Macroeconomic Dynamics, **25**, 2021, 2102–2127. Printed in the United States of America.
doi:10.1017/S1365100519000981

REGRESSIVE WELFARE EFFECTS OF HOUSING BUBBLES

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We analyze the welfare effects of asset bubbles in a model with income inequality and financial friction. We show that a bubble that emerges in the value of housing, a durable asset that is fundamentally useful for everyone, has regressive welfare effects. By raising the housing price, the bubble benefits high-income savers but negatively affects low-income borrowers. The key intuition is that, by creating a bubble in the market price, savers' demand for the housing asset for investment purposes imposes a negative externality on borrowers, who only demand the housing asset for utility purposes. The model also implies a feedback loop: high-income inequality depresses the interest rates, facilitating the existence of housing bubbles, which in turn has regressive welfare effects.

Keywords: Rational Bubble, Inequality, Housing, Financial Friction

1. INTRODUCTION

Many countries have experienced episodes of bubble-like booms in asset prices. Examples include the real estate booms in Japan in the 1980s, Southeast Asia in the 1990s, the U.S. in the 2000s, and more recently in China and Vietnam.[1] In general, when there is a high demand for savings but limited investment outlet, the rates of returns from investment are depressed and real estate investment can serve as a prominent store of value. Thus, a low interest rate environment, as seen in the recent decade, provides a fertile ground for the emergence of asset bubbles, including those in real estate. Given the prevalence of bubble episodes, a central question arises for academics and policymakers: What are the welfare effects of asset bubbles?

In this paper, we highlight the nuanced welfare effects of asset bubbles, especially those in real estate prices. We develop a simple overlapping generations (OLG) model of bubbles with intrageneration heterogeneity and financial friction. As described in Section 2 of the paper, households have identical preferences

We thank the editor, the associate editor, two anonymous referees, Julien Bengui, Ippei Fujiwara, Lutz Hendricks, Guido Menzio, and Andrii Parkhomenko for helpful suggestions. Toan Phan is funded by Vietnam National Foundation for Science and Technology Development under grant number 502.01-2017.12. The views expressed herein are those of the authors and not those of the Federal Reserve Bank of Richmond or the Federal Reserve System. Address correspondence to: Toan Phan, The Federal Reserve Bank of Richmond, 701 E Byrd St, Richmond, VA 23219, USA. e-mail: toanvphan@gmail.com.

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https://doi.org/10.1017/S1365100519000981 Published online by Cambridge University Press

over a perishable consumption good and a durable and perfectly divisible housing asset in fixed supply. Young agents receive endowments, and a fraction of them are *savers*, who are born with high endowments, and the remaining fraction are *borrowers*, who are born with low endowments. Young borrowers, given their low endowment, want to borrow to purchase the desired amount of housing that maximizes their utility. In contrast, young savers, given their high endowment, would instead like to save for old age. Thus, for savers, housing not only yields utility dividend, but also serves as a savings or investment vehicle. To highlight the difference between these two motives, we assume that the utility over housing has a satiation level $\bar{h}$. Any additional unit of housing above the satiation level yields no additional utility and thus purely serves as an investment vehicle.

In an economy without financial friction, households can achieve their first-best allocations by borrowing and lending in the credit market. However, in the presence of financial frictions, such as imperfect contract enforcement, young borrowers face a binding credit constraint, modeled as an exogenous limit on borrowers' debt capacity. In equilibrium, the constraint effectively limits how much savers can store their income by investing in the credit market. As we show in Section 3, in an economy with high-income inequality, there is a shortage of storage for savers, which can lead to an equilibrium interest rate that is below the economy's growth rate. The low interest rate environment in turn facilitates the emergence of asset bubbles.

In Section 4, the main part of our paper, we study housing bubbles. In a housing bubble equilibrium, young savers acquire housing in excess of the satiation level, because they would like to use the housing asset as an investment vehicle to save for old age. This "speculative" demand for the housing asset by savers is similar to the demand of a bubble asset in a standard rational bubbles model: an agent purchases an asset because he or she expects to be able to sell it to someone else in the future.

We then show that the housing bubble has opposite effects on borrowers and savers. The housing bubble increases the return from real estate investment for high-income savers, who demand storage of value, and hence increases their welfare (relative to the bubbleless benchmark). In contrast, the housing bubble *reduces* the welfare of borrowers, because it raises the price of housing and the speculative demand of savers for the housing asset crowds out the allocation of housing to borrowers, who in equilibrium have a relatively higher marginal utility from housing. By positively affecting high-income savers and negatively affecting low-income borrowers, the housing bubble thus has *regressive welfare effects*. Overall, our results imply a feedback loop on inequality: high-income inequality depresses the interest rate, thereby facilitating the existence of housing bubbles, which in turn has regressive welfare effects. The key insight is that, by creating a bubble in the market price of housing, savers' demand for the housing asset for investment purposes imposes a negative pecuniary externality on borrowers, who only demand the housing asset for utility purposes.

In comparison, Section 5.1 shows that the regressive welfare implications are lessened if the model considers pure bubbles, which are widely used in the

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rational bubble literature for their simplicity. A pure bubble is an asset that has no fundamental value, but which is traded at a positive price (such as fiat money or unbacked public debt). The pure bubble provides an additional and separate investment vehicle for savers: besides lending and investing in the housing market, savers can invest in the pure bubble asset (by purchasing the asset when young and reselling it when old). Unlike the housing bubble equilibrium, the pure bubble equilibrium is characterized by an *endogenous segmentation* in the bubble market. This is because only savers purchase the bubble asset for investment purposes, while credit-constrained borrowers have no demand for the asset. Furthermore, the option to invest in the pure bubble asset is a substitute for the option to use the housing asset purely as an investment vehicle. As a consequence, the crowding effect on housing that was prevalent in the housing bubble case is absent in the presence of a pure bubble. Consequently, the negative externality on borrowers' welfare is absent in the pure bubble equilibrium. Furthermore, because the presence of the pure bubble asset effectively enriches the available menu of investment vehicles, we can show that the allocations in the pure bubble equilibrium Pareto dominates the allocations in the housing bubble equilibrium.

Related literature. Our paper is related to the rational bubble literature, which has a long heritage dating back to Samuelson (1958), Diamond (1965), and Tirole (1985). For a survey of this literature, see Miao (2014) and Martin and Ventura (2017). Much of the literature has focused on a positive analysis of bubbles. A common theme in this literature is that rational bubbles emerge to reduce some inefficiency in the financial market, such as an aggregate shortage of assets for storage or a credit market imperfection, as in Martin and Ventura (2012), Hirano and Yanagawa (2017), Miao and Wang (2018), and Ikeda and Phan (2019). By departing from the pure bubble assumption and modeling a bubble as attached to a fundamentally useful durable asset such as housing, our paper is related to Arce and López-Salido (2011), Miao and Wang (2012), Wang and Wen (2012), Hillebrand and Kikuchi (2015), Zhao (2015), Kikuchi and Thepmongkol (2019), and Basco (2016).

To the best of our knowledge, among papers that analyze the welfare effects of bubbles, ours is the first to document regressive welfare effects of a housing bubble. Saint-Paul (1992), Grossman and Yanagawa (1993), and King and Ferguson (1993) show that if there is a positive externality in the accumulation of capital, the emergence of bubbles on an unproductive asset would inefficiently divert resources from investment. Similarly, Hirano et al. (2015) show that oversized bubbles inefficiently crowd out productive investment. On the other hand, Miao et al. (2015) show that bubbles can crowd in too much investment. Caballero and Krishnamurthy (2006) show that bubbles can marginally crowd out domestic savings and cause a shortage of liquid international assets in a small open economy framework. Focusing instead on risk, Ikeda and Phan (2016) and Bengui and Phan (2018) show that rational bubbles financed by credit can be

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excessively risky and Biswas et al. (2018) show that the collapse of bubbles can trigger inefficient recessions. The regressive welfare effects that we highlight are complementary to the welfare effects highlighted by these papers.

2. MODEL

Consider an endowment economy with OLG of agents who live for two periods. Time is discrete and infinite, with dates denoted by $t = 0, 1, 2, \ldots$. The population of young households in each period is constant at $L_t = 1$ for all t. There is a consumption good and a housing asset. The consumption good is perishable and cannot be stored. The housing asset is durable and perfectly divisible. The supply of housing is fixed to one. The consumption good is the numeraire, and the market price of a unit of the housing asset is denoted by p_t.

