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GROUPS WITH A GIVEN NUMBER OF NONPOWER SUBGROUPS

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Abstract

No group has exactly one or two nonpower subgroups. We classify groups containing exactly three nonpower subgroups and show that there is a unique finite group with exactly four nonpower subgroups. Finally, we show that given any integer k greater than 4, there are infinitely many groups with exactly k nonpower subgroups.

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1. Introduction

A subgroup H of a group G is called a *power subgroup* of G if there is a nonnegative integer m such that $H = \langle g^m : g \in G\rangle$. Any subgroup of G which is not a power subgroup is called a *nonpower subgroup* of G. Zhou *et al.* [3] proved that cyclic groups have no nonpower subgroups and infinite noncyclic groups have an infinite number of nonpower subgroups. They showed further that no group has either exactly one or exactly two nonpower subgroups and then asked: for each integer k greater than 2, does there exist at least one group possessing exactly k nonpower subgroups? This question was recently answered positively in [1], where it was also proved that for any integer k greater than 4 and composite, there are infinitely many groups with exactly k nonpower subgroups.

Let p be an odd prime. For each positive integer n, we define the group $G_{n,p}$ by

$$G_{n,p} := \langle x, y : x^{2^n} = 1 = y^p, yx = xy^{-1}\rangle.$$

We note that $G_{1,p}$ is the dihedral group of order $2p$ and $G_{2,p}$ is the generalised quaternion group of order $4p$ (setting $a = x^2y$ and $b = x$ gives its usual presentation $\langle a, b : a^{2p} = 1, b^2 = a^p, ba = a^{-1}b\rangle$). For any positive integer n, $G_{n,p}$ is the semidirect product $C_p \rtimes C_{2^n}$ and has order $2^n p$. We may now state our first result.

THEOREM 1.1. *There are infinitely many groups with an odd prime number of nonpower subgroups. In particular, for any odd prime p and each positive integer n, the group $G_{n,p}$ has exactly p nonpower subgroups.*

Theorem 1.1, combined with the fact that for composite k greater than 4 there are infinitely many groups with k nonpower subgroups [1, Theorem 5], gives the following immediate corollary.

COROLLARY 1.2. *Let k be an integer greater than* 4. *Then there are infinitely many groups with exactly k nonpower subgroups.*

The only unresolved cases are therefore $k = 3$ and $k = 4$. Our second main result deals with these cases.

THEOREM 1.3.

(a) *A group G contains exactly three nonpower subgroups if and only if G is isomorphic to one of $C_2 \times C_2$, Q_8 or $G_{n,3}$ for $n \in \mathbb{Z}^+$.*
(b) *Up to isomorphism, $C_3 \times C_3$ is the only group containing exactly four nonpower subgroups.*

For the rest of this section, we recall some preliminaries. We note that each power subgroup is characteristic and hence normal in G. Following [1], we write $s(G)$ for the number of subgroups in a group G, $ps(G)$ for the number of power subgroups of G and $nps(G)$ for the number of nonpower subgroups of G.

LEMMA 1.4 [1, Lemma 3]. *If A and B are finite groups such that $|A|$ and $|B|$ are coprime, then*

$$nps(A \times B) = nps(A)s(B) + ps(A)nps(B).$$

We denote by $\Phi(G)$ the Frattini subgroup of G, that is, the intersection of the maximal subgroups of G. It is a characteristic subgroup of G.

THEOREM 1.5 (Burnside's basis theorem). *Let G be a p-group and suppose that $[G : \Phi(G)] = p^d$.*

(a) *$G/\Phi(G)$ is elementary abelian of order p^d. Moreover, if $N \trianglelefteq G$ and G/N is elementary abelian, then $\Phi(G) \leq N$.*
(b) *Every minimal system of generators of G contains exactly d elements.*
(c) *$\Phi(G) = G^pG'$. In particular, if $p = 2$, then $\Phi(G) = G^2$.*

LEMMA 1.6 [2, Theorem 1.10(a)]. *Let G be a noncyclic p-group, where $p > 2$. Then the number of subgroups of order p in G is congruent to $1 + p$ modulo p^2.*

REMARK 1.7. It is well known that the only 2-groups with a unique involution are cyclic or generalised quaternion groups.

