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Steady streaming in a channel with permeable walls

KONSTANTIN ILIN

Department of Mathematics, University of York, Heslington, York YO10 5DD, UK
email: konstantin.ilin@york.ac.uk

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We study steady streaming in a channel between two parallel permeable walls induced by oscillating (in time) injection/suction of a viscous fluid at the walls. We obtain an asymptotic expansion of the solution of the Navier–Stokes equations in the limit when the amplitude of normal displacements of fluid particles near the walls is much smaller than both the width of the channel and the thickness of the Stokes layer. It is shown that the steady part of the flow in this problem is much stronger than the steady flow produced by vibrations of impermeable boundaries. Another interesting feature of this problem is that the direction of the steady flow is opposite to what one would expect if the flow was produced by vibrations of impermeable walls.

Key words: Boundary-layer theory; Steady streaming; Asymptotic methods; Oscillating viscous flows; Navier-Stokes equations

1 Introduction

It is well known that an oscillating (in time) body force or vibrations of the boundary of a domain occupied by a viscous fluid can produce not only an oscillating flow but also a (relatively) weak steady flow, which is usually called steady streaming (see [8, 11, 12]). In this paper, we present a theory of steady streaming in a channel with fixed but permeable walls produced by the given velocity at the walls which is oscillating in time with angular frequency ω. The basic parameters of the problem are the inverse Strouhal number ϵ^2 and the Reynolds number R, defined by

$$\epsilon^2 = \frac{1}{St} = \frac{V_0^*}{\omega d}, \quad R = \frac{V_0^* d}{\nu}, \tag{1.1}$$

where V_0^* is the amplitude of the oscillating velocity at the walls, d is the distance between the walls and ν is the kinematic viscosity of the fluid. Parameter ϵ^2 measures the ratio of the amplitude $a = V_0^*/\omega$ of the displacements of fluid particles in an oscillating velocity field with amplitude V_0^* to the distance between the walls, i.e. $\epsilon^2 = a/d$. If $\delta = \sqrt{\nu/\omega}$ is the thickness of the oscillatory boundary layer (the Stokes layer) near the wall, then the Reynolds number R can be written as $R = ad/\delta^2$. Another dimensionless parameter which is widely used in literature is the 'streaming Reynolds number', $R_s = V_0^{*2}/\omega\nu$. In terms of parameters R and ϵ, $R_s = \epsilon^2 R$. We are interested in the asymptotic behaviour of solutions of the Navier–Stokes equations in the limit $\epsilon \ll 1$ and $R \sim 1$. This means that the amplitude of displacements of fluid particles is much smaller than the thickness of the Stokes layer. Indeed, $R \sim 1$ implies that $a/\delta \sim \delta/d \sim \epsilon$. Also note that this limit corresponds to small R_s ($R_s \sim \epsilon^2 \ll 1$).

It is also worth mentioning that, in many papers on peristaltic pumping, a different definition of the Reynolds number is used. For example, in [2, 3] Reynolds number $\tilde{R}$ is defined as $\tilde{R} = cd/\nu$, where c is the wave speed of the peristaltic wave that travels along the channel. The relation between R and $\tilde{R}$ is given by $\tilde{R} = (L/2\pi d)R/\epsilon^2$, where L is the wavelength of the peristaltic wave. Evidently, $\tilde{R} \gg R$ when $L/2\pi d \sim 1$ and $\epsilon \ll 1$, so that flows at $R \sim 1$ in our approach correspond to high Reynolds number flows in [2, 3]. The definition of the Reynolds number accepted in the present paper is more general and is based on the physical velocity of the fluid at the walls rather than on the wave speed which is not related to the actual velocity of the fluid.

Early studies of the steady streaming in a channel induced by vibrations of the walls had been focused on the problem of peristaltic pumping in channels and pipes under the assumption of low Reynolds numbers ($R \ll 1$) and small amplitude-to-wavelength ratio (see, e.g. [2, 7, 17]). In recent years, there had been considerable renewed interest in the problem motivated by possible applications of steady streaming to micro-mixing [1, 14, 18] and to drag reduction in channel flows [3]. In all asymptotic theories of steady streaming produced by vibrating impermeable boundaries, the magnitude of steady velocity is $O(\epsilon^2)$ for $\epsilon \ll 1$. This is true not only for $R \sim 1$ but also for $R \ll 1$ and $R \gg 1$. The aim of the present study is to show that if the boundary is permeable, then the steady part of the velocity is $O(\epsilon)$ for small ϵ, i.e. asymptotically much bigger than in the case of an impermeable boundary.

To construct an asymptotic expansion, we use the Vishik–Lyusternik method rather than the method of matched asymptotic expansions (see, e.g. [10, 15]). Although there is certain similarity between these two techniques, they are different. In both methods, at each order of the expansion, the approximation consists of two terms. In the method of matched asymptotic expansions, the approximation is split into an inner (boundary layer) and outer (exterior flow) parts which are computed independently and then matched in an intermediate region (not too far and not too close to the boundary) where both parts are assumed to give a valid approximation to the solution. In the Vishik–Lyusternik method, the approximation at each order consists of a term associated with a regular expansion and a 'corrector' term which is non-zero only within a thin boundary layer and which gives a correction to the regular term, so that their sum satisfies required boundary conditions. In a present paper, the use of the Vishik–Lyusternik method allows us to construct a uniformly valid asymptotic expansion up to any order in the small parameter ϵ. Some recent application of this method in fluid dynamics can be found in [4–6, 13, 16]. The reason we employ the Vishik–Lyusternik method rather than the method of matched asymptotic expansions is that it is better suited for our problem and allows us to tackle the problem directly. Although it is also possible to construct an asymptotic solution using the latter method, it cannot be applied to our problem directly. It would require a reduction of the original problem to a problem with zero normal velocity at the walls. Although this can be done (and in many ways), it would make the solution less transparent physically.

The outline of the paper is as follows. In Section 2, we formulate the mathematical problem. In Section 3, the asymptotic expansion of the solution is described. In Section 4, we consider simple examples in which the leading-order asymptotic solution can be obtained analytically. Finally, conclusions are presented in Section 5.

2 Formulation of the problem

We consider a two-dimensional viscous incompressible flow in an infinite channel of width d. The walls of the channel are permeable for the fluid, and the flow is produced by a given velocity at the walls which is assumed to be periodic along the channel with period L^* and oscillating in time with angular frequency ω. We will use the following non-dimensional quantities:

$$\tau = \omega t^*, \quad \mathbf{x} = \frac{\mathbf{x}^*}{d}, \quad \mathbf{v} = \frac{\mathbf{v}^*}{V_0^*}, \quad p = \frac{p^*}{\rho d \omega V_0^*}.$$

Here t^* is the time; $\mathbf{x}^* = (x^*, y^*)$; x^* and y^* are Cartesian coordinates, the x^*-axis being parallel to the channel; $\mathbf{v}^* = (u^*, v^*)$ is the velocity of the fluid; p^* is the pressure; ρ is the density; V_0^* is the maximum of the given velocity at the walls over all x^* and t^*. In these variables, the Navier–Stokes equations take the form

$$\mathbf{v}_\tau + \epsilon^2(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \frac{\epsilon^2}{R}\nabla^2\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0, \tag{2.1}$$

where the dimensionless parameters ϵ^2 and R are defined by equation (1.1). Equations (2.1) are to be solved subject to the boundary conditions

$$\mathbf{v}\big|_{y=0} = \mathbf{V}^a(x,\tau,\epsilon), \quad \mathbf{v}\big|_{y=1} = \mathbf{V}^b(x,\tau,\epsilon). \tag{2.2}$$

