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COMMUTING VARIETIES FOR NILPOTENT RADICALS

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Abstract Let U be the unipotent radical of a Borel subgroup of a connected reductive algebraic group G, which is defined over an algebraically closed field k. In this paper, we extend work by Goodwin and Röhrle concerning the commuting variety of $\mathrm{Lie}(U)$ for $\mathrm{char}(k) = 0$ to fields whose characteristic is good for G.

Keywords: commuting varieties; nilpotent radicals; abelian Lie algebras

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Introduction

Let G be a connected reductive algebraic group, defined over an algebraically closed field k. Given a Borel subgroup $B \subseteq G$ with unipotent radical U, in this paper we investigate two closely related varieties associated with the Lie algebra $\mathfrak{u} := \mathrm{Lie}(U)$: the commuting variety $\mathcal{C}_2(\mathfrak{u})$, given by

$$\mathcal{C}_2(\mathfrak{u}) := \{(x, y) \in \mathfrak{u}\times\mathfrak{u};\ [x, y] = 0\}$$

and the variety

$$\mathbb{A}(2, \mathfrak{u}) := \{\mathfrak{a} \in \mathrm{Gr}_2(\mathfrak{u});\ [\mathfrak{a}, \mathfrak{a}] = (0)\},$$

of two-dimensional abelian subalgebras of $\mathfrak{u}$, which is a closed subset of the Grassmannian $\mathrm{Gr}_2(\mathfrak{u})$ of 2-planes of $\mathfrak{u}$.

For $\mathrm{char}(k) = 0$, the authors proved in [**6**] that $\mathcal{C}_2(\mathfrak{u})$ is equidimensional if and only if the adjoint action of B on $\mathfrak{u}$ affords only finitely many orbits. Being built on methods developed in [**14**, § 2] for $\mathrm{char}(k) = 0$, their arguments do not seem to readily generalize to fields of positive characteristic. In fact, most of Premet's paper [**14**] is devoted to the technically more involved case pertaining to fields of positive characteristic.

The purpose of this note is to extend the main result of [**6**] by employing techniques that work in good characteristics. For arbitrary G, this comprises the case $\mathrm{char}(k) = 0$ as well as $\mathrm{char}(k) \geq 7$. Letting $Z(G)$ and $\mathrm{mod}(B; \mathfrak{u})$ denote the centre of G and the modality of B on $\mathfrak{u}$, respectively, our main result reads as follows.

Theorem. *Suppose that* $\mathrm{char}(k)$ *is good for* G. *Then*

$$\dim \mathcal{C}_2(\mathfrak{u}) = \dim B - \dim Z(G) + \mathrm{mod}(B;\mathfrak{u}).$$

Moreover, $\mathcal{C}_2(\mathfrak{u})$ *is equidimensional if and only if* B *acts on* $\mathfrak{u}$ *with finitely many orbits.*

If $\mathrm{mod}(B;\mathfrak{u}) = 0$, then, by a theorem of Hille–Röhrle [7], the almost simple components of the derived group (G,G) of G are of type $(A_n)_{n\leq 4}$ or B_2. As in [6, 14], the irreducible components are parametrized by the so-called distinguished orbits.

Our interest in $\mathcal{C}_2(\mathfrak{u})$ derives from recent work [2] on the variety $\mathbb{E}(2,\mathfrak{u})$ of two-dimensional elementary abelian p-subalgebras of $\mathfrak{u}$, which coincides with $\mathbb{A}(2,\mathfrak{u})$ whenever $\mathrm{char}(k) \geq h(G)$, the Coxeter number of G.

Corollary. *Suppose that* $\mathrm{char}(k)$ *is good for a reductive group* G *of semisimple rank* $\mathrm{rk}_{\mathrm{ss}}(G) \geq 2$. *Then the following statements hold:*

(1) $\dim \mathbb{A}(2,\mathfrak{u}) = \dim B - \dim Z(G) + \mathrm{mod}(B;\mathfrak{u}) - 4$;

(2) $\mathbb{A}(2,\mathfrak{u})$ *is equidimensional if and only if* $\mathrm{mod}(B;\mathfrak{u}) = 0$;

(3) $\mathbb{A}(2,\mathfrak{u})$ *is irreducible if and only if every component of* (G,G) *has type* A_1 *or* A_2.

For the reader's convenience, we begin by collecting a number of subsidiary results in the first two sections, some of which are variants of results in the literature. Throughout this paper, all vector spaces over k are assumed to be finite dimensional.

1. Preliminaries

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k, and let $\mathrm{Aut}(\mathfrak{g})$ be its automorphism group. The commuting variety $\mathcal{C}_2(\mathfrak{g})$ is a conical closed subset of $\mathfrak{g}\times\mathfrak{g}$. Given a variety X, we denote by $\mathrm{Irr}(X)$ the set of irreducible components of X. Thus, each $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{g}))$ is a conical closed subset of the affine space $\mathfrak{g}\times\mathfrak{g}$.

Recall that the group $\mathrm{GL}_2(k)$ acts on the affine space $\mathfrak{g}\times\mathfrak{g}$ via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . (x,y) := (\alpha x + \beta y, \gamma x + \delta y),$$

with $\mathcal{C}_2(\mathfrak{g})$ being a $\mathrm{GL}_2(k)$-stable subset. In particular, the group $k^\times := k\smallsetminus\{0\}$ acts on $\mathcal{C}_2(\mathfrak{g})$ via

$$\alpha.(x,y) := \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} . (x,y) = (x,\alpha y).$$

We denote the two surjective projection maps by

$$\mathrm{pr}_i : \mathcal{C}_2(\mathfrak{g}) \longrightarrow \mathfrak{g} \qquad i \in \{1,2\}.$$

Given $x \in \mathfrak{g}$, we let $C_{\mathfrak{g}}(x)$ be the centralizer of x in $\mathfrak{g}$. Since

$$\mathrm{pr}_1^{-1}(x) = \{x\}\times C_{\mathfrak{g}}(x)$$

for all $x \in \mathfrak{g}$, the surjection $\mathrm{pr}_1 : \mathcal{C}_2(\mathfrak{g}) \longrightarrow \mathfrak{g}$ is a linear fibration $(\mathcal{C}_2(\mathfrak{g}), \mathrm{pr}_1)$ with total space $\mathcal{C}_2(\mathfrak{g})$ and base space $\mathfrak{g}$. For any (not necessarily closed) subvariety $X \subseteq \mathfrak{g}$, we denote by $\mathcal{C}_2(\mathfrak{g})|_X$ the subfibration given by $\mathrm{pr}_1 : \mathrm{pr}_1^{-1}(X) \longrightarrow X$.

Lemma 1.1. *Let $X \subseteq \mathfrak{g}$ be a subvariety. Suppose that $C \subseteq \mathcal{C}_2(\mathfrak{g})|_X$ is a $k^\times$-stable, closed subset. Then $\mathrm{pr}_1(C)$ is a closed subset of X.*

Proof. We consider the morphism

$$\iota : X \longrightarrow \mathcal{C}_2(\mathfrak{g})|_X; \qquad x \mapsto (x, 0).$$

Given $x \in \mathrm{pr}_1(C)$, we find $y \in \mathfrak{g}$ such that $(x, y) \in C$. By assumption, the map

$$f : k \longrightarrow \mathcal{C}_2(\mathfrak{g})|_X; \qquad \alpha \mapsto (x, \alpha y)$$

is a morphism such that $f(k^\times) \subseteq C$. Hence

$$(x, 0) = f(0) \in f(\overline{k^\times}) \subseteq \overline{f(k^\times)} \subseteq C,$$

so that $x \in \iota^{-1}(C)$. As a result, $\mathrm{pr}_1(C) = \iota^{-1}(C)$ is closed in X. □

Lemma 1.2. *Let $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{g}))$. Then the following statements hold:*

(1) $\mathrm{GL}_2(k).C = C$;

(2) *The set $\mathrm{pr}_i(C)$ is closed.*

Proof. (1) This well-known fact follows from $\mathrm{GL}_2(k)$ being connected.

(2) As C is $\mathrm{GL}_2(k)$-stable, Lemma 1.1 ensures that $\mathrm{pr}_1(C)$ is closed. By the same token, the map $(x, y) \mapsto (y, x)$ stabilizes C, so that $\mathrm{pr}_2(C)$ is closed as well. □

We next compute the dimension of $\mathcal{C}_2(\mathfrak{g})$ in terms of a certain invariant, which will be seen to coincide with the modality of certain group actions in our cases of interest.

Given $n \in \mathbb{N}_0$, lower semicontinuity of ranks ensures that

$$\mathfrak{g}_{(n)} := \{x \in \mathfrak{g};\ \mathrm{rk}(\mathrm{ad}\, x) = n\}$$

is a (possibly empty) locally closed subspace of $\mathfrak{g}$. We put $\mathbb{N}_0(\mathfrak{g}) := \{n \in \mathbb{N}_0;\ \mathfrak{g}_{(n)} \neq \emptyset\}$ and define

$$\mathrm{mod}(\mathfrak{g}) := \max_{n \in \mathbb{N}_0(\mathfrak{g})} \dim \mathfrak{g}_{(n)} - n.$$

Our next result elaborates on [6, (2.1)].

Proposition 1.3. *The following statements hold.*

(1) *Let $n \in \mathbb{N}_0(\mathfrak{g})$.*
   (a) *$(\mathcal{C}_2(\mathfrak{g})|_{\mathfrak{g}_{(n)}}, \mathrm{pr}_1)$ is a vector bundle of rank $\dim_k \mathfrak{g} - n$ over $\mathfrak{g}_{(n)}$. In particular, the morphism $\mathrm{pr}_1 : \mathcal{C}_2(\mathfrak{g})|_{\mathfrak{g}_{(n)}} \longrightarrow \mathfrak{g}_{(n)}$ is open;*

   (b) *if $X \in \mathrm{Irr}(\mathfrak{g}_{(n)})$, then $\overline{\mathrm{pr}_1^{-1}(X)} \subseteq \mathcal{C}_2(\mathfrak{g})$ is irreducible of dimension $\dim X + \dim_k \mathfrak{g} - n$.*

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(2) *We have* $\dim \mathcal{C}_2(\mathfrak{g}) = \dim_k \mathfrak{g} + \mathrm{mod}(\mathfrak{g})$.

