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TRANSITIONAL DYNAMICS AND THRESHOLDS IN ROMER'S ENDOGENOUS TECHNOLOGICAL CHANGE MODEL

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We obtain the transitional dynamics of the decentralized economy described by P.M. Romer and characterize the dynamic behavior of the most relevant variables. We determine the existence of a stable one-dimensional manifold containing a steady state with innovation, unique in ratios, and also find a threshold in the accumulation of physical capital below which the economy is not innovating. Finally, using simulations, we assess the significance of this threshold and analyze the influence that technological and utility parameters have on it.

Keywords: Endogenous Growth, Innovation Threshold, Transitional Dynamics

1. INTRODUCTION

Following the three real seminal works of Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992), an abundant literature has been generated aimed at explaining the economic growth through endogenous innovation. Most of this literature has focused on the analysis of the steady state, either because the emphasis has been placed on the long-term behavior, or because the proposed model has no transitory dynamics. Some papers have tackled the transitional dynamics but, in general, their scope has been rather limited. For instance, Benhabib et al. (1994) modify Romer's model to allow for some complementarity between the intermediate goods, but they restrict their attention to the dynamic stability analysis. Arnold (1998) includes the Uzawa-Lucas technology of human capital accumulation in the Grossman-Helpman model of horizontal innovation, but only presents a local analysis. A local analysis also is developed by Arnold (2000) for Romer's (1990) model.

We carry out a full characterization of the transitional dynamics of Romer's (1990) model. We have chosen this model for three reasons: first, because it is a benchmark model in the literature; second, because it includes physical capital accumulation in addition to innovation; and third, because it is suitable for (a) a

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dynamic analysis without linearization, (b) determining the behavior of the policy functions, and (c) a general consideration of all the possibilities that could appear in the transitory phase, even including the temporary inactivity of any of the accumulation sectors. Although the presence of the scale effect casts some doubts on its empirical relevance, the main results can be extended to the transition of innovation models without scale effect, such as those presented by Jones (1995) or Dinopoulos and Thompson (2000).

We draw attention to two main results: The first, is the characterization of the transitional dynamics as a compensation process between the two types of accumulable capital that are present in the model: physical capital and knowledge. This interpretation has already been pointed out in two-sector models that include both physical and human capital. However, up until now, this has not been possible in a context of decentralized innovation. When the stock of knowledge is high relative to the available stock of physical capital, it is more advantageous to produce more of each variety of capital goods than to create new varieties, and hence, the innovation will advance slowly. By contrast, when the stock of knowledge is low, the market will encourage the innovation activity.

The second main result is an immediate extension of the first. We find the existence of an innovation threshold in the ratio "physical-capital/varieties." When this physical-capital/knowledge ratio is smaller than a threshold, the allocation of resources to the R&D sector is not profitable. This implies that those economies with a history of low saving rates and low capital accumulation related to the available knowledge must pass through a transitory phase during which the accumulation of capital takes place without innovation.

We determine that these two results are possible because there is a stable one-dimensional manifold containing a steady state with innovation. Their relevance becomes clear if we consider the possibility that the returns of the innovation efforts could be subject to shocks. Thus, periods with positive innovation shocks will be followed by an acceleration of investment in physical capital. If the economy was close to the threshold, then the innovation activity would temporarily disappear. Notice that this point of view gathers, albeit in an exogenous form, the idea of alternance between innovation and accumulation of capital that is presented by Matsuyama (1999).

It is not easy to draw conclusions on the empirical relevance of the threshold because the stock of knowledge is a vague concept and one that is difficult to measure. Nevertheless, two approaches can be taken. First, in the case of economies characterized by the absence of long-term growth, the threshold is greater than or coincides with the steady-state ratio. Second, there is a relation between the physical-capital/knowledge ratio and the "physical-capital/output" ratio, and data on this second ratio are available.

In summary, our main objective is the full characterization of the transitional dynamics in Romer's model, including the possibility of structural change and the assessment of the importance of the innovation threshold. In relation to Benhabib et al.'s modification, we confirm the local uniqueness of the stable path, as Arnold

(2000) does. However, we also add the saddle-path stability along a one-dimensional manifold, the meaning of the corner solution, the corresponding existence of the innovation threshold, and the qualitative behavior of the variables in the two types of dynamic regimes.

The rest of the paper is organized as follows: In Section 2, we describe the institutional setup, the type of productive inputs, the agents' behavior, and the equilibrium conditions. In Section 3, we develop the transitional dynamics of the model, determining the type of stability and characterizing the solution. In Section 4, we obtain the innovation thresholds by means of numerical methods, and assess their significance. Finally, in Section 5, we present our main conclusions.

