For the base case, note that when $t=1$ we have that $U_{i,t} = U_{i,1} = W_{i,0}$ for every $i \in [k]$ so (10) holds by (9). Now assume the induction hypothesis holds for some $1 \leq t \leq T$. With (9) and (10) we can apply Lemma 17 to find a tiling $\mathcal{T}_t$ of size mk contained in $\bigcup_{i\in[k]} U_{i,t} \cup W_{i,t}$ such that, for every $i \in [k]$, we have that

$$U_{i,t+1} = \left(\bigcup_{s=0}^{t} W_{i,s}\right) \setminus \left(\bigcup_{s=1}^{t} V(\mathcal{T}_s)\right) = \left(U_{i,t} \cup W_{i,t}\right) \setminus V(\mathcal{T}_t)$$

satisfies (10) with t set to $t+1$. Therefore, the induction hypothesis holds for $t+1$. □

4. Proof of Theorem 6

Because Theorem 18 works for every n and the degree condition is weaker than Theorem 6, it might have independent interest. Note that Theorem 18 is stronger than the $k=3$ case of Conjecture 11, so Lemma 16 and Theorem 18 together imply Theorem 6.

Theorem 18. *The following holds for every $n \in \mathbb{N}$ and every subgraph G of the n-blow-up of C_3 with parts V_1, V_2, V_3. If there exist $\delta_1, \delta_2, \delta_3 \geq n/2$ such that $\delta_1 + \delta_2 + \delta_3 \geq 2n$ and*

$$\delta(G[V_i, V_{i+1}]) \geq \delta_i \qquad \textit{for every } i \in [3],$$

then G has a transversal C_3-tiling of size at least $n-1$.

Proof. For brevity, in this proof we call a transversal C_3-tiling a *tiling*. Let the size of a maximum tiling of G be m and let us assume for a contradiction that $m \leq n-2$. Call a pair of edges e and f *dissimilar* if $e \in E(G[V_i, V_{i+1}])$ and $f \in E\big(G\big[V_j, V_{j+1}\big]\big)$ for distinct $i, j \in [3]$. Call a set $F \subseteq E(G)$ a *dissimilar matching* if the edges in F are disjoint and the edges in F are pairwise dissimilar. For every maximum tiling $\mathcal{T}$, let $h(\mathcal{T})$ be the maximum size of a dissimilar matching $F \subseteq E(G[U(\mathcal{T})])$ such that every edge in F is uncovered by $\mathcal{T}$. Recall that an edge e is uncovered by $\mathcal{T}$ if both endpoints of e are disjoint from $V(\mathcal{T})$. Let $\{\alpha, \beta, \gamma\} = \{\delta_1/n, \delta_2/n, \delta_3/n\}$ and $\{A, B, C\} = \{V_1, V_2, V_3\}$ be labellings such that $\alpha \leq \beta \leq \gamma$ and

$$\delta(G[B,C]) \geq \alpha n, \quad \delta(G[A,C]) \geq \beta n, \quad \text{and} \quad \delta(G[A,B]) \geq \gamma n. \tag{11}$$

□

Claim 18.1. *Let $\mathcal{T}$ be a maximum tiling. If $e \in E(G)$ is uncovered by $\mathcal{T}$, then $d(e, U(\mathcal{T})) = 0$. Furthermore, if e and f are disjoint dissimilar edges that are uncovered by $\mathcal{T}$, then $d(e, T) + d(f, T) \leq 1$ for every $T \in \mathcal{T}$.*

Proof. If e is an edge uncovered by $\mathcal{T}$ and $x \in U(\mathcal{T})$ is such that $d(e, \{x\}) = 1$, then ex is triangle, and adding ex to $\mathcal{T}$ creates a tiling of size $m+1$, a contradiction. Similarly, if e and f are disjoint and dissimilar edges that are uncovered by $\mathcal{T}$ and $d(e,T) + d(f,T) \geq 2$ for some $T \in \mathcal{T}$, then, because e and f are dissimilar, there exist distinct $x, y \in T$ such that ex and fy are both triangles, so if we replace T with ex and fy in $\mathcal{T}$, then we have a tiling of size $m+1$, a contradiction. □

Claim 18.2. *Let $\mathcal{T}$ be a maximum tiling and let F be a dissimilar matching with $|F| = 3$. Then either there exists $e \in F$ such that $d(e, U(\mathcal{T})) \geq 1$ or there exists $T \in \mathcal{T}$ such that $\sum_{e\in F} d(e, T) \geq 2$. Consequently, $h(\mathcal{T}) \leq 2$ for every maximum tiling $\mathcal{T}$.*

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Proof. Let $\{a_1,\dots,a_n\}$, $\{b_1,\dots,b_n\}$ and $\{c_1,\dots,c_n\}$ be orderings of A, B and C, respectively, such that $a_ib_ic_i \in \mathcal{T}$ for every $i \in [m]$ (so, when $m+1 \le i \le n$, $a_ib_ic_i$ is not a triangle). By (11), we have

$$\sum_{i=1}^{n}\sum_{e\in F} d(e,a_ib_ic_i) = \sum_{e\in F} d(e,V(G)) \ge (\alpha+\beta-1)n + (\alpha+\gamma-1)n + (\beta+\gamma-1)n \ge n.$$

Therefore, if $0=\sum_{e\in F} d(e,U(\mathcal{T})) = \sum_{i=m+1}^{n}\sum_{e\in F} d(e,a_ib_ic_i)$, then there exists $1\le i\le m$ such that $\sum_{e\in F} d(e,a_ib_ic_i)=2$. This proves the first statement.