Here $\mathbf{V}^a = (U^a, V^a)$ and $\mathbf{V}^b = (U^b, V^b)$ are given functions which are 2π-periodic in τ and have zero mean value:

$$\bar{\mathbf{V}}^a \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathbf{V}^a(x,\tau,\epsilon)\,d\tau = \mathbf{0}, \quad \bar{\mathbf{V}}^b \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathbf{V}^b(x,\tau,\epsilon)\,d\tau = \mathbf{0}. \tag{2.3}$$

They are also assumed to be periodic in x with period $L = L^*/d$ and satisfy the condition

$$\int_0^L V^a(x,\tau,\epsilon)\,dx = \int_0^L V^b(x,\tau,\epsilon)\,dx \tag{2.4}$$

which follows from incompressibility of the fluid. In what follows, we are interested in the asymptotic behaviour of periodic (both in τ and x) solutions of equations (2.1) and (2.2) in the limit $\epsilon \to 0$ and $R = O(1)$. We assume that $\mathbf{V}^{a,b}(x,\tau,\epsilon)$ can be written as

$$\mathbf{V}^{a,b}(x,\tau,\epsilon) = \mathbf{V}_0^{a,b}(x,\tau) + \epsilon\,\mathbf{V}_1^{a,b}(x,\tau) + \epsilon^2\mathbf{V}_2^{a,b}(x,\tau) + \cdots \tag{2.5}$$

We seek a solution of (2.1) and (2.2) in the form

$$u = u^r + u^a + u^b, \quad v = v^r + \epsilon\,v^a + \epsilon\,v^b, \quad p = p^r + p^a + p^b. \tag{2.6}$$

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Here u^r, v^r and p^r are functions of x, y, τ and ϵ; u^a, v^a, and p^a depend on x, τ, ϵ and the boundary layer variable $\xi = y/\epsilon$; u^b, v^b and p^b depend on x, τ, ϵ and the boundary layer variable $\eta = (1-y)/\epsilon$. Functions u^r, v^r and p^r represent a regular expansion of the solution in power series in ϵ (an outer solution), and (u^a, v^a, p^a) and (u^b, v^b, p^b) correspond to boundary layer corrections to this regular expansion (inner solutions). Note that the vertical component of the velocity in the boundary layers is assumed to be proportional to ϵ. This is done just for convenience (this is the standard scaling in the boundary layer theory and, with this scaling, the continuity equation remains unchanged). If we had assumed the same scaling for both components of the velocity, i.e. $v = v^r + v^a + v^b$ rather than (2.6), we would arrive at the same asymptotic expansion (although the intermediate equations would be slightly different).

In what follows, we assume that the boundary layer parts of the expansion rapidly decay outside thin boundary layers:

$$u^a, v^a, p^a = o(\xi^{-s}) \quad \text{as} \quad \xi \to \infty \quad \text{and} \quad u^b, v^b, p^b = o(\eta^{-s}) \quad \text{as} \quad \eta \to \infty \tag{2.7}$$

for every $s > 0$. This means that the boundary layer part of the solution decays faster than any power of ξ (or η) as ξ (or η) goes to infinity. This assumption will be verified *a posteriori*.

3 Asymptotic expansion

In this section, we will describe the procedure of constructing the asymptotic expansion. We begin with the boundary conditions.

3.1 Boundary conditions

To obtain boundary conditions for u^r and u^a at the bottom wall, we insert (2.6) into the first condition (2.2) and ignore u^b and v^b because, according to (2.7), these are supposed to be small everywhere except a thin boundary layer near $y = 1$. This yields

$$u^r\big|_{y=0} + u^a\big|_{\xi=0} = U^a, \quad v^r\big|_{y=0} + \epsilon\, v^a\big|_{\xi=0} = V^a. \tag{3.1}$$

Similarly, at the upper wall we have

$$u^r\big|_{y=1} + u^b\big|_{\eta=0} = U^b, \quad v^r\big|_{y=1} + \epsilon\, v^b\big|_{\eta=0} = V^b. \tag{3.2}$$

Further, we assume that the regular and boundary layer parts of the solution can be presented in the form of power series in ϵ:

$$\mathbf{v}^r = \mathbf{v}_0^r + \epsilon\, \mathbf{v}_1^r + \cdots, \quad p^r = p_0^r + \epsilon\, p_1^r + \cdots, \tag{3.3}$$

$$u^a = u_0^a + \epsilon\, u_1^a + \cdots, \quad v^a = v_0^a + \epsilon\, v_1^a + \cdots, \quad p^a = p_0^a + \epsilon\, p_1^a + \cdots, \tag{3.4}$$

$$u^b = u_0^b + \epsilon\, u_1^b + \cdots, \quad v^b = v_0^b + \epsilon\, v_1^b + \cdots, \quad p^b = p_0^b + \epsilon\, p_1^b + \cdots \tag{3.5}$$

Now we substitute (2.6), (3.3)–(3.5) into (3.1) and (3.2) and collect terms of equal powers in ϵ. This leads to the following boundary conditions:

$$u_0^r\big|_{y=0} + u_0^a\big|_{\xi=0} = U_0^a, \quad v_0^r\big|_{y=0} = V_0^a, \tag{3.6}$$

$$u_k^r\big|_{y=0} + u_k^a\big|_{\xi=0} = U_k^a, \quad v_k^r\big|_{y=0} + v_{k-1}^a\big|_{\xi=0} = V_k^a \tag{3.7}$$

$$u_0^r\big|_{y=1} + u_0^b\big|_{\eta=0} = U_0^b, \quad v_0^r\big|_{y=1} = V_0^b, \tag{3.8}$$

$$u_k^r\big|_{y=1} + u_k^b\big|_{\eta=0} = U_k^b, \quad v_k^r\big|_{y=1} + v_{k-1}^b\big|_{\eta=0} = V_k^b \tag{3.9}$$

for $k \geqslant 1$.

In view of (2.7), we also require that for every $s > 0$ and for each $k = 0, 1, \ldots$,

$$u_k^a, v_k^a, p_k^a = o(\xi^{-s}) \quad \text{as} \quad \xi \to \infty \quad \text{and} \quad u_k^b, v_k^b, p_k^b = o(\eta^{-s}) \quad \text{as} \quad \eta \to \infty. \tag{3.10}$$

3.2 Regular part of the expansion

On substituting (3.3) into equation (2.1), we find that the successive approximations $\mathbf{v}_k^r$ and p_k^r satisfy the equations:

$$\partial_\tau \mathbf{v}_k^r = -\nabla p_k^r + \mathbf{F}_k, \quad \nabla \cdot \mathbf{v}_k^r = 0, \tag{3.11}$$

where $\mathbf{F}_0 \equiv 0$, $\mathbf{F}_1 \equiv 0$ and

$$\mathbf{F}_k = -\sum_{l=0}^{k-2} (\mathbf{v}_l^r \cdot \nabla)\mathbf{v}_{k-2-l}^r + \frac{1}{R}\nabla^2 \mathbf{v}_{k-2}^r \tag{3.12}$$

for $k \geqslant 2$. In what follows, we will use the following notation: For any 2π-periodic $f(\tau)$,

$$f(\tau) = \bar{f} + \tilde{f}(\tau), \quad \bar{f} = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\, d\tau, \tag{3.13}$$

where $\bar{f}$ is the mean value of $f(\tau)$ and, by definition, $\tilde{f}(\tau) = f(\tau) - \bar{f}$ is the oscillatory part of f that has zero mean value.