Heterogeneity. Each generation consists of two types of households—high-income savers and low-income borrowers (or debtors)—denoted by $i \in \{s, d\}$ correspondingly. Each group has an equal unit measure population. Each young household is endowed with e^i units of the consumption good, where $e^s > e^d$. In addition, each household receives an endowment of $e > 0$ when old.[2] Without loss of generality, we normalize $e^d = 1$. Thus, any increase in e^s leads to an increase in (within-generation) income inequality.

Preferences. Households derive utility from the housing asset and from the consumption good, consumed both when young and old. They have a separable utility function of the following form:

$$U\left(c^i_{t,y}, c^i_{t+1,o}, h^i_t\right) \equiv u\left(c^i_{t,y}\right) + \beta u\left(c^i_{t+1,o}\right) + v\left(h^i_t\right),$$

where $c^i_{t,y}$ and $c^i_{t+1,o}$ denote consumption in young and old age of a household of type $i \in \{s, d\}$ born in period t, β is the discount factor, and h^i_t denotes the housing. The consumption utility function satisfies $u' > 0, u'' < 0$, and the usual Inada condition $\lim_{c\to 0+} u'(c) = 0$.

We assume that the housing utility function has a *satiation level* $\bar{h} > 1/2$, that is

$$v'(h) \begin{cases} > 0 & \text{if } h < \bar{h} \\ = 0 & \text{if } h \geq \bar{h} \end{cases}.$$

This assumption will help us clearly illustrate the speculative motive: if a household only purchases housing for utility, then it should never purchase more than $\bar{h}$; however, if a household additionally wants to use housing as an investment vehicle, then it may purchase in excess of $\bar{h}$. The function also satisfies $v''(h) < 0$ if $h < \bar{h}$ and the Inada condition $\lim_{h\to 0+} v'(h) = 0$. We focus on the interesting parametric region where $\bar{h} > 1/2$ (savers only satiate their housing utility by consuming more than half of the housing supply).

Credit market friction. Households can borrow and lend to each other via a credit market. Let R_t denote the gross interest rate for debt between period t and

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$t+1$. As in Bewley (1977), Huggett (1993), and Aiyagari (1994), we model credit friction in the simplest possible way: an agent can commit to repay at most $\bar{d}$ units of the consumption good, where $\bar{d} \geq 0$ is an exogenous debt limit. This imperfection in the financial market will lead to a constraint on households' ability to borrow, as manifested in the optimization problem below.

The presence of the credit friction is important for the existence of asset price bubbles, which requires dynamic inefficiency. If the credit market were frictionless (e.g., $\bar{d} := \infty$), then the economy would be dynamically efficient and bubbles cannot arise (see Section 4).

We assume for simplicity that

$$\bar{d} = 0.$$

This assumption allows us to focus on the housing bubble's effect on the housing market rather on the credit market. We relax this assumption in Section 5.2 and show that, as long as $\bar{d}$ is sufficiently small so that the credit constraint binds for borrowers, our main results carry through.

Optimization. A household purchases housing, consumes, and borrows or lends when young, and then sells their housing asset and consumes when old. The optimization problem of a young household of type $i \in \{s, d\}$ born in period t consists of choosing housing asset position h_t^i, net financial asset position a_t^i, and old-age consumption c_{t+1}^i to maximize lifetime utility:

$$\max_{h_t^i, c_{t,y}^i, c_{t+1,o}^i, a_t^i} U\left(c_{t,y}^i, c_{t+1,o}^i, h_t^i\right)$$

subject to a budget constraint in young age:

$$p_t h_t^i + \frac{1}{R_t} a_t^i + c_{t,y}^i = e^i,$$

a budget constraint in old age:

$$c_{t+1,o}^i = p_{t+1} h_t^i + a_t^i + e,$$

nonnegativity constraints on housing and consumption:[3]

$$h_t^i, c_{t,y}^i, c_{t+1,o}^i \geq 0,$$

and the credit constraint:

$$a_t^i \geq -\bar{d}.$$

Throughout the paper, we focus on parameter regions where there is sufficient dispersion in endowment (e^s is sufficiently high) so that the credit constraint binds for borrowers but not for savers.

Finally, to close the model, without loss of generality assume that the old savers own the entire supply of housing in the initial period $t = 0$. We define an equilibrium as follows:

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DEFINITION 1. *An competitive equilibrium consists of allocation* $\{h_t^i, c_{t,y}^i, c_{t+1,o}^i, a_t^i\}_{t\geq 0}$ *and positive prices* $\{p_t, R_t\}_{t\geq 0}$ *such that:*

1. *Given prices, the allocations solve the optimization problem of households for all* $i \in \{s, d\}$ *and* $t \geq 0$.
2. *The consumption good market clears*
$$c_{t,y}^s + c_{t,y}^d + p_t(h_t^d + h_t^s) = e^s + e^d + e, \forall t \geq 0;$$
3. *The credit market clears*
$$a_t^s + a_t^d = 0, \forall t \geq 0;$$
4. *And the housing market clears*
$$h_t^s + h_t^d = 1, \forall t \geq 0.$$

It is straightforward that the first-order conditions of savers yield a standard asset pricing equation:

$$p_t = \frac{v'\left(h_t^s\right)}{u'\left(c_{y,t}^s\right)} + \beta \frac{p_{t+1}}{R_t}.$$

The equation states that the price of each unit of housing is equal to the marginal housing dividend, captured by the marginal rate of substitution $\frac{v'(h_t^s)}{u'(c_{y,t}^s)}$, plus the resale value discounted by the interest rate p_{t+1}/R_t. Recursions of this equation lead to a standard forward-looking asset pricing equation:

$$p_t = \underbrace{\sum_{j\geq 0} \frac{v'(h_t)/u'(c_t)}{\prod_{i=0}^{j} R_{t+i}}}_{\text{fundamental value}} + \underbrace{\lim_{j\to\infty} \frac{p_{t+j+1}}{\prod_{i=0}^{j} R_{t+i}}}_{\text{bubble value}}. \tag{1}$$

The first term on the right-hand side captures the discounted value of the marginal utility dividend stream (the *fundamental value*), and the second term captures the *bubble value*. We have the following standard definition of a bubbleless or bubble equilibrium:

DEFINITION 2. *A* bubbleless equilibrium (housing bubble equilibrium) *is an equilibrium where the bubble value in equation (1) is zero (positive).*

Throughout the rest of the paper, we focus on steady-state (or stationary) equilibria, where quantities and prices are time-invariant.

3. BUBBLELESS CASE

We start with the case of a stationary bubbleless equilibrium, or bubbleless steady state. There, equation (1) can be rewritten as:

$$p = \sum_{j\geq 0} \frac{v'(h^s)/u'\left(c_y^s\right)}{R^j}, \tag{2}$$

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which equates the (steady state) housing price to the discounted value of the dividend stream. This equation implies that

$$h^s < \bar{h}, \tag{3}$$

that is, savers do not satiate their utility for housing in the bubbleless equilibrium (as otherwise $v'(h^s)$ would be equal to zero, leading to a zero housing price, which is impossible in equilibrium). The equation also implies that

$$R > 1,$$

that is, the interest rate must be larger than the growth rate of the economy in a bubbleless equilibrium (as otherwise the discounted value of the dividend stream would not converge). Given $R > 1$, equation (2) can be rewritten more succinctly as:

$$p = \frac{v'(h^s)}{u'\left(c_y^s\right)} \frac{R}{R-1}, \tag{4}$$

As usual, the interest rate is determined by the intertemporal Euler equation of the unconstrained savers:

$$R = \frac{u'\left(c_y^s\right)}{\beta u'\left(c_o^s\right)}. \tag{5}$$

The corresponding Euler equation will not hold with equality for borrowers, as they are credit constrained. However, because of the Inada condition on the housing utility function v, the first-order condition for housing demand from borrowers necessarily holds with equality:

$$pu'\left(c_y^d\right) = v'\left(1 - h^s\right) + \beta pu'\left(c_o^d\right), \tag{6}$$

where we have imposed the housing market clearing condition $h^d = 1 - h^s$. Expression (6) equates the marginal cost (in terms of utility for a borrower) of investing in one unit of housing to the marginal gain, which consists of the marginal utility dividend v' and the marginal return from subsequently reselling the housing unit.