To see the second statement, assume for a contradiction that there exists a maximum tiling $\mathcal{T}$ such that $h(\mathcal{T})=3$. This means that there exists a dissimilar matching F such that $|F|=3$ and such that every edge in F is uncovered by $\mathcal{T}$. By the first part of the statement, there either exists $e\in F$ such $d(e,U(\mathcal{T}))\ge 1$, or there exist two edges $e,f\in F$ and $T\in\mathcal{T}$ such that $d(e,T)+d(f,T)\ge 1$. Because every edge in F is uncovered by $\mathcal{T}$, this contradicts Claim 18.1. □

Claim 18.3. *There exists a maximum tiling $\mathcal{T}$ such that $h(\mathcal{T})=2$, and for every maximum tiling $\mathcal{T}$ there does not exist $e\in E(G[B,C])$ which is uncovered by $\mathcal{T}$.*

Proof. Suppose for a contradiction that the statement is false and assume $\mathcal{T}$ and a dissimilar matching F in $G[U(\mathcal{T})]$ have both been selected so that

(A) there exists $e\in F$ such that $e\in E(G[B,C])$ if possible, and,

(B) subject to (A), $|F|$ is as large as possible.

Note that Claim 18.2, implies that $|F|\le h(\mathcal{T})\le 2$, so if there exists $e\in F$ such that $e\in E(G[B,C])$, then F has at most one edge that is contained in $E(G[A,B])\cup E(G[A,C])$. If there is no $e\in F$ that is in $E(G[B,C])$, then by the selection of $\mathcal{T}$ and F (c.f. (A)), for every maximum tiling $\mathcal{T}$ there does not exist $e\in E(G[B,C])$, so our contrary assumption implies $|F|\le h(\mathcal{T})\le 1$. Therefore, in all cases, F has at most one edge that is contained in $E(G[A,B])\cup E(G[A,C])$. Let $\{X,Y\}=\{B,C\}$ be a labelling such that F does not contain an edge in $E(G[A,X])$.

Let $W\subseteq U$ be the set of vertices that are incident to an edge in F. The fact that $|\mathcal{T}|\le n-2$, implies that there exist nonadjacent vertices $a\in A\setminus W$ and $x\in X\setminus W$ that are uncovered by $\mathcal{T}$. Let $\{a_1,\dots,a_n\}$, $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$ be orderings of A, X and Y, respectively such that $a_n=a$, $x_n=x$ and $a_ix_iy_i\in\mathcal{T}$ for every $i\in[m]$. We can assume that the orderings are such that W is contained in the set $\left\{a_{n-1},x_{n-1},y_{n-1},y_n\right\}$ with $x_{n-1}y_{n-1}\in F$ if $F\cap E(G[X,Y]) = F\cap E(G[B,C])\neq\emptyset$.

Since a and x are nonadjacent, $d(x,a_n)+d(a,x_n)+d(a,y_n)=d(x,a)+d(a,x)+d(a,y_n)\le 1$, and, by (11) and the fact that $\alpha\le\beta\le\gamma$, we have

$$\sum_{i=1}^{n} d(x,a_i)+d(a,x_i)+d(a,y_i) = d(x,A)+d(a,X)+d(a,Y)\ge \beta n+\beta n+\gamma n\ge 2n,$$

so there must exist $i\in[n-1]$ such that $d(x,a_i)+d(a,x_i)+d(a,y_i)=3$. Note that $i\neq n-1$, because if ax_{n-1} is an edge, then the fact that $F\cap E(G[A,X])=\emptyset$ and the maximality of F imply that $x_{n-1}y_{n-1}\in F$, but, because $\mathcal{T}$ is a maximum tiling, $ax_{n-1}y_{n-1}$ is not a triangle. Since $F\cap E(G[A,X])=\emptyset$, the maximality of F also implies that that $d(x,a_j)=0$ for every $m+1\le j\le n-2$, so it must be that $i\in[m]$, that is, that $T=a_ix_iy_i$ is a triangle in $\mathcal{T}$. Therefore, we can swap T for the triangle ax_iy_i in $\mathcal{T}$ to form the maximum tiling $\mathcal{T}'$. Because the edge a_ix is uncovered by $\mathcal{T}'$ and $W\subseteq U(\mathcal{T}')$, we have a contradiction to the selection of $\mathcal{T}$ and F (c.f. (B)). □

Claim 18.3 implies that there exists a maximum tiling $\mathcal{T}$ such that $h(\mathcal{T})=2$. By Claim 18.3, we can assume that $\mathcal{T}$ leaves no edge in $E(G[B,C])$ uncovered by $\mathcal{T}$. This means that there are disjoint edges $ab\in E(G[A,B])$ and $a'c\in E(G[A,C])$ with $a,a'\in A$ that are uncovered by $\mathcal{T}$. Since $|\mathcal{T}|=$

$m \le n-2$, there also exists $b' \in B \setminus \{b\}$ and $c' \in C \setminus \{c\}$ that are uncovered by $\mathcal{T}$. Furthermore, the fact that no edge in $E(G[B, C])$ is uncovered implies that

$$\sum_{T\in\mathcal{T}} d\big(b', T\cap C\big) + d\big(c', T\cap B\big) = d\big(b', C\big) + d\big(c', B\big) \ge 2\delta(G[B, C]) \ge n > m = |\mathcal{T}|,$$

so there exists $T \in \mathcal{T}$ such that $d\big(b', T\cap C\big) + d\big(c', T\cap B\big) = 2$. Let e be the edge incident to c' and $T\cap B$ and let e' be the edge incident to b' and $T\cap C$. If we define $F := \{ab, a'c, e\}$, then F is a dissimilar matching, so Claim 18.2 implies that we are in one of the following two cases.