(3) *If* $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{g}))$, *then*

$$\dim C = \dim \mathrm{pr}_1(C) + \dim_k \mathfrak{g} - n_C,$$

where $n_C := \max\{n \in \mathbb{N}_0;\ \mathfrak{g}_{(n)} \cap \mathrm{pr}_1(C) \neq \emptyset\}$.

(4) *Let* $X \in \mathrm{Irr}(\mathfrak{g}_{(n)})$ *be such that* $\overline{\mathrm{pr}_1^{-1}(X)} \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{g}))$. *Then we have*

$$C_{\mathfrak{g}}(x) \subseteq \overline{X} \subseteq \overline{\mathfrak{g}_{(n)}} \subseteq \bigsqcup_{m \leq n} \mathfrak{g}_{(m)} \quad \text{for all } x \in X.$$

(5) *If* $n \in \mathbb{N}_0$ *is such that* $\mathrm{mod}(\mathfrak{g}) = \dim \mathfrak{g}_{(n)} - n$, *then* $\overline{\mathrm{pr}_1^{-1}(X)} \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{g}))$ *for every* $X \in \mathrm{Irr}(\mathfrak{g}_{(n)})$ *such that* $\dim X = \dim \mathfrak{g}_{(n)}$.

Proof. (1a) If V, W are k-vector spaces and $\mathrm{Hom}_k(V,W)_{(n)} := \{f \in \mathrm{Hom}_k(V,W);\ \mathrm{rk}(f) = n\}$, then the map

$$\mathrm{Hom}_k(V,W)_{(n)} \longrightarrow \mathrm{Gr}_{\dim_k V - n}(V); \qquad f \mapsto \ker f$$

is a morphism. Consequently,

$$C_{\mathfrak{g}} : \mathfrak{g}_{(n)} \longrightarrow \mathrm{Gr}_{\dim_k \mathfrak{g} - n}(\mathfrak{g}); \qquad x \mapsto C_{\mathfrak{g}}(x)$$

is a morphism as well and general theory implies that

$$E_{C_{\mathfrak{g}}} := \{(x,y) \in \mathfrak{g}_{(n)} \times \mathfrak{g};\ y \in C_{\mathfrak{g}}(x)\}$$

is a vector bundle of rank $\dim_k \mathfrak{g} - n$ over $\mathfrak{g}_{(n)}$, which coincides with $\mathcal{C}_2(\mathfrak{g})|_{\mathfrak{g}_{(n)}}$, see [**15**, (VI.1.2)].

(1b) Given an irreducible component $X \in \mathrm{Irr}(\mathfrak{g}_{(n)})$, we consider the subbundle $\mathcal{C}_2(\mathfrak{g})|_X = \mathcal{C}_2(\mathfrak{g}) \cap (X \times \mathfrak{g})$ together with its surjection $\mathrm{pr}_1 : \mathcal{C}_2(\mathfrak{g})|_X \longrightarrow X$.

Let $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{g})|_X)$ be an irreducible component. Since $\mathcal{C}_2(\mathfrak{g})|_X$ is $k^\times$-stable, so is C. In view of Lemma 1.1, we conclude that $\mathrm{pr}_1(C)$ is closed in X. It now follows from [**3**, (1.5)] that the variety $\mathrm{pr}_1^{-1}(X)$ is irreducible. Hence its closure enjoys the same property. Consequently,

$$\mathrm{pr}_1 : \overline{\mathrm{pr}_1^{-1}(X)} \longrightarrow \overline{X}$$

is a dominant morphism of irreducible affine varieties such that $\dim \mathrm{pr}_1^{-1}(x) = \dim_k \ker(\mathrm{ad}\, x) = \dim_k \mathfrak{g} - n$ for every $x \in X$. Since X is locally closed, it is an open subset of $\overline{X}$. The fibre dimension theorem thus yields

$$\dim \overline{\mathrm{pr}_1^{-1}(X)} = \dim \overline{X} + \dim_k \mathfrak{g} - n = \dim X + \dim_k \mathfrak{g} - n,$$

as desired.

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(2) We have

$$(*) \qquad \mathcal{C}_2(\mathfrak{g}) = \bigcup_{n\in\mathbb{N}_0(\mathfrak{g})} \bigcup_{X\in\operatorname{Irr}(\mathfrak{g}_{(n)})} \overline{\operatorname{pr}_1^{-1}(X)},$$

whence

$$\begin{aligned}\dim \mathcal{C}_2(\mathfrak{g}) &= \max_{n\in\mathbb{N}_0(\mathfrak{g})} \max_{X\in\operatorname{Irr}(\mathfrak{g}_{(n)})} \dim X + \dim_k \mathfrak{g} - n = \max_{n\in\mathbb{N}_0(\mathfrak{g})} \dim \mathfrak{g}_{(n)} \\ &\quad + \dim_k \mathfrak{g} - n = \dim_k \mathfrak{g} + \operatorname{mod}(\mathfrak{g}),\end{aligned}$$

as asserted.

(3) In view of (1b) and $(*)$, there are $n_C \in \mathbb{N}_0$ and $X_C \in \operatorname{Irr}(\mathfrak{g}_{(n_C)})$ such that

$$C = \overline{\operatorname{pr}_1^{-1}(X_C)}.$$

Since pr_1 is surjective, we have $X_C = \operatorname{pr}_1(\operatorname{pr}_1^{-1}(X_C))$. Consequently, $\operatorname{pr}_1(C) = \operatorname{pr}_1(\overline{\operatorname{pr}_1^{-1}(X_C)}) \subseteq \overline{X_C}$, while $X_C \subseteq \operatorname{pr}_1(C)$ in conjunction with Lemma 1.1 yields $\overline{X_C} \subseteq \operatorname{pr}_1(C)$. Thus, lower semicontinuity of the rank function yields

$$\operatorname{pr}_1(C) \subseteq \overline{\mathfrak{g}_{(n_C)}} \subseteq \bigsqcup_{n\leq n_C} \mathfrak{g}_{(n)},$$

so that $\max\{n \in \mathbb{N}_0;\ \operatorname{pr}_1(C) \cap \mathfrak{g}_{(n)} \neq \emptyset\} \leq n_C$. On the other hand, $\emptyset \neq X_C \subseteq \operatorname{pr}_1(C) \cap \mathfrak{g}_{(n_C)}$ implies $n_C \leq \max\{n \in \mathbb{N}_0;\ \operatorname{pr}_1(C) \cap \mathfrak{g}_{(n)} \neq \emptyset\}$. Hence we have equality and (1b) yields

$$\begin{aligned}\dim C &= \dim X_C + \dim_k \mathfrak{g} - n_C = \dim \overline{X_C} + \dim_k \mathfrak{g} - n_C \\ &= \dim \operatorname{pr}_1(C) + \dim_k \mathfrak{g} - n_C,\end{aligned}$$

as desired.

(4) Let $x \in X$. Then we have $\{x\} \times C_{\mathfrak{g}}(x) = \operatorname{pr}_1^{-1}(x) \subseteq \overline{\operatorname{pr}_1^{-1}(X)}$. By assumption, the latter set is $\operatorname{GL}_2(k)$-stable, so that in particular $C_{\mathfrak{g}}(x) \times \{x\} \subseteq \overline{\operatorname{pr}_1^{-1}(X)}$. It follows that

$$C_{\mathfrak{g}}(x) \subseteq \overline{X} \qquad \forall x \in X.$$

Since $\overline{X} \subseteq \overline{\mathfrak{g}_{(n)}} \subseteq \bigsqcup_{m\leq n} \mathfrak{g}_{(m)}$, our assertion follows.

(5) This follows from (1b) and (2). □

Corollary 1.4. *The following statements hold.*

(1) *The subset* $\overline{\operatorname{pr}_1^{-1}(\mathfrak{g}_{(\max \mathbb{N}_0(\mathfrak{g}))})}$ *is an irreducible component of* $\mathcal{C}_2(\mathfrak{g})$ *of dimension* $2\dim_k \mathfrak{g} - \max \mathbb{N}_0(\mathfrak{g})$.

(2) *Suppose that* $\mathcal{C}_2(\mathfrak{g})$ *is equidimensional. Then we have* $\operatorname{mod}(\mathfrak{g}) = \dim_k \mathfrak{g} - \max \mathbb{N}_0(\mathfrak{g})$.

(3) *Suppose that* $\mathcal{C}_2(\mathfrak{g})$ *is irreducible. Then we have* $\dim \mathfrak{g}_{(n)} - n = \operatorname{mod}(\mathfrak{g})$ *if and only if* $n = \max \mathbb{N}_0(\mathfrak{g})$.

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Proof. (1) Let $n_0 := \max \mathbb{N}_0(\mathfrak{g})$. By lower semicontinuity of the function $x \mapsto \operatorname{rk}(\operatorname{ad} x)$, $\mathfrak{g}_{(n_0)}$ is an open, and hence irreducible and dense, subset of $\mathfrak{g}$. Hence $\operatorname{pr}_1^{-1}(\mathfrak{g}_{(n_0)})$ is open in $\mathcal{C}_2(\mathfrak{g})$, and Proposition 1.3 shows that $C_{(n_0)} := \overline{\operatorname{pr}_1^{-1}(\mathfrak{g}_{(n_0)})}$ is irreducible of dimension $\dim \mathfrak{g}_{(n_0)} + \dim_k \mathfrak{g} - n_0 = 2\dim_k \mathfrak{g} - n_0$. Let $C \in \operatorname{Irr}(\mathcal{C}_2(\mathfrak{g}))$ be such that $C_{n_0} \subseteq C$. Then $\operatorname{pr}_1^{-1}(\mathfrak{g}_{(n_0)})$ is a non-empty open subset of C, so that $C_{n_0} = C \in \operatorname{Irr}(\mathcal{C}_2(\mathfrak{g}))$.