Equations (4), (5), and (6) characterize the bubbleless steady state. Let R_n, p_n, and h_n^s denote the solution to this system of equations (the subscript n refers to "no bubble"). Similarly, $\{c_{y,n}^i, c_{o,n}^i\}_{i\in\{s,d\}}$ and U_n^i denote the associated consumption profiles and lifetime utility.

Figure 1 illustrates the determination of housing allocation h_n^s and housing price p_n. The curve labeled "foc savers" plots the sets of allocations and prices that solve the first-order condition $pu'(c_y^s) = v'(h^s) + \beta pu'(c_o^s)$, which is the combination of (4) and (5). Similarly, the curve labeled "foc borrowers" plots first-order condition (6) of borrowers. The intersection of the two curves determines h_n^s and p_n. The dashed line plots what happens when there is an exogenous increase in the endowment e^s of savers, while all other parameters stay the same (and thus an increase in inequality). The curve associated with the Euler equation of savers

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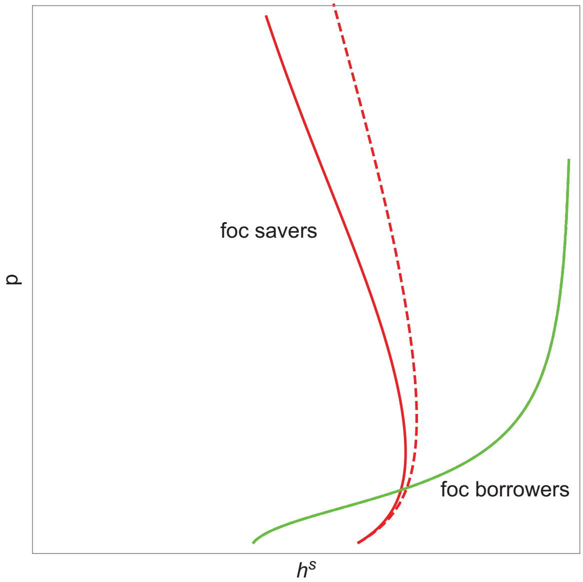

FIGURE 1. Determination of bubbleless steady state's housing allocation and price.

shifts out to the right, leading to an increase in h_n^s and p^n (as there is an increase in the demand for housing from savers). Thus, higher inequality is associated with a higher housing price.

Recall from the beginning of the section that for the conjectured prices and allocations characterized above to indeed constitute a bubbleless equilibrium, the interest rate must satisfy $R_n > 1$. The following lemma summarizes this section:

LEMMA 1 (Bubbleless steady state). *The bubbleless steady state is characterized by interest rate R_n, housing price p_n, and housing allocation $h_n^s < \bar{h}$, which are implicitly defined by (4), (5), and (6). The bubbleless steady state exists if and only if $R_n > 1$.*

Proof. Appendix A.1. □

It is straightforward to show that R_n is strictly decreasing in e^s, as a higher endowment of young savers implies a higher demand for storage, which leads to a lower interest rate in equilibrium. For convenience, we can implicitly define $\bar{e}$ as the savers' endowment threshold such that

$$R_n|_{e^s=\bar{e}} = 1. \tag{7}$$

It is then immediate that $R_n > 1$ if and only if $e^s < \bar{e}$. In other words, the interest rate R_n is larger than the growth rate of the economy if and only if there is not too much inequality. What happens when $e^s \geq \bar{e}$ and $R_n \leq 1$? As we have shown, the bubbleless steady state does not exist. Instead, a housing bubble steady state will arise, as we will show in the next section.

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4. HOUSING BUBBLE

We now characterize a stationary housing bubble equilibrium, where the bubble value is positive. Recall that the steady-state version of asset pricing equation (1) is

$$p = \sum_{j\geq 0} \frac{v'(h^s)/u'\left(c_y^s\right)}{R^j} + \lim_{j\to\infty} \frac{p}{R^{j+1}}. \tag{8}$$

Since p is necessarily finite and positive, the steady-state bubble value $\frac{p}{R^{j+1}}$ is positive if and only if $R = 1$. In other words, the interest rate is equal to the growth rate of the economy, as in a standard rational bubble equilibrium (e.g., Tirole (1985)). Interestingly, this implies that savers must satiate their utility for housing, as formalized in the following lemma:

LEMMA 2. *In any housing bubble steady state, savers satiate their utility for housing:*

$$h^s \geq \bar{h}.$$

Proof. We already know $R = 1$ in any housing bubble steady state. However, this implies that the fundamental value $\sum_{j\geq 0} \frac{v'(h^s)/u'(c_y^s)}{R^j}$ converges to a finite value if and only if the utility dividend $v'(h^s)/u'(c_y^s)$ is zero, that is, $h^s \geq \bar{h}$. □

Note that given $h^s \geq \bar{h}$, there is another intuitive way to understand the identity $R = 1$ as a no-arbitrage condition for savers. For these agents, the investment in each additional unit of housing in excess of the satiation level yields a marginal return rate of one in steady state, as savers purchase each unit of housing and subsequently resell it for a return rate of $p/p = 1$. In equilibrium, savers must be indifferent between investing in housing and lending. For this to be the case, the interest rate on lending R must be equal to 1.

With $R = 1$, the housing bubble steady state can be characterized by housing allocation h^s and housing price p that solve a system of two equations. The system consists of intertemporal Euler equation (5) for savers, which, given $R = 1$, is simplified to

$$u'\left(c_y^s\right) = \beta u'\left(c_0^s\right), \tag{9}$$

and first-order condition (6) for the housing demand from borrowers. Let p_b and h_b^s denote the solution to this system of equations (the subscript b refers to "bubble"). Similarly, let $\{c_{y,b}^i, c_{o,b}^i\}$ and U_b^i denote the associated consumption profiles and lifetime utility.

Figure 2 illustrates the determination of h_b^s and p_b. Here, the curve labeled "foc borrowers" plots the housing first-order condition (6) of borrowers on the $h^s \times p$ plane, while the curve labeled "foc savers" plots the lending first-order condition (9) of savers. Their intersection determines h_b^s and p_b. Furthermore, as in Figure 1, the dashed line in Figure 2 plots what happens when there is an exogenous increase in e^s while all other parameters stay the same. The curve

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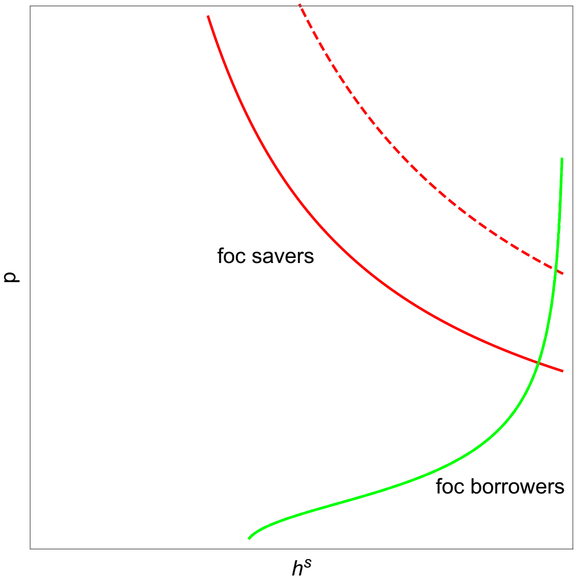

FIGURE 2. Determination of housing bubble steady state's housing allocation and price.

associated with the Euler equation of savers shifts outward to the right, leading to an increase in housing price p_b and housing allocation to savers h_b^s. Intuitively, an increase in the endowment of young savers leads to an increase in the demand from savers for housing as a storage of value.

The following lemma summarizes this section. It also states a standard low interest rate condition for the existence of the housing bubble steady state.

LEMMA 3 (Housing bubble steady state). *The housing bubble steady state is characterized by interest rate $R_b = 1$, housing allocation $h_b^s \geq \bar{h}$, and housing price p_b, which are implicitly defined by (6) and (9). The housing bubble steady state exists if and only if $R_n \leq 1$ (or equivalently there is sufficient inequality that $e^s \geq \bar{e}$).*

Proof. Appendix A.2. □

REMARK 1. An interesting feature of our model that differs from the standard rational bubbles models (e.g., Samuelson (1958), Diamond (1965), Tirole (1985)) is that the housing asset is *not* a pure bubbly asset—defined as an asset that does not have any fundamental value but is traded at a positive price (see Section 5.1). In our framework, the housing asset is fundamentally useful as it yields utility to households. However, the pricing equation (2) of the housing asset is similar to that of a pure bubbly asset in the sense that, from the perspective of savers, each *marginal* unit of additional investment in the housing asset beyond the satiation point $\bar{h}$ does not yield any additional *marginal* utility.