Case 1: *There exists $f \in F$ such that $d(f, U(\mathcal{T})) \ge 1$.* By Claim 18.1, the fact that ab and $a'c$ are uncovered by $\mathcal{T}$ implies that $f = e$. So, there exists a triangle T' that contains e and a vertex in $U \cap A$. This means that we can create a maximum tiling by replacing T with T' in $\mathcal{T}$ that leaves the edge $e' \in E(G[B, C])$ uncovered, contradicting Claim 18.3.

Case 2: *There exists $T' \in \mathcal{T}$ such that $\sum_{f\in F} d(f, T') \ge 2$.* By Claim 18.1, $d(ab, T') + d(a'c, T') \le 1$, so there is $f \in \{ab, a'c\}$ such that $d(e, T') + d(f, T') \ge 2$. This means that there exist two triangles, say T'', T''', in the graph induced by the vertices incident to e, f and T'. Therefore, we can create a new tiling, say $\mathcal{T}'$, by removing T and T' from $\mathcal{T}$ and replacing them with T'' and T'''. Since $\mathcal{T}$ is maximum tiling, we have that $T \ne T'$ and that $\mathcal{T}'$ is a maximum tiling. Because $T \ne T'$, the edge $e' \in E(G[B, C])$ is uncovered by $\mathcal{T}'$ which contradicts Claim 18.3.

5. Proof of Theorem 7

The following example proves the second part of the theorem

Example 19. For the case $\gamma = 2/3$, it can be checked that Example 4 for $k = 3$ satisfies the second claim of Theorem 7.

For the case $\gamma \in \left(\frac{3}{4}, \frac{7}{9}\right]$, we assume n satisfies the following: $\gamma \ge \frac{3}{4} + \frac{1}{n}$ and $(1-\beta)n/2$ is an integer. Clearly, since β is rational and $\gamma > 4/3$, there are infinitely many choices for such n. Let us fix $\varepsilon \in \left(0, \frac{1}{n}\right]$ such that $(1-\gamma+\varepsilon)n$ is an integer.

Take sets $A = A_0 \cup A_1 \cup A_2 \cup A_3$, $B = B_0 \cup B_1 \cup B_2 \cup B_3$ and $C = C_0 \cup C_1 \cup C_2 \cup C_3$, such that:

- $|B_i| = (1-\gamma+\varepsilon)n$, $|A_i| = |C_i| = (1-\beta)n/2$, for $i \in [3]$;
- $|B_0| = n - 3(1-\gamma+\varepsilon)n = (3\gamma-2)n - 3\varepsilon n$; and
- $|A_0| = |C_0| = n - 3(1-\beta)n/2 = (3\beta-1)n/2$.

Let G be the 3-partite graph with parts A, B and C, where $E(G)$ consists of the union of the edges in the following graphs:

- the complete bipartite graphs with parts $A_0, B \cup C$ and $B_0, A \cup C$ and $C_0, A \cup B$.
- the complete bipartite graphs with parts $A_1, B_2 \cup B_3$ and $A_2, B_1 \cup B_3$ and $A_3, B_1 \cup B_2$.
- the complete bipartite graphs with parts B_i, C_i and A_i, C_i for each $i \in [3]$.

Since $\gamma \le \frac{7}{9}$ and $\varepsilon > 0$, we have $(1-\gamma+\varepsilon)n > (\gamma - 1/3)n/2 = (1-\beta)n/2$. So,

- $\delta(G[A, B]) = n - (1-\gamma+\varepsilon)n = \gamma n - \varepsilon n$,
- $\delta(G[B, C]) = n - 2(1-\gamma+\varepsilon)n = (2\gamma-1)n - 2\varepsilon n$, and
- $\delta(G[A, C]) = n - 2(1-\beta)n/2 = \beta n$.

Recall that $\varepsilon \le \frac{1}{n}$ and $\gamma \ge \frac{3}{4} + \frac{1}{n}$, so $\delta(G[A, B]) \ge \gamma n - 1$ and $\delta(G[B, C]) \ge (2\gamma-1)n - 2 \ge n/2$.

Note that $A_0 \cup B_0 \cup C_0$ is a triangle cover and

$$|A_0| + |B_0| + |C_0| = (3\beta-1)n/2 + (3\gamma-2)n - 3\varepsilon n + (3\beta-1)n/2 = (1-3\varepsilon)n < n.$$

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We now proceed with the proof of the first part of the Theorem 7. We will use the following definition throughout the proof.

Definition 20. For $U, W, U' \subseteq V(G)$, let $P_3(U, W, U')$ be the set of paths on 3 vertices in which the middle vertex is in W, one endpoint is in U and the other endpoint is in U'. When a set $\{u\}$ is a singleton, we sometimes replace $\{u\}$ with u in this notation.