In the leading order ($k = 0$), equation (3.11) reduces to

$$\partial_\tau \mathbf{v}_0^r = -\nabla p_0^r, \quad \nabla \cdot \mathbf{v}_0^r = 0. \tag{3.14}$$

The general solution of (3.14), which is periodic in τ, can be written as

$$\mathbf{v}_0^r = \bar{\mathbf{v}}_0^r + \tilde{\mathbf{v}}_0^r, \quad \tilde{\mathbf{v}}_0^r = \nabla\phi_0, \tag{3.15}$$

where ϕ_0 has zero mean value and is the solution of the boundary value problem

$$\nabla^2\phi_0 = 0, \quad \partial_y \phi_0\big|_{y=0} = V_0^a, \quad \partial_y \phi_0\big|_{y=1} = V_0^b, \quad \phi_0(x+L, y) = \phi_0(x, y). \tag{3.16}$$

Equations (3.16) guarantee that the leading-order boundary conditions for normal velocity

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are satisfied. However, the conditions for the tangent component of the velocity cannot be satisfied at this stage. They will be satisfied later when the boundary layer part of the expansion is taken into account.

To find the leading-order averaged velocity $\bar{\mathbf{v}}_0^r$, we need to consider the equation for $\mathbf{v}_2^r$ (the first equation (3.11) for $k = 2$). It can be written as

$$\partial_\tau \mathbf{v}_2^r + (\mathbf{v}_0^r \cdot \nabla)\mathbf{v}_0^r = -\nabla p_2^r + \frac{1}{R}\nabla^2 \mathbf{v}_0^r. \tag{3.17}$$

Averaging this equation and the incompressibility condition for $\mathbf{v}_0^r$ and using the fact that $\tilde{\mathbf{v}}_0^r$ is irrotational, we obtain

$$(\bar{\mathbf{v}}_0^r \cdot \nabla)\bar{\mathbf{v}}_0^r = -\nabla \Pi_0 + \frac{1}{R}\nabla^2 \bar{\mathbf{v}}_0^r, \quad \nabla \cdot \bar{\mathbf{v}}_0^r = 0, \tag{3.18}$$

where $\Pi_0 = \bar{p}_2^r + \overline{|\nabla \phi_0|^2/2}$. Equations (3.18) represent the time-independent Navier–Stokes equations. It will be shown later that, in the leading order, the boundary layers are purely oscillatory, i.e. $\bar{u}_0^a = 0$, $\bar{v}_0^a = 0$, $\bar{u}_0^b = 0$ and $\bar{v}_0^b = 0$. Keeping this in mind, we average boundary conditions (3.6) and (3.8). This leads to the zero boundary conditions for $\bar{\mathbf{v}}_0^r$:

$$\bar{\mathbf{v}}_0^r\big|_{y=0} = \bar{\mathbf{v}}_0^r\big|_{y=1} = \mathbf{0}. \tag{3.19}$$

The only solution of (3.18) that is periodic in x and satisfies (3.19) is zero solution:

$$\bar{\mathbf{v}}_0^r \equiv \mathbf{0}. \tag{3.20}$$

This means that *there is no steady streaming in the leading order of the expansion.*

Consider now the first-order equations. Equations (3.11) for $k = 1$ have the same form as equation (3.14), and the general solution can be written as

$$\mathbf{v}_1^r = \bar{\mathbf{v}}_1^r + \tilde{\mathbf{v}}_1^r, \quad \tilde{\mathbf{v}}_1^r = \nabla \phi_1, \tag{3.21}$$

where ϕ_1 has zero mean value and is the solution of the boundary value problem

$$\begin{aligned} &\nabla^2 \phi_1 = 0, \quad \phi_1(x+L, y) = \phi_1(x, y), \\ &\partial_y \phi_1\big|_{y=0} = V_1^a - \tilde{v}_0^r\big|_{y=0}, \quad \partial_y \phi_1\big|_{y=1} = V_1^b - \tilde{v}_0^r\big|_{y=1}. \end{aligned} \tag{3.22}$$

Boundary conditions for $\partial_y \phi_1$ ensure that the oscillatory part of v_1^r satisfies the boundary conditions (3.7) and (3.9) for $k = 1$. Again, boundary conditions for u_1^r will be satisfied later when the boundary layers are taken into account.

To obtain equations for the averaged velocity $\bar{\mathbf{v}}_1^r(\mathbf{x})$, we average the incompressibility condition for $\bar{\mathbf{v}}_1^r$ and the equation for $\mathbf{v}_3^r$ (the first equation (3.11) for $k = 3$) and then take account of equation (3.20) and the fact that $\tilde{\mathbf{v}}_0^r$ and $\tilde{\mathbf{v}}_1^r$ are both irrotational. This yields

$$\mathbf{0} = -\nabla \Pi_1 + \frac{1}{R}\nabla^2 \bar{\mathbf{v}}_1^r, \quad \nabla \cdot \bar{\mathbf{v}}_1^r = 0, \tag{3.23}$$

where $\Pi_1 = \bar{p}_3^r + \overline{(\nabla \phi_0 \cdot \nabla \phi_1)}$. Thus, the first-order averaged outer flow is described by the Stokes equations.

Boundary conditions for $\bar{\mathbf{v}}_1^r$ are obtained by averaging boundary conditions (3.7) and (3.9) for $k = 1$:

$$\bar{u}_1^r\big|_{y=0} = -\bar{u}_1^a\big|_{\xi=0}, \quad \bar{v}_1^r\big|_{y=0} = 0, \tag{3.24}$$

$$\bar{u}_1^r\big|_{y=1} = -\bar{u}_1^b\big|_{\eta=0}, \quad \bar{v}_1^r\big|_{y=1} = 0. \tag{3.25}$$

Here we have used the fact that $\bar{\mathbf{V}}_1^{a,b} = 0$ and our assumption (which will be verified later) that $\bar{v}_0^{a,b} = 0$. Thus, the first-order averaged flow can be determined by solving the Stokes equations (3.23) subject to boundary conditions (3.24) and (3.25), provided that $\bar{u}_1^a$ and $\bar{u}_1^b$ are known.

The regular expansion described above can continue to give us equations and boundary conditions up to any order in ϵ, but we will restrict out attention only to the first two terms of the expansion.

