Robotica (2010) volume 28, pp. 341–348. © Cambridge University Press 2009
doi:10.1017/S0263574709005633

New advances in automatic selection of eligible surface elements for grasping and fixturing

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(Received in Final Form: April 3, 2009. First published online: May 5, 2009)

SUMMARY
Many object surfaces involve a number of pieces, expressed by different equations. Previous methods of optimal grasp planning can hardly cope with such cases. Ding *et al.* solve this problem by characterizing the object surface with convex facets and discrete points, then selecting the eligible ones for force-closure, and finally seeking the optimal contact positions on the selected elements. So far, however, no point contact with friction (PCwF) but only frictionless point contacts (FPC) can be used on the facets, while soft finger contacts (SFC) are excluded at all. In this paper, to the above two surface elements we add line segments. Moreover, the limitations on the contact types are completely removed. A general condition and a quantitative criterion of eligibility are presented, followed by a heuristic algorithm and an iterative algorithm for finding the better eligible elements. Three common examples show: the new advances make the formerly tough problems smoothly solvable.

KEYWORDS: Multifingered grasping; Optimal grasp planning; Fixture; Force-closure; Eligibility.

1. Introduction

Multifingered robot hands have been ardently explored for over two decades due to the potential in dexterous manipulation. In this field, a fundamental problem is optimal grasp planning—finding ideal contact positions on the object surfaces to construct a good force-closure grasp.[1–11] If the contacts are frictionless, the property relates only to the object geometry, and the term "form-closure" is often used instead. Likewise fixture is an important tool in manufacturing for locating and immobilizing the object. In its design, one also encounters the problem of seeking proper contact positions. The difference is that the localization accuracy[12–15] has priority over the ratio of grasp capability to costs. Accordingly, different number and types of contacts are adopted in the two cases. For 3D grasping, even three point contacts with friction (PCwFs) or two soft finger contacts (SFCs) usually suffice if the contact forces can be as large as we wish. As for fixturing, seven frictionless point contacts (FPCs) are required in principle.

Currently, the most advanced algorithms for grasp[3–11] or fixture[12–14] planning can search smooth areas[3–10] or discrete point domains[11–14] on the object surface for good contact positions. However, if the object surface consists of several pieces, the search is more prone to fall into an undesired local optimum. If it comprises many pieces, one has to pick the eligible pieces having force-closure contact positions before the search. In addition, the contact positions selected in discrete points can be further optimized in their adjacent areas. This idea was not mentioned in refs. [11–14]. Considering all of the above, we put forward a hybrid approach, which first selects the eligible surface elements and then finds the optimal contact positions on them or nearby. Compared with the previous work, our work makes the following advances:

1. The surface elements used to represent or characterize an object surface involve not only convex facets[15] and discrete points[11–14] but also line segments, which are particularly useful for describing ruled surfaces (a ruled surface is a surface that can be swept out by moving a line in space).
2. Ding *et al.*[15] deduced an eligibility condition for checking if a set of convex facets can provide force-closure locations for seven FPCs. We generalize this condition to include PCwFs and SFCs on any kind of surface elements with elaborate discussions on its necessity and sufficiency.
3. Imitating Ferrari–Canny second criterion,[2] we present a load capacity criterion for a set of contact positions and accordingly an eligibility criterion for a set of surface elements to estimate their goodness.
4. Inspired by the work of Ding *et al.*,[15] we put forward a heuristic algorithm for seeking eligible elements based on our criterion. To improve the solution quality, an iterative algorithm is also suggested.

The present work is based on the following assumptions:

1. All the points on the surface elements are regular, where the normal and tangent vectors are well defined. There are only point contacts at the regular points.
2. Like a convex facet, either a discrete point or a line segment on a smooth surface has a unique normal, so that three surface elements can be used uniquely[11–15] or together. Although some ruled surfaces cannot be described by such line segments, they can be described by discrete points.

2. Definitions of Force-Closure and Eligibility

Consider a 3D object to be grasped or fixtured by m contacts. Let $\boldsymbol{n}_i$, $\boldsymbol{o}_i$, and $\boldsymbol{t}_i$ denote the unit inward normal at contact

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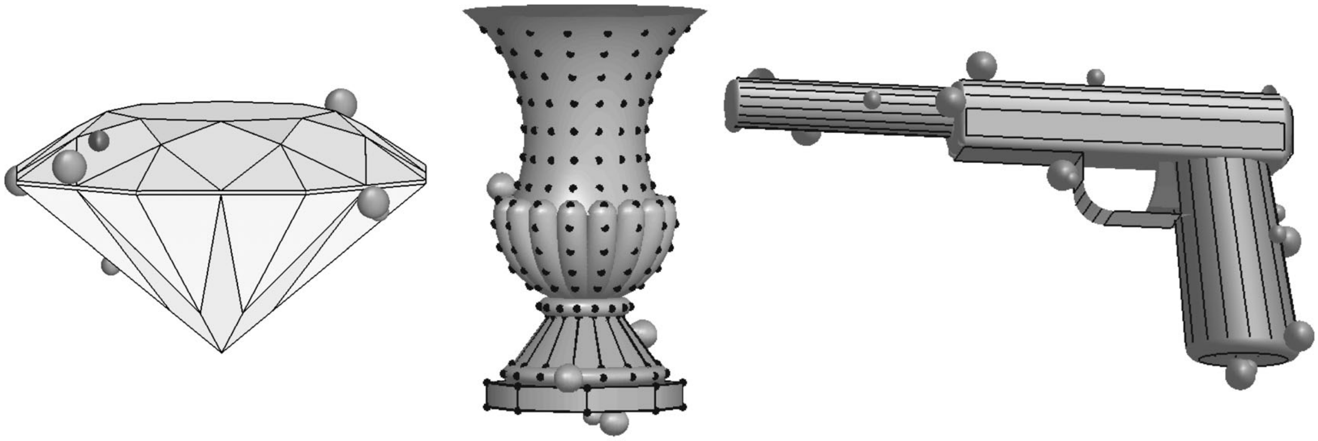

(a) (b) (c)

Fig. 1. Representation of object surfaces using different surface elements. Small and large balls depict initial and optimal contact positions on the selected surface elements, respectively. (a) A diamond-shaped lamp with 57 convex facets. (b) A cup characterized by 9 convex facets, 16 line segments, and 224 discrete points. (c) A pistol portrayed by 7 facets and 42 line segments.