Let $\{\alpha, \beta, \gamma\} = \{\delta_1/n, \delta_2/n, \delta_3/n\}$ and $\{A, B, C\} = \{V_1, V_2, V_3\}$ be labellings such that $\alpha \leq \beta \leq \gamma$ and

$$\delta(G[B, C]) \geq \alpha n, \quad \delta(G[A, C]) \geq \beta n, \quad \text{and} \quad \delta(G[A, B]) \geq \gamma n.$$

We can assume $\gamma + \beta = 4/3$ and $\alpha = 1/2$. Therefore,

$$1/2 \leq \beta \leq 2/3 \qquad \text{and} \qquad 2/3 \leq \gamma \leq 5/6. \tag{12}$$

Let U be a triangle cover and let $x = |A \cap U|/n$, $y = |B \cap U|/n$ and $z = |C \cap U|/n$. For a contradiction, assume that

$$x + y + z < 1. \tag{13}$$

Let $A' = A \setminus U$, $B' = B \setminus U$ and $C' = C \setminus U$.

Using this notation, we now give an informal sketch of the proof. We use the fact that $G[A', B', C']$ is triangle-free to iteratively improve bounds on x, y and z until we derive a contradiction.

We start with the simple observation that the common neighbourhood of the endpoints of every edge in $G[A', B', C']$ must be in U, so $x \geq \gamma + \beta - 1 = 1/3$, $y \geq \gamma + \alpha - 1 = \gamma - 1/2$ and $z \geq \beta + \alpha - 1 = \beta - 1/2$ (Claim 20.1). Since $1 > x + y + z$ and $\gamma \geq 2/3$, these lower bounds imply that $x, y < \gamma$ and $\beta < 1/2$. We derive that $y < 1/2$ by noting that if $y \geq 1/2$, then for every $b \in B'$ and every $a \in N_{\overline{G}}(b, A')$ there exists $b' \in B'$ and $a' \in A'$ such that $ab'a'b$ is a 4-vertex path. This then implies that $|P_3(b, C', a)| \leq |C'| - d(a', C) - d\big(b', C\big)$, which further implies an upper bound on $|P_3(b, C', A')|$ that contradicts the fact that

$$\big|P_3\big(b, C', A'\big)\big| = \sum_{c \in N(b, C')} d(c, A') \geq (\alpha - z)n \cdot (\beta - x)n.$$

So $y < 1/2$ (Claim 20.2). This means every vertex in C' has a neighbour in B'. We use this to get an upper bound on $G[A', C']$ which we compare to the lower bound given by the minimum degree. This inequality is then used to prove that there is a 4-vertex path in $G[A', B']$ between every nonadjacent $a \in A'$ and $b \in B'$ (Claim 20.3). Using these paths in the same manner as before, we derive upper and lower bounds on $|P_3(a, C', B')|$ and $|P_3(A', B', b)|$ for every $a \in A'$ and $b \in B'$. These bounds and the previous inequality imply that $y < 1/3$ and $x < \beta$ (Claim 20.4). A similar argument implies that there exist non-adjacent $a_1 \in A'$ and $c_1 \in C'$ such that there does not exist a 4-vertex path between a_1 and c_1 (Claim 20.5). We fix $a_2 \in N(c_1, A')$ and $c_2 \in N(a_1, C')$. We then observe that, by the selection of a_1 and c_1, the sets $N(a_1, C')$ and $N(a_2, C')$ are disjoint and the sets $A_1 = N(c_1, A')$ and $A_2 = N(c_2, A')$ are disjoint. Since a_1 and a_2 must have a common neighbour in B', we get that $z \geq \beta - 1/4$ (Claim 20.6). We can now deduce that, in $G[A', C']$, every $a \in A'$ has a 4-vertex path to either c_1 or c_2 (Claim 20.8). We let B_1 and B_2 be subsets of $N(c_1, B')$ and $N(c_2, B')$ with cardinality exactly $\lceil (1/2 - y)n \rceil$. Note that for every $a \in A_i$ every vertex in B_i is a non-neighbour of a and this yields an upper bound on the number of non-neighbours of a in $B \setminus (B_1 \cup B_2)$. Using the fact that for every $a \in A \setminus (A_1 \cup A_2)$ there is 4-vertex path in $G[A', C']$ from a to c_i, we can get an upper bound on the number of non-neighbours of a in $B \setminus (B_1 \cup B_2)$ (Claim 20.9). There observations lead to a upper bound on the number of non-neighbours between A' and $B' \setminus (B_1 \cup B_2)$. To get a contradiction, we note that every vertex in B' has a neighbour in C', which implies a lower bound on the number of non-neighbours between A' and $B' \setminus (B_1 \cup B_2)$ that is in conflict with the previous upper bound.

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We now proceed with the formal proof.

Claim 20.1. *$x \geq 1/3$, $y \geq \gamma - 1/2$ and $z \geq \beta - 1/2$.*

Proof. Since $y + z \leq x + y + z < 1$, one of y or z is less than $1/2$, so there exists an edge $bc \in E(G[B', C'])$. Because $G[A', B', C']$ is triangle-free,

$$0 = |N(b, A') \cap N(c, A')| \geq d(b, A) + d(c, A) - |A| - xn \geq \gamma n + \beta n - n - xn = n/3 - xn,$$

so $x \geq 1/3$. By considering an edge in $E(G[A', B'])$ and an edge in $E(G[A', C'])$ the same argument yields $z \geq \beta - 1/2$ and $y \geq \gamma - 1/2$, respectively. □

Claim 20.2. *$x < \gamma$, $y < 1/2$ and $z < 1/2$.*

Proof. We first show that both $x < \gamma$ and $y < \gamma$. To this end, note that if $x \geq \gamma$, then $x + y \geq \gamma + \gamma - 1/2 = 2\gamma - 1/2$. If $y \geq \gamma$, then, because $\gamma - 1/2 \leq 5/6 - 1/2 = 1/3$, we also have $x + y \geq 1/3 + \gamma \geq 2\gamma - 1/2$. So, in either case, we have the following contradiction

$$1 > x + y + z \geq 2\gamma - 1/2 + \beta - 1/2 = 1/3 + \gamma \geq 1.$$

Similarly, it is clear that $z < 1/2$, since otherwise $x + y + z \geq 1/3 + (\gamma - 1/2) + 1/2 \geq 1$, a contradiction.