3.3 Boundary layers

To derive boundary layer equations near the bottom wall ($y = 0$), we ignore u^b, v^b and p^b because they are supposed to be small everywhere except a thin boundary layer near $y = 1$, and assume that

$$u = u_0^r + u_0^a + \epsilon(u_1^r + u_1^a) + \cdots, \quad v = v_0^r + \epsilon(v_1^r + v_0^a) + \cdots, \quad p = p_0^r + p_0^a + \epsilon(p_1^r + p_1^a) + \cdots$$

We substitute these into equation (2.1) and take into account that u_k^r, v_k^r and p_k^r satisfy (3.11). Then we make the change of variables $y = \epsilon\,\xi$ in u_k^r, v_k^r and p_k^r, expand every function of $\epsilon\,\xi$ in Taylor's series at $\epsilon = 0$ and, finally, collect terms of the equal powers in ϵ. As a result, we obtain

$$\partial_\tau\, u_k^a + \partial_x\, p_k^a - \frac{1}{R}\,\partial_\xi^2\, u_k^a = F_k^a, \quad \partial_\xi\, p_k^a = G_k^a, \quad \partial_x\, u_k^a + \partial_\xi\, v_k^a = 0 \tag{3.26}$$

for $k = 0, 1, \ldots$ In equation (3.26), functions F_k^a and G_k^a depend on $\mathbf{v}_0^r, \ldots, \mathbf{v}_{k-1}^r$, $u_0^a, \ldots, u_{k-1}^a$, $v_0^a, \ldots, v_{k-1}^a$. For $k = 0, 1$, these are given by

$$F_0^a = 0, \quad F_1^a = -V_0^a(x,\tau)\,\partial_\xi\, u_0^a, \quad G_0^a = 0, \quad G_1^a = 0. \tag{3.27}$$

A similar procedure leads to the equations of the boundary layer near the upper wall:

$$\partial_\tau\, u_k^b + \partial_x\, p_k^b - \frac{1}{R}\,\partial_\eta^2\, u_2^b = F_k^b, \quad \partial_\eta\, p_k^b = -G_k^b, \quad \partial_x\, u_k^b - \partial_\eta\, v_k^b = 0 \tag{3.28}$$

for $k = 0, 1, \ldots$. Functions F_k^b and G_k^b for $k = 0, 1$ are given by

$$F_0^b = 0, \quad F_1^b = V_0^b(x,\tau)\,\partial_\eta\, u_0^b, \quad G_0^b = 0, \quad G_1^b = 0. \tag{3.29}$$

3.3.1 *Leading-order equations*

Boundary layer at $y = 0$: In the leading order ($k = 0$), equation (3.26) simplify to

$$\partial_\tau u_0^a + \partial_x p_0^a - \frac{1}{R}\partial_\xi^2 u_0^a = 0, \quad \partial_\xi p_0^a = 0, \quad \partial_x u_0^a + \partial_\xi v_0^a = 0.$$

The second equation and the condition of decay at infinity (in variable ξ) for p_0^a imply that $p_0^a \equiv 0$. Hence, the first equation reduces to the heat equation

$$\partial_\tau u_0^a = \frac{1}{R}\partial_\xi^2 u_0^a. \tag{3.30}$$

Boundary condition for u_0^a at $\xi = 0$ follows from (3.6) and is given by

$$u_0^a\big|_{\xi=0} = U_0^a - u_0^r\big|_{y=0} = U_0^a - \partial_x \phi_0\big|_{y=0}. \tag{3.31}$$

We note in passing that for simple harmonic oscillations, when $\mathbf{V}^a$ and $\mathbf{V}^b$ in equation (2.2) can be written as

$$\mathbf{V}^a(x,\tau,\epsilon) = Re\left(\hat{\mathbf{V}}^a(x,\epsilon)e^{i\tau}\right), \quad \mathbf{V}^a(x,\tau,\epsilon) = Re\left(\hat{\mathbf{V}}^a(x,\epsilon)e^{i\tau}\right),$$

the boundary condition (3.31) can be presented in the form

$$u_0^a\big|_{\xi=0} = Re(\hat{h}(x)\,e^{i\tau})$$

for a suitable function $\hat{h}(x)$. In this case, the periodic (in τ) solution of equation (3.30) that satisfies (3.31) and the condition of decay at infinity is given by the simple formula

$$u_0^a = Re(\hat{h}(x)\,e^{i\tau - \sqrt{R/2}(1+i)\xi}).$$

Let us show that $\bar{u}_0^a \equiv 0$. Averaging equation (3.30), we find that $\partial_\xi^2 \bar{u}_0^a = 0$. The only solution of this equation that satisfies the decay condition at infinity and the boundary condition $\bar{u}_0^a|_{\xi=0} = 0$ (which follows from (3.31) and the fact that $\bar{U}_0^a = 0$ and $\bar{\phi}_0 = 0$) is zero solution. Thus, in the leading order the boundary layer at $y = 0$ is purely oscillatory: $\bar{u}_0^a \equiv 0$. This partly justifies our earlier assumption.

Integration of the incompressibility condition in variable ξ gives us the normal velocity v_0^a:

$$v_0^a(x,\xi,\tau) = \partial_x \int_\xi^\infty u_0^a(x,\xi',\tau)\,d\xi'. \tag{3.32}$$

Here the constant of integration is chosen so as to guarantee that $v_0^a \to 0$ as $\xi \to \infty$. Note that equation (3.32) together with the fact that $\bar{u}_0^a \equiv 0$ implies that $\bar{v}_0^a \equiv 0$.

Boundary layer at $y = 1$: Exactly the same arguments as above lead to the problem

$$\partial_\tau u_0^b = \frac{1}{R}\partial_\eta^2 u_0^b, \tag{3.33}$$

$$u_0^b\big|_{\eta=0} = U_0^b - \partial_x \phi_0\big|_{y=1}, \quad u_0^b \to 0 \ \text{ as } \ \eta \to \infty. \tag{3.34}$$

Again, it follows from (3.33) and (3.34) that $\bar{u}_0^b \equiv 0$, and this justifies our assumption that, in the leading order, the boundary layers are purely oscillatory. The normal velocity v_0^b is given by

$$v_0^b(x,\eta,\tau) = -\partial_x \int_\eta^\infty u_0^b(x,\eta',\tau)\, d\eta'. \tag{3.35}$$

Equation (3.35) implies that $\bar{v}_0^b \equiv 0$. Now $u_0^{a,b}$ and $v_0^{a,b}$ are known, and the oscillatory part of the first-order outer flow can be found by solving problem (3.22).

3.3.2 *First-order equations*

Boundary layer at $y = 0$: Consider now equation (3.26) for $k = 1$. Again, the condition of decay at infinity for p_1^a and the second equation (3.26) imply that $p_1^a \equiv 0$. Hence, we have

$$\partial_\tau u_1^a = \frac{1}{R}\partial_\xi^2 u_1^a - V_0^a(x,\tau)\,\partial_\xi u_0^a. \tag{3.36}$$

Averaging this equation, we find that $\partial_\xi^2 \bar{u}_1^a = R\,\overline{V_0^a(x,\tau)\,\partial_\xi u_0^a}$. Integration in ξ yields

$$\bar{u}_1^a = -R \int_\xi^\infty \overline{V_0^a(x,\tau)u_0^a(x,\xi',\tau)}\, d\xi'. \tag{3.37}$$

Here the constants of integration are chosen so as to satisfy the condition of decay at infinity. The oscillatory part of u_1^a as well as both averaged and oscillatory parts of v_1^a can also be found but are not needed in what follows.

Boundary layer at $y = 1$: A similar analysis leads to

$$\bar{u}_1^b = R \int_\eta^\infty \overline{V_0^b(x,\tau)u_0^b(x,\eta',\tau)}\, d\eta'. \tag{3.38}$$

Now we know both $\bar{u}_1^a$ and $\bar{u}_1^b$, so that these can be inserted into the boundary conditions (3.24) and (3.25). Then the first-order averaged outer flow can be determined by solving the Stokes problems (3.23)–(3.25).