i ($i = 1, 2, \ldots, m$) and two unit tangent vectors such that $\boldsymbol{n}_i = \boldsymbol{o}_i \times \boldsymbol{t}_i$. The contact force can be expressed in the local coordinate frame $\{\boldsymbol{n}_i, \boldsymbol{o}_i, \boldsymbol{t}_i\}$ by

$$\boldsymbol{f}_i = [f_{in}\, f_{io}\, f_{it}\, f_{is}]^T,$$

where f_{in} is the normal force component, f_{io} and f_{it} are two tangential force components, and f_{is} is the spin moment about $\boldsymbol{n}_i$. To avoid separation and slip at contact, $\boldsymbol{f}_i$ must be within one of the following convex cones, known as the friction cones:

$$\text{FPC}: \quad F_i = \{\boldsymbol{f}_i | f_{in} \geq 0,\ f_{io} = f_{it} = f_{is} = 0\} \tag{1}$$

$$\text{PCwF}: \quad F_i = \left\{\boldsymbol{f}_i | f_{in} \geq 0, \sqrt{f_{io}^2 + f_{it}^2} \leq \mu_i f_{in},\ f_{is} = 0\right\} \tag{2}$$

$$\text{SFCl}: \quad F_i = \left\{\boldsymbol{f}_i | f_{in} \geq 0, \frac{\sqrt{f_{io}^2 + f_{it}^2}}{\mu_i} + \frac{|f_{is}|}{\mu_{si}} \leq f_{in}\right\} \tag{3}$$

$$\text{SFCe}: \quad F_i = \left\{\boldsymbol{f}_i | f_{in} \geq 0, \sqrt{\frac{f_{io}^2 + f_{it}^2}{\mu_i^2} + \frac{f_{is}^2}{\mu'^2_{si}}} \leq f_{in}\right\} \tag{4}$$

where μ_i is the Coulomb friction coefficient, and μ_{si} and μ'_{si} are the coefficients of spin moment for SFC with linear (SFCl) and elliptic (SFCe) models,[17] respectively.

Limited by material strength or actuator power, the magnitude of f_{in} has an upper bound f_i^U. Under the above friction cone constraint confining the contact force direction, the contact force magnitude is restricted by

$$\Omega_i = \left\{\boldsymbol{f}_i | f_{in} \leq f_i^U\right\}. \tag{5}$$

From Eqs. (1)–(5), $F_i \cap \Omega_i$ is the set consisting of *allowable* contact forces at contact i, which is a compact and convex set. The grasp matrix $\boldsymbol{G}_i$ transforms $F_i \cap \Omega_i$ into a compact convex set in $\mathbb{R}^6$, namely $\boldsymbol{G}_i(F_i \cap \Omega_i)$, which consists of all allowable wrenches generated by contact i on the object, where

$$\boldsymbol{G}_i = \begin{bmatrix} \boldsymbol{n}_i & \boldsymbol{o}_i & \boldsymbol{t}_i & \boldsymbol{0} \\ \boldsymbol{r}_i \times \boldsymbol{n}_i & \boldsymbol{r}_i \times \boldsymbol{o}_i & \boldsymbol{r}_i \times \boldsymbol{t}_i & \boldsymbol{n}_i \end{bmatrix}, \tag{6}$$

where $\boldsymbol{r}_i$ is the position vector of contact i and $\boldsymbol{0}$ denotes the zero vector or origin of a space. Then the Minkowski sum $\sum_{i=1}^m \boldsymbol{G}_i(F_i \cap \Omega_i) = \boldsymbol{G}(F \cap \Omega)$ is a compact convex set in $\mathbb{R}^6$, containing all the allowable resultant wrenches that can be generated by the m contacts, where $\boldsymbol{G} = [\boldsymbol{G}_1 \boldsymbol{G}_2 \cdots \boldsymbol{G}_m]$, $F = \prod_{i=1}^m F_i$, and $\Omega = \prod_{i=1}^m \Omega_i$.

Definition 1. *A set of contacts is said to be force-closure if* $\text{int}\{\boldsymbol{G}(F \cap \Omega)\} \neq \emptyset$ *and* $\boldsymbol{0} \in \text{int}\{\boldsymbol{G}(F \cap \Omega)\}$, *where* $\text{int}(\bullet)$ *denotes the interior of a set.*

The object surface is depicted by three kinds of elements, i.e. convex facets, line segments, and discrete points (see Fig. 1), which are listed in order of preference. First, many practical 3D objects contain planar surfaces, which can be described by convex polygons. Nonplanar surfaces can also be described with sufficient accuracy by small facets or triangles. Second, line segments are very effective for representing ruled surfaces, such as cylindrical and conical surfaces, which are familiar to us. Any surface not appropriate to be portrayed by the first two elements can be described by discrete points. On any of the three kinds of elements, the contact position $\boldsymbol{r}_i$ can be uniformly written as

$$\boldsymbol{r}_i = \sum_{j=1}^{n_i} c_{ij} \boldsymbol{v}_{ij} \quad \text{with} \quad \sum_{j=1}^{n_i} c_{ij} = 1 \ \text{ and } \ c_{ij} \geq 0 \quad \text{for } j = 1, 2, \ldots, n_i, \tag{7}$$

where $\boldsymbol{v}_{ij}$ is vertex j ($j = 1, 2, \ldots, n_i$) of the surface element that contact i lies in and n_i is the number of its vertices.

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Definition 2. *A set of surface elements is said to be eligible if it can provide force-closure contact positions; i.e. there are c_{ij} satisfying Eq. (7) such that the set of contacts $\boldsymbol{r}_i$, $i = 1, 2, \ldots, m$ is force-closure.*

Often the number of elements used to represent an object surface is quite considerable. Then an eligible element set needs to be selected prior to the determination of force-closure contact positions. This is the problem we shall solve in the sequel.

3. Generalization of the Force-Closure and Eligibility Conditions

In this section, we first restate the well-known force-closure condition of Mishra *et al.*[16] in a unified form, including the three contact types. This will help to understand the extension of the eligibility condition of Ding *et al.*[15] afterwards and the derivation of formulas for computing the quality criteria in the next section. Hereinafter, we often refer to the Minkowski sum of two sets, which is the result of adding every element of one set to every element of the other.