Assume $y \geq 1/2$ and let $b \in B'$. Note that there are at most $(1 - \gamma)n$ vertices $a \in N_{\overline{G}}(b, A')$. For every such a we can find a 4-vertex path $ab'a'b$ in $G[A', B']$. Indeed, since $y < \gamma$, there exists $b' \in N(a, B')$. Then, because $2\gamma + \beta \geq 2$ and $1 > x + y + z$,

$$x < 1 - y - z \leq 1/2 - z \leq 1/2 - (\beta - 1/2) = 1 - \beta \leq 2\gamma - 1.$$

Since $|N(b, A) \cap N(b', A)| \geq 2\gamma n - n > xn$, there exists $a' \in |N(b, A') \cap N(b', A')|$, giving us the 4-path $ab'a'b$. Note that $N(a', C')$ and $N(b', C')$ are disjoint and that every $c \in C'$ that is adjacent to both a and b is not adjacent to a' and not adjacent to b'. Therefore, $|P_3(b, C', a)|$ is at most

$$|C' \setminus \big(N(a', C') \cup N(b', C')\big)| \leq (1 - z)n - (\beta - z)n - (1/2 - z)n = (z - \beta + 1/2)n,$$

and $|P_3(b, C', A')|/n^2 \leq (1 - \gamma)(z - \beta + 1/2)$. On the other hand,

$$|P_3(b, C', A')| \geq \sum_{c \in N(b, C')} d(c, A') \geq (1/2 - z)n \cdot (\beta - x)n.$$

The claim then follows because there are no solutions to

$$(1 - \gamma)(z - \beta + 1/2) \geq (1/2 - z)(\beta - x),$$

when $x \geq 1/3$, $y \geq 1/2$ and $z \geq \beta - 1/2$. (See Appendix B in [2] for a proof of this fact.) □

Note that Claim 20.2 implies that $\delta(G[A', B']) \geq 1$, $\delta(G[B', C']) \geq 1$, and that every vertex in A' has a neighbour in C'. (We do not yet know if every vertex in C' has a neighbour in A'.) We will use these facts in the rest of the argument without comment.

In particular, the fact that every $c \in C'$ has a neighbour $b \in B'$ implies that

$$d(c, A') \leq |A' \setminus N(b, A')| \leq |A \setminus N(b, A)| \leq (1 - \gamma)n,$$

so $|E(A', C')| = \sum_{c \in C'} d(c, A') \leq |C'|(1 - \gamma)n = (1 - z)(1 - \gamma)n^2$. On the other hand, we have that $|E(A', C')| = \sum_{a \in A'} d(a, C') \geq |A'|(\beta - z)n = (1 - x)(\beta - z)n^2$. This yields the following useful inequality

$$(1 - \gamma)(1 - z) \geq (1 - x)(\beta - z). \tag{14}$$

Claim 20.3. *For every $a \in A'$ and $b \in B'$, there is an (a, b)-path in $G[A', B']$ with at most 4 vertices.*

Proof. Assume the contrary and let $a \in A'$ and $b \in B'$ be such that there is no (a, b)-path in $G[A', B']$ with at most 4 vertices. Let $b' \in N(a, B')$ and $a' \in N(b, A')$. By our contrary assumption,

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we have that $N(a,B')\cap N(a',B')=\emptyset$ so $y\geq |N(a,B)\cap N(a',B)|/n\geq 2\gamma-1$, By the same argument, $N(b,A')\cap N(b',A')=\emptyset$ and $x\geq 2\gamma-1$. Since $\beta\leq 2/3$, we have $2\gamma-1=2(4/3-\beta)-1=5/3-2\beta\geq 1-\beta$, so

$$z<1-x-y\leq 1-2(2\gamma-1)\leq 1-2(1-\beta)=2\beta-1\leq |N(a,C)\cap N(a',C)|/n,$$

therefore there exists $c\in N(a,C')\cap N(a',C')$. But then $N(c,B')$ cannot intersect $N(a,B')\cup N(a',B')$, so

$$(1/2-y)+2(\gamma-y)\leq |N(c,B')\cup N(a,B')\cup N(a',B')|/n\leq 1-y$$

which implies that $y\geq\gamma-1/4$, therefore $y\geq 5/12$. But (14) has no solutions when $y\geq 5/12$, $x\geq 1/3$ and $z\geq\beta-1/2$. (See Appendix B in [2] for a proof of this fact.) This is a contradiction. □