(2) This follows directly from (1) and Proposition 1.3(2).

(3) Suppose that $n \in \mathbb{N}_0(\mathfrak{g})$ is such that $\operatorname{mod}(\mathfrak{g}) = \dim \mathfrak{g}_{(n)} - n$. Let $X \in \operatorname{Irr}(\mathfrak{g}_{(n)})$ be an irreducible component such that $\dim X = \dim \mathfrak{g}_{(n)}$. Thanks to Proposition 1.3(5), $C_X := \overline{\operatorname{pr}_1^{-1}(X)}$ is an irreducible component of $\mathcal{C}_2(\mathfrak{g})$, so that $C_X = \mathcal{C}_2(\mathfrak{g})$. Consequently,

$$\mathfrak{g} = \operatorname{pr}_1(\mathcal{C}_2(\mathfrak{g})) = \operatorname{pr}_1(C_X) \subseteq \overline{X} \subseteq \bigcup_{m \leq n} \mathfrak{g}_{(m)},$$

so that $\max \mathbb{N}_0(\mathfrak{g}) \leq n$. Hence we have equality. □

In general, the value of $\operatorname{mod}(\mathfrak{g})$ is hard to compute. For certain Lie algebras of algebraic groups and for those having suitable filtrations, the situation is somewhat better.

Example. Let $\operatorname{char}(k) = p \geq 5$ and consider the p-dimensional Witt algebra $W(1) := \operatorname{Der}_k(k[X]/(X^p))$, see [18, (IV.2)] for more details. This simple Lie algebra affords a canonical descending filtration

$$W(1) = W(1)_{-1} \supsetneq W(1)_0 \supsetneq \cdots \supsetneq W(1)_{p-2} \supsetneq (0),$$

where $\dim_k W(1)_i = p - 1 - i$. By way of illustration, we shall verify the following statements.

(1) The variety $\mathcal{C}_2(W(1))$ has dimension $p+1$ and is not equidimensional, with

$$\operatorname{Irr}(\mathcal{C}_2(W(1))) = \left\{\overline{\operatorname{pr}_1^{-1}(W(1)_{(\ell)})};\ \frac{p+1}{2} \leq \ell \leq p-1\right\}.$$

(2) Let $\mathfrak{b} := W(1)_0$. The variety $\mathcal{C}_2(\mathfrak{b})$ has pure dimension p, with

$$\operatorname{Irr}(\mathcal{C}_2(\mathfrak{b})) = \left\{\overline{\operatorname{pr}_1^{-1}(\mathfrak{b}_{(\ell)})};\ \frac{p-1}{2} \leq \ell \leq p-2\right\}.$$

(3) (cf. [20, (4.3)]) Let $\mathfrak{u} := W(1)_1$. The variety $\mathcal{C}_2(\mathfrak{u})$ has pure dimension p, with

$$\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u})) = \left\{\overline{\operatorname{pr}_1^{-1}(\mathfrak{u}_{(\ell)})};\ \frac{p-3}{2} \leq \ell \leq p-4\right\}.$$

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(4) (cf. [**20**, (3.6)]) Let $\mathcal{N} := \{x \in W(1);\ (\operatorname{ad} x)^p = 0\}$ be the p-nilpotent cone of $W(1)$. The variety $\mathcal{C}_2(\mathcal{N}) := \mathcal{C}_2(W(1)) \cap (\mathcal{N}\times\mathcal{N})$ has pure dimension p, with

$$\operatorname{Irr}(\mathcal{C}_2(\mathcal{N})) = \left\{\overline{\operatorname{pr}_1^{-1}(W(1)_{(\ell)})};\ \ell \in \left\{\frac{p+1}{2}, \ldots, p-2\right\}\right\} \cup \{\overline{\operatorname{pr}_1^{-1}(W(1)_{(p-1)} \cap \mathcal{N})}\}.$$

Proof. (1) Let $x \in W(1)\smallsetminus\{0\}$ and consider the Jordan–Chevalley–Seligman decomposition $x = x_s + x_n$, with x_s semisimple, x_n p-nilpotent and $[x_s, x_n] = 0$ (cf. [**18**, (II.3.5)]). Since every maximal torus $\mathfrak{t} \subseteq W(1)$ is one-dimensional and self-centralizing, the assumption $x_s \neq 0$ entails $x_n \in C_{W(1)}(x_s) = kx_s$, so that $x_n = 0$. As a result, every $x \in W(1)\smallsetminus\{0\}$ is either p-nilpotent or semisimple, and [**20**, (2.3)] implies

$$\ker(\operatorname{ad} x) = \begin{cases} W(1)_{p-1-i}, & x \in W(1)_i \smallsetminus W(1)_{i+1},\ \dfrac{p-1}{2} \le i \le p-2 \\ kx\oplus W(1)_{p-1-i}, & x \in W(1)_i \smallsetminus W(1)_{i+1},\ 1 \le i \le \dfrac{p-3}{2} \\ kx, & x \in W(1)\smallsetminus W(1)_1. \end{cases}$$

This in turn yields

$$W(1)_{(\ell)} = \begin{cases} W(1)_{p-\ell}\smallsetminus W(1)_{p-\ell+1}, & 2 \le \ell \le \dfrac{p-1}{2} \\ W(1)_{(p-3)/2}\smallsetminus W(1)_{(p+1)/2}, & \ell = \dfrac{p+1}{2} \\ W(1)_{p-\ell-1}\smallsetminus W(1)_{p-\ell}, & \dfrac{p+3}{2} \le \ell \le p-2 \\ W(1)\smallsetminus W(1)_1, & \ell = p-1 \\ \{0\}, & \ell = 0 \\ \emptyset & \text{else.} \end{cases}$$

We thus have $\operatorname{mod}(W(1)) = 1$, so that $\dim \mathcal{C}_2(W(1)) = p+1$. Moreover, each of the varieties $W(1)_{(\ell)}$ is irreducible, with $\overline{W(1)_{(\ell)}} = W(1)_{p-\ell}$ for $2 \le \ell \le (p-1)/2$. Proposition 1.3(4) in conjunction with the above now shows that $\overline{\operatorname{pr}_1^{-1}(W(1)_{(\ell)})} \notin \operatorname{Irr}(\mathcal{C}_2(W(1)))$ for $2 \le \ell \le (p-1)/2$. Consequently,

$$(*) \qquad \mathcal{C}_2(W(1)) = \bigcup_{((p+1)/2)\le\ell\le p-1} \overline{\operatorname{pr}_1^{-1}(W(1)_{(\ell)})}.$$

According to Corollary 1.4,

$$\overline{\operatorname{pr}_1^{-1}(W(1)_{(p-1)})} = \overline{\bigcup_{x\in W(1)\smallsetminus W(1)_1} \{x\}\times kx} \subseteq \{(x,y) \in \mathcal{C}_2(W(1));\ \dim_k kx + ky \le 1\}$$

is an irreducible component of dimension $p+1$. Let $\ell \in \{((p+1)/2), \ldots, p-2\}$. Given $x \in W(1)_{(\ell)}$, it thus follows that

$$\{x\}\times C_{W(1)}(x) \subseteq \overline{\operatorname{pr}_1^{-1}(W(1)_{(\ell)})} \ \text{ while } \ \{x\}\times C_{W(1)}(x) \not\subseteq \overline{\operatorname{pr}_1^{-1}(W(1)_{(p-1)})},$$

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whence

$$\overline{\mathrm{pr}_1^{-1}(W(1)_{(\ell)})} \not\subseteq \overline{\mathrm{pr}_1^{-1}(W(1)_{(p-1)})}.$$

Thanks to Proposition 1.3(3) we have

$$\dim \overline{\mathrm{pr}_1^{-1}(W(1)_{(\ell)})} = \dim_k W(1)_{p-\ell-1} + \dim_k W(1) - \ell = p,$$

so that there are no containments among the irreducible sets $(\overline{\mathrm{pr}_1^{-1}(W(1)_{(\ell)})})_{((p+1)/2)\leq \ell \leq p-2}$. As a result, $(*)$ is the decomposition of $\mathcal{C}_2(W(1))$ into its irreducible components.

(2) We now consider the 'Borel subalgebra' $\mathfrak{b} := W(1)_0$ of dimension $p-1$. Writing $W(1) = ke_{-1} \oplus \mathfrak{b}$ with $C_{W(1)}(e_{-1}) = ke_{-1}$, we have $(\mathrm{ad}\, x)(W(1)) = k[x, e_{-1}] \oplus (\mathrm{ad}\, x)(\mathfrak{b})$ for all $x \in \mathfrak{b}$, whence $\mathfrak{b}_{(\ell)} = W(1)_{(\ell+1)}$ for $1 \leq \ell \leq p-3$, while $\mathfrak{b}_{(p-2)} = \mathfrak{b} \smallsetminus W(1)_1$. Consequently,

$$\dim \mathfrak{b}_{(\ell)} = \begin{cases} \ell, & 1 \leq \ell \leq \dfrac{p-3}{2}, \\ \ell + 1, & \dfrac{p-1}{2} \leq \ell \leq p-2, \\ 0, & \ell = 0, \\ -1 & \text{else}, \end{cases}$$

where we put $\dim \emptyset = -1$. Thus, $\mathrm{mod}(\mathfrak{b}) = 1$ and $\dim \mathcal{C}_2(\mathfrak{b}) = p$. The arguments above show that $\overline{\mathrm{pr}_1^{-1}(\mathfrak{b}_{(\ell)})} \not\in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{b}))$, whenever $1 \leq \ell \leq ((p-3)/2)$. In view of the irreducibility of $\mathfrak{b}_{(\ell)}$, Proposition 1.3(5) shows that $\overline{\mathrm{pr}_1^{-1}(\mathfrak{b}_{(\ell)})}$ is an irreducible component of dimension p for $\ell \in \{((p-1)/2), \ldots, p-2\}$.