3.1. Force-closure condition

The allowable contact force set $F_i \cap \Omega_i$ can be rewritten as the convex hull of an extreme set U_i with $\mathbf{0}$, i.e.

$$F_i \cap \Omega_i = \mathrm{conv}\{U_i, \mathbf{0}\}, \tag{8}$$

where conv(•) denotes the convex hull of a set and U_i takes one of the following forms:

$$\text{FPC}: \; U_i = \left\{\boldsymbol{f}_i | f_{in} = f_i^U, f_{io} = f_{it} = f_{is} = 0\right\} \tag{9}$$

$$\text{PCwF}: \; U_i = \left\{\boldsymbol{f}_i | f_{in} = f_i^U, \sqrt{f_{io}^2 + f_{it}^2} = \mu_i f_i^U, f_{is} = 0\right\} \tag{10}$$

$$\text{SFCl}: \; U_i = \left\{\boldsymbol{f}_i | f_{in} = f_i^U, \frac{\sqrt{f_{io}^2 + f_{it}^2}}{\mu_i} + \frac{|f_{is}|}{\mu_{si}} = f_i^U\right\} \tag{11}$$

$$\text{SFCe}: \; U_i = \left\{\boldsymbol{f}_i | f_{in} = f_i^U, \sqrt{\frac{f_{io}^2 + f_{it}^2}{\mu_i^2} + \frac{f_{is}^2}{\mu'^2_{si}}} = f_i^U\right\} \tag{12}$$

The contact forces in U_i reach the extremes of both inclination angle and magnitude, given by the constraints (1)–(4) and (5), respectively; thus U_i is called the *extreme contact force set*. From Eq. (8) we obtain

$$\boldsymbol{G}_i(F_i \cap \Omega_i) = \mathrm{conv}\{\boldsymbol{G}_i(U_i), \mathbf{0}\} = \mathrm{conv}\{W_i, \mathbf{0}\}, \tag{13}$$

where W_i is called the *extreme contact wrench set*:

$$W_i = \boldsymbol{G}_i(U_i). \tag{14}$$

Then

$$\boldsymbol{G}(F \cap \Omega) = \sum_{i=1}^{m} \mathrm{conv}\{W_i, \mathbf{0}\} = \mathrm{conv}\left\{\sum_{i=1}^{m}\{W_i, \mathbf{0}\}\right\}. \tag{15}$$

Let W^{M} be the union of all the Minkowski sums of k of $W_1, W_2, \ldots, W_m$ for $k = 1, 2, \ldots, m$

$$W^{\mathrm{M}} = \bigcup_{k=1}^{m} \bigcup_{i_1<i_2<\cdots<i_k=1}^{m} (W_{i_1} + W_{i_2} + \cdots + W_{i_k}) = \sum_{i=1}^{m}\{W_i, \mathbf{0}\} \backslash \{\mathbf{0}\}. \tag{16}$$

Combining Eqs. (15) and (16) yields

$$\boldsymbol{G}(F \cap \Omega) = \mathrm{conv}\left\{W^{\mathrm{M}}, \mathbf{0}\right\} = \mathrm{conv}\left\{W_c^{\mathrm{M}}, \mathbf{0}\right\}, \tag{17}$$

where $W_c^{\mathrm{M}} = \mathrm{conv} W^{\mathrm{M}}$. From Eq. (17), if $\mathbf{0} \in W_c^{\mathrm{M}}$, then $W_c^{\mathrm{M}} = \boldsymbol{G}(F \cap \Omega)$; otherwise $W_c^{\mathrm{M}} \subset \boldsymbol{G}(F \cap \Omega)$. Hence it turns out that $\mathbf{0} \in \mathrm{int}\{\boldsymbol{G}(F \cap \Omega)\}$ if and only if $\mathbf{0} \in \mathrm{int} W_c^{\mathrm{M}}$; that is, $\mathbf{0} \in \mathrm{int} W_c^{\mathrm{M}}$ is a sufficient and necessary condition of force-closure. We use W_c^{M} instead of $\boldsymbol{G}(F \cap \Omega)$ in force-closure test and quantitative evaluation in the next section, because $\mathbf{0}$ lies outside the interior of W_c^{M} if the set of contacts is not force-closure and then we may measure how far the set of contacts is from being force-closure by the distance between $\mathbf{0}$ and W_c^{M}. More importantly, this distance gives a useful guide to the optimization of contact positions for force-closure; that is, changing the contact positions to reduce the distance can lead them to be force-closure. If using $\boldsymbol{G}(F \cap \Omega)$, however, we will lose this guide, since $\mathbf{0}$ is always contained in $\boldsymbol{G}(F \cap \Omega)$ and the distance is zero for all non-force-closure contact sets.

3.2. Eligibility condition

Substituting Eq. (7) into Eq. (6), we obtain

$$\boldsymbol{G}_i = \sum_{j=1}^{n_i} c_{ij}\boldsymbol{G}_{ij} \;\text{ with} \sum_{j=1}^{n_i} c_{ij} = 1 \;\text{ and }\; c_{ij} \geq 0 \quad \text{for } j = 1, 2, \ldots, n_i, \tag{18}$$

where $\boldsymbol{G}_{ij}$ is called the *vertex grasp matrix*

$$\boldsymbol{G}_{ij} = \begin{bmatrix} \boldsymbol{n}_i & \boldsymbol{o}_i & \boldsymbol{t}_i & 0 \\ \boldsymbol{v}_{ij} \times \boldsymbol{n}_i & \boldsymbol{v}_{ij} \times \boldsymbol{o}_i & \boldsymbol{v}_{ij} \times \boldsymbol{t}_i & \boldsymbol{n}_i \end{bmatrix}. \tag{19}$$

Substituting Eq. (18) into Eq. (14) produces

$$W_i = \sum_{j=1}^{n_i} c_{ij}\boldsymbol{G}_{ij}(U_i) = \sum_{j=1}^{n_i} c_{ij} W_{ij} \quad \text{with} \quad \sum_{j=1}^{n_i} c_{ij} = 1 \quad \text{and } c_{ij} \geq 0 \;\text{ for } j = 1, 2, \ldots, n_i, \tag{20}$$