Claim 20.4. *$y<1/3$ and $x<\beta$.*

Proof. Let $a\in A'$. We first get an upper bound on $|P_3(a,C',B')|$. Note that there are at most $(1-\gamma)n$ ways to select $b\in B'$ that is not adjacent to a. By Claim 20.3, there exists $a'\in A'$ and $b'\in B'$ such that $ab'a'b$ is a path. Note that every vertex $c\in C'$ that is adjacent to both a and b cannot be in $N(a',C')\cup N(b',C')$. Since $N(a',C')$ and $N\big(b',C\big)$ are disjoint, we have the cardinality of $P_3(a,C',b)$ is at most

$$|C'|-d(a',C')-d(b',C')\leq (1-z)n-(\beta-z)n-(1/2-z)n=(z-\beta+1/2)n$$

Therefore, $|P_3(a,C',B')|/n^2\leq (1-\gamma)(z-\beta+1/2)$. We also have that $|P_3(a,C',B')|\geq \sum_{c\in N(a,C')} d(c,B')\geq (\beta-z)n(1/2-y)n$, so

$$(1-\gamma)(z-\beta+1/2)\geq |P_3(a,C',B')|/n^2\geq (\beta-z)(1/2-y). \tag{15}$$

By considering $b\in B'$ and estimating $P_3(A',C',b)$, the same arguments yield that

$$(1-\gamma)(z-\beta+1/2)\geq |P_3(A',C',b)|/n^2\geq \sum_{c\in N(b,C')} d(c,A')/n^2\geq (\beta-x)(1/2-z). \tag{16}$$

But (15), (16) and (14) cannot hold simultaneously when $x\geq 1/3$, $y\geq 1/3$ and $z\geq\beta-1/2$. (See Appendix B in [2] for a proof of this fact.) Therefore $y<1/3$.

Now we will show that $x<\beta$. Indeed, if $\beta\leq x$, we have

$$y\geq\gamma-1/2=5/6-\beta\geq 5/6-x>y+z-1/6,$$

so $z<1/6$. With (15) we get $(\beta-1/3)(1/6-\beta+1/2)\geq(\beta-1/6)(1/2-y)$. Plugging $y<1/3$ we get that $-\beta^2+(5/6)\beta-1/4>0$ which does not have a solution, a contradiction. □

Note that Claims 20.2 and 20.4 together imply $\delta(G[A',B']),\delta(G[B',C']),\delta(G[C',A'])\geq 1$.

Claim 20.5. *There exists $a_1\in A'$ and $c_1\in C'$ such that there is no (a_1,c_1)-path in $G[A',C']$ with at most 4-vertices.*

Proof. Assume the contrary and let $a_1\in A'$. Then, for every $c_1\in C'\setminus N(a',C')$, there exists $a_2\in A'$ and $c_2\in C'$ such that $a_1c_2a_2c_1$ is a path, so, since $G[A',B',C']$ is triangle-free, $|P_3(a_1,B',c_1)|$ is at most

$$|B'\setminus(N(a_2,B')\cup N(c_2,B')|\leq |B'|-(d(a_2,B)-yn+d(c_2,B)-yn)\leq\big(y-\gamma+1/2\big)n.$$

Since a_1 has at most $(1-\beta)n$ non-neighbours in C', we have that

$$(1-\beta)\left(y-\gamma+1/2\right)\geq |P_3(a_1,B',C')|/n^2=\sum_{b\in N(a_1,B')} d(b,C')/n^2\geq(\gamma-y)\,(1/2-z)$$

which is impossible when $x\geq 1/3$, $1/3>y\geq\gamma-1/2$ and $z\geq\beta-1/2$. (See Appendix B in [2] for a proof of this fact.) □

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By Claim 20.5, there exists $a_1 \in A'$ and $c_1 \in C'$ such that there is no (a_1, c_1)-path in $G[A', C']$ with at most 4-vertices. Fix such vertices a_1 and c_1. By Claims 20.2 and 20.4, we can also fix $c_2 \in N(a_1, C')$ and $a_2 \in N(c_1, A')$. Note that, by the selection of a_1 and c_1,

$$N(a_1, C') \cap N(a_2, C') = \emptyset \text{ and } N(c_1, A') \cap N(c_2, A') = \emptyset. \tag{17}$$

Claim 20.6. $z \geq \beta - 1/4$.

Proof. Since $|N(a_1, B) \cap N(a_2, B)|/n \geq 2\gamma - 1 \geq 1/3 > y$, there exists $b \in B'$ that is adjacent to both a_1 and a_2. Since $G[A', B', C']$ is triangle-free (17) implies that

$$1/2 - z \leq d(b, C')/n \leq |C' \setminus (N(a_1, C') \cup N(a_2, C'))|/n \leq 1 - z - 2(\beta - z) = 1 - 2\beta + z,$$

so $z \geq \beta - 1/4$. □

Claim 20.7. *At least one of the following statements is true.*

- *For every $a \in A'$, we have that $N(a, C')$ intersects $N(a_1, C') \cup N(a_2, C')$.*
- *For every $c \in C'$, we have that $N(c, A')$ intersects $N(c_1, A') \cup N(c_2, A')$.*