(3) We next consider $\mathfrak{u} := W(1)_1$ and observe that $\mathfrak{u}_{(\ell)} = \mathfrak{b}_{(\ell+1)} \cap \mathfrak{u}$ for $0 \leq \ell \leq p-3$. Consequently,

$$\dim \mathfrak{u}_{(\ell)} = \begin{cases} \ell + 1, & 0 \leq \ell \leq \dfrac{p-5}{2}, \\ \ell + 2, & \dfrac{p-3}{2} \leq \ell \leq p-4, \\ -1 & \text{else}, \end{cases}$$

so that $\mathrm{mod}(\mathfrak{u}) = 2$ and $\dim \mathcal{C}_2(\mathfrak{u}) = p$. The remaining assertions follow as in (2).

(4) In view of [20, (2.3)], we have $C_{\mathfrak{g}}(x) \subseteq \mathcal{N}$ for all $x \in \mathcal{N} \smallsetminus \{0\}$. This implies

$$\mathcal{C}_2(\mathcal{N}) = \bigcup_{2 \leq \ell \leq p-1} \overline{\mathrm{pr}_1^{-1}(W(1)_{(\ell)} \cap \mathcal{N})} = \bigcup_{2 \leq \ell \leq p-2} \overline{\mathrm{pr}_1^{-1}(W(1)_{(\ell)})} \cup \overline{\mathrm{pr}_1^{-1}(W(1)_{(p-1)} \cap \mathcal{N})}.$$

By the arguments above, we have $\mathrm{pr}_1^{-1}(W(1)_{(\ell)}) \subseteq \bigcup_{((p+1)/2) \leq n \leq p-2} \overline{\mathrm{pr}_1^{-1}(W(1)_{(n)})}$ for $\ell \in \{1, \ldots, ((p-1)/2)\}$, so that

$$\mathcal{C}_2(\mathcal{N}) = \bigcup_{((p+1)/2) \leq \ell \leq p-2} \overline{\mathrm{pr}_1^{-1}(W(1)_{(\ell)})} \cup \overline{(\mathrm{pr}_1^{-1}(W(1)_{(p-1)} \cap \mathcal{N})}.$$

By work of Premet [13], the variety $\mathcal{N}$ is irreducible of dimension $\dim \mathcal{N} = p-1$. It follows that the dense open subset $W(1)_{(p-1)} \cap \mathcal{N}$ is irreducible as well. Lemma 1.1 implies that

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$\mathrm{pr}_1(C)$ is closed in $W(1)_{(p-1)} \cap \mathcal{N}$ for every $C \in \mathrm{Irr}(\mathcal{C}_2(\mathcal{N})|_{W(1)_{(p-1)}\cap\mathcal{N}})$. Using [3, (1.5)], we conclude that the variety

$$\mathrm{pr}_1^{-1}(W(1)_{(p-1)} \cap \mathcal{N}) = \mathcal{C}_2(\mathcal{N})|_{W(1)_{(p-1)}\cap\mathcal{N}}$$

is irreducible of dimension p. □

Remarks.

(1) In [11, (Theorem 5)], P. Levy has shown that commuting varieties of Lie algebras of reductive algebraic groups are irreducible, provided the characteristic of k is good for $\mathfrak{g}$. For $p = 3$, we have $W(1) \cong \mathfrak{sl}(2)$, so that $\mathcal{C}_2(W(1))$ is in fact irreducible. Our example above shows that commuting varieties of Lie algebras, all whose maximal tori are self-centralizing, may not even be equidimensional. In contrast to $W(1)$, the Borel subalgebra $\mathfrak{b} \subseteq W(1)$, whose maximal tori are also self-centralizing, is an algebraic Lie algebra.

(2) A consecutive application of (4) and [2, (2.5.1), (2.5.2)] implies that the variety $\mathbb{E}(2, W(1))$ of two-dimensional elementary abelian subalgebras of $W(1)$ has pure dimension $p-4$ as well as $|\,\mathrm{Irr}(\mathbb{E}(2, W(1)))| = (p-3)/2$.

2. Algebraic Lie algebras

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of a connected algebraic group G. The adjoint representation

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$$

induces an action

$$g.(x, y) := (\mathrm{Ad}(g)(x), \mathrm{Ad}(g)(y))$$

of G on the commuting variety $\mathcal{C}_2(\mathfrak{g})$ such that the surjections

$$\mathrm{pr}_i : \mathcal{C}_2(\mathfrak{g}) \longrightarrow \mathfrak{g}$$

are G-equivariant. In the sequel, we will often write $g.x := \mathrm{Ad}(g)(x)$ for $g \in G$ and $x \in \mathfrak{g}$.

Let $T \subseteq G$ be a maximal torus with character group $X(T)$,

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha\in R_T} \mathfrak{g}_\alpha$$

be the root space decomposition of $\mathfrak{g}$ relative to T. Here $R_T \subseteq X(T)\smallsetminus\{0\}$ is the set of roots of G relative to T, while $\mathfrak{g}^T := \{x \in \mathfrak{g};\ t.x = x \ \ \forall t \in T\}$ denotes the subalgebra of points of $\mathfrak{g}$ that are fixed by T. Given $x = x_0 + \sum_{\alpha\in R_T} x_\alpha \in \mathfrak{g}$, we let

$$\mathrm{supp}(x) := \{\alpha \in R_T;\ x_\alpha \neq 0\}$$

be the *support* of x. For any subset $S \subseteq X(T)$, we denote by $\mathbb{Z}S$ the subgroup of $X(T)$ generated by S. The group $\mathbb{Z}R_T$ is the called the *root lattice* of G relative to T.

If $H \subseteq G$ is a closed subgroup and $x \in \mathfrak{g}$, then $C_H(x) := \{h \in H;\ h.x = x\}$ is the centralizer of x in H.

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2.1. Centralizers, supports and components

Lemma 2.1.1. *Let $T \subseteq G$ be a maximal torus, $x \in \mathfrak{g}$. Then we have*

$$\dim C_T(x) = \dim T - \mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x)).$$

Proof. Writing

$$x = \sum_{\alpha \in R_T \cup \{0\}} x_\alpha,$$

we see that $C_T(x) = \bigcap_{\alpha \in \mathrm{supp}(x)} \ker \alpha = \bigcap_{\alpha \in \mathbb{Z}\,\mathrm{supp}(x)} \ker \alpha$. Since T is a torus, its coordinate ring $k[T]$ is the group algebra $kX(T)$ of $X(T) \subseteq k[T]^\times$. By the above, the centralizer $C_T(x)$ coincides with the zero locus $Z(\{\alpha - 1;\ \alpha \in \mathbb{Z}\,\mathrm{supp}(x)\})$. Thus, letting $(k\mathbb{Z}\,\mathrm{supp}(x))^\dagger$ denote the augmentation ideal of $k\mathbb{Z}\,\mathrm{supp}(x)$, we obtain the ensuing equalities of Krull dimensions

$$\begin{aligned}\dim k[C_T(x)] &= \dim k[T]/k[T]\{\alpha - 1;\ \alpha \in \mathbb{Z}\,\mathrm{supp}(x)\}\\ &= \dim kX(T)/kX(T)(k\mathbb{Z}\,\mathrm{supp}(x))^\dagger\\ &= \dim k(X(T)/\mathbb{Z}\,\mathrm{supp}(x)),\end{aligned}$$

so that [17, (3.2.7)] yields

$$\dim C_T(x) = \dim k[C_T(x)] = \mathrm{rk}(X(T)/\mathbb{Z}\,\mathrm{supp}(x)) = \dim T - \mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x)),$$

as desired. □

Let $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie algebra of a connected algebraic group G, and let $\mathfrak{n} \subseteq \mathfrak{g}$ be a G-stable subalgebra. Then $\mathcal{C}_2(\mathfrak{n}) \subseteq \mathcal{C}_2(\mathfrak{g})$ is a closed, G-stable subset. For $x \in \mathfrak{n}$, we define

$$\mathfrak{C}(x) := \overline{G.(\{x\} \times C_{\mathfrak{n}}(x))} \subseteq \mathcal{C}_2(\mathfrak{n}).$$

Then $\mathfrak{C}(x) = \overline{\mathrm{pr}_1^{-1}(G.x)}$ is a closed irreducible subset of $\mathcal{C}_2(\mathfrak{n})$ such that $\mathfrak{C}(x) = \mathfrak{C}(g.x)$ for all $g \in G$.

It will be convenient to have the following three basic observations at our disposal.

Lemma 2.1.2. *Let $\mathfrak{x}, \mathfrak{y} : k \longrightarrow \mathfrak{n}$ be morphisms, and let $\mathcal{O} \subseteq k$ be a non-empty open subset such that*

(a) $[\mathfrak{x}(\alpha), \mathfrak{y}(\alpha)] = 0$ *for all $\alpha \in k$ and*

(b) $\mathfrak{x}(\alpha) \in G.\mathfrak{x}(1)$ *for all $\alpha \in \mathcal{O}$.*

Then we have $(\mathfrak{x}(0), \mathfrak{y}(0)) \in \mathfrak{C}(\mathfrak{x}(1))$.

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Proof. In view of (a), there is a morphism

$$\varphi : k \longrightarrow \mathcal{C}_2(\mathfrak{n}); \qquad \alpha \mapsto (\mathfrak{x}(\alpha), \mathfrak{y}(\alpha)).$$

Let $\alpha \in \mathcal{O}$. Then (b) provides $g \in G$ such that $\mathfrak{x}(\alpha) = g.\mathfrak{x}(1)$. Thus,

$$\varphi(\alpha) = g.(\mathfrak{x}(1), g^{-1}.\mathfrak{y}(\alpha)) \in \mathfrak{C}(\mathfrak{x}(1)) \qquad \forall \alpha \in \mathcal{O},$$

so that

$$(\mathfrak{x}(0), \mathfrak{y}(0)) = \varphi(0) \in \varphi(\overline{\mathcal{O}}) \subseteq \overline{\varphi(\mathcal{O})} \subseteq \mathfrak{C}(\mathfrak{x}(1)),$$

as desired. □

Lemma 2.1.3. *Let $T \subseteq G$ be a maximal torus, $x \in \mathfrak{n}$. Suppose that $c \in \mathfrak{n} \cap \mathfrak{g}_{\alpha_0}$ (for some $\alpha_0 \in R_T$) is such that*

(a) $\mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x+c)) > \mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x))$ *and*

(b) $k[c, x] = [c, C_{\mathfrak{n}}(x)]$.