where W_{ij} is called the *vertex contact wrench set*

$$W_{ij} = \boldsymbol{G}_{ij}(U_i). \tag{21}$$

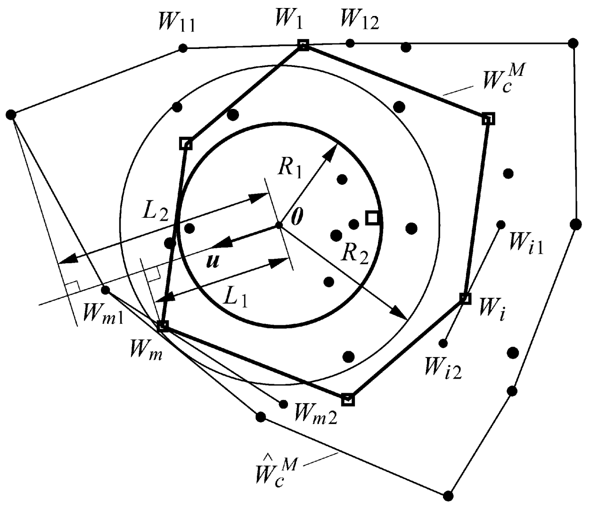

Fig. 2. Illustration of W_c^{M} and $\hat{W}_c^{\mathrm{M}}$. The squares (resp. circular dots) denote the sets W_i (resp. W_{ij}) and their Minkowski sums. W_i is a convex combination of W_{ij}, $j = 1, 2, \ldots, n_i$. Bounded by the thick line, the convex hull W_c^{M} of the squares is always contained in the convex hull $\hat{W}_c^{\mathrm{M}}$ of the circular dots. The functions $p_{W_c^{\mathrm{M}}}(\mathbf{u})$ and $p_{\hat{W}_c^{\mathrm{M}}}(\mathbf{u})$ w.r.t. the unit vector $\mathbf{u}$ equal L_1 and L_2, respectively. $\rho(\mathbf{0}, W_c^{\mathrm{M}})$ and $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$ equal the radii of the largest balls centered at the origin $\mathbf{0}$ contained in W_c^{M} and $\hat{W}_c^{M}$, respectively, denoted by R_1 and R_2.

Let $\hat{W}_i$ be the union of W_{ij}, $j = 1, 2, \ldots, n_i$:

$$\hat{W}_i = \bigcup_{j=1}^{n_i} W_{ij}. \tag{22}$$

From Eqs. (7) and (20) we see that the convex hull of $\hat{W}_i$ contains all possible choices of the extreme contact wrench set W_i when the contact position $\boldsymbol{r}_i$ varies in the surface element. Let

$$\begin{aligned}\hat{W}^{\mathrm{M}} &= \bigcup_{k=1}^{m} \bigcup_{i_1<i_2<\cdots<i_k=1}^{m} (\hat{W}_{i_1} + \hat{W}_{i_2} + \cdots + \hat{W}_{i_k}) \\ &= \sum_{i=1}^{m} \{\hat{W}_i, \mathbf{0}\} \setminus \{\mathbf{0}\}.\end{aligned} \tag{23}$$

Let $\hat{W}_c^{\mathrm{M}} = \operatorname{conv} \hat{W}^{\mathrm{M}}$. From Eqs. (20) and (22), $W_i \subset \operatorname{conv} \hat{W}_i$. Next, from Eqs. (16) and (23), $W_c^{\mathrm{M}} \subset \hat{W}_c^{\mathrm{M}}$. Hence, along with the variation of c_{ij} in Eq. (7), W_c^{M} varies in $\hat{W}_c^{\mathrm{M}}$, as shown in Fig. 2. From this we see that $\mathbf{0} \in \operatorname{int} \hat{W}_c^{\mathrm{M}}$ is a necessary condition of the existence of W_c^{M} in $\hat{W}_c^{\mathrm{M}}$ such that $\mathbf{0} \in \operatorname{int} W_c^{\mathrm{M}}$; thus $\mathbf{0} \in \operatorname{int} \hat{W}_c^{\mathrm{M}}$ is a necessary condition of eligibility.

However, the condition $\mathbf{0} \in \operatorname{int} \hat{W}_c^{\mathrm{M}}$ is not sufficient, since $\hat{W}_c^{\mathrm{M}}$ being 6D does not necessarily mean that W_c^{M} is also 6D. It only ensures that there exists c_{ij} satisfying Eq. (7) such that $\mathbf{0} \in \operatorname{ri} W_c^{\mathrm{M}}$ (see Appendix A for the proof), where ri(•) denotes the relative interior of a set. The condition is sufficient if and only if W_c^{M} determined by such c_{ij} also has a nonempty interior.

For m PCwFs, W_c^{M} has a nonempty interior if and only if $\boldsymbol{r}_i$, $i = 1, 2, \ldots, m$ are noncollinear. Then an element set with $\mathbf{0} \in \operatorname{int} \hat{W}_c^{\mathrm{M}}$ is not eligible if and only if $\boldsymbol{r}_i$ calculated by Eq. (7) are collinear for any c_{ij} achieving $\mathbf{0} \in \operatorname{ri} W_c^{\mathrm{M}}$. However, this case rarely happens to practical objects.

For m SFCs, W_c^{M} has a nonempty interior if and only if $\boldsymbol{r}_i$, $i = 1, 2, \ldots, m$, are noncollinear or $\boldsymbol{n}_i^T(\boldsymbol{r}_1 - \boldsymbol{r}_2) \neq \mathbf{0}$ for some i where $\boldsymbol{r}_1 - \boldsymbol{r}_2 \neq \mathbf{0}$. Then an element set with $\mathbf{0} \in \operatorname{int} \hat{W}_c^{\mathrm{M}}$ is not eligible if and only if neither condition is satisfied for c_{ij} achieving $\mathbf{0} \in \operatorname{ri} W_c^{\mathrm{M}}$. This case is almost impossible.

For m FPCs, the interior of W_c^{M} may be empty in many cases. It is hard to enumerate all the cases in which an element set with $\mathbf{0} \in \operatorname{int} \hat{W}_c^{\mathrm{M}}$ is not eligible. Nevertheless, we ascertained that such element sets are probably eligible, which is also demonstrated in the work of Ding *et al.*[15]

In general, therefore, $\mathbf{0} \in \operatorname{int} \hat{W}_c^{\mathrm{M}}$ can also be used as a sufficient condition for an element set to be eligible.

4. Quantitative Criteria

In this section, we offer quantitative criteria of force-closure and eligibility defined in terms of the L_2 distance between the origin and a set (see Appendix B for the mathematical definition). If the origin lies outside the set, the L_2 distance is equal to the L_2 norm of the closest point in the set to the origin. If the origin lies inside the interior of the set, the L_2 distance is equal to the negative value of the L_2 norm of the closest point on the boundary of the set to the origin. Otherwise the L_2 distance equals zero.