Proof. Assume the contrary, so there exists $a_3 \in A'$ such that $N(a_1, C')$, $N(a_2, C')$ and $N(a_3, C')$ are pairwise disjoint and that there exists $c_3 \in C'$ such that $N(c_1, A')$, $N(c_2, A')$ and $N(c_3, A')$ are pairwise disjoint. This implies that

$$(1 - x)n = |A'| \geq d(c_1, A') + d(c_2, A') + d(c_3, A') \geq 3(\beta - x)n,$$

so $x \geq (3\beta - 1)/2$, and, by considering the sets $N(a_1, C')$, $N(a_2, C')$ and $N(a_3, C')$, we similarly have that $z \geq (3\beta - 1)/2$. This implies that

$$y < 1 - x - z \leq 1 - (3\beta - 1) = 2 - 3(4/3 - \gamma) = 3\gamma - 2.$$

Note that $|N(a_1, B) \cap N(a_2, B) \cap N(a_3, B)| \geq 3\gamma n - 2|B| = (3\gamma - 2)n > yn$, so there exists $b \in N(a_1, B') \cap N(a_2, B') \cap N(a_3, B')$. Note that $N(b, C')$ must be disjoint from $N(a_1, C') \cup N(a_2, C') \cup N(a_3, C')$ so, since $N(a_1, C')$, $N(a_2, C')$ and $N(a_3, C')$ are pairwise disjoint,

$$(1 - z)n = |C'| \geq d(b, C') + d(a_1, C') + d(a_2, C') + d(a_3, C') \geq (1/2 - z + 3(\beta - z))n,$$

so $z \geq \beta - 1/6$. But then $1 > x + y + z \geq 1/3 + \gamma - 1/2 + \beta - 1/6 = 1$, a contradiction. □

Claim 20.8. *For every $a \in A'$ there exists $i \in \{1, 2\}$ such that there is an (a, c_i)-path in $G[A', C']$ with at most 4 vertices.*

Proof. Since c_1a_2 and c_2a_1 are edges, we have the desired path if $N(a, C')$ intersects either $N(a_1, C')$ or $N(a_2, C')$. So assume otherwise, that is, assume that the sets $N(a, C')$, $N(a_1, C')$ and $N(a_2, C')$ are pairwise disjoint. By Claim 20.2, there exists $c \in N(a, C')$. Because $N(c_1, A')$ and $N(c_2, A')$ are disjoint, Claim 20.7 implies that $N(c, A')$ must intersect one of $N(c_1, A')$ or $N(c_2, A')$ and this gives us the desired path. □

For $i \in \{1, 2\}$, let $A_i = N(c_i, A')$, let $B_i \subseteq N(c_i, B')$ such that $|B_i| = \lceil (1/2 - y)n \rceil$, and let $B_0 = B' \setminus (B_1 \cup B_2)$. Define $\zeta = |B_1|/n = |B_2|/n$, so $|B_0| \geq (1 - y - 2\zeta)n$.

Claim 20.9. *Every $a \in A_1 \cup A_2$ has at most $(1 - \gamma - \zeta)n$ non-neighbours in B_0. Every $a \in A' \setminus (A_1 \cup A_2)$ has at most $2(1 - \gamma - \zeta)n$ non-neighbours in B_0.*

Proof. Let $a \in A'$. First suppose $a \in A_i = N(c_i, A')$ for some $i \in \{1, 2\}$, then a has no neighbours in B_i, so

$$|N_{\overline{G}}(a, B_0)| \leq |N_{\overline{G}}(a, B)| - |B_i| \leq (1 - \gamma - \zeta)n.$$

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Now assume that $a \in A' \setminus (A_1 \cup A_2)$. By Claim 20.8, there exists $i \in \{1, 2\}$, $c' \in C'$ and $a' \in A'$ such that $ac'a'c_i$ is a path. Because ac' is an edge, a has no neighbours in $N_G(c', B')$, thus the number of non-neighbours of a in $B \setminus N_G(c', B')$ is at most

$$|N_{\overline{G}}(a, B)| - |N_G(c', B')| \leq |N_{\overline{G}}(a, B)| - \lceil (1/2 - y)n \rceil \leq (1 - \gamma - \zeta)n,$$

so the number of non-neighbours of a in $B_0 \setminus N_G(c', B_0) \subseteq B \setminus N_G\big(c', B\big)$ is at most $(1 - \gamma - \zeta)n$. To see that the number of non-neighbours of a in $N_G(c', B_0)$ is at most $(1 - \gamma - \zeta)n$ (which proves the claim), note that $N_G(c', B_0) \subseteq N_{\overline{G}}(a', B_0)$ (because G is triangle-free) and, by the first part of the claim, the fact that $a' \in N(c_i, A') = A_i$ implies that $|N_{\overline{G}}(a', B_0)| \leq (1 - \gamma - \zeta)n$. □

Now we will estimate $e\big(\overline{G}[A', B_0]\big)$ from both sides. Recall that A_1 and A_2 are disjoint, so $|A_1 \cup A_2| \geq 2(\beta - x)n$. This with Claim 20.9 implies

$$\begin{aligned} e\big(\overline{G}[A', B_0]\big) &\leq |A_1 \cup A_2| \cdot (1 - \gamma - \zeta)n + |A' \setminus (A_1 \cup A_2)| \cdot 2(1 - \gamma - \zeta)n \\ &\leq 2(\beta - x)(1 - \gamma - \zeta)n^2 + (1 - 2\beta + x) \cdot 2(1 - \gamma - \zeta)n^2 \\ &= 2(1 - \beta)(1 - \gamma - \zeta)n^2. \end{aligned} \tag{18}$$

(In (18), we used that $1 - \gamma - \zeta \geq 0$, which is implied by Claim 20.9.) By Claim 20.2, for every $b \in B_0$ there exists $c \in N(b, C')$. Since $N_{\overline{G}}(b, A') \supseteq N(c, A')$,

$$e\big(\overline{G}[A', B_0]\big) \geq |B_0|(\beta - x)n \geq (1 - y - 2\zeta)(\beta - x)n^2. \tag{19}$$

The conclusion then follows because (18) and (19) together yield

$$(1 - y - 2\zeta)(\beta - x) \leq 2(1 - \beta)(1 - \gamma - \zeta)$$

which has no solutions when $x \geq 1/3$, $1/3 > y \geq \gamma - 1/2$, $z \geq \beta - 1/4$ and $\zeta \geq 1/2 - y$. (See Appendix B in [2] for a proof of this fact.)