Then $\mathfrak{C}(x) \subseteq \mathfrak{C}(x+c)$.

Proof. Note that

$$x + \alpha_0(t)c = t.(x+c) \in G.(x+c) \qquad \forall t \in C_T(x).$$

In view of Lemma 2.1.1, condition (a) ensures that $\dim C_T(x+c)^\circ < \dim C_T(x)^\circ$, so that $\dim \overline{\mathrm{im}\,\alpha_0(C_T(x)^\circ)} = 1$. Chevalley's theorem (cf. [**12**, (I.§ 8)]) thus provides a dense open subset $\mathcal{O} \subseteq k$ such that $\mathcal{O} \subseteq \alpha_0(C_T(x)^\circ)$. As a result,

$$(*) \qquad x + \lambda c \in G.(x+c) \qquad \text{for all } \lambda \in \mathcal{O}.$$

Condition (b) provides a linear form $\eta \in C_{\mathfrak{n}}(x)^*$ such that

$$[y, c] = \eta(y)[x, c] \qquad \forall y \in C_{\mathfrak{n}}(x).$$

Given $y \in C_{\mathfrak{n}}(x)$, we define morphisms $\mathfrak{x}, \mathfrak{y} : k \longrightarrow \mathfrak{n}$ via

$$\mathfrak{x}(\alpha) = x + \alpha c \quad \text{and} \quad \mathfrak{y}(\alpha) := \begin{cases} y + \eta(x)^{-1}\eta(y)\alpha c, & \eta(x) \neq 0, \\ y, & \eta(x) = 0. \end{cases}$$

In view of $(*)$, we may apply Lemma 2.1.2 to obtain

$$(x, y) = (\mathfrak{x}(0), \mathfrak{y}(0)) \in \mathfrak{C}(x+c).$$

As a result, $\{x\} \times C_{\mathfrak{n}}(x) \subseteq \mathfrak{C}(x+c)$, whence $\mathfrak{C}(x) \subseteq \mathfrak{C}(x+c)$. □

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Lemma 2.1.4. *Given $x \in \mathfrak{n}$, let $\mathfrak{v} \subseteq \mathfrak{n}$ be a G-submodule such that $G.x \subseteq \mathfrak{v}$. Then the following statements hold:*

(1) *if $\mathfrak{C}(x) \in \operatorname{Irr}(\mathcal{C}_2(\mathfrak{n}))$, then $C_{\mathfrak{n}}(x) \subseteq \mathfrak{v}$;*

(2) *if $C_{\mathfrak{n}}(\mathfrak{v}) \not\subseteq \mathfrak{v}$, then $\mathfrak{C}(x) \notin \operatorname{Irr}(\mathcal{C}_2(\mathfrak{n}))$.*

Proof. (1) Since the component $\mathfrak{C}(x)$ is $\mathrm{GL}_2(k)$-stable, we have $C_{\mathfrak{n}}(x)\times\{x\} \subseteq \mathfrak{C}(x)$. Thus,

$$C_{\mathfrak{n}}(x) \subseteq \mathrm{pr}_1(\mathfrak{C}(x)) \subseteq \overline{G.x} \subseteq \mathfrak{v}.$$

(2) Let $y \in C_{\mathfrak{n}}(\mathfrak{v})\smallsetminus\mathfrak{v}$. Since $x \in \mathfrak{v}$, we have $y \in C_{\mathfrak{n}}(x)\smallsetminus\mathfrak{v}$, and our assertion follows from (1). □

2.2. Distinguished elements

Let $\mathfrak{g} = \operatorname{Lie}(G)$ be the Lie algebra of a connected algebraic group G. In the following, we denote by $T(G)$ the maximal torus of $Z(G)$. Note that $T(G)$ is contained in any maximal torus $T \subseteq G$.

An element $x \in \mathfrak{g}$ is *distinguished* (*for* G) provided every torus $T \subseteq C_G(x)$ is contained in $T(G)$. If x is distinguished, so is every element of $G.x$. In that case, we say that $G.x$ is a *distinguished orbit*.

Lemma 2.2.1. *Let $x \in \mathfrak{g}$. Then x is distinguished if and only if $C_T(x)^\circ = T(G)$ for every maximal torus $T \subseteq G$.*

Proof. Suppose that x is distinguished. If $T \subseteq G$ is a maximal torus, then $C_T(x)^\circ \subseteq C_G(x)$ is a torus, so that $C_T(x)^\circ \subseteq T(G)$. On the other hand, we have $T(G) \subseteq T$, whence $T(G) \subseteq C_T(x)^\circ$.

For the reverse direction, we let $T' \subseteq C_G(x)$ be a torus. Then there is a maximal torus $T \supseteq T'$ of G, so that

$$T' \subseteq C_T(x)^\circ = T(G).$$

Hence x is distinguished. □

Lemma 2.2.2. *Let $B \subseteq G$ be a Borel subgroup with unipotent radical U. We write $\mathfrak{b} := \operatorname{Lie}(B)$ and $\mathfrak{u} := \operatorname{Lie}(U)$.*

(1) *If $x \in \mathfrak{b}$ is distinguished for G, then it is distinguished for B.*

(2) *If $\mathcal{O} \subseteq \mathfrak{g}$ is a distinguished G-orbit, then $\mathcal{O} \cap \mathfrak{u}$ consists of distinguished elements for B.*

Proof. (1) Since B is a Borel subgroup, [**17**, (6.2.9)] yields $Z(G)^\circ = Z(B)^\circ$, whence $T(G) = T(B)$. Let $T' \subseteq C_B(x)$ be a torus. Since x is distinguished for G, we obtain $T' \subseteq T(G) = T(B)$, so that x is also distinguished for B.

(2) This follows directly from (1). □

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Lemma 2.2.3. *Let G be a connected algebraic group with maximal torus T such that $Z(G) = \bigcap_{\alpha \in R_T} \ker \alpha$.*

(1) *If $x \in \mathfrak{g}$ is distinguished, then $\mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x)) = \mathrm{rk}(\mathbb{Z}R_T)$.*

(2) *If $(T \cap C_G(x))^\circ$ is a maximal torus of $C_G(x)$ and $\mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x)) = \mathrm{rk}(\mathbb{Z}R_T)$, then x is distinguished.*

Proof. Let $\hat{x} \in \mathfrak{g}$ be an element such that $\mathrm{supp}(\hat{x}) = R_T$. By assumption, we have $Z(G) = C_T(\hat{x})$, and Lemma 2.1.1 implies that

$$\dim Z(G) = \dim T - \mathrm{rk}(\mathbb{Z}R_T).$$

By the same token,

$$\dim C_T(x) - \dim Z(G) = \mathrm{rk}(\mathbb{Z}R_T) - \mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x))$$

for every $x \in \mathfrak{g}$.

(1) Let $x \in \mathfrak{g}$ be distinguished. Observing $Z(G) \subseteq T$, we have $Z(G)^\circ = C_T(x)^\circ$. Hence $\mathrm{rk}(\mathbb{Z}R_T) = \mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x))$.

(2) We put $\hat{T} := (T \cap C_G(x))^\circ$. Since $\hat{T} \subseteq C_T(x)^\circ$, we obtain $\hat{T} = C_T(x)^\circ$. Hence $\mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x)) = \mathrm{rk}(\mathbb{Z}R_T)$ yields $\hat{T} = Z(G)^\circ$, so that $Z(G)^\circ$ is a maximal torus of $C_G(x)$. As a result, the element x is distinguished. □

Recall that the semisimple rank $\mathrm{rk}_{\mathrm{ss}}(G)$ of a reductive group G coincides with the rank of its derived group (G, G).

Corollary 2.2.4. *Let $B \subseteq G$ be a Borel subgroup of a reductive group G, and let $T \subseteq B$ be a maximal torus. If $x \in \mathfrak{b}$ is distinguished for B, then*

$$\mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x)) = \mathrm{rk}_{\mathrm{ss}}(G).$$

Proof. Let $T \subseteq B$ be a maximal torus. Then T is a maximal torus for G such that $Z(G) = \bigcap_{\alpha \in R_T} \ker \alpha$, cf. [**8**, (§ 26, Ex. 4)]. In view of [**17**, (6.2.9)], we have $\dim Z(G)^\circ = \dim Z(B)^\circ$. Lemma 2.1.1 implies that

$$\begin{aligned}\dim C_T(x) - \dim Z(B) = \dim C_T(x) - \dim Z(G) &= \mathrm{rk}(\mathbb{Z}R_T) - \mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x)) \\ &= \mathrm{rk}_{\mathrm{ss}}(G) - \mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x))\end{aligned}$$

for every $x \in \mathfrak{b}$, cf. [**9**, (II.1.6)].

Let $x \in \mathfrak{b}$ be distinguished for B. Then $Z(B)^\circ \subseteq T$ is a maximal torus of $C_B(x)$ and $Z(B)^\circ \subseteq C_T(x) \subseteq C_B(x)$. Thus, $C_T(x)^\circ = Z(B)^\circ$, and the identity above yields $\mathrm{rk}(\mathbb{Z}\,\mathrm{supp}(x)) = \mathrm{rk}_{\mathrm{ss}}(G)$. □

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2.3. Modality

Let G be a connected algebraic group acting on an algebraic variety X. Given $i \in \mathbb{N}_0$, we put

$$X_{[i]} := \{x \in X;\ \dim G.x = i\}.$$

Since $X_{[i]} = \emptyset$ whenever $i > \dim X$, the set $\mathbb{N}_0(X) := \{i \in \mathbb{N}_0;\ X_{[i]} \neq \emptyset\}$ is finite.