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REMARK 2. As is standard in the rational bubbles literature, the housing bubble steady state is saddle-path stable, as shown in Appendix A.7.

REMARK 3. The low interest rate condition ($R_n < 1$) for the existence of bubbles is related to the dynamic inefficiency condition in Diamond (1965) and Tirole (1985). There has been an active debate over the empirical validity of this condition. For instance, Abel et al. (1989) argue that the U.S. (between 1929 and 1985) and six other advanced economies (between 1960 and 1985) satisfy a sufficient condition for dynamic efficiency that aggregate investment falls short of capital income. However, using more updated data on mixed income and land rents, Geerolf (2017) finds evidence to the contrary, providing strong support for the hypothesis that these major economies are dynamically inefficient.[4] Recent theoretical models have also pointed out that dynamic inefficiency is not always a necessary condition for the existence of rational bubbles if there are financial frictions (e.g., Farhi and Tirole (2012), Martin and Ventura (2012)) or international capital inflows (e.g., Ikeda and Phan (2019)).

4.1. Comparative Statics

We now conduct a comparative statics exercise and compare the bubble and the bubbleless equilibria. Throughout, we vary the endowment of savers e^s, while keeping all other parameters fixed. Recall that an increase in e^s leads to an increase in the income of the top earners (savers) relative to the bottom earners (borrowers).

Figure 3 plots the equilibrium interest rate R, housing price p, and housing allocation h^s as functions of e^s (the thick lines).[5] It also plots $h_n = \bar{h}, R_n, p_n$ (the dotted lines) and h_b^s, R_b, p_b (the thin lines). From Lemmas 1 and 3, we know that when $e^s < \bar{e}$, the equilibrium is bubbleless and savers do not satiate their housing utility, that is, $h^s < \bar{h}$, as illustrated in the top panel. The interest rate and housing price are determined by R_n and p_n. As shown in the middle and bottom panels, R_n is decreasing while p_n is increasing in e^s.

When $e^s \geq \bar{e}$, we know from the lemmas that the housing bubble equilibrium arises and replaces the bubbleless equilibrium, explaining the kinks in the functions for the equilibrium quantity and prices. Because of the investment motive, savers purchase housing beyond the satiation level $\bar{h}$ (the top panel). The interest rate is equal to the marginal return rate of buying and selling a unit of housing: $R_b = 1$ (the middle panel). As seen in the bottom and top panels, the housing price p_b and housing allocation h_b^s are increasing in e^s. Note further that the presence of the bubble in the housing market when $e^s > \bar{e}$ raises the equilibrium interest rate and the housing price. The following lemma summarizes this observation:

LEMMA 4 (Housing bubble prices). *If $e^s > \bar{e}$ (so that the housing bubble equilibrium exists), then $p_b > p_n$ and $R_b > R_n$.*

Proof. See Appendix A.3. □

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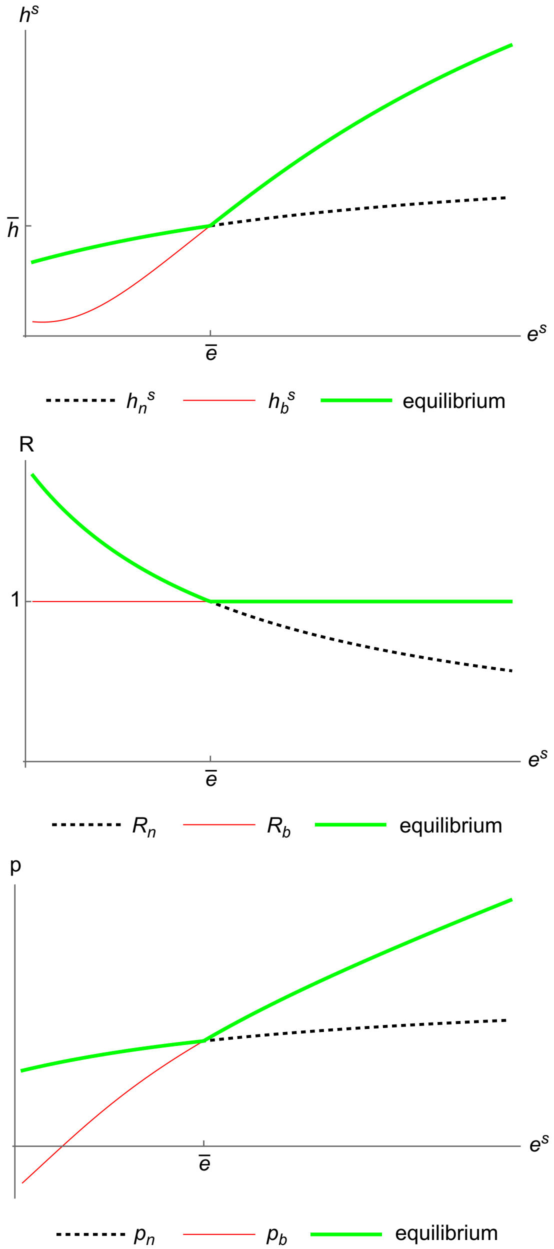

FIGURE 3. Equilibrium housing allocation, interest rate, and housing price as functions of savers' endowment.

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Intuitively, by providing an additional investment vehicle for savers, the housing bubble raises savers' demand for the housing asset, leading to an increase in its price. Furthermore, the investment by savers in the bubbly housing market crowds out their lending in the credit market, raising the interest rate.

4.2. Welfare Analysis

We can now establish the main result of the paper that the presence of the housing bubble has regressive welfare effects and exacerbates welfare inequality. Recall that $U_n^i, i \in \{s, d\}$, denotes the lifetime utility of households from the bubbleless consumption profile $\{c_{y,n}^i, c_{o,n}^i\}$ and housing allocation $h^s = 1 - h^d = h_n^s$ derived in Section 3.[6] Similarly, U_b^i denotes the lifetime utility from the consumption profile $\{c_{y,b}^i, c_{o,b}^i\}$ and housing allocation $h^s = 1 - h^d = h_b$ derived in Section 4. We will compare U_n^i and U_b^i.

The presence of the housing bubble has heterogeneous effects on savers and borrowers. For savers, who want to save for old age, the housing bubble improves their welfare by improving the return from investing in the housing asset. In contrast, the housing bubble has a *negative* effect on the welfare of borrowers. This is because it increases the price of housing, hence reducing the amount of housing that borrowers purchase and consequently their housing utility. The following proposition summarizes these regressive welfare effects of the bubble and the main result of our paper:

PROPOSITION 5 (Housing bubble benefits savers but not borrowers). *If $e^s > \bar{e}$ (so that the housing bubble equilibrium exists), then $U_b^s > U_n^s$ but $U_b^d < U_n^d$.*

Proof. Appendix A.4. □

Figure 4 illustrates the welfare effects of the housing bubble. It plots U_b^i and U_n^i as functions of e^s. When $e^s > \bar{e}$, the housing bubble arises and it raises the welfare of savers (reflected by the fact that the thick solid line representing savers' lifetime utility U_b^s lies above the dotted line representing U_n^s) but reduces the welfare of borrowers (reflected by the fact that the thick solid line representing borrowers' lifetime utility U_b^d lies below the dotted line representing U_n^d).