4.1. Quantitative criterion of force-closure

The performance quality of a set of contacts can be assessed by the L_2 distance between $\mathbf{0}$ and W_c^{M}, denoted by $\rho(\mathbf{0}, W_c^{\mathrm{M}})$. If the contacts do not achieve force-closure, then $\rho(\mathbf{0}, W_c^{\mathrm{M}})$ is nonnegative and its value implies how far the contacts are from being force-closure; otherwise $\rho(\mathbf{0}, W_c^{\mathrm{M}})$ is negative and its negative value equals the magnitude of the minimum, over all wrench directions (because the direction of the external wrench is unknown), of the maximum resultant wrench that can be generated by the contact forces in $F \cap \Omega$ in that direction. Thus, the sign of $\rho(\mathbf{0}, W_c^{\mathrm{M}})$ indicates the force-closure property and its quantity signifies the *load capacity* of the contacts.

By Theorem 1 of Appendix B we have

$$\rho\left(\mathbf{0}, W_c^{\mathrm{M}}\right) = -\min_{\boldsymbol{u}^T\boldsymbol{u}=1} p_{W_c^{\mathrm{M}}}(\boldsymbol{u}), \tag{24}$$

where $p_{W_c^{\mathrm{M}}}$ is the support function of W_c^{M}, defined according to the convex analysis book[18] as

$$p_{W_c^{\mathrm{M}}}(\boldsymbol{u}) = \sup_{\boldsymbol{w} \in W_c^{\mathrm{M}}} \boldsymbol{u}^T \boldsymbol{w}. \tag{25}$$

Calculating Eq. (24) requires the use of a nonlinear optimization procedure. Here we call the function *fmincon* of Matlab and choose one of the three algorithms it provides, called the active-set algorithm, to compute (24). The active-set algorithm uses a sequential quadratic programming (SQP) method, which solves a quadratic programming (QP) subproblem at each iteration and updates the quasi-Newton approximation to the Hessian of the Lagrangian using the

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Table I. Operation count of computing $p_{W_i}(\boldsymbol{u})$.

Contact types	Multiplications	Additions	Square roots	Etc.	Expressions
FPC	7	5	0	None	(28) and (29)
PCwF	22	17	1	None	(28) and (30)
SFCl	29	22	1	1 absolute value, 1 comparison	(28) and (31)
SFCe	30	23	1	None	(28) and (32)

BFGS formula.[19] Nevertheless, *fmincon* has the limitation that it may fall into a local minimum. To avoid the local minima of $p_{W_c^M}(\boldsymbol{u})$ and attain its global minimum, we run *fmincon* repeatedly with scattered initial values of $\boldsymbol{u}$ to minimize $p_{W_c^M}(\boldsymbol{u})$ subject to $\boldsymbol{u}^T\boldsymbol{u} = 1$ and adopt the minimum result as $\rho(\mathbf{0}, W_c^M)$. The values of $\rho(\mathbf{0}, W_c^M)$ and $p_{W_c^M}(\boldsymbol{u})$ are illustrated in Fig. 2. From Eq. (16) and Theorem 2(a)–(c) in Appendix B it follows that

$$p_{W_c^M}(\boldsymbol{u}) = p_{W^M}(\boldsymbol{u}) = \max_{1\le k\le m}\max_{1\le i_1<i_2<\cdots<i_k\le m} p_{W_{i_1}+W_{i_2}+\cdots+W_{i_k}}(\boldsymbol{u}) = \max_{1\le k\le m}\max_{1\le i_1<i_2<\cdots<i_k\le m}\left(p_{W_{i_1}}(\boldsymbol{u}) + p_{W_{i_2}}(\boldsymbol{u}) + \cdots + p_{W_{i_k}}(\boldsymbol{u})\right). \tag{26}$$

From Eq. (14) and Theorem 2(d) we obtain

$$p_{W_i}(\boldsymbol{u}) = p_{U_i}\left(\boldsymbol{G}_i^T\boldsymbol{u}\right) = p_{U_i}(\boldsymbol{d}_i), \tag{27}$$

where

$$\boldsymbol{d}_i = \boldsymbol{G}_i^T\boldsymbol{u} \in \mathbb{R}^{d_i}. \tag{28}$$

Combining Eqs. (9)–(12), (27), and (28), we figure out

$$\text{FPC}: \quad p_{W_i}(\boldsymbol{u}) = f_i^U d_{in} \tag{29}$$

$$\text{PCwF}: \quad p_{W_i}(\boldsymbol{u}) = f_i^U d_{in} + \mu_i f_i^U\sqrt{d_{io}^2 + d_{it}^2} \tag{30}$$

$$\text{SFCl}: \quad p_{W_i}(\boldsymbol{u}) = f_i^U d_{in} + f_i^U \times \max\left\{\mu_i\sqrt{d_{io}^2 + d_{it}^2},\, \mu_{si}\,|d_{is}|\right\} \tag{31}$$

$$\text{SFCe}: \quad p_{W_i}(\boldsymbol{u}) = f_i^U d_{in} + f_i^U\sqrt{\mu_i^2\left(d_{io}^2 + d_{it}^2\right) + \mu_{si}'^2 d_{is}^2} \tag{32}$$

where d_{in}, d_{io}, d_{it}, and d_{is} are the components of $\boldsymbol{d}_i$. From Eqs. (28)–(32), computing $p_{W_i}(\boldsymbol{u})$ takes only a few basic operations, as listed in Table I. Then computing $p_{W_c^M}(\boldsymbol{u})$ by Eq. (26) needs additional $2^m - m - 1$ additions and $2^m - 2$ comparisons. Unlike the methods for formulating and computing the quality criterion,[2,7,10] the method presented here utilizes the original friction cones and includes three contact types. It also differs from the method,[9] which describes the criterion as a min–max problem requiring more computation cost.

4.2. *Quantitative criterion of eligibility*

Similarly, the performance quality of surface elements is assessed by the L_2 distance between $\mathbf{0}$ and $\hat{W}_c^M$, denoted by $\rho(\mathbf{0}, \hat{W}_c^M)$. $\mathbf{0} \in \mathrm{int}\hat{W}_c^M$ if and only if $\rho(\mathbf{0}, \hat{W}_c^M) < 0$. If $\rho(\mathbf{0}, \hat{W}_c^M) < 0$, its negative value implies the radius of the largest ball centered at $\mathbf{0}$ contained in $\hat{W}_c^M$. Recall that W_c^M is always contained in $\hat{W}_c^M$ for any c_{ij} satisfying Eq. (7). For finding a large W_c^M in $\hat{W}_c^M$, we wish $\hat{W}_c^M$ to be as large as possible, or the value of $\rho(\mathbf{0}, \hat{W}_c^M)$ to be as small as possible.