The set $X_{[i]}$ is locally closed and G-stable. If $x \in X_{[i]}$, then $G.x$ is closed in $X_{[i]}$. Suppose that G acts on X. Then

$$\mathrm{mod}(G; X) := \max_{i \in \mathbb{N}_0(X)} \dim X_{[i]} - i$$

is called the *modality of G on X*.

For ease of reference, we record the following well-known fact.

Lemma 2.3.1. *Suppose that the connected algebraic group G acts on X. Then* $\mathrm{mod}(G; X) = 0$ *if and only if G acts on X with finitely many orbits. In this case, $X_{[i]}$ has pure dimension i for every $i \in \mathbb{N}_0(X)$.*

Proposition 2.3.2. *Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$ and such that* $\mathrm{Lie}(C_G(x)) = C_{\mathfrak{g}}(x)$ *for all $x \in \mathfrak{g}$. Then we have*

$$\dim \mathcal{C}_2(\mathfrak{g}) = \dim G + \mathrm{mod}(G; \mathfrak{g}).$$

Proof. Given $x \in \mathfrak{g}$, the identity $\mathrm{Lie}(C_G(x)) = C_{\mathfrak{g}}(x)$ implies that the differential

$$\mathfrak{g} \longrightarrow T_x(G.x); \qquad y \mapsto [y, x]$$

of the orbit map $g \mapsto g.x$ is surjective, cf. [**10**, (2.2)]. In particular, $\mathrm{rk}(\mathrm{ad}\, x) = \dim G.x$, so that

$$\mathfrak{g}_{(n)} = \mathfrak{g}_{[n]}.$$

Hence $\mathrm{mod}(\mathfrak{g}) = \mathrm{mod}(G; \mathfrak{g})$, and our assertion follows from Proposition 1.3(2). □

3. Springer isomorphisms

The technical condition of Proposition 2.3.2 automatically holds in the case where $\mathrm{char}(k) = 0$. In this section, we are concerned with its verification for the unipotent radicals of Borel subgroups for good characteristics of G. Throughout, we assume that G is a connected reductive group. Following [**10**, (2.6)], we say that the characteristic $\mathrm{char}(k)$ is *good for G* provided $\mathrm{char}(k) = 0$ or the prime $p := \mathrm{char}(k) > 0$ is a good prime for G, see *loc. cit.* for more details.

Lemma 3.1. *Let G be semisimple with almost simple factors $G_1, \ldots, G_n$. For $i \in \{1, \ldots, n\}$, we let $B_i = U_i \rtimes T_i$ be a Borel subgroup of G_i with unipotent radical U_i and maximal torus T_i. Then the following statements hold.*

(1) *$B := B_1 \cdots B_n$ is a Borel subgroup of G with unipotent radical $U := U_1 \cdots U_n$ and maximal torus $T := T_1 \cdots T_n$.*

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(2) *The product morphism*

$$\mu_U : \prod_{i=1}^{n} U_i \longrightarrow U; \qquad (u_1, \ldots, u_n) \mapsto u_1 \cdot u_2 \cdots u_n$$

is an isomorphism of algebraic groups.

Proof. We consider the direct product $\hat{G} := \prod_{i=1}^{n} G_i$ along with the multiplication

$$\mu_G : \hat{G} \longrightarrow G; \qquad (g_1, \ldots, g_n) \mapsto g_1 \cdot g_2 \cdots g_n.$$

Since $(G_i, G_j) = e_k$ for $i \neq j$, it follows that μ_G is a surjective homomorphism of algebraic groups, cf. [8, (27.5)].

(1) We put $\hat{B} := \prod_{i=1}^{n} B_i$, $\hat{U} := \prod_{i=1}^{n} U_i$ and $\hat{T} := \prod_{i=1}^{n} T_i$. These three subgroups of $\hat{G}$ are closed and connected. Moreover, they are solvable, unipotent and diagonalizable, respectively. Direct computation shows that $\hat{U}$ is normal in $\hat{B}$, as well as $\hat{B} = \hat{U} \rtimes \hat{T}$.

Let $H \supseteq \hat{B}$ be a connected, closed solvable subgroup of $\hat{G}$. Since the ith projection $\mathrm{pr}_i : \hat{G} \longrightarrow G_i$ is a homomorphism of algebraic groups for $1 \leq i \leq n$, it follows that $H_i := \mathrm{pr}_i(H) \supseteq B_i$ is a closed, connected, solvable subgroup of G_i. Hence $H_i = B_i$, so that

$$H \subseteq \prod_{i=1}^{n} H_i = \hat{B}.$$

As a result, $\hat{B}$ is a Borel subgroup of $\hat{G}$. In view of [8, (21.3C)], $B = \mu_G(\hat{B})$ is a Borel subgroup of G. Similarly, $T = \mu_G(\hat{T})$ is a maximal torus of B. In addition, $B = \mu_G(\hat{B}) = \mu_G(\hat{U} \rtimes \hat{T}) = U \cdot T$. It follows that the unipotent closed normal subgroup $U = \mu_G(\hat{U}) \trianglelefteq B$ is the unipotent radical of B.

(2) According to [8, (27.5)], the product morphism

$$\mu_G : \hat{G} \longrightarrow G$$

has a finite kernel. Since $\hat{G}$ is connected, it follows that $\ker \mu_G \subseteq Z(\hat{G})$, while $\hat{G}$ being semisimple forces $Z(\hat{G})$ to be diagonalizable, cf. [9, (II.1.6)]. As a result, the kernel $\ker \mu_U$ is diagonalizable and unipotent, so that $\ker \mu_U = \{1\}$. Since μ_U is surjective, the map μ_U is a bijective morphism of algebraic varieties.

Note that $\mathrm{Lie}(\hat{U}) = \bigoplus_{i=1}^{n} \mathrm{Lie}(U_i)$ and that the differential $\mathrm{d}(\mu_U) : \mathrm{Lie}(\hat{U}) \longrightarrow \mathrm{Lie}(U)$ is given by

$$(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} x_i.$$

Let $i \neq j$. Since $(T_i, U_j) = \{1\}$, we have $\mathrm{Ad}(t_i)|_{\mathrm{Lie}(U_j)} = \mathrm{id}_{\mathrm{Lie}(U_j)}$, $\forall t_i \in T_i$. Thus, if $(x_1, \ldots, x_n) \in \ker \mathrm{d}(\mu_U)$, then $\mathrm{Ad}(t)(x_i) = x_i$ for all $t \in T$ and $i \in \{1, \ldots, n\}$. Using the root space decomposition of $\mathrm{Lie}(U)$ relative to T, we conclude that $x_i = 0$ for $i \in \{1, \dots, n\}$. As a result, the map $\mathrm{d}(\mu_U)$ is injective. Since μ_U is bijective, we have

$$\dim_k \mathrm{Lie}(U) = \dim_k \mathrm{Lie}(\hat{U}) = \sum_{i=1}^{n} \dim_k \mathrm{Lie}(U_i)$$

so that $\mathrm{d}(\mu_U)$ is an isomorphism. We may now apply [**17**, (5.3.3)] to conclude that μ_U is an isomorphism as well. □

Let $B \subseteq G$ be a Borel subgroup with unipotent radical $U \trianglelefteq B$. A B-equivariant isomorphism

$$\varphi : U \longrightarrow \mathrm{Lie}(U)$$

will be referred to as a *Springer isomorphism for* B.

Springer isomorphisms first appeared in [**16**] in the context of semisimple algebraic groups, providing a homeomorphism between the unipotent variety of a group and the nilpotent variety of its Lie algebra. Our next result extends [**4**, (2.2), (4.2)] to the context of reductive groups.

Proposition 3.2. *Suppose that* $\mathrm{char}(k)$ *is good for* G. *Let* $B \subseteq G$ *be a Borel subgroup with unipotent radical* U *and put* $\mathfrak{u} := \mathrm{Lie}(U)$.

(1) *There is a Springer isomorphism* $\varphi : U \longrightarrow \mathfrak{u}$.

(2) *We have* $\mathrm{Lie}(C_U(x)) = C_{\mathfrak{u}}(x)$ *for every* $x \in \mathfrak{u}$.

Proof. (1) We first assume that G is semisimple, so that $G = G_1 \cdots G_n$, where $G_i \trianglelefteq G$ is almost simple. As before, we put $\hat{G} := \prod_{i=1}^{n} G_i$. Then every Borel subgroup of $\hat{G}$ is of the form $\hat{B} = \prod_{i=1}^{n} B_i$ for some Borel subgroups $B_i \subseteq G_i$. Hence [**8**, (21.3C)] ensures that there exist Borel subgroups $B_i = U_i \rtimes T_i$ of G_i such that $B = B_1 \cdots B_n$ and $U = U_1 \cdots U_n$. We put $\mathfrak{u}_i := \mathrm{Lie}(U_i)$. As noted in [**4**, (2.2)], there are Springer isomorphisms $\varphi_i : U_i \longrightarrow \mathfrak{u}_i$ for $1 \leq i \leq n$.