2. INSTITUTIONAL SETUP, AGENTS' BEHAVIOR, AND EQUILIBRIUM CONDITIONS IN ROMER'S MODEL

Two types of agents exist in the economy described in Romer's model: households and firms. Each of the H households, of infinite life and constant size, is endowed with one unit of human capital, inelastically supplied in the labor market, taking the wage as given, and each owns a stock of assets, the rental rate of which is the market interest rate. From the income so obtained, one fraction is devoted to the consumption of final goods and the other to the acquisition of new assets.

Three types of firms exist, each pertaining to a different productive sector: the final-goods sector, the R&D sector, and the capital-goods sector. Final-goods-sector firms sell their production in a perfectly competitive market and employ two types of inputs: human capital, traded in a competitive market, taking the wage as given; and a set of differentiated goods, which they hire from capital-goods-sector firms, taking the rent as given. R&D-sector firms create new designs of differentiated capital goods, the patents of which are sold in a competitive market, using only human capital. Capital-goods-sector firms buy patents of new varieties, thus obtaining the monopoly of their production, and produce the existing varieties of capital goods with a technology similar to that used to produce the final goods.

Each household derives utility from the consumption of final goods, according to a constant elasticity-of-substitution function. Denoting the time fractions devoted to final goods and R&D sectors by u_Y and u_N, respectively, the wages in each of these sectors by w_Y and w_N, the stock of assets by b, the interest rate by r, and the consumption by c, we find the problem faced by households lies in deciding the time allocation between the two productive sectors and the level of consumption, in such a way that

$$\operatorname*{Max}_{c,u_Y} \int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma}\, dt,$$

subject to the budget constraint[1] $\dot{b} = br + w_Y u_Y + w_N u_N - c$. The current-value Lagrangian (generalized Hamiltonian) for this problem is

$$\frac{c^{1-\sigma} - 1}{1-\sigma} + \theta[br + w_Y u_Y + w_N(1 - u_Y) - c] + \lambda(1 - u_Y),$$

where θ is the costate variable associated with the assets and λ is the Lagrange multiplier corresponding to the constraint $u_Y \leq 1$. The necessary conditions are

$$\theta = c^{-\sigma}, \qquad \textbf{(1a)}$$

$$\theta(w_Y - w_N) = \lambda, \qquad \textbf{(1b)}$$

$$\dot{\theta} = \rho\theta - \theta r, \qquad \textbf{(1c)}$$

$$\lambda(1 - u_Y) = 0, \qquad \textbf{(1d)}$$

$$\lim_{t\to\infty} \theta b e^{-\rho t} = 0. \qquad \textbf{(1e)}$$

All firms of the final-goods sector use the same technology of constant returns to scale, which can be represented in the aggregated form as

$$Y = H_Y^{\alpha} \int_0^N x_j^{1-\alpha}\, dj, \qquad 0 < \alpha < 1, \qquad \textbf{(2)}$$

where x_j is the amount used for each one of the N different varieties of capital goods, $H_Y(= u_Y H)$ is the human capital employed, and Y is the output of the final-goods sector. The firms of this sector maximize their profit, taking the wage w_Y and the hiring price of the productive inputs p_j as given, choosing the amount of human capital and the demand for each input. The problem has no temporal dimension and the following conditions must be met:

$$w_Y = \alpha Y / H_Y, \qquad \textbf{(3)}$$

$$p_j = (1 - \alpha) H_Y^{\alpha} x_j^{-\alpha}. \qquad \textbf{(4)}$$

One firm wishing to begin the production of one variety of capital good must have acquired the patent giving it the exclusive right to production. Calling the market price of a patent P_N, the firm will issue infinite life bonds to finance its acquisition. The intermediate goods are durables rented by the firms producing final goods. We assume that the production of one unit of each variety of input requires η units of final good. Thus, the cost of producing x units of a variety will be $r\eta x$. Firms producing capital goods obtain the necessary resources to finance the production by issuing bonds for a value equal to the planned production variation $(\eta\dot{x})$.

The problem for these firms has no temporal dimension and is, every time, to choose p_j in order to maximize their profit $\pi_j = p_j x_j - r\eta x_j$ subject to (4). The solution to this problem leads to a hiring price of each capital good $p_j = r\eta/(1-\alpha)$. Introducing this last expression of p_j in the demand function, we have the quantity of each capital good that is used:

$$x_j = [(1 - \alpha)^2 / r\eta]^{1/\alpha} H_Y. \qquad \textbf{(5)}$$

As $x_j = x$, using (5) we can write $Y = [(1-\alpha)^2/r\eta]^{(1-\alpha)/\alpha} H_Y N$, and the firm's profit then will be

$$\pi = (p - r\eta)x = \alpha(1-\alpha)Y/N. \tag{6}$$

A measure of the stock of physical capital will be the amount of final good equivalent to the existing stock of productive inputs; that is, $K = \eta x N$. Thus, the final-good production function can be written as

