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# THE INSPECTION PARADOX

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We relate two variants of the inspection paradox by showing that both represent the length of a renewal interval that contains a chosen point, the difference being in the way that point is chosen. We show, in both cases, that the length of the interval is likelihood ratio larger than that of an ordinary renewal interval.

## 1. AN INSPECTION PARADOX

A total of $n$ families have children who attend a certain school. Arbitrarily number these families, and for $i = 1,\dots,n$, let $X_i$ denote the number of children of family $i$ who attend the school. Assume that the $X_i$ are independent and identically distributed with mass function $p_j = P\{X_i = j\}$, $j \geq 1$, and finite mean $\mu = \sum_j jp_j$.

Suppose that one of the school children is randomly chosen. We are interested in the distribution of the number of school children in the selected child's family; that is, if $I$ is the family to which the selected child belongs, we are interested in $X_I$.

THEOREM 1.1:

*(a)* $X_I$ *is likelihood ratio ordered larger than* $X_1$.
*(b)* $P\{X_I \geq r\} \geq \sum_{j\geq r} p_j$.
*(c)* $P\{X_I = j\} \geq \dfrac{jp_j}{\mu} \dfrac{n\mu}{j + (n-1)\mu}$
*(d)* $\lim_{n\to\infty} P\{X_I = j\} = \dfrac{jp_j}{\mu}$.

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PROOF:

$$P\{X_I = j | I = k\} = P\{X_k = j | I = k\} = \frac{P\{I = k | X_k = j\} p_j}{P\{I = k\}}. \tag{1}$$

Now,

$$P\{I = k | X_k = j, X_i, i \neq k\} = \frac{j}{j + \sum_{i \neq k} X_i},$$

implying that

$$P\{I = k | X_k = j\} = E\left[\frac{j}{j + \sum_{i \neq k} X_i} | X_k = j\right] = E\left[\frac{j}{j + \sum_{i \neq k} X_i}\right].$$

Therefore, using the preceding and the fact that $P\{I = k\} = 1/n$, we obtain from Eq. (1) that

$$P\{X_I = j | I = k\} = jp_j E\left[\frac{n}{j + \sum_{i \neq k} X_i}\right].$$

Because the right-hand side does not depend on $k$, the preceding gives

$$P\{X_I = j\} = jp_j E\left[\frac{n}{j + \sum_{i \neq k} X_i}\right], \qquad j \geq 1, \tag{2}$$

or

$$\frac{P\{X_I = j\}}{p_j} = nE\left[\frac{1}{1 + \sum_{i \neq k} X_i / j}\right].$$

Part (a) now follows because the right-hand side of the preceding is clearly increasing in $j$. Part (b) follows from part (a) because likelihood ratio ordering implies stochastic ordering.

Applying Jensen's inequality to the right-hand side of Eq. (2) gives

$$P\{X_I = j\} \geq jp_j \frac{n}{j + (n-1)\mu},$$

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which is part (c). To prove part (d), note that, by the strong law of large numbers,

$$\frac{1}{j/n + \sum_{i \neq k} X_i/n} \to \frac{1}{\mu} \quad \text{as } n \to \infty$$

and, because $X_i \geq 1$,

$$\frac{1}{j/n + \sum_{i \neq k} X_i/n} \leq 1.$$

Therefore, part (d) follows from Eq. (2) upon applying the strong law and Lebesgue's dominated convergence theorem. ■

The result (a) (or the weaker (b)) that the family size of the randomly chosen student tends to be larger than that of a regular family is known as the *inspection paradox*.

## 2. RELATION TO THE INSPECTION PARADOX OF RENEWAL THEORY

Let $N(t), t \geq 0$, be a renewal process with interarrival distribution $F$, and let $X_i$ denote its $i$th interarrival time. The random variable $X_{N(t)+1}$ represents the length of the renewal interval that contains the time point $t$. The inspection paradox of renewal theory states that its distribution is stochastically larger than that of $X_1$; that is,

$$P\{X_{N(t)+1} > x\} \geq 1 - F(x).$$

The preceding is easily proven by conditioning on $A(t)$, the age of the renewal process at $t$ (equal to the time at $t$ since the last renewal, where we suppose that a renewal has occurred at time 0). Given that $A(t) = s$, the length of the renewal interval that contains $t$ is distributed as an interarrival time conditioned to be at least $s$; that is, (see [3]),

$$P\{X_{N(t)+1} > x | A(t) = s\} = P\{X_1 > x | X_1 > s\} \geq P\{X_1 > x\}. \tag{3}$$

The model considered in Section 1 is analogous to asking for the length of the renewal interval containing a point that is uniformly distributed over the first $n$ interarrival times. To see this, let $T$, conditional on $\sum_{i=1}^{n} X_i$, be uniformly distributed over $(0, \sum_{i=1}^{n} X_i)$. Then, with $I$ denoting the interarrival interval that contains $T$, the random variable of interest is $X_I$. Letting $X_i$ be the number of school children in family $i$ shows the equivalence. It is interesting to note that the analog of Eq. (3) does not hold when the time is chosen uniformly over the first $n$ interarrival times; that is, if $A_r$ is the event that the randomly selected child is the $r$th oldest child in his or her family, then $X_I$ is not distributed, as $X_1$ conditional on $X_1 \geq r$. (For instance, suppose

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$n = 1, p_1 = p_{10} = \frac{1}{2}$. Then, given that the randomly chosen child is the youngest in her family, the probability that she is the only child is 10/11.) Indeed, for $j \geq r$,

$$P\{X_I = j | A_r\} = \frac{P(A_r | X_I = j) P\{X_I = j\}}{\sum_{j \geq r} P(A_r | X_I = j) P\{X_I = j\}}$$

$$= \frac{\frac{1}{j} P\{X_I = j\}}{\sum_{j \geq r} \frac{1}{j} P\{X_I = j\}}.$$

Of course, as $n$ goes to infinity,

$$P\{X_I = j | A_r\} \rightarrow \frac{p_j}{\sum_{j \geq r} p_j} = P\{X_1 = j | X_1 \geq r\}.$$

*Remarks:*

1. For $T$ being uniformly chosen on the interval from 0 to $\sum_{i=1}^{n} X_i$, the age of the renewal process at time $T$ and the excess at time $T$ (equal to the time from $T$ until the next renewal) have the same distribution. When the renewal process is a Poisson process with rate $\lambda$, neither the age nor the excess at time $T$ is exponential with rate $\lambda$. (This last statement is easily seen by taking $n = 1$ and noting that the age plus the excess at $T$ is equal to an exponential with rate $\lambda$.)
2. Although the inspection paradox of renewal theory is usually taken to be that the length of the renewal interval that contains the fixed time $t$ is stochastically larger than that of an ordinary renewal interval (see [1–3]), it can be shown, as in Theorem 1, that it is larger in the likelihood ratio sense. To verify this claim (which is apparently new), let $f$ be the density of an interarrival time. Then,

$$f_{X_{N(t)+1}}(x) = E[\, f_{X_{N(t)+1}}(x) | A(t)]$$

$$= \int_0^x \frac{f(x)}{1 - F(s)} \, dH_{A(t)}(s),$$

where $H_{A(t)}(s)$ is the distribution function of $A(t)$. Consequently,

$$\frac{f_{X_{N(t)+1}}(x)}{f(x)} = \int_0^x \frac{1}{1 - F(s)} \, dH_{A(t)}(s)$$

is nondecreasing in $x$.

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