We define $\hat{B}$ and $\hat{U}$ as in the proof of Lemma 3.1 and consider the product morphisms

$$\mu_B : \hat{B} \longrightarrow B \quad \text{and} \quad \mu_U : \hat{U} \longrightarrow U.$$

Then $\mathrm{Lie}(\hat{U}) = \bigoplus_{i=1}^{n} \mathfrak{u}_i$ and

$$\hat{\varphi} : \hat{U} \longrightarrow \mathrm{Lie}(\hat{U}); \qquad (u_1, \dots, u_n) \mapsto (\varphi_1(u_1), \dots, \varphi_n(u_n))$$

is a $\hat{B}$-equivariant isomorphism of varieties. Lemma 3.1 implies that $\mu_U : \hat{U} \longrightarrow U$ is an isomorphism of algebraic groups such that

$$\mu_U(\hat{b}\hat{u}\hat{b}^{-1}) = \mu_B(\hat{b})\mu_U(\hat{u})\mu_B(\hat{b})^{-1}$$

for all $\hat{b} \in \hat{B}$ and $\hat{u} \in \hat{U}$. Moreover, the differential

$$\mathrm{d}(\mu_U) : \mathrm{Lie}(\hat{U}) \longrightarrow \mathfrak{u}$$

is an isomorphism such that

$$\mathrm{d}(\mu_U)(\mathrm{Ad}\,\hat{b}(x)) = \mathrm{Ad}(\mu_B(\hat{b}))(\mathrm{d}(\mu_U)(x))$$

for all $\hat{b} \in \hat{B}$ and $x \in \mathrm{Lie}(\hat{U})$. Consequently, $\varphi := \mathrm{d}(\mu_U) \circ \hat{\varphi} \circ \mu_U^{-1}$ defines an isomorphism

$$\varphi : U \longrightarrow \mathfrak{u}.$$

For $b = \mu_B(\hat{b}) \in B$ and $u \in U$, we obtain, writing $b.x := \mathrm{Ad}(b)(x)$,

$$\varphi(bub^{-1}) = (\mathrm{d}(\mu_U) \circ \hat{\varphi})(\hat{b}\mu_U^{-1}(u)\hat{b}^{-1}) = \mathrm{d}(\mu_U)(\hat{b}.\hat{\varphi}(\mu_U^{-1}(u))) = b.\varphi(u),$$

as desired.

Now let G be reductive. Then $G' := (G, G)$ is semisimple, while $G = G' \cdot Z(G)^\circ$, with $Z(G)^\circ$ being a torus. Let $B \subseteq G$ be a Borel subgroup. Since $Z(G)^\circ \subseteq B$, we obtain $B = (B \cap G')Z(G)^\circ$, and B being connected implies that $B = (B \cap G')^\circ Z(G)^\circ$. Let $B' \supseteq (B \cap G')^\circ$ be a Borel subgroup of G'. Then $B'Z(G)^\circ$ is a closed, connected, solvable subgroup of G containing B, whence $B = B'Z(G)^\circ$. As a result, $B' \subseteq B \cap G'$, so that $B' = (B \cap G')^\circ$.

Let U be the unipotent radical of B. Since $Z(G)^\circ \twoheadrightarrow G/G'$ is onto, the latter group is diagonalizable, so that the canonical morphism $U \longrightarrow G/G'$ is trivial. As a result, $U \subseteq G'$, whence $U \subseteq (B \cap G')^\circ$. If U' is the unipotent radical of $(B \cap G')^\circ$, then $B = (B \cap G')^\circ Z(G)^\circ$ implies that U' is normal in B, whence $U' \subseteq U$. It follows that U is the unipotent radical of the Borel subgroup $(B \cap G')^\circ$ of G'. The first part of the proof now provides a $(B \cap G')^\circ$-equivariant isomorphism $\varphi : U \longrightarrow \mathfrak{u}$. Since $Z(G)$ acts trivially on both spaces, this map is also B-equivariant.

(2) In view of (1), the arguments of [4, (4.2)] apply. □

4. Commuting varieties of unipotent radicals

Throughout this section, G denotes a connected reductive algebraic group. If B is a Borel subgroup of G with unipotent radical U, then B acts on $\mathfrak{u} := \mathrm{Lie}(U)$ via the adjoint representation. Hence B also acts on the commuting variety $\mathcal{C}_2(\mathfrak{u})$, and for every $x \in \mathfrak{u}$ we consider

$$\mathfrak{C}(x) := \overline{B.(\{x\} \times C_\mathfrak{u}(x))}.$$

As observed earlier, we have

$$\mathfrak{C}(x) = \mathfrak{C}(b.x) \qquad \forall b \in B, x \in \mathfrak{u}.$$

4.1. The dimension formula

Lemma 4.1.1. *Let $B \subseteq G$ be a Borel subgroup with unipotent radical $U \subseteq B$, $x \in \mathfrak{u} := \mathrm{Lie}(U)$.*

(1) *There exists a maximal torus $T \subseteq B$ such that*
 (a) *$C_B(x)^\circ = C_U(x)^\circ \rtimes C_T(x)^\circ$ and*

(b) $\mathfrak{C}(x)$ *is irreducible of dimension*

$$\dim \mathfrak{C}(x) = \dim B - \dim C_T(x)$$

whenever $\operatorname{char}(k)$ *is good for* G.

(2) *If* $\operatorname{char}(k)$ *is good for* G*, then we have*

$$\dim \mathfrak{C}(x) = \dim B - \dim Z(G)$$

if and only if x *is distinguished for* B.

Proof. (1a) Let $T' \subseteq C_B(x)^\circ$ be a maximal torus, and let $T \supseteq T'$ be a maximal torus of B. We write $B = U \rtimes T$ and recall that $U = B_u$ is the set of unipotent elements of B, see [**17**, (6.3.3), (6.3.5)]. Thus, $C_U(x)^\circ = C_B(x)^\circ_u = B_u \cap C_B(x)^\circ$ is the unipotent radical of $C_B(x)^\circ$.

Since $T' \subseteq C_T(x)^\circ$, while the latter group is a torus of $C_B(x)^\circ$, it follows that $T' = C_T(x)^\circ$. General theory (cf. [**17**, (6.3.3), (6.3.5)]) now yields

$$C_B(x)^\circ = C_B(x)^\circ_u \rtimes T' = C_U(x)^\circ \rtimes C_T(x)^\circ,$$

as asserted.

(1b) Since $\{x\} \times C_{\mathfrak{u}}(x)$ is irreducible, so is the closure $\mathfrak{C}(x)$ of its B-saturation. Consider the dominant morphism

$$\omega : B \times C_{\mathfrak{u}}(x) \longrightarrow \mathfrak{C}(x); \qquad (b, y) \mapsto (b.x, b.y).$$

We fix $(b_0.x, b_0.y_0) \in \operatorname{im} \omega$. Then

$$\zeta : C_B(x) \longrightarrow \omega^{-1}(b_0.x, b_0.y_0); \qquad c \mapsto (b_0 c, c^{-1}.y_0)$$

is a morphism with inverse morphism

$$\eta : \omega^{-1}(b_0.x, b_0.y_0) \longrightarrow C_B(x); \qquad (b, y) \mapsto b_0^{-1} b.$$

As a result, $\dim \omega^{-1}(b_0.x, b_0.y_0) = \dim C_B(x)$, and the fibre dimension theorem gives

$$\dim \mathfrak{C}(x) = \dim B + \dim C_{\mathfrak{u}}(x) - \dim C_B(x).$$

In view of Proposition 3.2(2), we have $\operatorname{Lie}(C_U(x)) = C_{\mathfrak{u}}(x)$. Consequently,

$$\dim \mathfrak{C}(x) = \dim B + \dim C_U(x)^\circ - \dim C_B(x)^\circ,$$

and the assertion now follows from (1a).

(2) Suppose that $\dim \mathfrak{C}(x) = \dim B - \dim Z(G)$. Part (1) provides a maximal torus $T \subseteq B$ such that $\dim C_T(x) = \dim Z(G)$. This readily implies $C_T(x)^\circ = Z(G)^\circ$, so that $C_B(x)^\circ = Z(G)^\circ \ltimes C_U(x)^\circ$. In particular, $Z(G)^\circ$ is the unique maximal torus of $C_B(x)^\circ$, so that x is distinguished for B.

Suppose that x is distinguished for B. Let $T \subseteq B$ be a maximal torus such that $C_T(x)^\circ$ is a maximal torus of $C_B(x)^\circ$. It follows that $C_T(x)^\circ = Z(G)^\circ$, whence $\dim \mathfrak{C}(x) = \dim B - \dim Z(G)$. □

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Theorem 4.1.2. *Suppose that* $\operatorname{char}(k)$ *is good for* G. *Let* $B \subseteq G$ *be a Borel subgroup of* G, *and let* $U \subseteq B$ *be its unipotent radical,* $\mathfrak{u} := \operatorname{Lie}(U)$. *Then we have*

$$\dim \mathcal{C}_2(\mathfrak{u}) = \dim B - \dim Z(G) + \operatorname{mod}(B; \mathfrak{u}).$$

Proof. We first assume that G is almost simple, so that $\dim Z(G) = 0$. Thanks to [5, Theorem 10], we have

$$\operatorname{mod}(U; \mathfrak{u}) = \operatorname{mod}(B; \mathfrak{u}) + \operatorname{rk}(G),$$

so that a consecutive application of Propositions 3.2 and 2.3.2 implies

$$\dim \mathcal{C}_2(\mathfrak{u}) = \dim U + \operatorname{mod}(U; \mathfrak{u}) = \dim U + \operatorname{rk}(G) + \operatorname{mod}(B; \mathfrak{u}) = \dim B + \operatorname{mod}(B; \mathfrak{u}).$$

Next, we assume that G is semisimple with almost simple constituents $G_1, \ldots, G_n$, say. There are Borel subgroups $B_i \subseteq G_i$ of G_i with unipotent radicals U_i such that $B = B_1 \cdots B_n$ and $U = U_1 \cdots U_n$. Let $\mathfrak{u} := \operatorname{Lie}(U)$ and $\mathfrak{u}_i := \operatorname{Lie}(U_i)$. Lemma 3.1 provides an isomorphism $U \cong \prod_{i=1}^n U_i$, so that $\mathfrak{u} = \bigoplus_{i=1}^n \mathfrak{u}_i$. If $x = \sum_{i=1}^n x_i \in \mathfrak{u}$, then $B.x = \prod_{i=1}^n B_i.x_i$, so that $\dim B.x = \sum_{i=1}^n \dim B_i.x_i$. This readily implies

$$\mathfrak{u}_{[j]} := \{x \in \mathfrak{u};\ \dim B.x = j\} = \bigcup_{\{m \in \mathbb{N}_0^n; |m| = j\}} \prod_{i=1}^n (\mathfrak{u}_i)_{[m_i]} \qquad \forall j \in \mathbb{N}_0,$$

where we put $|m| := \sum_{i=1}^n m_i$ for $m \in \mathbb{N}_0^n$. Consequently,

$$\dim \mathfrak{u}_{[j]} = \max\left\{ \sum_{i=1}^n \dim(\mathfrak{u}_i)_{[m_i]};\ m \in \mathbb{N}_0^n\ ,\ |m| = j \right\} \qquad \forall j \in \mathbb{N}_0.$$