Furthermore, the figure also illustrates that the housing bubble exacerbates the welfare inequality in the economy. As the parameter moves from the low-inequality region $e^s < \bar{e}$ to the high-inequality region $e^s \geq \bar{e}$, the economy switches from a bubbleless equilibrium to a housing bubble equilibrium, and the equilibrium lifetime utility switches from U_n^i to U_b^i (represented by the kinks at $\bar{e}$ in the solid curves). In this region, an increase in e^s raises savers' speculative demand for and the price of the housing asset, crowding out the allocation of housing to borrowers. Hence, borrowers' lifetime utility decreases more rapidly, as reflected by the steep downward-sloping portion of the solid curve representing U_b^d. Therefore, an interesting implication arises on the feedback loop between inequality and bubble: high-income inequality facilitates the existence

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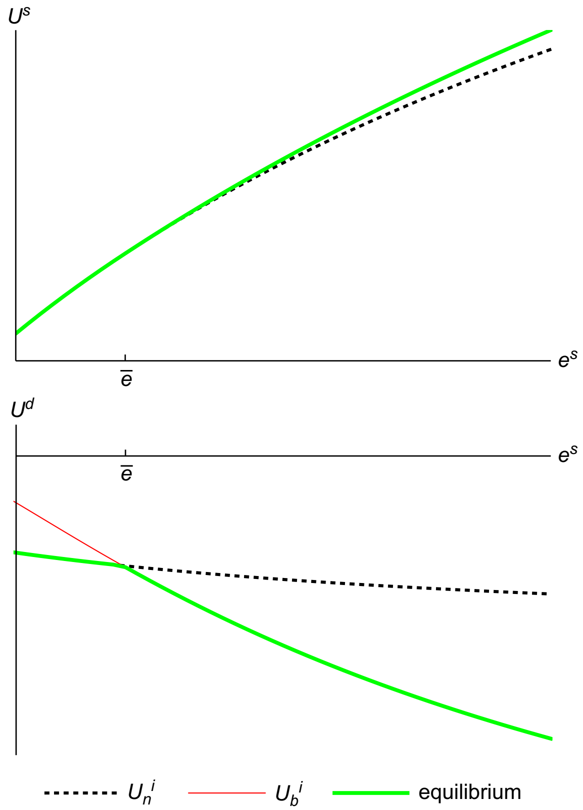

FIGURE 4. Comparative statics on welfare.

of a housing bubble, which in turn has regressive welfare effects and exacerbates welfare inequality.

5. DISCUSSIONS

5.1. Comparison with a Pure Bubble

To appreciate the welfare results established in the previous section, we compare them against the welfare effects of a pure bubble, which is an asset that pays no dividend but has a positive market price and which has been analyzed extensively in the literature (e.g., Samuelson (1958), Diamond (1965), Sargent and Wallace (1982), Tirole (1985)). The pure bubble asset can be useful as a savings instrument; however, unlike housing, the pure bubble asset does *not* give households any direct utility. As a consequence, there will be an *endogenous segmentation* of the pure bubble market, as only savers purchase the asset as an investment vehicle. Consequently, the bubble will have much less effect on borrowers, as we show below.

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5.1.1. Housing asset does not yield any utility. First, let us consider the benchmark where households do not derive utility from housing, that is, $v(h) \equiv 0$. Then, the housing asset provides no fundamental value. The model is then similar to the models of pure rational bubbles (e.g., fiat money or unbacked government debt) of Samuelson (1958), Diamond (1965), and Sargent and Wallace (1982).

As is standard and well known in this environment, there is always a bubbleless equilibrium where the housing asset is not traded. The lifetime utility in this equilibrium is given by $U_n^s = u(e^s) + \beta u(e)$ and $U_n^d = u(e^d) + \beta u(e)$. As usual, the bubbleless interest rate is determined by the Euler equation of savers: $R_n = \frac{u'(c_y^s)}{\beta u'(c_o^s)} = \frac{u'(e^s)}{\beta u'(e)}$.

Furthermore, when $R_n < 1$, the economy is dynamically inefficient and there exists another bubble equilibrium, where the housing asset is traded at a positive price. The key difference compared to the equilibrium in Section 4 is that, because here households do not derive utility from housing, borrowers will *not* have any incentive to purchase the housing asset. Thus, the equilibrium features an endogenous segmentation in the housing market: $h^d = 0$ and $h^s = 1$. As investing in the bubble allows savers to transfer more resources from young age to old age, it improves their welfare: savers' lifetime utility with the bubble is $U_b^s = u(e^s - p_b) + \beta u(e + p_b) > U_n^s$, where p_b is the bubble equilibrium price of the housing asset. As borrowers do not participate in the housing market, the rise in housing price due to the bubble does *not* affect them: borrowers' lifetime utility with the bubble is also $U_b^d = u(e^d) + \beta u(e) = U_n^d$.

In summary, when households do not derive utility from housing ($v \equiv 0$), the negative effect on borrowers of the increase in the price of housing due to the bubble is absent. The endogenous segmentation of the housing market effectively shields borrowers from the effect of the housing price bubble.

5.1.2. Pure bubble as another asset. In this subsection, we will show that the insight from the previous subsection also carries through to an alternative environment where households still derive utility from housing (i.e., v satisfies the same properties as in Section 2), but an additional pure bubble asset is introduced.

Formally, assume that there is an asset in fixed unit supply that pays no dividend but is traded at price b_t per unit. Given prices, each household of type i chooses its holding $x_t^i \geq 0$ of the bubble asset. Their optimization problem is

$$\max_{h_t^i, c_{t,y}^i, c_{t+1,o}^i, x_t^i, a_t^i} \quad U\left(h_t^i, c_{t,y}^i, c_{t+1,o}^i\right), \tag{10}$$

subject to budget constraints:

$$p_t h_t^i + b_t x_t^i + a_t^i \frac{1}{R_t} + c_{y,t}^i = e^i, \tag{11}$$

$$c_{o,t+1}^i = p_{t+1} h_t^i + b_{t+1} x_t^i + a_t^i + e, \tag{12}$$

the nonnegativity constraints:

$$x_t^i, h_t^i, c_{t,y}^i, c_{t+1,o}^i \geq 0,$$

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and the credit constraint:

$$a_t^i \geq -\bar{d}.$$

To close the model, assume that old savers own the entire supply of housing and the bubble in the initial period $t = 0$.

The definition of a pure bubble equilibrium is similar to Definition 1, except that we have an additional condition that the pure bubble price is positive:

$$b_t > 0, \forall t \geq 0,$$

and an additional market clearing condition of the bubble asset:

$$x_t^s + x_t^d = 1, \forall t \geq 0.$$

As before, we focus on steady-state equilibria where the quantities and prices are time-invariant.

Similar to the housing bubble case, savers in equilibrium must be indifferent between investing in the pure bubble asset and lending. The former yields a return of $b/b = 1$ in steady state, and the latter yields R. Hence, the interest rate in the pure bubble steady state must also be $R = 1$ as in the housing bubble case. Appendix A.5 characterizes the steady state in more details.

Welfare Analysis. Assume $e^s > \bar{e}$ so that the pure bubble equilibrium exists. Are the welfare implications of a pure bubble different from those of a housing bubble? Let U_x^i denote the lifetime utility in the pure bubble equilibrium.

Like the housing bubble, the pure bubble allows savers to store their income into old age more efficiently and hence improves their welfare relative to the bubbleless case, that is, $U_x^s > U_n^s$.

However, as the pure bubble market absorbs savers' demand for storage, savers no longer need to use the housing asset for an investment purpose. Thus, the pure bubble does not affect the housing price. As a consequence, it does not affect the welfare of borrowers, that is, $U_x^d = U_n^d$. In summary, the negative externality of savers' investment in the housing market is absent in the pure bubble equilibrium.

Furthermore, we can compare welfare across the pure bubble equilibrium and the housing bubble equilibrium. On the one hand, because the negative externality on borrowers is absent in the pure bubble case, it is straightforward to see that borrowers are better off in this case: $U_x^d > U_b^d$. On the other hand, because the pure bubble is similar to the housing bubble in providing storage for savers, it can be shown that savers get the same welfare in the two equilibria: $U_x^s = U_b^d$. The intuition above is summarized by the following result:

LEMMA 6. *The pure bubble steady state Pareto dominates the housing bubble steady state:* $U_x^d > U_b^d$ *and* $U_x^s = U_b^s$.