The computation of $\rho(\mathbf{0}, \hat{W}_c^M)$ is similar to Eq. (24) for computing $\rho(\mathbf{0}, W_c^M)$

$$\rho\left(\mathbf{0}, \hat{W}_c^M\right) = -\min_{\boldsymbol{u}^T\boldsymbol{u}=1} p_{\hat{W}_c^M}(\boldsymbol{u}), \tag{33}$$

where $p_{\hat{W}_c^M}$ is the support function of $\hat{W}_c^M$, defined similarly to (25) as

$$p_{\hat{W}_c^M}(\boldsymbol{u}) = \sup_{\boldsymbol{w}\in\hat{W}_c^M} \boldsymbol{u}^T\boldsymbol{w}. \tag{34}$$

To calculate Eq. (33), we also call the function *fmincon* with different initial values of $\boldsymbol{u}$ to minimize $p_{\hat{W}_c^M}(\boldsymbol{u})$ subject to $\boldsymbol{u}^T\boldsymbol{u} = 1$. Figure 2 depicts $\rho(\mathbf{0}, \hat{W}_c^M)$ and $p_{\hat{W}_c^M}(\boldsymbol{u})$. The function $p_{\hat{W}_c^M}(\boldsymbol{u})$ can be computed by

$$p_{\hat{W}_c^M}(\boldsymbol{u}) = p_{\hat{W}^M}(\boldsymbol{u}) = \max_{1\le k\le m}\max_{1\le i_1<i_2<\cdots<i_k\le m} p_{\hat{W}_{i_1}+\hat{W}_{i_2}+\cdots+\hat{W}_{i_k}}(\boldsymbol{u}) = \max_{1\le k\le m}\max_{1\le i_1<i_2<\cdots<i_k\le m}\left(p_{\hat{W}_{i_1}}(\boldsymbol{u}) + p_{\hat{W}_{i_2}}(\boldsymbol{u}) + \cdots + p_{\hat{W}_{i_k}}(\boldsymbol{u})\right) \tag{35}$$

where

$$p_{\hat{W}_i}(\boldsymbol{u}) = \max_{1\le j\le n_i} p_{W_{ij}}(\boldsymbol{u}) \tag{36}$$

and $p_{W_{ij}}(\boldsymbol{u})$ can be calculated likewise by Eqs. (29)–(32) with $\boldsymbol{G}_i$ in Eq. (28) replaced by $\boldsymbol{G}_{ij}$. From Eq. (36), computing $p_{\hat{W}_i}(\boldsymbol{u})$ needs n_i times the operation count of computing $p_{W_i}(\boldsymbol{u})$ plus additional $n_i - 1$ comparisons. Equation (35) for computing $p_{\hat{W}_c^M}(\boldsymbol{u})$ has the same operation count as Eq. (26) for computing $p_{W_c^M}(\boldsymbol{u})$.

5. Algorithms for Selecting Eligible Elements

Ding *et al.*[15] gave a qualitative heuristic algorithm based on the ray-shooting approach.[20] We present a quantitative one according to the proposed criterion in the following.

Heuristic algorithm. Starting with a set Γ of elements for locating i_0 ($i_0 < m$) contacts, this algorithm seeks the other $m - i_0$ elements in sequence. Computing $\rho(\mathbf{0}, \hat{W}_c^M)$ for Γ by Eq. (33), we obtain the optimal solution $\boldsymbol{u}^*$ for which $p_{\hat{W}_c^M}(\boldsymbol{u})$ attains the minimum. $\boldsymbol{u}^*$ defines a hyperplane

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passing $\mathbf{0}$ by $\boldsymbol{w}^T\boldsymbol{u}^* = 0$. If $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}}) \geq 0$, then $p_{W_{ij}}(\boldsymbol{u}^*) \leq 0$ for all W_{ij} of Γ, which implies that all these W_{ij} lie on one side of the hyperplane. Note that $\mathbf{0} \in \mathrm{int}\hat{W}_c^{\mathrm{M}}$ if and only if W_{ij} spreads over both sides of any hyperplane passing through $\mathbf{0}$. Hence the element for locating contact $i_0 + 1$ should be selected so as to introduce $W_{i_0+1,j}$ situated on the other side of the hyperplane, namely $p_{W_{i_0+1,j}}(\boldsymbol{u}^*) > 0$ for some j. Using this heuristic, the algorithm can be described as follows:

Step 1. Select the elements for i_0 contacts randomly by computers or deliberately by planners. Let Γ be the set of these elements and set $i = i_0$.
Step 2. Compute $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$ for Γ by Eq. (33) and set $\boldsymbol{u}^*$ to be the optimal solution.
Step 3. Let $i = i + 1$. If $i > m$, then go to Step 5.
Step 4. Add the element for which the maximum value of $p_{W_{ij}}(\boldsymbol{u}^*)$ for $j = 1, 2, \ldots, n_i$ is maximal to Γ and return to Step 2.
Step 5. If $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}}) \geq \mathbf{0}$, then the obtained element set is not eligible and go back to Step 1 for a new round of search; otherwise, an eligible element set is found and the algorithm ends.

The prominent advantage of the heuristic algorithm is high efficiency, since Γ grows in a promising direction towards an eligible element set. Usually, an eligible element set is found after one round of search, and only $m - i_0$ iterations are required. However, the heuristic may miss the best set. Thus we supply another algorithm as follows.

Iterative algorithm. The algorithm changes an element in a set Γ to one of its adjacent elements at an iteration causing the descent of $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$, until no such adjacency exists for any element in Γ.

Step 1. Select an element set Γ for all m contacts or use the heuristic result. Compute $\rho^* = \rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$ for Γ by Eq. (33). Set $i = 0$ and count $= 0$.
Step 2. Let $i = i + 1$. If $i > m$, then let $i = i - m$.
Step 3. If $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$ for Γ is less than ρ^* when element i is replaced by one of its adjacencies, then change it to its adjacent element leading to the greatest descent of $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$; update ρ^* and set count $= 0$. Otherwise, let count $=$ count $+ 1$.
Step 4. If count $= m$, then no descent adjacencies exist and the algorithm ends; otherwise, return to Step 2.

The iterative algorithm often needs more than m iterations, depending on the goodness of the initial element set given in Step 1. Once the number count is annihilated in Step 3, at least additional m iterations are required to terminate the algorithm.