$$Y = \eta^{\alpha-1}(H_Y N)^\alpha K^{1-\alpha}, \tag{7}$$

and, taking into account the expression of x in (5), the interest rate will be

$$r = (1-\alpha)^2 Y/K. \tag{8}$$

R&D firms develop designs of new goods according to the following linear technology:

$$\dot{N} = \delta H_N N, \tag{9}$$

where H_N is the human capital employed in the sector ($H_Y + H_N = H$ and $H_N = u_N H$). These firms operate in a competitive market, taking the patent price and the wage w_N as given. Thus, the amount of human capital employed should satisfy the condition

$$w_N = P_N \delta N. \tag{10}$$

Because the patents market is competitive, firms producing the varieties of inputs will be disposed to offer an amount equal to the present value of the future profits for the exclusive right to produce a new input. Thus, the price of the patents will be

$$P_N(t) = \int_t^\infty \pi(s) e^{-\tilde{r}(t,s)}\, ds,$$

with

$$\tilde{r}(t,s) = \int_t^s r(v)\, dv.$$

Differentiating the previous expression with respect to time, we obtain

$$\dot{P}_N = r P_N - \pi. \tag{11}$$

3. TRANSITIONAL DYNAMICS

Given the existence of an innovation threshold, full characterization of the transitional dynamics requires a separate consideration of that part where the economy devotes resources to the R&D sector (interior solution) and of that where there is capital accumulation without innovation (corner solution).

3.1. Transitional Dynamics in the Interior Solution

The national income will be the flow of payments perceived by the households as compensation for their contribution to the production process. Human-capital payments come given by $wH = \alpha YH/H_Y$, where $w = w_Y = w_N$. Non-human-capital payments have their source in the bonds that have financed the resources required by the firms producing the varieties of productive inputs. These firms issue bonds each time for an aggregate value of $P_N \dot{N}$ in order to finance the acquisition of patents, with the interest payments corresponding to their net incomes for each period. Thus, bondowners perceive a flow of rents equal to πN. Moreover, firms must finance capital accumulation by issuing bonds that require the corresponding interest payments, considered by these firms as current costs. The amount of bonds issued each period to finance the creation of physical capital will be $\dot{K} = \eta x \dot{N} + \eta N \dot{x}$. The first term corresponds to the resources required for the production of new varieties; the second represents those needed to adjust the production of each variety to changes in demand. The total sum of the bondowners' rents will be $r\eta x N$. Thus, households obtain human-capital payments $\alpha YH/H_Y$, interest payments on bonds issued to finance the acquisition of patents $\pi N = \alpha(1-\alpha)Y$, and interest payments on bonds issued to finance physical-capital accumulation in an amount $r\eta x N = rK$. A part of these incomes is saved to subscribe for the issue of bonds required to finance the acquisition of patents in an amount that, in equilibrium, will be $P_N \dot{N} = \alpha YH_N/H_Y$. Another part is saved to subscribe for bonds issued to finance physical-capital accumulation ($\dot{K}$) and the rest is devoted to consumption. Thus, the aggregated budget constraint of the households can be written in equilibrium as

$$\dot{K} = rK + wH + \pi N - P_N \dot{N} - cH = rK + \alpha(2-\alpha)Y - cH.$$

Taking into account the expression for r in (8), we can write

$$\dot{K} = \eta^{\alpha-1} H_Y^{\alpha} N^{\alpha} K^{1-\alpha} - cH. \qquad \textbf{(12)}$$

In the interior solution, wages must be equal in both sectors. Thus, by substituting the expression (11) for P_N and the expression (6) for π in (10), the wage paid by R&D firms can be written[2] as $w_N = \delta\alpha(1-\alpha)Y[r - g(P_N)]^{-1}$. Equalizing this expression to the wage paid in the final-goods sector, which comes given by (3), we have $g(P_N) = r - \delta(1-\alpha)H_Y$.

For wages to be the same at all times, their rates of change also must be equal. From (3), and taking into account (7), the rate of change of the final-goods-sector wage will be $g(w_Y) = \alpha g(N) + (1-\alpha)[g(K) - g(H_Y)]$. From (10), we have the growth rate of the wage paid by R&D firms: $g(w_N) = g(P_N) + g(N)$. Equalizing both rates, taking into account the previous expressions for $\dot{K}$ and $g(P_N)$, we can obtain with some algebra the dynamic equation for the labor employed in the final-goods sector:

$$g(H_Y) = 2\delta H_Y + \alpha Y/K - \delta H - cH/K. \qquad \textbf{(13)}$$

Using (1a) and (1c), and taking into account (8), the consumption growth rate will be

$$g(c) = \sigma^{-1}[(1-\alpha)^2 Y/K - \rho]. \tag{14}$$

Equations (9), (12), (13), and (14) form a system of four differential equations in N, K, H_Y, and c that characterizes the dynamics of the interior equilibrium of the economy. However, it is more convenient to reformulate the system in terms of variables that are constant in the steady state. If we define $z = K/N$ and $q = cH/K$, the preceding system can be rewritten as