Proof. Appendix A.6. □

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5.2. Role of the Credit Constraint

So far, we have assumed that the credit limit is $\bar{d} = 0$. However, our main result (in particular Proposition 5 of the regressive welfare effects) carries through if $\bar{d} > 0$, as long as we continue to assume that e^s is sufficiently large so that the credit constraint $a_t^i \geq -\bar{d}$ binds for borrowers in equilibrium. In fact, the negative effect of the housing bubble on borrowers will be strengthened when $\bar{d} > 0$. This is because the bubble exerts an additional negative effect on borrowers through the credit market. To see this, recall that the budget constraint of credit-constrained young borrowers is given by

$$c_y^d + ph^d = e^d + \frac{\bar{d}}{R}.$$

Since the bubble raises the interest rate R (relative to the bubbleless rate), it reduces the total resources available for young borrowers $e^d + \frac{\bar{d}}{R}$ and thus reduces their ability to consume and purchase housing. In short, the housing bubble not only raises the cost of housing, but also raises the cost of borrowing for credit-constrained borrowers. On the flip side, by raising the interest rate on lending, the bubble provides an additional benefit to savers who want save for old age. Thus, the presence of a positive binding credit constraint $\bar{d}$ amplifies the regressive welfare effect of the housing bubble.

5.3. Policy Discussions and Rental Market

Given the externality of savers' demand in housing for investment purposes, policy interventions may be warranted. In fact, recent policy debates in China and other emerging economies (e.g., Bloomberg (2016), Nikkei Asian Review (2017), The Economist (2018)), can be discussed within the context of our model.[7] The well-documented shortage of assets for storage of wealth, such as safe government bonds, in these economies could be a reason why bubbles tend to arise in the real estate markets. A restriction on speculative investment in housing could prevent housing bubbles but would entail trade-offs, because it would help low-income households but hurt high-income households, as highlighted in our welfare analysis.

Furthermore, we have so far abstracted away from the presence of a rental market. In Appendix A.8, we extend the model to relax this assumption. We show that if the rental market is frictionless, then borrowers can rent from savers, reducing the demand for borrowing and thus raising the interest rates. As a consequence, the housing bubble may not arise. However, the housing bubble will arise again just as in the benchmark model if there are sufficient frictions in the rental market. In practice, many problems, including the presence of moral hazard and adverse selection, are prevalent in the rental markets. It has been documented that such markets are quite underdeveloped in emerging economies, including even in major cities in China (The Economist (2018)). An implication of the model is that

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the development of a functioning rental market, as discussed in recent proposals by the Chinese government to develop a market for good-quality rental housing, could help mitigate the negative externality of the housing bubble on low-income households.

6. CONCLUDING REMARKS

We have shown that a housing bubble, or, more generally, a bubble attached to a fundamentally useful asset, has heterogeneous welfare effects on households, depending on their demand for savings and borrowing. By providing an additional investment vehicle, it raises the returns from investment for savers and thus improves their welfare. However, by raising the interest rate on debt and raising the housing price, the housing bubble negatively affects the welfare of borrowers, who need debt to finance their purchase of housing. Our model also implies a feedback loop on inequality: high-income inequality leads to an environment with low interest rates, which facilitate housing bubbles, which in turn has regressive welfare effects. The key insight is that savers' demand for the housing asset for investment purposes imposes a negative externality on borrowers, who only demand the housing asset for utility purposes, by creating a bubble in the market price.

NOTES

1. See, for example, Hunter (2005), Mian and Sufi (2014), Fang et al. (2015).

2. We thus focus on the heterogeneity of endowments in young age. One can interpret the young-age endowment as wage income and the old-age endowment as payment from, for example, Social Security. Furthermore, we have effectively set the economic growth rate to be zero. It is straightforward to extend our framework with exogenous endowment growth.

3. Because of the Inada conditions, these constraints do not bind in equilibrium.

4. See also Giglio et al. (2016), who find no evidence of bubbles that violate the transversality condition in housing markets in the U.K. and Singapore. Also see Engsted et al. (2016) who provide econometric evidence for explosive housing bubbles in OECD countries.

5. To be consistent with our analysis, the minimum value of e^s in the figure is set to be sufficiently large (relative to e^d, which was normalized to one) so that borrowers are credit constrained and cannot achieve the satiation housing level $\bar{h}$ in equilibrium.

6. Also recall that when $e^s > \bar{e}$, even though the bubbleless equilibrium no longer exists, U_n^i can still be thought of as the lifetime utility in the constrained economy where households cannot use housing for speculation.

7. As an anecdote, the Chinese leader Xi Jinping said in his address at the 19th Party Congress in Beijing that "Houses are built to be inhabited, not for speculation" (Bloomberg (2017)).

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A: APPENDIX

A.1. Proof of Lemma 1

As shown in the main text, R_n is determined by (5) while p_n and h_n^s are implicitly determined by (4) and (6). Equivalently, p_n and h_n^s are the solutions to the following system of two first-order conditions for housing for savers and borrowers:

$$u'\underbrace{\left(e^s - p_n h_n^s\right)}_{c_{y,n}^s} = v'(h_n^s) + \beta p_n u'\underbrace{\left(e + p_n h_n^s\right)}_{c_{o,n}^s} \tag{A1}$$

$$p_n u'\underbrace{\left(e^d - p_n\left(1 - h_n^s\right)\right)}_{c_{y,n}^d} = v'\left(1 - h_n^s\right) + \beta p_n u'\underbrace{\left(e + p_n\left(1 - h_n^s\right)\right)}_{c_{o,n}^d}. \tag{A2}$$

Equation (A2) uniquely determines p_n as a function of h_n^s. To see this, define the difference between the two sides of (A2) as:

$$\Delta\left(p, h_n^s\right) \equiv p_n u'\left(e^d - p\left(1 - h_n^s\right)\right) - v'(1 - h_n^s) - \beta p u'\left(e + p\left(1 - h_n^s\right)\right).$$

Because u is strictly concave, this function is strictly increasing in p. Also, $\Delta(0) < 0$. Furthermore, since $\lim_{c\to 0+} u'(c) = \infty$, it follows that $\lim_{p\to e^d/(1-h_n^s)-} \Delta(p) = \infty$. Hence, by continuity, for each h_n^s, there exists a unique $p_n \in (0, e^d/(1 - h_n^s))$ such that $\Delta(p_n, h_n^s) = 0$. The curve labeled "foc borrowers" in Figure 1 plots p_n as a function of h_n^s. It is straightforward to see that the curve intersects the horizontal axis (associated with $p = 0$) at $h^s = 1 - \bar{h}$ (where $v'(1 - h^s) = 0$).

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Similarly, we can show that equation (A1) determines h_n^s as a function of p_n, as plotted by the curve labeled "foc savers" in Figure 1, which intersects the horizontal axis at $h^s = \bar{h}$. Given $\bar{h} > 1/2$ and the Inada conditions, the two first-order curves intersect at a unique point, which determines $h_n^s \in (0, 1)$ and p_n. Given h_n^s and p_n, the interest rate R_n is simply given by (5), which is always positive.

Finally, the allocations and prices associated with h_n^s, p_n, and R_n constitute a stationary equilibrium if and only if p_n is positive. From (4), this is the case if and only if $R_n > 1$.

A.2. Proof of Lemma 3

Recall that h_b^s and p_b solve (6) and (9), which can be rewritten, respectively, as:

$$u'\underbrace{\left(e^s - p_b h_b^s\right)}_{c_{y,b}^s} = \beta u' \underbrace{\left(e + p_b h_b^s\right)}_{c_{o,b}^s} \tag{A3}$$

$$p_b u' \underbrace{\left(e^d - p_b\left(1 - h_b^s\right)\right)}_{c_{y,b}^d} = v'\left(1 - h_b^s\right) + \beta p_b u' \underbrace{\left(e + p_b\left(1 - h_b^s\right)\right)}_{c_{o,b}^d}. \tag{A4}$$

Using similar arguments to those used in Appendix A.1, it is straightforward to show that $h_b^s \in (0, 1)$ and p_b are uniquely determined by these two equations, as illustrated in Figure 2.