In addition to the computation of $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$, the above algorithms need only to calculate and compare the values of $p_{W_{ij}}(\boldsymbol{u}^*)$ for different elements (Step 4 of the heuristic algorithm) or compare the values of $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$ for different element sets (Step 3 of the iterative algorithm), which involve merely some basic operations. Therefore, the stability of the two algorithms is entirely dependent on the method used to compute $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$. From Eq. (33), the computation of $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$ is a simple optimization problem, which can be solved by commercially available tools, such as Matlab, to ensure the stability. After the eligible elements are selected, the optimal contact positions can be sought in their occupied and adjacent areas using the known methods[3–10] or the optimization toolbox of Matlab.

6. Numerical Examples

We implement the proposed algorithms using Matlab on a notebook with Pentium-M 1.86GHz CPU and 512MB RAM. As shown in Fig. 1, objects (a) and (b) are gripped by four PCwFs and four SFCs, respectively, whereas (c) is fixtured by seven FPCs. Assume $\mu = 0.2$ and $\mu_s = 0.2$ mm.

Equations (24) and (33) are computed using the function *fmincon* of Matlab with the initial values taken to be the elements of the standard basis for $\mathbb{R}^6$ and their opposites. The maximum number of iterations is set to 20,000, the termination tolerance on $p_{W_c^{\mathrm{M}}}(\boldsymbol{u})$ and $p_{\hat{W}_c^{\mathrm{M}}}(\boldsymbol{u})$ is set to 10^{-8}, and the termination tolerance on $\boldsymbol{u}$ is also set to 10^{-8}. The other optimization parameters in Matlab adopt their default values.

For the three objects, randomly starting with an element, the heuristic algorithm turns out the elements sets with the CPU times of 19.12 s, 20.86 s, and 32.52 s, respectively. The values of $\rho(0, \hat{W}_c^{\mathrm{M}})$ are -0.0806, -0.1124, and -0.2705. By the iterative algorithm, their values are further reduced to -0.8654, -0.3131, and -0.9971. The CPU times are 462.82 s, 645.84 s, and 704.32 s. Figure 3 shows the iterative processes and the required number of iterations.

Finally, the contact positions are determined by the function *fmincon*, where the maximum number of iterations is set to 10. Figure 1 depicts the results, for which $\rho(\mathbf{0}, W_c^{\mathrm{M}})$ are -0.5682, -0.2268, and -0.4843. The required CPU times for the three cases are 1072.42 s, 1280.64 s, and 1956.83 s, respectively. All these object surfaces are composed of so many pieces that the optimal grasp planning approaches prior to the one by Ding *et al.*[15] can hardly handle these. Using the algorithm,[15] handling such cases is still very difficult, because line segments, PCwFs on facets, and SFCs are excluded before the new advances presented in this paper.

7. Conclusions

First selecting an eligible set of surface elements from those representing the object surface and then seeking the optimal contact positions on the selected elements increase the efficiency of optimal grasp planning and help to attain grasps with superior quality, especially when the object is bounded by many pieces of smooth surfaces. In this paper, we use not only convex facets and discrete points but also line segments as the elements to characterize an object surface. Each element has a unique normal. All the points of the first two elements can be taken as the convex combinations of their vertices. These nice properties facilitate the search for eligible surface elements. Moreover, all the three contact types (FPC, PCwF, and SFC) are available now uniformly or miscellaneously. Altogether, this paper may be regarded as an essential complement to the creative work of Liu, Wang, and Ding.[11–15]

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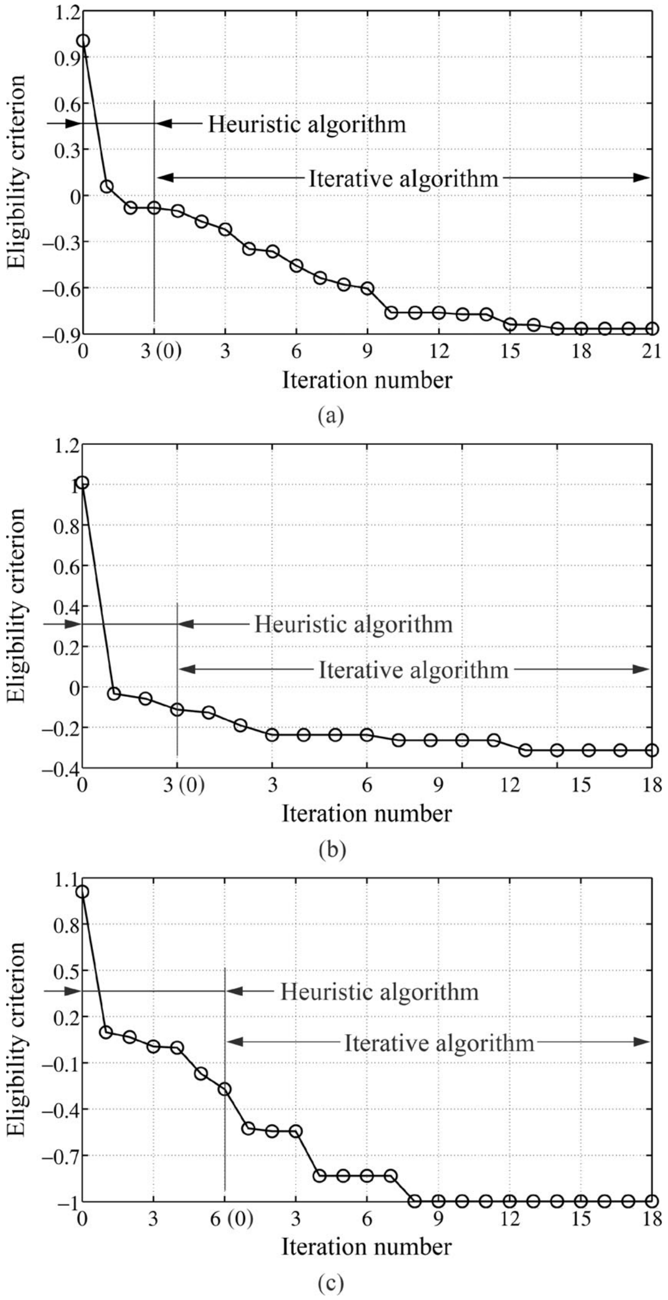

Fig. 3. $\rho(\mathbf{0}, \hat{W}_c^{\mathrm{M}})$ versus the iteration number.