3.4 Steady streaming

In the leading order, the steady streaming is described by the first non-zero term in the expansion for the averaged flow. The averaged velocity field has the form

$$\bar{u} = \epsilon\left(\bar{u}_1^r + \bar{u}_1^a + \bar{u}_1^b\right) + O(\epsilon^2), \quad \bar{v} = \epsilon\,\bar{v}_1^r + O(\epsilon^2),$$

where boundary layer contributions $\bar{u}_1^a$ and $\bar{u}_1^b$ are given by equations (3.37) and (3.38) and $\bar{\mathbf{v}}_1^r$ is the solution of the Stokes equations that satisfies boundary conditions (3.24) and (3.25). If we introduce the stream function for the averaged flow $\bar{\psi}$ defined by the

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standard relations $\bar{u} = \bar{\psi}_y$ and $\bar{v} = -\bar{\psi}_x$, then the corresponding expansion of $\bar{\psi}$ will have the form

$$\bar{\psi} = \epsilon\, \bar{\psi}_1^r + O(\epsilon^2),$$

where $\bar{\psi}_1^r$ is the stream function for $\bar{\mathbf{v}}_1^r = (\bar{u}_1^r, \bar{v}_1^r)$. Note that boundary layer terms do not appear in the leading order of the expansion for $\bar{\psi}$. This is because to get the $O(\epsilon)$ boundary layer velocity one needs the $O(\epsilon^2)$ stream function.

The fact that the steady streaming is described by quantities that are linear in ϵ is in sharp contrast with the asymptotic theories of steady streaming produced by transverse vibrations of the solid impermeable walls where it is an effect of second order in ϵ. To clarify this point, let us discuss it in more detail.

First, we note that the case of vibrating impermeable walls is also covered by the present theory. The only difference from what has already been discussed comes from the boundary conditions, which now take the form:

$$\mathbf{v}\big|_{y=\epsilon^2 f(x,\tau)} = f_\tau(x,\tau)\,\mathbf{e}_y, \quad \mathbf{v}\big|_{y=1+\epsilon^2 g(x,\tau)} = g_\tau(x,\tau)\,\mathbf{e}_y. \tag{3.39}$$

Here $y = \epsilon^2 f(x,\tau)$ and $y = 1 + \epsilon^2 g(x,\tau)$ represent vibrating walls, f and g are given functions which are periodic in both τ and x with periods 2π and L respectively. Assuming that ϵ is small, we expand the left sides of (3.39) in Taylor's series about $y = 0$ and $y = 1$:

$$\mathbf{v}\big|_{y=0} + \epsilon^2 f(x,\tau)\,\partial_y\,\mathbf{v}\big|_{y=0} + O(\epsilon^4) = f_\tau(x,\tau)\,\mathbf{e}_y, \tag{3.40}$$

$$\mathbf{v}\big|_{y=1} + \epsilon^2 g(x,\tau)\,\partial_y\,\mathbf{v}\big|_{y=1} + O(\epsilon^4) = g_\tau(x,\tau)\,\mathbf{e}_y. \tag{3.41}$$

Now, according to (2.6), (3.3)–(3.5), near the bottom wall, we have

$$u = u_0^r + u_0^a + \epsilon\left(u_1^r + u_1^a\right) + O(\epsilon^2), \quad v = v_0^r + \epsilon\left(v_1^r + v_0^a\right) + O(\epsilon^2).$$

Substituting these in equation (3.40) and collecting the terms of equal powers in ϵ, we obtain the following conditions:

$$u_0^r\big|_{y=0} + u_0^a\big|_{\xi=0} = 0, \quad v_0^r\big|_{y=0} = f_\tau, \tag{3.42}$$

$$u_1^r\big|_{y=0} + u_1^a\big|_{\xi=0} + f\,\partial_\xi\,u_0^a\big|_{\xi=0} = 0, \quad v_1^r\big|_{y=0} + v_0^a\big|_{\xi=0} = 0 \quad \text{etc.} \tag{3.43}$$

Note the presence of a term involving $\partial_\xi\, u_0^a$ in equation (3.43). It comes from the second term on the left side of (3.40) after we take into account that $\partial_y\, u_0^a = \epsilon^{-1}\,\partial_\xi\, u_0^a$. Similarly, near the upper wall,

$$u = u_0^r + u_0^b + \epsilon\left(u_1^r + u_1^b\right) + O(\epsilon^2), \quad v = v_0^r + \epsilon\left(v_1^r + v_0^b\right) + O(\epsilon^2).$$

Substitution of these in equation (3.41) yields

$$u_0^r\big|_{y=1} + u_0^b\big|_{\eta=0} = 0, \quad v_0^r\big|_{y=1} = g_\tau, \tag{3.44}$$

$$u_1^r\big|_{y=1} + u_1^b\big|_{\eta=0} - g\,\partial_\eta\,u_0^b\big|_{\eta=0} = 0, \quad v_1^r\big|_{y=1} + v_0^b\big|_{\eta=0} = 0 \quad \text{etc.} \tag{3.45}$$

Evidently, (3.42)–(3.45) are similar to (3.6)–(3.9). Moreover, if we define functions $U_0^{a,b}$, $V_0^{a,b}$, $U_1^{a,b}$ and $V_1^{a,b}$ as

$$U_0^{a,b} = 0, \quad V_0^a = f_\tau, \quad V_0^b = g_\tau, \quad V_1^{a,b} = 0,$$
$$U_1^a = -f\,\partial_\xi\, u_0^a\big|_{\xi=0}, \quad U_1^b = g\,\partial_\eta\, u_0^b\big|_{\eta=0}, \tag{3.46}$$

then (3.42)–(3.45) can be written in exactly the same form as boundary conditions (3.6)–(3.9) for $k = 0, 1$. There are, however, two differences: First, U_1^a and U_1^b are not given functions but depend on u_0^a and u_0^b, and second, $\bar{U}_1^a$ and $\bar{U}_1^b$ may be non-zero functions.

Now our aim is to find boundary conditions for the first-order averaged outer flow (governed by the Stokes equations (3.23)). Consider first boundary conditions at $y = 0$. Averaging equation (3.43), we obtain

$$\bar{u}_1^r\big|_{y=0} = -\,\bar{u}_1^a\big|_{\xi=0} - \overline{f(x,\tau)\,\partial_\xi\, u_0^a}\,\big|_{\xi=0}, \tag{3.47}$$

$$\bar{v}_1^r\big|_{y=0} = 0. \tag{3.48}$$

It follows from equations (3.37) and (3.46) that

$$\bar{u}_1^a\big|_{\xi=0} = -R\int_0^\infty \overline{\partial_\tau f(x,\tau)\, u_0^a(x,\xi,\tau)}\, d\xi = R\int_0^\infty \overline{f(x,\tau)\,\partial_\tau\, u_0^a(x,\xi,\tau)}\, d\xi$$
$$= \int_0^\infty \overline{f(x,\tau)\,\partial_\xi^2\, u_0^a(x,\xi,\tau)}\, d\xi = -\overline{f(x,\tau)\,\partial_\xi\, u_0^a(x,\xi,\tau)}\,\big|_{\xi=0}\,. \tag{3.49}$$

Here we used the facts that $\overline{A'(\tau)B(\tau)} = -\overline{A(\tau)B'(\tau)}$ for any periodic $A(\tau)$ and $B(\tau)$ and that $u_0^a(x,\xi,\tau)$ satisfies the heat equation (3.30). On substituting (3.49) into (3.47), we find that

$$\bar{u}_1^r\big|_{y=0} = 0. \tag{3.50}$$

Thus, in spite of the presence of a non-zero averaged boundary layer near the bottom wall in the first order of the expansion, the boundary conditions at the bottom wall for the outer flow are such that both components of the averaged velocity must be zero. A similar analysis of the boundary conditions at the upper wall lead to the same conclusion: Both components of the averaged velocity must also be zero at the upper wall. Since the only solution of the Stokes equations (3.23) subject to zero boundary conditions is zero solution, we conclude that, in the problem with vibrating impermeable walls, there is no steady streaming in the first order in ϵ, which is consistent with all earlier studies of this problem.