Now, we show h_b^s and p_b are increasing in e^s. Consider two arbitrary endowment levels $e_1^s > e_2^s$, with the corresponding solutions $(h_{b,1}^s, p_{b,1})$ and $(h_{b,2}^s, p_{b,2})$. We need to show that $h_{b,1}^s > h_{b,2}^s$ and $p_{b,1} > p_{b,2}$. Because of the concavity of u, equation (A3) determines ph^s as an increasing function of e^s, implying $p_{b,1}h_{b,1}^s > p_{b,2}h_{b,2}^s$. Equation (A4) can be further rewritten as:

$$p\left[u'\left(e^d - p + ph^s\right) - \beta u'\left(e + p - ph^s\right)\right] = v'(1 - h^s).$$

Thus,

$$\frac{p_{b,1}\left[u'\left(e^d - p_{b,1} + p_{b,1}h_{b,1}^s\right) - \beta u'\left(e + p_{b,1} - p_{b,1}h_{b,1}^s\right)\right]}{p_{b,2}\left[u'\left(e^d - p_{b,2} + p_{b,2}h_{b,2}^s\right) - \beta u'\left(e + p_{b,2} - p_{b,2}h_{b,2}^s\right)\right]} = \frac{v'\left(1 - h_{b,1}^s\right)}{v'\left(1 - h_{b,2}^s\right)}. \tag{A5}$$

Suppose on the contrary that $p_{b,1} \le p_{b,2}$. Then, because $p_{b,1}h_{b,1}^s > p_{b,2}h_{b,2}^s$, it must be that $h_{b,1}^s > h_{b,2}^s$. Then the left-hand side of (A5) is smaller than 1, while the right-hand side is larger than 1, leading to a contradiction. Therefore, $p_{b,1} > p_{b,2}$. Similarly, $h_{b,1}^s > h_{b,2}^s$. In other words, both h_b^s and p_b as implicitly determined by (A3) and (A4).

From Lemma 2, we know that for the allocations and prices associated with h_s^b and p_b to constitute a housing bubble equilibrium, it is necessary that $h_b^s \ge \bar{h}$. Note that when $e^s = \bar{e}$, the systems (A1)–(A2) and (A3)–(A4) yield the same solution: $R_n = R_b = 1$, $h_n^s = h_b^s = \bar{h}$, and $p_n = p_b$. Furthermore, we already proved that h_b^s is increasing in e^s. Hence, $h_b^s \ge \bar{h}$ if and only if $e^s \ge \bar{e}$, or equivalently, $R_n \le 1$. Conversely, if $e \ge \bar{e}$, then $h_s^b \ge \bar{h}$ and $p_b \ge p_n > 0$, hence the allocations and prices above constitute a housing bubble equilibrium.

A.3. Proof of Lemma 4

Recall that R_n is decreasing in e^s. The fact that if $e^s > \bar{e}$, then $R_b > R_n$ immediately follows from Lemma 1 and the proof in Appendix A.1. From Lemma 3 and the proof in

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Appendix A.2, we know that p_b is increasing in e^s, and that $p_b = p_n$ when $e^s = \bar{e}$. Hence, it immediately follows that if $e^s > \bar{e}$, then $p_b > p_n$.

A.4. Proof of Proposition 5

We start by showing that $U_b^s > U_n^s$. Recall that

$$U_b^s = u\underbrace{\left(e^s - p_b h_b^s\right)}_{c_{y,b}^s} + \beta u \underbrace{\left(e + p_b h_b^s\right)}_{c_{o,b}^s} + v(\bar{h}),$$

where the three terms on the right-hand side correspond to the utility over consumption in young age, the discounted utility over consumption in old age, and the utility over (satiated) housing consumption, respectively. Similarly, recall that

$$U_n^s = u\underbrace{\left(e^s - p_n h_n^s\right)}_{c_{y,n}^s} + \beta u \underbrace{\left(e + p_n h_n^s\right)}_{c_{o,n}^s} + v(h_n^s).$$

Since $v(\bar{h}) > v(h_n^s)$, it suffices to show that

$$u\left(e^s - p_b h_b^s\right) + \beta u\left(e + p_b h_b^s\right) > u\left(e^s - p_n h_n^s\right) + \beta u\left(e + p_n h_n^s\right),$$

or equivalently

$$F\left(p_b h_b^s\right) > F\left(p_n \bar{h}\right), \tag{A6}$$

where $F(x) \equiv u(e^s - x) + \beta u(e + x)$. Note that the solution to $\max_x F(x)$ is the solution to the first-order condition $u'(e^s - x) = \beta u'(e + x)$. From (A3), it follows that $x = p_b h_b^s$ solves $\max_x F(x)$. From (A1) and the fact that $R_n < 1$, it follows that $x = p_n h_n^s$ does *not* solve $\max_x F(x)$. Therefore, (A6) automatically follows. This completes the proof of $U_b^s > U_n^s$.

Now we show $U_n^d > U_b^d$. Recall that

$$U_b^d = u\underbrace{\left(e^d - p_b h_b^d\right)}_{c_{y,b}^d} + \beta u \underbrace{\left(e + p_b h_b^d\right)}_{c_{o,b}^d} + v\left(h_b^d\right),$$

and

$$U_n^d = u\underbrace{\left(e^d - p_n h_n^d\right)}_{c_{y,n}^d} + \beta u \underbrace{\left(e + p_n h_n^d\right)}_{c_{o,n}^d} + v\left(h_n^d\right),$$

where $h_b^d = 1 - h_b^s$ and $h_n^d = 1 - h_n^s$. Also recall that $h_b^d < h_n^d$. Hence, under the bubbleless steady-state prices, the housing allocation h_b^d (along with the net asset position $a_b^d = 0$) is feasible for borrowers. Thus, by the definition of h_n^d and $a_n^d = 0$ as the solution to the optimization problem of borrowers given the bubbleless steady-state prices, it follows that borrowers must be at least better off with the bubbleless steady-state allocations than with the bubble steady-state allocations:

$$U_n^d \geq u(e^d - p_n h_b^d) + \beta u(e + p_n h_b^d) + v(h_b^d). \tag{A7}$$

Furthermore, combining the fact that $p_n h_b^d < p_b h_b^d$ (because the housing price is higher in the housing bubble steady state) and the fact that $u'(e^d - p_b h_b^d) > \beta u'(e + p_b h_b^d)$ (borrowers in the housing bubble steady state are constrained in their ability to tilt consumption forward) yields

$$u\left(e^d - p_n h_b^d\right) + \beta u\left(e + p_n h_b^d\right) > u\left(e^d - p_b h_b^d\right) + \beta u\left(e + p_b h_b^d\right). \tag{A8}$$

Inequalities (A7) and (A8) then imply $U_n^d > U_b^d$, as desired.

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A.5. Pure Bubble Steady State

The pure bubble steady state is characterized by interest rate $R = 1$, pure bubble asset allocation $x^s = 1$, housing allocation $h^s = h^s_n$, along with housing price $p = p_n$ and pure bubble price b that is implicitly determined by the Euler equation of savers:

$$u'\left(e^s - p_n h^s_n - b\right) = \beta u'\left(e + p_n h^s_n + b\right). \tag{A9}$$

It is straightforward that given the conjectured prices, the conjectured allocations solve the optimization problems of individual borrowers and savers.

Thus, by definition, to verify whether the allocations and prices specified in the lemma constitute a pure bubble steady state, it is equivalent to verify that $b > 0$ (the pure bubble has a positive price). Recall that b solves equation (A9). Because u is strictly concave, it follows from this equation that $b > 0$ if and only if

$$u'(e^s - p_n\bar{h}) < \beta u'(e + p_n\bar{h}).$$

Because of equation (A1), this inequality is equivalent to

$$R_n < 1.$$

A.6. Proof of Lemma 6

Recall from Proposition 5 that $U^d_n > U^d_b$. Furthermore, from Appendix A.5, we know that the bubbleless and pure bubble steady states yield the same allocation to borrowers, and hence $U^d_x = U^d_n$. Thus, it immediately follows that $U^d_x > U^d_b$.

It remains to show that $U^s_x = U^s_b$. Since the interest rate is $R = 1$ in both the housing bubble and the pure bubble steady states, the Euler equation for savers in both steady states is

$$u'\left(c^s_y\right) = \beta u'\left(c^s_o\right).$$

Furthermore, in both cases, we have

$$c^s_y + c^s_o = e^s + e.$$

In other words, both $(c^s_{y,b}, c^s_{o,b})$ and $(c^s_{y,x}, c^s_{o,x})$ are solutions to the system of two equations above. As u is strictly concave, the system has only one solution. Thus the two consumption allocations are the same. Furthermore, savers achieve the same housing utility $v(\bar{h})$ in both steady states. Hence, $U^s_x = U^s_b$, as desired.