$$g(z) = \eta^{\alpha-1} H_Y^{\alpha} z^{-\alpha} - \delta(H - H_Y) - q, \tag{15a}$$

$$g(H_Y) = 2\delta H_Y + \alpha\eta^{\alpha-1} H_Y^{\alpha} z^{-\alpha} - \delta H - q, \tag{15b}$$

$$g(q) = \left[\frac{(1-\alpha)^2}{\sigma} - 1\right]\eta^{\alpha-1} H_Y^{\alpha} z^{-a} + q - \frac{\rho}{\sigma}. \tag{15c}$$

Equating these three equations to zero, we obtain the steady-state values of the three variables:

$$H_Y^* = \frac{\sigma\delta H + \rho}{\delta(\sigma + 1 - \alpha)}, \qquad q^* = \frac{\rho}{\sigma} - \left[\frac{1}{\sigma} - \frac{1}{(1-\alpha)^2}\right](1-\alpha)\delta H_Y^*,$$

and

$$z^* = \left[\frac{1-\alpha}{\delta\left(\eta H_Y^*\right)^{1-\alpha}}\right]^{1/\alpha}.$$

With respect to this system, we can state and prove the following three propositions.

PROPOSITION 1. *Defining the vector of parameters* $(\alpha, \delta, \eta, H, \rho, \sigma)$ *as* $\psi \in R_+^6$, *the system of differential equations* (15) *presents local saddle-path stability in the steady state for the set of parameters*

$$\Psi_i \equiv \left\{\psi \in R_+^6 \,|\, (1-\alpha)\delta H \geq \rho > \frac{(1-\sigma)(1-\alpha)}{2-\alpha}\delta H\right\}.$$

Proof. See Appendix. ■

PROPOSITION 2. *The stable manifold M containing the steady state of system* (15) *is one-dimensional in such a way that the solution tends to the steady state for any value of z if the initial point pertains to M.*

Proof. See Appendix. ■

PROPOSITION 3.[3] *When* $\sigma > (1-\alpha)^2$, *the optimal solution for system* (15) *can be represented by the policy functions* $H_Y = H_{Yi}(z)$ *and* $q = q_i(z)$, *with* $H_{Yi} < 0$ *and* $q_i < 0$.

Proof. See Appendix. ■

3.2. Transitional Dynamics in the Corner Solution

Balanced growth is only possible when the human-capital endowment is greater than the threshold $H_u = \rho/\delta(1-\alpha)$; otherwise, growth is possible only in the transitory phase. However, we have determined the existence of a second threshold, associated with the physical-capital/varieties ratio. Thus, as the policy function $H_{Yi}(z)$ is decreasing, a value z_u will exist for which $H_{Yi}(z_u) = H$. If the ratio z of an economy is below this threshold, a previous accumulation of capital is required before the start of the innovation process. In this case, the Lagrange multiplier is positive and condition (1b) allows us to conclude that the value of the labor marginal productivity in the final-goods sector is greater than that corresponding to the R&D sector and, hence, $H_Y = H$ and $\dot{N} = 0$. The human-capital income will be αY. Non-human-capital income continues to have two sources: the interest payments on the bonds issued to finance the physical-capital accumulation (rK) and those paid on the bonds issued to finance past innovations (πN). However, given that there is no innovation, savings can now be materialized only in the form of subscriptions to new bonds issued to finance the physical-capital accumulation. Consequently, equation (12) for $\dot{K}$ does not alter. Now, taking into account the previous expression for $\dot{K}$ and the definitions of z and q, we can write

$$g(z) = \eta^{\alpha-1} H^{\alpha} z^{-\alpha} - q, \qquad \textbf{(16a)}$$

$$g(q) = q - [1 - (1-\alpha)^2/\sigma]\eta^{\alpha-1} H^{\alpha} z^{-\alpha} - \rho/\sigma. \qquad \textbf{(16b)}$$

Equations (16) constitute a system of differential equations in z and q that characterize the transitional dynamics of the economy when innovation does not exist. For $\sigma > (1-\alpha)^2$, we can immediately represent the saddle-path stable behavior[4] toward a steady state with values $q_e^* = \rho(1-\alpha)^{-2}$, and $z_e^* = [\eta^{\alpha-1}(1-\alpha)^2/\rho]^{1/\alpha} H$ by means of a policy function $q = q_e(z)$, with $q_e^1 < 0$.

3.3. Full Characterization of Transitional Dynamics

As a consequence of the innovation threshold, both the policy functions and the dynamic paths of the variables experience a change of regime and their characterization requires separate consideration of the different regimes.