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2. Proof of main results

PROOF OF THEOREM 1.1. Let p be an odd prime. Our goal is to show that for any positive integer n and any odd prime p, the group $G_{n,p} = \langle x, y : x^{2^n} = 1 = y^p, yx = xy^{-1}\rangle$ contains exactly p nonpower subgroups. We have $|G_{n,p}| = 2^n p$. We first obtain a count of the number of subgroups in $G_{n,p}$. Since the Sylow 2-subgroup $\langle x\rangle$ is not a normal subgroup, the number of Sylow 2-subgroups of $G_{n,p}$ must be p. On the other hand, since $y^x = y^{-1}$, there is a unique normal Sylow p-subgroup, namely the cyclic subgroup $\langle y\rangle$ of order p. Since x^2 is central in $G_{n,p}$ and each Sylow 2-subgroup of $G_{n,p}$ is cyclic, there is a unique subgroup of order 2^k (for each $k \in \{0, \dots, n-1\}$) and a unique subgroup of order $2^k p$ (for each $k \in \{1, \dots, n\}$). Along with the p subgroups of order 2^n, we see that $s(G_{n,p}) = 2n + p + 1$. As the subgroups of order 2^n are not normal, we know immediately that they are nonpower subgroups. Hence $nps(G_{n,p}) \geq p$. We now show that any subgroup of $G_{n,p}$ that is not a Sylow 2-subgroup of $G_{n,p}$ is a power subgroup of $G_{n,p}$. First, the unique subgroup of order p is $G_{n,p}^{2^n}$. Secondly, for each $k \in \{0, \dots, n-1\}$, the subgroup of order 2^k is $G_{n,p}^{2^{n-k}p}$. Finally, for each $k \in \{1, \dots, n\}$, the subgroup of order $2^k p$ is $G_{n,p}^{2^{n-k}}$. Therefore, $ps(G_{n,p}) = 2n + 1$, whence $nps(G_{n,p}) = p$. □

We now move on to the proof of Theorem 1.3. Let G be a finite noncyclic group. Then G falls into one of the following three categories:

(i) a noncyclic p-group;
(ii) a noncyclic nilpotent group that is not a p-group;
(iii) a nonnilpotent group.

For each of these cases above, we classify all the finite groups with exactly three or four nonpower subgroups.

PROPOSITION 2.1. *Let G be a finite noncyclic p-group. Then $nps(G) = 3$ if and only if G is $C_2 \times C_2$ or Q_8, and $nps(G) = 4$ if and only if G is $C_3 \times C_3$.*

PROOF. Let G be noncyclic of order p^n. It was shown in [3] that if $N \trianglelefteq G$ and A/N is a nonpower subgroup of G/N, then A is a nonpower subgroup of G. Suppose G has exactly k nonpower subgroups, where $k \in \{3, 4\}$. Then $G/\Phi(G) \cong C_p \times \cdots \times C_p$ (d times) and $d \geq 2$ as G is not cyclic. The $(p^d - 1)/(p - 1)$ cyclic subgroups of order p in C_p^d are nonpower subgroups. It follows that $G/\Phi(G)$, and hence G, has at least $1 + p + \cdots + p^{d-1}$ nonpower subgroups. Hence, $d = 2$, either $p = 2$ or $p = 3$, and G has $p + 1$ maximal subgroups that are nonpower subgroups.

The power subgroups of G are $G^1 = G, G^p, G^{p^2}, \dots, G^{p^m}$, where p^m is the exponent of G. There are thus at most $m + 1$ distinct power subgroups. Since G is not cyclic, this means $m < n$; so $ps(G) \leq n$.

What about $s(G)$? There is at least one subgroup of order p^i for $0 \leq i \leq n$ (just take any composition series). This gives at least $n + 1$ subgroups. But there are $p + 1$ maximal subgroups (of order p^{n-1}) arising from the $p + 1$ nontrivial proper subgroups of $G/\Phi(G)$. Thus $s(G) \geq n + p + 1$.

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Suppose $p = 2$. If G is not generalised quaternion (and by assumption G is not cyclic), then G has at least three involutions and hence at least three subgroups of order 2. So, if $n > 2$, then $s(G) \geq n + 5$, meaning that $nps(G) \geq 5$, a contradiction. Thus, either G is generalised quaternion or $n = 2$, which means $G \cong C_2 \times C_2$, and in this case $nps(G) = 3$. If G is generalised quaternion, then G has $2^{n-1} + 2$ elements of order 4, resulting in $2^{n-2} + 1$ subgroups of order 4. On the other hand, if $n > 3$, then $s(G) \geq n + 1 + 2^{n-2} \geq n + 5$. Again, this means that $nps(G) \geq 5$. Thus, $n = 3$ and then $G \cong Q_8$. Again, $nps(Q_8) = 3$.

The remaining case is $p = 3$. By Lemma 1.6, there are at least four subgroups of order 3 in G. If $n > 2$, then these are distinct from the four maximal subgroups and so $s(G) \geq n + 7$. This forces $nps(G) \geq 7$, a contradiction. The only possibility is that $n = 2$. A quick check shows that $nps(C_3 \times C_3) = 4$.