It had been understood long ago (see, e.g. [9]) that, in oscillatory flows, the averaged Lagrangian velocity (the velocity of fluid particles) may be different from the averaged Eulerian velocity. This difference is usually referred to as the Stokes drift. So far we have discussed only the Eulerian velocity. However, it is the velocity of fluid particles that is observed in experiments and is responsible for mass transport in oscillating flows. Therefore, our study would be incomplete if we did not discuss the Stokes drift. It turns out that, in our problem, the Stokes drift is the effect of higher order and does not appear in the first order in ϵ. Therefore, up to terms of the first order in ϵ, the averaged Eulerian velocity coincides with the averaged Lagrangian velocity. This is shown in Appendix A.

4 Examples

Below we consider a few simple examples in which the velocity at the walls oscillates harmonically in time. Before presenting these examples, let us discuss general properties of the equations that describe steady streaming (equations (3.23)–(3.25), (3.37) and (3.38)). It follows from (3.37) and (3.38) that if $V_0^a = 0$ and $V_0^b = 0$, then $\bar{u}_1^a = 0$ and $\bar{u}_1^b = 0$, so that the first-order boundary layers are purely oscillatory. This means that the boundary conditions for $\bar{u}_1^r$, given by (3.24) and (3.25), become zero conditions and therefore *there is no steady streaming in the first order in* ϵ. In this case, steady streaming appears in higher order approximations (for a half space this problem had been treated in [16]).

Even if V_0^a and V_0^b are non-zero, this does not guarantee the appearance of steady streaming (in the first order in ϵ). To show this, we first observe that the leading-order outer oscillatory flow is uniquely determined by V_0^a and V_0^b and does not use U_0^a and U_0^b. Now, let us write U_0^a and U_0^b in the form

$$U_0^a = \partial_x \phi_0 \big|_{y=0} + Q^a(x,\tau), \quad U_0^b = \partial_x \phi_0 \big|_{y=1} + Q^b(x,\tau)$$

for some given functions $Q^a(x,\tau)$ and $Q^b(x,\tau)$. Then, it follows from (3.31) and (3.34) that u_0^a and u_0^b are completely determined by Q^a and Q^b and do not depend on V_0^a and V_0^b. Therefore, if Q^a and Q^b were such that

$$\bar{u}_1^a \big|_{\xi=0} = -R \int_0^\infty \overline{V_0^a(x,\tau) u_0^a(x,\xi',\tau)}\, d\xi' = 0, \quad \bar{u}_1^b \big|_{\eta=0} = R \int_0^\infty \overline{V_0^b(x,\tau) u_0^b(x,\eta',\tau)}\, d\eta' = 0,$$

then there would be no steady streaming. The simplest choice of Q^a and Q^b that ensures the absence of steady streaming is $Q^a(x,\tau) \equiv 0$ and $Q^b(x,\tau) \equiv 0$. Thus, given $V_0^a(x,\tau)$ and $V_0^b(x,\tau)$, it is always possible to choose $U_0^a(x,\tau)$ and $U_0^b(x,\tau)$ such that there is no steady streaming in the first order in ϵ.

It is also true that, for a wide class of functions $U_0^a(x,\tau)$ and $U_0^b(x,\tau)$, there is steady streaming in the first order in ϵ. Below we present three examples which show the existence of steady streaming in the first order in ϵ.

4.1 Example 1: Standing waves

Let $\mathbf{V}^a = \cos kx \cos \tau\, \mathbf{e}_y$ and $\mathbf{V}^b = \alpha \mathbf{V}^a$, where $k = 2\pi/L$ and $\alpha = \pm 1$. This choice corresponds to standing waves of injection/suction applied at the boundaries of the channel ($\alpha = 1$ if the waves are in phase, and $\alpha = -1$ if they have opposite phase). After substitution of $\mathbf{V}^a$ and $\mathbf{V}^b$ in the general formulae of Section 3, we find that

$$\bar{u}_1^r \big|_{y=0} = \bar{u}_1^r \big|_{y=1} = -\sqrt{R}\, \frac{A(k)}{4\sqrt{2}} \sin(2kx), \quad A(k) \equiv \frac{\cosh(k) - \alpha}{\sinh(k)}. \tag{4.1}$$

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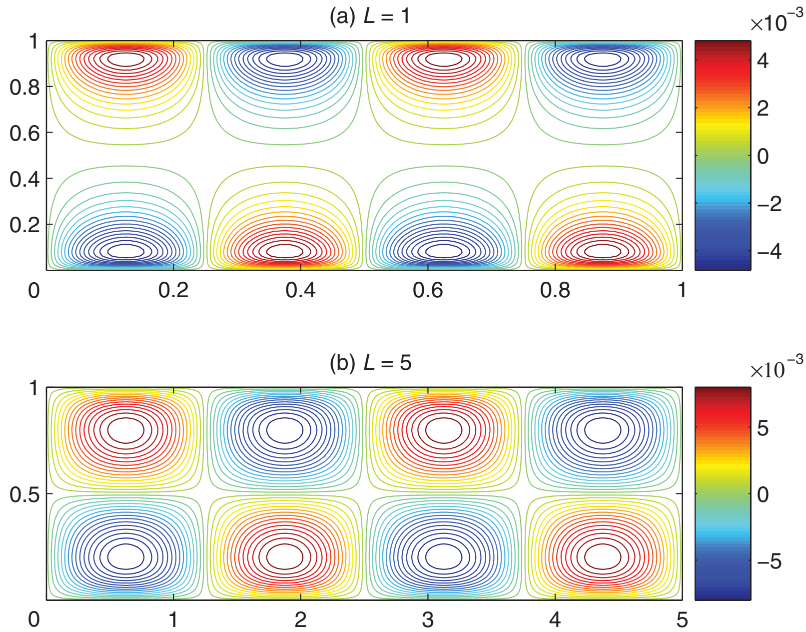

FIGURE 1. (Colour online) The streamlines $\bar{\psi}_1^r = const$ for standing waves for $R = 1$ and $\alpha = 1$: (a) $L = 1$; (b) $L = 5$.