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Appendix A. Proof of a Statement

Herein we claim that there exists c_{ij} satisfying Eq. (7) such that $\mathbf{0} \in \mathrm{ri}\, W_c^{\mathrm{M}}$ if $\mathbf{0} \in \mathrm{int}\, \hat{W}_c^{\mathrm{M}}$. Its proof relies on the theorem of convex analysis: Let S_i, $i = 1, 2, \ldots, m$ be the nonempty compact convex subsets of $\mathbb{R}^d$, and $I = \bigcup_{\lambda_1,\lambda_2,\ldots,\lambda_m>0,\ \sum_{i=1}^m \lambda_i=1} \{\sum_{i=1}^m \lambda_i(\mathrm{ri} S_i)\}$ and $S = \mathrm{conv}(\bigcup_{i=1}^m S_i)$. Then $\mathrm{ri} S = I$.

Thus, if $\mathbf{0} \in \mathrm{int}\, \hat{W}_c^{\mathrm{M}}$, then we have

$$\mathbf{0} \in \sum_{k=1}^{m} \sum_{i_1<i_2<\cdots<i_k=1}^{m} \sum_{j_{i_1}=1, j_{i_2}=1,\ldots,j_{i_k}=1}^{n_{i_1},n_{i_2},\ldots,n_{i_k}} \alpha_{j_{i_1} j_{i_2} \cdots j_{i_k}} \left\{ \mathrm{ri}\left(\mathrm{conv} \hat{W}_{i_1 j_{i_1}}\right) + \mathrm{ri}\left(\mathrm{conv} \hat{W}_{i_2 j_{i_2}}\right) + \cdots + \mathrm{ri}\left(\mathrm{conv} \hat{W}_{i_k j_{i_k}}\right) \right\}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n_i} \alpha_{ij} \left\{ \mathrm{ri}\left(\mathrm{conv} \hat{W}_{ij}\right) \right\}$$

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$$= \sum_{i=1}^{m} \left\{ \sigma_i \sum_{j=1}^{n_i} c_{ij} \left\{ \mathrm{ri}\left(\mathrm{conv}\hat{W}_{ij}\right) \right\} \right\} = \sum_{i=1}^{m} \sigma_i \mathrm{ri}\left(\mathrm{conv} W_i\right)$$

$$= \sum_{k=1}^{m} \sum_{i_1<i_2<\cdots<i_k=1}^{m} \beta_{i_1 i_2 \ldots i_k} \left\{ \mathrm{ri}\left(\mathrm{conv} W_{i_1}\right) + \mathrm{ri}\left(\mathrm{conv} W_{i_2}\right) + \cdots + \mathrm{ri}\left(\mathrm{conv} W_{i_k}\right) \right\}$$

where $\sigma_i = \sum_{j=1}^{n_i} \alpha_{ij}$, $c_{ij} = \alpha_{ij}/\sigma_i$, $\beta_{i_1 i_2 \ldots i_k} = \sum_{j_{i_1}=1, j_{i_2}=1, \ldots, j_{i_k}=1}^{n_{i_1}, n_{i_2}, \ldots, n_{i_k}} \alpha_{j_{i_1} j_{i_2} \ldots j_{i_k}}$, $W_i = \sum_{j=1}^{n_i} c_{ij} W_{ij}$. Since $\sum_{k=1}^{m} \sum_{i_1<i_2<\cdots<i_k=1}^{m} \beta_{i_1 i_2 \ldots i_k} = 1$ and $\beta_{i_1 i_2 \ldots i_k} > 0$, we obtain $\mathbf{0} \in \mathrm{ri} W_c^{\mathrm{M}}$.

Appendix B. Introduction to a Distance Function

Let S be a nonempty compact convex set in $\mathbb{R}^d$. The L_2 distance between the origin $\mathbf{0}$ and S is defined by

$$\rho(\mathbf{0}, S) = \begin{cases} \min\limits_{\mathbf{x} \in S} \|\mathbf{x}\|, & \text{if } \mathbf{0} \notin \mathrm{int} S \\ -\min\limits_{\mathbf{x} \in \mathrm{bd} S} \|\mathbf{x}\|, & \text{if } \mathbf{0} \in \mathrm{int} S \end{cases}$$

where $\|\bullet\|$ denotes the L_2 norm of a vector and $\mathrm{bd}(\bullet)$ denotes the boundary of a set. The value of $\rho(\mathbf{0}, S)$ equals the radius of the largest open ball centered at $\mathbf{0}$ without intersecting S if $\mathbf{0} \notin \mathrm{int} S$ or contained in S if $\mathbf{0} \in \mathrm{int} S$. $\rho(\mathbf{0}, S) < 0$ if and only if $\mathbf{0} \in \mathrm{int} S$; otherwise, $\rho(\mathbf{0}, S) \geq 0$. The compactness of S ensures that $\rho(\mathbf{0}, S)$ is bounded. From our previous work,[10] $\rho(\mathbf{0}, S)$ can be computed as follows:

Theorem 1. $\rho(\mathbf{0}, S) = -\min_{\boldsymbol{u}^T \boldsymbol{u}=1} p_S(\boldsymbol{u})$, *where* p_S *is the support function of* S *defined by* $p_S(\boldsymbol{u}) = \sup_{\boldsymbol{x} \in S} \boldsymbol{u}^T \boldsymbol{x}$.

From the definition of p_S, we may readily derive the following:

Theorem 2. *Let* S_1 *and* S_2 *be two nonempty compact sets and* $\boldsymbol{u}$ *a point in* $\mathbb{R}^d$. *Then the following statements are true:*

(a) $p_{\mathrm{conv} S_1}(\boldsymbol{u}) = p_{S_1}(\boldsymbol{u})$.
(b) $p_{S_1 \cup S_2}(\boldsymbol{u}) = \max\{p_{S_1}(\boldsymbol{u}), p_{S_2}(\boldsymbol{u})\}$.
(c) $p_{\alpha_1 S_1 \pm \alpha_2 S_2}(\boldsymbol{u}) = \alpha_1 p_{S_1}(\boldsymbol{u}) + \alpha_2 p_{S_2}(\pm \boldsymbol{u})$ for $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$.
(d) $p_{\boldsymbol{R}(S_1)}(\boldsymbol{u}) = p_{S_1}(\boldsymbol{R}^T \boldsymbol{u})$, *where* $\boldsymbol{R} \in \mathbb{R}^{l \times d}$.

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