Thus, $nps(G) = 3$ if and only if G is $C_2 \times C_2$ or Q_8, and $nps(G) = 4$ if and only if G is $C_3 \times C_3$. □

LEMMA 2.2. *Let G be a finite noncyclic nilpotent group. If G is not a p-group, then $nps(G) \geq 6$.*

PROOF. Recall that a finite group is nilpotent if and only if it is the direct product of its Sylow subgroups, each of which is normal. Since G is noncyclic, at least one of these Sylow subgroups is noncyclic. Let $p_1, \ldots, p_r$ be the primes dividing $|G|$ and let $P_1, \ldots, P_r$ be the respective Sylow subgroups. Assume, without loss of generality, that P_1 is noncyclic. Write $Q = P_2 \times \cdots \times P_r$; so $G \cong P_1 \times Q$. Since G is not a p-group, we have $Q \neq \{1\}$. Therefore, by Lemma 1.4,

$$nps(G) = nps(P_1)s(Q) + ps(P_1)nps(Q) \geq nps(P_1)s(Q).$$

As $Q \neq \{1\}$, we have $s(Q) \geq 2$. As P_1 is not cyclic, $nps(P_1) \geq 3$. Hence $nps(G) \geq 6$. □

LEMMA 2.3. *If G is a finite nonnilpotent group such that $nps(G) \in \{3, 4\}$, then $nps(G) = 3$ and $G \cong G_{n,3} = \langle x, y : x^{2^n} = 1 = y^3, yx = xy^{-1} \rangle$, for some positive integer n.*

PROOF. Suppose G is finite, nonnilpotent and $nps(G) = k \in \{3, 4\}$. If G had a unique Sylow p-subgroup for each p dividing $|G|$, then G would be nilpotent. So there is at least one such p for which G has more than one Sylow p-subgroup. For any such p, the number, n_p, of Sylow p-subgroups is congruent to 1 mod p. So $n_p \geq p + 1$. These groups are not normal, so are not power subgroups. Therefore, as $nps(G) \in \{3, 4\}$, either $p = 2$ and $n_2 = 3$, or $p = 3$ and $n_3 = 4$. For all other primes q dividing $|G|$, there must be a unique Sylow q-subgroup. If any subgroup of G, other than the Sylow p-subgroups, were nonnormal, then it and its conjugates could not be power subgroups. Thus there would be at least two further nonpower subgroups, forcing $nps(G) \geq 5$, a contradiction. Therefore, every subgroup of G, other than the Sylow p-subgroups, is normal.

Let P be one of the Sylow p-subgroups. Let $q_1, \ldots, q_r$ be the primes other than p dividing $|G|$. Let $Q_1, \ldots, Q_r$ be the corresponding normal Sylow subgroups. Each Q_i is normal and the Q_i intersect trivially. Therefore, defining $H = Q_1 Q_2 \cdots Q_r$,

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we have that $H \cong Q_1 \times Q_2 \times \cdots \times Q_r$ is a normal subgroup of G, with $G = PH$. Now $P \trianglelefteq N_G(P)$ and, setting $K = H \cap N_G(P)$, we have $K \trianglelefteq G$ (because certainly K is not a Sylow p-subgroup). But P is normal in $N_G(P) = PK$; so $N_G(P) \cong P \times K$. Let $h \in H - N_G(P)$. Then $(PK)^h = P^h K \neq PK$. This means that PK is not normal in G, a contradiction unless $K = \{1\}$. Therefore, $K = \{1\}$ and $P = N_G(P)$. In particular, $n_p = |G : P| = |H|$.

Suppose first that $p = 3$. Then $|H| = 4$. If $H \cong C_2 \times C_2$, then each of its cyclic subgroups would be normal, and hence the involutions they contain would be central. But that would imply that P is normal in G, a contradiction. Therefore $H \cong C_4$. Let z be a generator of H. We have $H \leq C_G(z) \leq G$. Thus, $|z^G| = 3^i$ for some i with $0 \leq i \leq n$. But $z^G \subseteq \{z, z^{-1}\}$. The only possibility is that $z^G = \{z\}$, and z is central in G. Again, this implies that P is normal in G, a contradiction. Therefore, $p \neq 3$.

The remaining case is when $p = 2$. In this case, $H \cong C_3$. Let A_1, A_2 and A_3 be the three Sylow 2-subgroups. Every proper subgroup of P is not one of A_1, A_2 and A_3, so is normal in G and hence contained in all of A_1, A_2 and A_3. If P were not cyclic, then each of its generators would generate a proper cyclic subgroup, and would hence be contained in A_1, A_2 and A_3. This implies $P \leq A_1 \cap A_2 \cap A_3$, a contradiction. Therefore, P is cyclic of order 2^n. Write $P = \langle x \rangle$ and $H = \langle y \rangle$. Certainly, $y^x \neq y$; so the only possibility is that $y^x = y^{-1}$. Therefore,

$$G = \langle x, y : x^{2^n} = 1, y^3 = 1, yx = xy^{-1} \rangle$$

for some integer $n \geq 1$. That is, $G \cong G_{n,3}$. By Theorem 1.1, we have $nps(G) = 3$. □

Theorem 1.3 follows immediately from Proposition 2.1 and Lemmas 2.2 and 2.3.

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