We have seen that the optimal behavior comes determined by the ratio z. When $H > H_u$, if $z \geq z_u (< z^*)$, the policy functions $H_{Yi}(z)$ and $q_i(z)$ corresponding to system (15) provide two values, H_Y and q, that fulfill all the constraints, and the variables move according to Proposition 3. If $z < z_u (< z_e^*)$,[5] the constraint $H_Y \leq H$ is exhausted and the economy will follow the path corresponding to system (16), with z increasing and q decreasing until the moment at which the threshold z_u is reached.

In the special case in which $H = H_u$, it can be verified that $z^* = z_e^* = z_u$, in such a way that if $z < z^*$, the economy follows the dynamic of system (16), with $H_Y = H$, z increasing and q decreasing. When $z > z^*$, the economy follows the dynamics of system (15).

What type of intuition lies behind this transitional dynamic? If an economy is not sufficiently endowed with physical capital, the interest rate will be high.[6] Because the price of each differentiated input is proportional to its cost, and this comes determined by the interest rate, this price also will be high, leading to a low demand and a low production of each capital good and, consequently, to a low level of profit for each monopolist. However, a high interest rate not only affects the current profit but also implies a high discount rate in the present value of the future profits, leading to a low market price of the patents. Thus, the value of the labor marginal productivity in the R&D sector is low and, eventually, lower than the value of the labor marginal productivity in the final-goods sector. In such a situation, the labor devoted to innovation will be zero and the economy will be moving along the path of the corner solution.

As long as physical capital is growing, the output will grow $[g(Y) = (1-\alpha)g(K)]$ at a lower rate than that of physical capital, and the wage and the current profits of the monopolists also will grow. Nevertheless, the interest rate is decreasing $[g(r) = -\alpha g(Y)/(1-\alpha)]$. If it was constant, then the price of the patents would be growing at the same rate as the current profits (or the wage). However, because it is decreasing, the patent price will be increasing at a higher rate than the wage. Thus, the condition $\delta P_N N = w_Y = \alpha Y/H$ eventually will be fulfilled, and then the agents will devote human capital to R&D. From this moment on, the economy will move along the transitional path leading to the steady state with innovation.

In the case in which the human-capital endowment is low $(H < H_u)$, the demand for differentiated inputs, as well as the monopolists' profits, always will be insufficient. The interest rate will never be low enough to raise the patents price to the level required in order to make the value of the labor marginal productivity in the R&D sector equal that of the final-goods sector. Consequently, the economy will continue moving along the path of the corner solution toward a steady state without innovation and without growth.

The transitional path can be roughly characterized as a compensation process between the physical-capital stock and the varieties of capital goods. Initially, once the physical-capital accumulation has reached the innovation threshold, a low level of human capital will be allocated to the R&D sector. The innovation is expensive, given that it requires the explicit allocation of labor and because the application of each innovation requires additional accumulation of physical capital. Furthermore, when the output level and the use of each input is under the steady-state value, it will be necessary to accumulate not only to produce the new inputs of each period, but also to increase the available stock of the preexisting inputs. Under these conditions, the time fraction devoted to innovation will be close to zero, as is derived from the policy functions, and innovation will increase slowly.

However, from the moment at which the economy begins to devote resources to R&D, the number of varieties of capital goods will increase and, as a consequence of the externality, the costs of innovation will decrease. The production of each variety, as well as their available stock, also will increase. These two effects enhance the incentives to innovate, causing the amount of resources devoted to the R&D sector to increase more and more. Strengthening the previous argument, as the available stock of each input approaches the steady-state value, that part of the investment destined to increasing the stock of preexisting inputs will be losing its relative importance, and the investment in new varieties will tend to be the sole component.

An analogous process, but one that takes places in the opposite direction, will occur when the initial situation of the economy is characterized by an excess of physical capital relative to the level of available knowledge. There, the relative abundance of physical capital leads to a low interest rate, which provides incentives that favor innovation. Under such conditions, the allocation of resources to R&D will be high. As the physical-capital/varieties ratio decreases, so the incentives to innovate decrease, until the steady state is reached.

4. INNOVATION THRESHOLD

We have already seen that from the policy functions it is possible to derive the existence of a threshold in z related to the existence of innovation. Because we have not been able to obtain an analytical determination of the threshold, we have used the time elimination method [see Mulligan and Sala-i-Martin (1993)] to obtain the policy functions and the threshold values.

4.1. Policy Functions and Numerical Determination of the Threshold

Figure 1 shows the policy functions $u_Y = H_Y(z)/H$ for three different values of H (1, 1.5, and 2) and a given combination of parameters (exclusively for illustrative purposes, we have taken $\sigma = 2.5$, $\alpha = 0.5$, $\eta = 1$, $\delta = 0.2$, and $\rho = 0.04$). The value of H_u is 0.4 and $z_u(H_u) = 15.6$. The steady-state values z^{*j}, u_Y^{*j}, and the threshold values z_u^j, for $j = 1, 1.5, 2$ are indicated in the figure. The arrows show the dynamic of the variables. For example, for $H = 1$, if $z < z_u^1$, then the economy evolves without innovation ($u_Y = 1$) until reaching z_u^1. From there on, the policy function indicates the stable path toward the steady state (point C).