A.7. Stability of the Housing Bubble Steady State

The housing bubble steady-state values (p_b, h_b, R_b) constitute the fixed point of the following dynamic system, which can be summarized by two difference equations in the price of bubble p_t and the allocation of housing to savers h^s_t:

$$p_t = \frac{v'\left(1 - h^s_t\right)}{u'\left(e^d - p_t\left(1 - h^s_t\right)\right)} + \beta \frac{u'\left(e + p_{t+1}\left(1 - h^s_t\right)\right)}{u'\left(e^d - p_t\left(1 - h^s_t\right)\right)} p_{t+1} \tag{focD}$$

$$p_t = \beta \frac{u'\left(e + p_{t+1} h^s_t\right)}{u'\left(e^s - p_t h^s_t\right)} p_{t+1}, \tag{focS}$$

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along with the following Euler equation that determines the interest rate R_t:

$$R_t = \frac{u'\left(e^s - p_t h_t^s\right)}{\beta u'\left(e + p_{t+1} h_t^s\right)}.$$

Equations (focD) and (focS) are simply the first-order conditions of borrowers and savers with respect to housing, where we have substituted the interest rate R_t by the previous Euler equation. Given the state variable p_t, these two equations implicitly define h_t^s and p_{t+1}, denoted by $\mathcal{H}(p_t)$ and $\mathcal{P}(p_t)$, respectively. From h_t^s and p_{t+1}, we also obtain R_t as a function $\mathcal{R}(p_t)$ of p_t. Let $z_t := (p_t, h_t^s, R_t)$. Then these functions define a dynamical system $z_{t+1} = \Phi(z_t) \equiv (\mathcal{P}(p_t), \mathcal{H}(\mathcal{P}(p_t)), \mathcal{R}(\mathcal{P}(p_t)))$. It can be verified from the assumptions on the utility functions that $\mathcal{P}$ and $\mathcal{R}$ are increasing functions of p_t.

We now show that the housing bubble steady state is saddle-path stable. Suppose $p_0 > p_b$. Then, because of the monotonicity of $\mathcal{R}$, we have $R_0 = \mathcal{R}(p_0) > \mathcal{R}(p_b) = 1$. It then follows that $p_1 = p_0 R_0 > p_0$. Hence $p_1 > p_0 > p_b$. By recursion, we get $p_t > \cdots > p_1 > p_0 > p_b$, for all $t > 0$. Hence, $\{p_t\}$ does not converge to p_b. Similarly, suppose $p_0 < p_b$. Then $p_t < \cdots < p_1 < p_0 < p_b$, for all $t > 0$ and $\{p_t\}$ again does not converge to p_b. Only when $p_0 = p_b$ does the system converge to the housing bubble steady state. Hence, the housing bubble steady state is saddle-path stable. This saddle-path stability is standard in the rational bubbles literature (e.g., Tirole, 1985).

A.8. Extension with Rental Market

Frictionless rental market

In this extension, we introduce a perfectly competitive and frictionless rental market to the model in the main text. Let $h^i \geq 0$ continue to denote a household of type i's purchase of the housing asset, and let $\hat{h}^i$ denote the household's net rental housing, where a positive position means the household is a renter and a negative position means the household is a landlord. The purchasing and renting prices of housing are denoted by p and $\hat{p}$, respectively.

Taking (steady-state) prices as given, the optimization problem of a representative household of type i is to maximize $U(c_y^i, c_o^i, h^i + \hat{h}^i)$ subject to the credit constraint $a^i \geq -\bar{d}$, to nonnegative constraints $c_y^i, c_o^i, h^i, h^i + \hat{h}^i \geq 0$, and to the following budget constraints:

$$ph^i + \hat{p}\hat{h}^i + \frac{a^i}{R} + c_y^i = e_y^i$$
$$c_o^i = ph^i + a^s + e.$$

In an equilibrium with the competitive rental market, the rental market must clear: $\hat{h}^s + \hat{h}^d = 0$. We again focus on the parameter region where e^s is sufficiently high such that in equilibrium, borrowers will be credit constrained.

The first-order condition with respect to rental housing of a representative borrower yields

$$\hat{p} = v'(h^d + \hat{h}^d)/u'(c_y^d). \qquad \textbf{(A10)}$$

Intuitively, the price of housing must be equal to the marginal rate of substitution between housing services and consumption for borrowers.

As in the main model, the first-order conditions with respect to borrowing/lending of the unconstrained savers and the constrained borrowers yield

$$\frac{1}{R} = \frac{\beta u'\left(c_o^s\right)}{u'\left(c_y^s\right)} > \frac{\beta u'\left(c_o^d\right)}{u'\left(c_y^d\right)}. \qquad \textbf{(A11)}$$

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Furthermore, by letting $h_c^s \equiv h^s + \hat{h}^s$ denote the consumption of housing services of a representative saver (the amount of housing that enters the utility function), we can rewrite the optimization problem of savers as maximizing $U(c_y^s, c_o^s, h_c^s)$ subject to the following budget constraints:

$$\begin{aligned} ph^s + \frac{a^s}{R} + c_y^s &= e_y^s + \hat{p}\left(h^s - h_c^s\right) \\ c_o^s &= ph^s + a^s + e, \end{aligned}$$

and the nonnegativity constraints $c_y^s, c_o^s, h^s, h_c^s \geq 0$. The first-order condition with respect to h^s of this problem then yields

$$(p - \hat{p})u'\left(c_y^s\right) = \beta p u'\left(c_o^s\right),$$

or equivalently:

$$p = \hat{p} + \frac{p}{R}. \tag{A12}$$

Intuitively, the price of housing is equal to the dividend, measured by the rental price $\hat{p}$, plus the resale value $\frac{p}{R}$. In summary, the equilibrium prices $p, \hat{p}$, and R must satisfy (A10), (A11), and (A12).

Equations (A10), (A11), and (A12) imply two results. First, the fact that the credit constraint binds for borrowers implies that borrowers do not buy housing and are net renters. To see this, let λ^d be the Lagrange multiplier associated with the nonnegativity constraint $h^d \geq 0$. Then, the first-order condition with respect to h^d yields

$$\begin{aligned} p &= \frac{v'\left(h^d + \hat{h}^d\right)}{u'\left(c_y^d\right)} + p\frac{\beta u'\left(c_o^d\right)}{u'\left(c_y^d\right)} + \lambda^d \\ &= \hat{p} + p\frac{\beta u'\left(c_o^d\right)}{u'\left(c_y^d\right)} + \lambda^d. \end{aligned}$$

Combined with the inequality in (A11) due to the binding credit constraint, we get

$$p < \hat{p} + \frac{p}{R} + \lambda^d.$$

Combined with asset pricing equation (A12), we get

$$0 < \lambda^d,$$

that is, the constraint $h^d \geq 0$ binds. Thus, in equilibrium $h^d = 0$ and $h^s = 1$. Because v satisfies the Inada conditions, it follows that $\hat{d} > 0$, that is, borrowers will rent in equilibrium.

Second, since $\hat{p}$ and p must be positive in any equilibrium with the rental market, equation (A12) implies that $R > 1$. Therefore, the presence of the rental market effectively rules out the possibility of bubbles, including housing bubbles. In fact, equation (A12) implies that in equilibrium the price of housing is simply equal to the present value of rental dividends:

$$p = \frac{\hat{p}}{1 - \frac{1}{R}},$$

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that is, the housing price is equal to the fundamental value of housing, and the bubble component is necessarily zero.

Frictional rental market

The previous subsection assumes a frictionless rental market. However, in practice, rental markets tend to have many frictions, especially in developing economies. In this subsection, we show a simple way to further extend the model to capture, in a reduced form, rental market frictions.

Specifically, assume that there is a transaction (or maintenance) cost Ψ associated with renting. The budget constraint of young households would then become:

$$ph^i + \left(1 + \Psi\left(\hat{h}^i\right)\right)\hat{p}\hat{h}^i + c_y^i = e_y^i,$$

where for simplicity we assume $\Psi(h) \equiv \frac{\psi}{2}h^2$, with $\psi \geq 0$ being an exogenous constant. Also assume for simplicity that the maintenance cost is a dead-weight loss to the economy (i.e., the cost is not paid to anyone).

When $\psi = 0$, the model collapses to the frictionless rental market in the previous subsection. However, when $\psi \to \infty$, the cost of renting is too high and the model collapses to the main model with no rental market in the main text. It can be shown that, for a sufficiently large ψ, the rental market is sufficiently frictional, housing bubbles can exist, and the main effects as studied in the main text continue to apply.

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