Solving the Stokes equations (3.23) with boundary conditions (3.24) and (3.25), we obtain

$$\bar{\psi}_1^r = -\frac{\sqrt{R}}{4\sqrt{2}}\,\frac{A(k)}{\sinh(2k) - 2k}\,\{(y-1)\sinh(2ky) + y\sinh[2k(1-y)]\}\,\sin(2kx). \tag{4.2}$$

Typical streamlines are shown in Figure 1. It is clear from (4.2) that the flow picture is the same for $\alpha = 1$ and –1, the only difference is in the magnitude of the flow. The latter is determined by $A(k)$. For $\alpha = 1$, A is an increasing function, $A(k) \sim k$ for $k \ll 1$ and $A(k) \to 1$ as $k \to \infty$. For $\alpha = -1$, A is a decreasing function, $A(k) \sim 1/k$ for $k \ll 1$ and $A(k) \to 1$ as $k \to \infty$. So, for all $k > 0$, the magnitude of the steady streaming for $\alpha = -1$ is greater than for $\alpha = 1$. Thus, the steady streaming is stronger when standing waves of injection/suction applied at the walls have opposite phase. Also, in this case the magnitude of the steady streaming increases with the wavelength of the waves.

4.2 Example 2: Waves travelling in the same direction

Now, let the velocity at both walls be purely normal and have the form of waves travelling in the direction of the x-axis: $\mathbf{V}^a = \cos(kx - \tau)\,\mathbf{e}_y$ and $\mathbf{V}^b = \alpha\,\mathbf{V}^a$, where k and α are the same as in Example 1. Boundary conditions (3.24) and (3.25) take the form

$$\bar{u}_1^r\big|_{y=0} = \bar{u}_1^r\big|_{y=1} = -\sqrt{R}\,\frac{A(k)}{2\sqrt{2}}, \tag{4.3}$$

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where $A(k)$ is given by (4.1). The Stokes equations (3.23) subject to (3.24) and (3.25) lead to the constant solution[1]: $\bar{u}_1^r = -\sqrt{R}\,A(k)/2\sqrt{2}$, $\bar{v}_1^r = 0$. Thus, the waves travelling in the same direction produce a constant mean flow whose direction is opposite to the direction in which the waves advance. This is a surprising result which is in sharp contrast with the steady streaming produced by vibrating impermeable walls in the form of waves travelling in the same direction (peristaltic pumping) where the induced flow is in the direction in which the waves travel. This result agrees with numerical simulations reported earlier in [3]. The magnitude of the mean flow is determined by $A(k)$. Properties of $A(k)$ imply that the most efficient way to generate the mean unidirectional flow is to apply injection/suction in the form of waves travelling in the same direction and having opposite phase.

4.3 Example 3: Waves travelling in the opposite directions

Now, let the normal velocity at the walls has the form of waves travelling in opposite directions: $\mathbf{V}^a = \cos(kx - \tau)\,\mathbf{e}_y$ and $\mathbf{V}^b = \alpha\,\cos(kx + \tau)\,\mathbf{e}_y$, where k and α are the same as in Examples 1 and 2. Boundary conditions for $\bar{u}_1^r$ reduce to

$$\bar{u}_1^r\big|_{y=0} = -\frac{\sqrt{R}}{2\sqrt{2}}\,\frac{B^-(k,x)}{\sinh(k)}, \quad \bar{u}_1^r\big|_{y=1} = \frac{\sqrt{R}}{2\sqrt{2}}\,\frac{B^+(k,x)}{\sinh(k)}, \tag{4.4}$$

where $B^{\pm} = \cosh(k) + \alpha[\cos(2kx) \pm \sin(2kx)]$. The corresponding solution of the Stokes equations is given by

$$\bar{\psi}_1^r = -\frac{\sqrt{R}}{2\sqrt{2}}\left[\frac{\cosh(k)}{\sinh(k)}\,y(1-y) + D^-\,y\sinh[2k(1-y)] + D^+(1-y)\sinh(2ky)\right], \tag{4.5}$$

where

$$D^{\pm}(k,x) = \frac{\alpha}{\sinh(k)}\left[\frac{\cos(2kx)}{\sinh(2k)+2k} \pm \frac{\sin(2kx)}{\sinh(2k)-2k}\right].$$

In the short wave limit ($k \to \infty$),

$$\bar{\psi}_1^r = -\frac{\sqrt{R}}{2\sqrt{2}}\,y(1-y) + O(e^{-k}),$$

and the averaged flow is well approximated by the plane parallel flow with linear velocity profile. For long waves ($k \to 0$),

$$\bar{\psi}_1^r = \frac{\sqrt{R}}{2\sqrt{2}}\,\frac{1}{k}\,y(1-y)\,[1 + \alpha\cos(2kx) - \alpha(1-2y)\sin(2kx)] + O(k),$$

and the averaged flow is non-parallel. For moderate k, it is a superposition of a shear flow with a linear velocity profile and a periodic array of vortices ('cat's eyes') in the middle of the channel. When the wave length increases, these vortices grow and eventually fill the

[1] The Stokes equations (3.23) also admit solutions with a non-zero pressure gradient $\nabla\Pi_1 = c_0\mathbf{e}_x$ ($c_0 = const$), which are not considered here, because this would be equivalent to a modification of our problem, allowing the presence of a weak $O(\epsilon^3)$ pressure gradient.

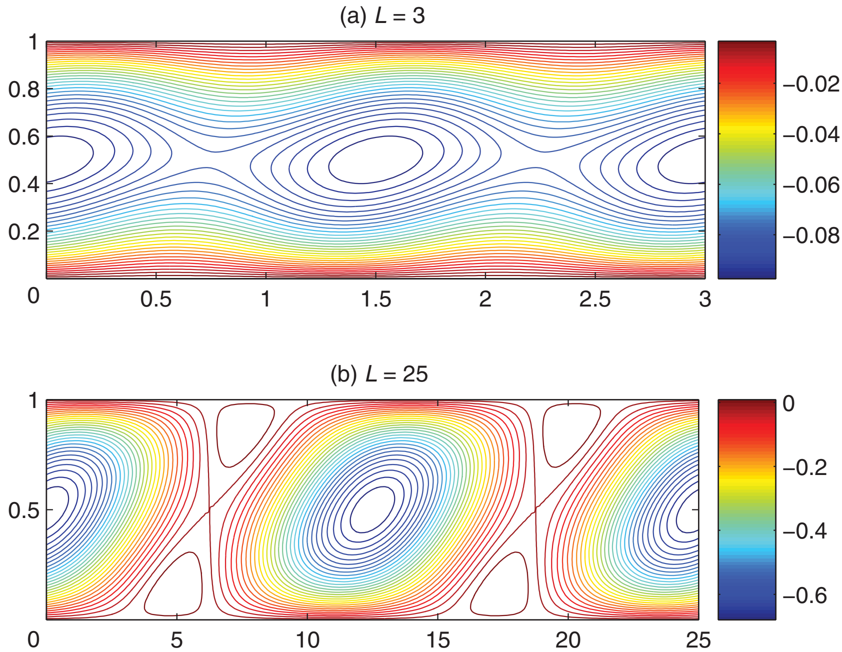

FIGURE 2. (Colour online) The streamlines $\bar{\psi}_1^r = const$ for waves travelling in opposite directions for $R = 1$ and $\alpha = 1$: (a) $L = 3$; (b) $L = 25$.

whole channel, and at the same time smaller vortices of opposite sign appear near the walls. Typical streamlines of the flow (4.5) for $\alpha = 1$ are shown in Figure 2. In the case of $\alpha = -1$, the flow picture is the same except that it is shifted along x-axis by $L/4$ (this can be deduced directly from (4.5)).