The figure allows us to consider the effects of shocks on the parameters. For example, let us suppose that the economy is in the steady state when $H = 1$ (point C) and a shock results in $H = 1.5$. This shock will cause an instantaneous fall in u_Y (point C′), increasing the innovation activity. From this point on, there will be an increase in u_Y, while z will decrease until reaching a new steady state in point B. By contrast, if the initial point is A and a shock reduces H to 1, the shock will cause a marked and sudden change in u_Y (point A′), increasing the physical-capital investment. Then, a progressive reduction in u_Y will take place as the ratio z increases until the new steady state (point C) is reached. If the adverse shock simultaneously affects both K and H (for instance, in the case of a war) in such a way that z falls below z_u^1, the economy will move, for example, to A″, and the savings initially will be devoted only to physical-capital accumulation until the threshold is reached.

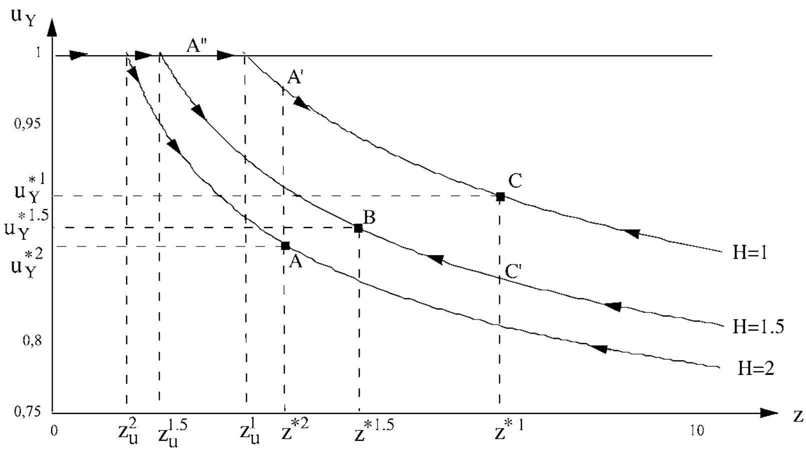

FIGURE 1. Policy functions $u_Y(z)$.

It is also interesting to consider the effects of shocks on the number of varieties, since the results of the innovating effort have a strong random behavior. There, we could represent a new discovery, one that opens up great opportunities for investment, such as a sudden increase in N. This shock causes the physical-capital stock to become reduced in relation to the varieties available, and the economy increases the resources devoted to its accumulation, abandoning the innovation if the horizontal $u_Y = 1$ is reached.

Having determined the existence of a minimum level in the necessary stock of physical capital in order for there to be innovation, it would therefore seem advisable to examine how the parameters affect the value of the threshold and the relation between z_u and z^*.

4.2. Value of the Innovation Threshold and Its Relation to the Steady-State Value

To study the effects of changes in the value of the parameters, we have carried out numerical simulations in which we have calculated the threshold when only one of the parameters is changing, and the others are maintained fixed. The results can be represented by means of the following expression, indicating the relation between every parameter and the value of the threshold:

$$z_u = z_u(\delta,\ H,\ \eta,\ \alpha,\ \sigma,\ \rho)$$
$$-,\ -,\ -,\ \pm,\ +,\ +$$

The intuition for this relation is as follows: The solution without innovation appears when the payment that human capital would receive in the innovative activity ($w_N = \delta P_N N$) would be inferior to that obtained in the production of final goods [$w_Y = \alpha(\eta H/z)^{\alpha-1} N$]. In this transitory phase, both the wage obtained in the final-goods sector and the patents price increase as long as there is physical-capital accumulation, but the patents price increases at a faster rate than that of

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the wage. The threshold value can be characterized by the patents price, satisfying $w_Y = w_N$, at which level the decision of whether or not to innovate is indifferent.

Thus, if δ is high, the value of the patents price required to make the innovation profitable will be low, as will be the required capital accumulation (z_u). In the same sense, when H is high, the labor marginal productivity in the final-goods sector will be low as, consequently, will be the wage. Therefore, the value of the patents price required to make innovation profitable also will be low, as will be the threshold value. An analogous argument can be used in the case of the relation between η and the threshold. High values of ρ and σ mean that the agents wish to consume as much as possible and will have a low rate of saving, which is equivalent to a slow physical capital accumulation and a high value of the threshold. In the case of α, it is not possible to reach a clear conclusion because of the contradictory role played by the parameter in the model: On the one hand, this is the human-capital elasticity of the output and, on the other hand, it determines the value of the markup in the intermediate-goods price.