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A. Proof of Lemma 13

Let $V_1, \ldots, V_k$ be the parts of G, let $\ell := k(t+1)$, and let $\mathcal{A}$ be the set of all n^ℓ sequences $a_1, \ldots, a_\ell$ such that $a_j \in V_i$ if j is equivalent to i modulo k. Note that we do not require the vertices $a_1, \ldots, a_\ell$ to be distinct in this definition, so $|\mathcal{A}| = n^\ell$.

For every transversal U, define $\mathcal{A}_U$ to be the set of sequences in $\mathcal{A}$ such that if A is the set of vertices in $\mathcal{A}_U$, the graph induced by A and the graph induced by $A \cup U$ both have a transversal C_k-factor. The probabilistic argument below relies critically on the fact that, for every transversal U, the set $\mathcal{A}_U$ is sufficiently large, and this follows from the fact that G is $(2\eta, t)$-linked. To see this, first label the vertices in U as $u_1, \ldots, u_k$ so that $u_i \in V_i$ for $i \in [k]$. Because G is $(2\eta, t)$-linked we easily have that there are at least ηn^k ways to select vertices $c_1, \ldots, c_k$ where $c_i \in V_i \setminus \{u_i\}$ for $i \in [k]$ that induce a transversal C_k in G. Because G is $(2\eta, t)$-linked, iteratively, for i from 1 to k, we can select a (c_i, u_i, t)-linking sequence L_i that avoids all previously selected vertices in at least ηn^t ways. Since the graph induced by c_i and the vertices in L_i contains a transversal C_k-factor we have that $(t+1)$ is divisible by k and that there exists S_i an ordering of these $(t+1)$ vertices so that the jth vertex is in V_i if j is equivalent to i modulo k. Finally, because each L_i is a (c_i, u_i, t)-linking sequence, the concatenation of the sequences $S_1, \ldots, S_k$ is in $\mathcal{A}_U$, and we have that $|\mathcal{A}_U| \geq \eta n^k \cdot \left(\eta n^t\right)^k = \eta^{k+1} n^\ell$.

Let $p := 0.2 \cdot \sigma \cdot n^{-\ell+1}$ and select the elements of $\mathcal{A}$ independently with probability p to form the random set $\mathcal{A}_{\text{rand}}$. The Chernoff and union bounds imply that with high probability

$$|\mathcal{A}_{\text{rand}}| \leq \sigma n \qquad \text{and} \qquad |\mathcal{A}_{\text{rand}} \cap \mathcal{A}_U| \geq 0.1 \cdot \sigma \eta^{k+1} n \geq \left(\ell^2 + 1\right) \sigma^2 n \tag{20}$$

for every transversal $U \subseteq V(G)$. Note that the number of pairs of sequences in $\mathcal{A}$ in which a vertex is repeated is less than $n \cdot \binom{2\ell}{2} \cdot n^{2\ell-2} = \binom{2\ell}{2} n^{2\ell-1}$, so the expected number of pairs of sequences in $\mathcal{A}_{\text{rand}}$ in which a vertex is repeated is less than $p^2 \cdot \binom{2\ell}{2} n^{2\ell-1} \leq (\ell^2\sigma^2 n)/4$. So, by Markov's inequality, with probability at least $1/2$, if we add both elements from every such pair to form the set $\mathcal{A}_{\text{rep}}$ we have that

$$|\mathcal{A}_{\text{rep}}| \leq \ell^2 \sigma^2 n. \tag{21}$$

Therefore, there exists an outcome in which both (20) and (21) hold. We form the collection $\mathcal{A}'$ from $\mathcal{A}_{\text{rand}}$ by removing all sequences that are in $\mathcal{A}_{\text{rep}}$ and all sequences for which there does not exists a transversal U for which it is a linking sequence. Note that, by (20), $z := |\mathcal{A}'| \leq \sigma n$ and, by (20) and (21), $|\mathcal{A}' \cap \mathcal{A}_U| \geq \sigma^2 n$ for every transversal $U \subseteq V(G)$. Let A be the vertices that appear in a sequence of $\mathcal{A}'$. Because no vertex is repeated in $\mathcal{A}'$ we have that $|A \cap V_i| = z$ for every $i \in [k]$, nd, because every sequence in $\mathcal{A}'$ is a linking sequence for some transversal U, for every sequence in $\mathcal{A}'$ the graph induced by the vertices in the sequence has a transversal C_k-factor.

Suppose that there exists a transversal C_k-tiling of $G - A$ that covers all of the vertices in $V(G - A)$ except a set W such that $|W| \leq k\sigma^2 n$. We can arbitrarily partition W into transversals $U_1, \ldots, U_m$ where $m = |W|/k \leq \sigma^2 n$. Since for every $i \in [m]$, we have that $|\mathcal{A}_{U_i} \cap \mathcal{A}'| \geq \sigma^2 n \geq m$, we can greedily select distinct sequences $A_1, \ldots, A_m$ such that $A_i \in \mathcal{A}_{U_i} \cap \mathcal{A}'$ for every $i \in [m]$. This implies that there is a transversal C_k-factor of $G[W \cup A]$ and, therefore, a transversal C_k-factor of G.

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