5 Conclusions

We have considered incompressible flows in a channel between two parallel permeable walls and constructed an asymptotic expansion of solutions of the Navier–Stokes equations in the limit when the amplitude of displacements of fluid particles near the walls is much smaller than both the width of the channel and the thickness of the Stokes layer. The asymptotic procedure is based on the Vishik–Lyusternik method and can be used to construct as many terms of the expansion as necessary. In the leading order, the averaged flow is described by the stationary Stokes equations subject to the boundary conditions that are determined by the boundary layers near the walls. The key difference between the present expansion and the asymptotic theories of steady streaming induced by vibrating impermeable boundaries is that, in our study, the magnitude of the averaged velocity is $O(\epsilon)$, which is much bigger than $O(\epsilon^2)$ averaged velocity in the case of impermeable walls. Another important difference is that the direction of the steady part of the flow is opposite to the direction one would expect if the normal vibrations of impermeable walls were used to generate it.

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The general formulae have been applied to three particular examples of steady streaming induced by blowing/suction at the walls in the form of standing and travelling plane waves. In the case of standing waves, the averaged flow has the form of a double array of vortices (see Figure 1). For short waves the vortices are concentrated near the walls, while long waves produce vortices that fill the entire channel.

If the normal velocity at the walls have the form of plane harmonic waves which travel in the same direction, the induced steady flow is a constant unidirectional flow whose direction is opposite to the direction in which the waves travel. This is different from the case of the steady streaming generated by vibrations of impermeable walls where the induced flow has the same direction as the travelling wave. As far as we are aware, this was first observed in numerical simulations in [3].

If the normal velocities at the walls have the form of plane waves travelling in opposite directions, the averaged flow is a superposition of a shear flow with a linear velocity profile and a periodic array of vortices ('cat's eyes') in the middle of the channel. When the wavelength is small, the vortices are weak. Their intensity and size monotonically grow with the wavelength and eventually they fill the entire gap between the walls.

There are many open problems in this area. In particular, it is not quite clear how the present theory can be extended to the case of $R_s \sim 1$. Although the problem does not involve a moving boundary, which, in general, simplifies things, there is a technical difficulty of a different sort. It is related to the leading-order boundary layer equations for $R_s \sim 1$ which are difficult to solve analytically.[2] This is a subject of a continuing investigation.

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Appendix A

Below we show that, in the present problem, the averaged Largangian and Eulerian velocities coincide up to the terms of first order in ϵ. The motion of fluid particles is governed by the ordinary differential equation

$$\frac{d\mathbf{x}}{d\tau} = \epsilon^2 \mathbf{v}(\mathbf{x}, \tau, \epsilon), \tag{A 1}$$

which should be solved subject to initial condition $\mathbf{x} = \mathbf{B}_0$. The velocity field $\mathbf{v}(\mathbf{x}, \tau, \epsilon)$ is the solution of the Navier–Stokes equations, which is 2π-periodic in τ and has a non-zero average. We have already computed first two terms in the uniformly valid asymptotic expansion of $\mathbf{v}(\mathbf{x}, \tau, \epsilon)$. Now we are interested in constructing an asymptotic expansion of the solution of (A 1) for small ϵ. Since the steady streaming appears in the first order in ϵ, this fact and equation (A 1) suggest that there is a slow drift of fluid particles over times of order $\epsilon^3\tau$. Therefore, we introduce the slow time $t = \epsilon^3\tau$ and assume that $\mathbf{x} = \mathbf{x}(\tau, t, \epsilon)$.

[2] Note that in the case of vibrating impermeable walls and $R_s \sim 1$, the same equations (but with different boundary conditions) can be solved analytically [5].

This assumption results in the equation

$$\partial_\tau \mathbf{x} + \epsilon^3 \partial_t \mathbf{x} = \epsilon^2 \mathbf{v}(\mathbf{x}, \tau, \epsilon).$$

In what follows, we restrict our analysis to the outer flow and ignore the boundary layers because (i) we are interested mostly in the flow away from the walls, and (ii) this considerably simplifies the analysis. We have

$$\partial_\tau \mathbf{x} + \epsilon^3 \partial_t \mathbf{x} = \epsilon^2 \left[\tilde{\mathbf{v}}_0^r(\mathbf{x}, \tau) + \epsilon \left(\bar{\mathbf{v}}_1^r(\mathbf{x}) + \tilde{\mathbf{v}}_1^r(\mathbf{x}, \tau)\right)\right] + O(\epsilon^2). \tag{A 2}$$

Assuming that

$$\mathbf{x} = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \cdots,$$

we substitute this into (A 2) and expand $\tilde{\mathbf{v}}_0^r(\mathbf{x}_0 + \epsilon \mathbf{x}_1 + \cdots, \tau)$, $\bar{\mathbf{v}}_1^r(\mathbf{x}_0 + \epsilon \mathbf{x}_1 + \cdots)$ and $\tilde{\mathbf{v}}_1^r(\mathbf{x}_0 + \epsilon \mathbf{x}_1 + \cdots, \tau)$ in Taylor's series at $\mathbf{x}_0$. As a result, we obtain the sequence of equations

$$\partial_\tau \mathbf{x}_0 = 0, \quad \partial_\tau \mathbf{x}_1 = 0, \tag{A 3}$$

$$\partial_\tau \mathbf{x}_2 = \tilde{\mathbf{v}}_0^r(\mathbf{x}_0, \tau), \tag{A 4}$$

$$\partial_\tau \mathbf{x}_3 + \partial_t \mathbf{x}_0 = (\mathbf{x}_1 \cdot \nabla)\, \tilde{\mathbf{v}}_0^r(\mathbf{x}_0, \tau) + \bar{\mathbf{v}}_1^r(\mathbf{x}_0) + \tilde{\mathbf{v}}_1^r(\mathbf{x}_0, \tau) \quad \text{etc.} \tag{A 5}$$

Equations (A 3) imply that $\mathbf{x}_0(t, \tau) = \bar{\mathbf{x}}_0(t)$ and $\mathbf{x}_1(t, \tau) = \bar{\mathbf{x}}_1(t)$. We do not need to solve equation (A 4) since $\mathbf{x}_2$ does not appear there. Finally, averaging equation (A 5), we find that

$$\partial_t \bar{\mathbf{x}}_0 = \bar{\mathbf{v}}_1^r(\bar{\mathbf{x}}_0). \tag{A 6}$$

Thus, the Lagrangian velocity (the right side of equation (A 6)) coincides with the Eulerian velocity, $\bar{\mathbf{v}}_1^r(\bar{\mathbf{x}}_0)$.

If we continued the expansion, we would find that there is a non-zero Stokes drift velocity in the next approximation given by

$$\mathbf{v}^s(\bar{\mathbf{x}}_0) = \overline{(\mathbf{w}(\bar{\mathbf{x}}_0(t), \tau) \cdot \nabla)\tilde{\mathbf{v}}_1^r(\bar{\mathbf{x}}_0(t), \tau)},$$

where, by definition, $\mathbf{w}$ is the unique function such that $\partial_\tau \mathbf{w}(\bar{\mathbf{x}}_0(t), \tau) = \tilde{\mathbf{v}}_1^r(\bar{\mathbf{x}}_0(t), \tau)$ and $\bar{\mathbf{w}} = 0$. The last formula represents the standard expression for the Stokes drift velocity (see, e.g. [9]).

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