If the value of the threshold is informative on the level of physical capital required to start the innovation, its relation to the steady-state value of z gives some indication of the significance of the corner solution. We have observed, by means of numerical simulations, that the ratio z_u/z^* is a decreasing function of the ratio $\delta H/\rho$, so that, when $\delta H/\rho$ is low, the ratio z_u/z^* is close to 1, indicating that the threshold value is close to the steady-state value. Even in the extreme case in which $H = H_u$, the innovation threshold and the steady-state value of the physical-capital/varieties ratio coincide ($z_u = z^* = z_e^*$). In general, parameter combinations leading to low growth rates imply a value of the threshold close to the steady state. The intuition is straightforward: A low growth rate is associated to a low saving rate and a low level of resources devoted to the R&D sector. If we suppose that H_N^* is close to zero, then any exogenous shock that increases the number of varieties will have the accumulation of physical capital without innovation as its consequence.

It is difficult to find an observable index for the ratio K/N. However, taking into account that $(Y/K) = \eta^{\alpha-1} H_Y^{\alpha} z^{-\alpha}$, we can obtain $(K/Y)^*$ and $(K/Y)_u$ for each combination of parameters. Fortunately, we have empirical information about the physical-capital/output ratio. For example, Mankiw et al. (1992) obtain estimates for K/Y between 1 and 3 from the Summers and Heston database. Angus Maddison (1991) estimates a value of 2 for the United States. In the case of the United Kingdom, the value varies from 1 at the beginning of the century to 2 in 1987. The values for Germany are 2.07 in 1950 and 2.99 in 1987. For the parameter values used in the preceding section with $H = 1$, we obtain a ratio $(K/Y)^*$ of 2.77, while the value $(K/Y)_u$ corresponding to the threshold is 1.72. If we choose a physical-capital/output elasticity value closer to 1/3 (e.g., $\alpha = 0.7$, the value for $(K/Y)^*$ is 1.56, and for $(K/Y)_u$ it is 1.23), in such a way that the value of the threshold represents a proportion of 79% of the value in the steady state.

Although possibly an inaccurate index, the physical-capital/output ratio indicates that, for low values, innovation is slow, which can be confirmed by the data, and that countries without innovation would have a very low value, which is an admissible intuition.

5. CONCLUSIONS

We have demonstrated that Romer's (1990) model has a stable one-dimensional manifold containing a steady state with innovation. The transitory dynamics have been characterized as a compensation process between physical capital and number of varieties. This interpretation can be extended to models of innovation without scale effect, in which the innovation technology presents decreasing returns in N and there is growth in human capital (or population). In these cases, in which the dynamics are much more complex, the compensation takes place between per-capita physical capital and varieties.

We also have determined the existence of a minimum level in the stock of physical capital that is necessary for innovation. This threshold implies that those economies with a history of low rates of saving and physical-capital accumulation related to the level of knowledge must experience a transitory phase during which they concentrate their efforts exclusively on physical-capital accumulation. The value of this threshold is close to the steady-state value for those parameter combinations that lead to low rates of economic growth.

NOTES

1. We omit the time reference in order to simplify the notation.
2. We represent the growth rate of any variable as $g(\cdot)$.
3. We only consider this case because it is the most feasible.
4. The determinant of the Jacobian of the system (16) at the steady state is $-\alpha q_e^* \rho/\sigma$. Because this is always negative, the system is locally saddle-path stable. One can immediately verify that this is also a global property, by means of an argument similar to that of Proposition 2.
5. From Proposition 3, when $H = H_Y$ and $z = z_u$ in system (15), we have that $g(z_u) > 0$, $g(H) > 0$, and $g(q_u) < 0$. The two first inequalities require that $z_u < \tilde{z} = [(1-\alpha)\eta^{\alpha-1}(\delta H)^{-1}]^{1/\alpha} H$. As $\tilde{z} < z_e^*$ when $H > H_u$, then $z_u < z_e^*$.
6. According to Proposition 3, r and z follow an inverse relation in the saddle path.

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APPENDIX

PROOF OF PROPOSITION 1

First, condition (1e) requires that $\lim_{t\to\infty} g(\theta) + g(b) - \rho < 0$. However, $g(\theta) = \rho - (1-\alpha)\delta H_Y^*$ and $g(b) = \delta(H - H_Y^*)$ in steady state. Thus, the transversality condition will be fulfilled when

$$\rho > \frac{(1-\sigma)(1-\alpha)\delta H}{2-\alpha}.$$

Second, the condition $H_Y \leq H$ requires $\rho \leq (1-\alpha)\delta H$. All of these conditions determine the parameter space Ψ_i. The Jacobian of system (15) at the steady state will be

$$J^* = \begin{pmatrix} \dfrac{-\alpha\delta H_Y^*}{(1-\alpha)} & \dfrac{\delta z^*}{(1-\alpha)} & -z^* \\ -\dfrac{\alpha^2\delta H_Y^{*2}}{(1-\alpha)z^*} & \left(2+\dfrac{\alpha^2}{1-\alpha}\right)\delta H_Y^* & -H_Y^* \\ (q^*-\rho/\sigma)\alpha q^*/z^* & -(q^*-\rho/\sigma)\alpha q^*/H_Y^* & q^* \end{pmatrix}.$$

The determinant and the trace of the Jacobian are

$$D(J^*) = \frac{-\alpha(\sigma+1-\alpha)}{\sigma} q^*(\delta H_Y^*)^2,$$

and $Tr(J^*) = q^* + (2-\alpha)\delta H_Y^*$.
As $\forall \psi \in \Psi_i$, $H_Y^* > 0$ and $q^* > 0$, we can see that $D(J^*) < 0$ and $\mathrm{Tr}(J^*) > 0$, which means that system (15) presents local saddle-path stability. ■

PROOF OF PROPOSITION 2

We make use of the equivalent system (A.1), which is a reformulation of system (15) in terms of r, H_Y, and q, where $r^* = (1-\alpha)\delta H_Y^*$ is the steady-state value of the interest rate:

$$g(r) = \alpha\delta(H_Y - H_Y^*) - \frac{\alpha}{1-\alpha}(r - r^*), \quad \textbf{(A.1a)}$$

$$g(H_Y) = 2\delta(H_Y - H_Y^*) + \alpha(1-\alpha)^{-2}(r - r^*) - (q - q^*), \quad \textbf{(A.1b)}$$

$$g(q) = [\sigma^{-1} - (1-\alpha)^{-2}](r - r^*) + (q - q^*). \quad \textbf{(A.1c)}$$

By Theorem 5.20 of Beavis and Dobbs (1990), representing system (A.1) as $\dot{\chi} = A(\chi - \chi^*) + \boldsymbol{h}(\chi, t)$, where χ is the three-dimensional vector (r, L_Y, q), χ^* is the steady state value, and the first term of the right-hand side is the linear approximation, then, since A has one root with negative real part, the system has a one-dimensional manifold M containing χ^* if $h(\chi^*, t) = 0$. Because this condition is satisfied, saddle-path stability is a global property. Thus, we can consider any value of the state variable z, with it always being guaranteed that there will be one, and only one, point going toward the steady state. ■

PROOF OF PROPOSITION 3

When $\sigma > (1-\alpha)^2$, the coefficient of $r - r^*$ in equation (A.1c) is negative. Let us suppose that $r > r^*$ in this equation. Thus, if q were smaller than q^*, $g(q)$ would be negative. Because the stable path requires a positive growth rate, when $r > r^*$ we must have $q > q^*$ and $g(q) < 0$. When $r < r^*$, we must have $q < q^*$ and $g(q) > 0$.

Figure A.1 shows the planes $\dot{q} = 0$ and $\dot{H}_Y = 0$ corresponding to this case. According to equation (A.1b), q is increasing at the points over the plane $\dot{q} = 0$ and decreasing at the points under that plane. According to equation (A.1c), H_Y is decreasing at the points over the plane $\dot{H}_Y = 0$, and increasing at the points under it. Thus, the stable path has the following shape:

(a) It moves under the plane $\dot{q} = 0$ and over the plane $\dot{H}_Y = 0$ with q and H_Y decreasing.

(b) It moves over the plane $\dot{q} = 0$ and under the plane $\dot{H}_Y = 0$ with q and H_Y increasing.

Thus, a stable saddle path exists along which, if $r > r^*$, then $q > q^*$, $H_Y > H_Y^*$, and if $r < r^*$, then $q < q^*$, $H_Y < H_Y^*$. Equation (15c) can be rewritten as

$$g(z) = \delta(H_Y - H_Y^*) + \frac{1}{\sigma}(r - r^*) - g(q),$$

and from this expression, we can deduce that, in the stable path, if $r > r^*$, then $z < z^*$, and if $r < r^*$, then $z > z^*$. ■

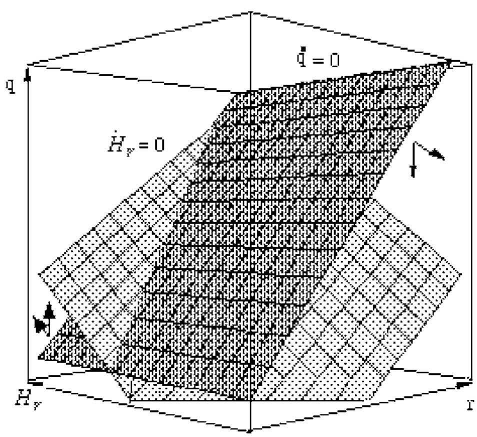

FIGURE A.1. Location of the stable path.

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