As a result,

$$\begin{aligned} \operatorname{mod}(B; \mathfrak{u}) &= \max_{j \geq 0} \max\left\{ \sum_{i=1}^n \dim(\mathfrak{u}_i)_{[m_i]};\ m \in \mathbb{N}_0^n;\ |m| = j \right\} - j \\ &= \max_{j \geq 0} \max\left\{ \sum_{i=1}^n \dim(\mathfrak{u}_i)_{[m_i]} - m_i;\ m \in \mathbb{N}_0^n;\ |m| = j \right\} \\ &= \max_{m \in \mathbb{N}_0^n} \sum_{i=1}^n (\dim(\mathfrak{u}_i)_{[m_i]} - m_i) = \sum_{i=1}^n \max_{m_i \geq 0} (\dim(\mathfrak{u}_i)_{[m_i]} - m_i) \\ &= \sum_{i=1}^n \operatorname{mod}(B_i; \mathfrak{u}_i). \end{aligned}$$

Since $\mathcal{C}_2(\mathfrak{u}) \cong \prod_{i=1}^n \mathcal{C}_2(\mathfrak{u}_i)$, we arrive at

$$\dim \mathcal{C}_2(\mathfrak{u}) = \sum_{i=1}^n \dim \mathcal{C}_2(\mathfrak{u}_i) = \sum_{i=1}^n \dim B_i + \operatorname{mod}(B_i; \mathfrak{u}_i) = \dim B + \operatorname{mod}(B; \mathfrak{u}),$$

as desired.

If G is reductive, then $G = Z(G)^\circ G'$, with $G' := (G,G)$ being semisimple and $Z(G)^\circ$ being a torus. By the arguments of Proposition 3.2, $B' := (B \cap G')^\circ$ is a Borel subgroup of G' with unipotent radical U and such that $B = B'Z(G)^\circ$ with $Z(G) \cap B'$ being finite. It follows that

$$B.x = B'.x$$

for all $x \in \mathfrak{u}$, and the identities

$$\dim \mathcal{C}_2(\mathfrak{u}) = \dim B' + \operatorname{mod}(B';\mathfrak{u}) = \dim B - \dim Z(G) + \operatorname{mod}(B;\mathfrak{u})$$

verify our claim. □

We denote by $\mathcal{O}_{\mathrm{reg}} \subseteq \mathfrak{g}$ the regular nilpotent G-orbit.

Lemma 4.1.3. *Suppose that* $\operatorname{char}(k)$ *is good for* G*. Given* $x \in \mathcal{O}_{\mathrm{reg}} \cap \mathfrak{u}$, $\mathfrak{C}(x)$ *is an irreducible component of* $\mathcal{C}_2(\mathfrak{u})$ *of dimension* $\dim B - \dim Z(G)$.

Proof. By general theory, $\mathcal{O}_{\mathrm{reg}} \cap \mathfrak{u}$ is an open B-orbit of $\mathfrak{u}$, cf. [1, (5.2.3)]. Consequently, $\mathcal{O}_{\mathrm{reg}} \cap \mathfrak{u}_{(\max \mathbb{N}_0(\mathfrak{u}))}$ is a non-empty subset of $\mathfrak{u}$. Since $\mathfrak{u}_{(\max \mathbb{N}_0(\mathfrak{u}))}$ is a B-stable subset of $\mathfrak{u}$, it follows that $\mathcal{O}_{\mathrm{reg}} \cap \mathfrak{u} \subseteq \mathfrak{u}_{(\max \mathbb{N}_0(\mathfrak{u}))}$.

Let $x \in \mathcal{O}_{\mathrm{reg}} \cap \mathfrak{u}$. Then $B.x \subseteq \mathfrak{u}_{(\max \mathbb{N}_0(\mathfrak{u}))}$ is open in $\mathfrak{u}$, so that $\operatorname{pr}_1^{-1}(B.x)$ is open in $\mathcal{C}_2(\mathfrak{u})$. Corollary 1.4 now shows that $\operatorname{pr}_1^{-1}(B.x)$ is an open subset of the irreducible component $\overline{\operatorname{pr}_1^{-1}(\mathfrak{u}_{(\max \mathbb{N}_0(\mathfrak{g}))})}$ of $\mathcal{C}_2(\mathfrak{u})$. Consequently,

$$\mathfrak{C}(x) = \overline{\operatorname{pr}_1^{-1}(B.x)} = \overline{\operatorname{pr}_1^{-1}(\mathfrak{u}_{(\max \mathbb{N}_0(\mathfrak{u}))})}$$

is an irreducible component of $\mathcal{C}_2(\mathfrak{u})$. Since the element x is distinguished for G, Lemma 2.2.2 shows that it is also distinguished for B. We may now apply Lemma 4.1.1 to see that $\dim \mathfrak{C}(x) = \dim B - \dim Z(G)$. □

Remarks.

(1) The foregoing result in conjunction with Theorem 4.1.2 implies that $\mathcal{C}_2(\mathfrak{u})$ is equidimensional only if B acts on $\mathfrak{u}$ with finitely many orbits.

(2) It also follows from the above and Corollary 1.4 that $\max \mathbb{N}_0(\mathfrak{u}) = \dim_k \mathfrak{u} - \operatorname{rk}_{\mathrm{ss}}(G)$.

4.2. Minimal supports

As before, we let G be a connected reductive algebraic group, with Borel subgroup $B = U \rtimes T$. The corresponding Lie algebras will be denoted $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{u}$. Let R_T be the root system of G relative to T, and let $\Delta := \{\alpha_1, \ldots, \alpha_n\} \subseteq R_T$ be a set of simple roots. Given $\alpha = \sum_{i=1}^n m_i \alpha_i \in R_T$, we denote by $\operatorname{ht}(\alpha) = \sum_{i=1}^n m_i$ the *height* of α (relative to

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Δ), and put for $x \in \mathfrak{u}\smallsetminus\{0\}$

$$\deg(x) := \min\{\mathrm{ht}(\alpha);\ \alpha \in \mathrm{supp}(x)\}$$

as well as

$$\mathrm{msupp}(x) := \{\alpha \in \mathrm{supp}(x);\ \mathrm{ht}(\alpha) = \deg(x)\}.$$

Given $n \in \mathbb{N}_0$, we put

$$\mathfrak{u}^{(\geq n)} := \langle\{x \in \mathfrak{u};\ \deg(x) \geq n\}\rangle.$$

Lemma 4.2.1. *Given* $x \in \mathfrak{u}\smallsetminus\{0\}$, *we have* $\deg(b.x) = \deg(x)$ *and* $\mathrm{msupp}(b.x) = \mathrm{msupp}(x)$ *for all* $b \in B$.

Proof. For $u \in U$ we consider the morphism

$$\Phi_u : U \longrightarrow U; \qquad v \mapsto [u, v],$$

where $[u, v] := uvu^{-1}v^{-1}$ denotes the commutator of u and v. According to [**17**, (4.4.13)], we have

$$\mathrm{d}(\Phi_u)(x) = u.x - x \qquad \forall x \in \mathfrak{u}.$$

Given a positive root $\alpha \in R_T^+$, we consider the root subgroup U_α of U. For $u \in U_\alpha$ and $\beta \in R_T^+$, an application of [**17**, (8.2.3)] shows that

$$\Phi_u(U_\beta) \subseteq \prod_{i,j>0} U_{i\alpha+j\beta}.$$

Let $x \in \mathfrak{u}\smallsetminus\{0\}$ and put $d := \deg(x)$. Since $\mathfrak{u}_\beta = \mathrm{Lie}(U_\beta)$, the foregoing observations in conjunction with [**17**, (8.2.1)] yield

$$\mathrm{Ad}(u)(x) \equiv x \mod(\mathfrak{u}^{(\geq d+1)}) \qquad \forall u \in U.$$

Thus, $\mathfrak{u}^{(\geq n)}$ is a U-submodule of $\mathfrak{u}$ for all $n \geq 1$ such that U acts trivially on $\mathfrak{u}^{(\geq n)}/\mathfrak{u}^{(\geq n+1)}$.

Now write $x = \sum_{\alpha\in\mathrm{msupp}(x)} x_\alpha + x'$, where $x' \in \mathfrak{u}^{(\geq d+1)}$. Given $b \in B$, there are $t \in T$ and $u \in U$ such that $b = tu$. By the above, we obtain

$$b.x \equiv \sum_{\alpha\in\mathrm{msupp}(x)} \alpha(t)x_\alpha \mod(\mathfrak{u}^{(\geq d+1)}),$$

whence $\deg(b.x) = \deg(x)$ and $\mathrm{msupp}(b.x) = \mathrm{msupp}(x)$. □

Let $\mathcal{O} \subseteq \mathfrak{u}$ be a B-orbit. In view of Lemma 4.2.1, we may define

$$\mathrm{msupp}(\mathcal{O}) := \mathrm{msupp}(x) \qquad (x \in \mathcal{O}).$$

4.3. The case $\mathrm{mod}(B; \mathfrak{u}) = 0$

The case where B acts on $\mathfrak{u}$ with finitely many orbits is governed by the theorem of Hille–Röhrle [**7**, (1.1)], which takes on the following form in our context.

Proposition 4.3.1. *Suppose that* $\mathrm{char}(k)$ *is good for* G. *Then* $\mathrm{mod}(B; \mathfrak{u}) = 0$ *if and only if every almost simple constituent of* (G, G) *is of type* $(A_n)_{n\leq 4}$ *or* B_2.

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Proof. Returning to the proof of Theorem 4.1.2, we let $G_1, \ldots, G_n$ be the simple constituents of (G, G) and pick Borel subgroups B_i of G_i, with unipotent radicals U_i. Then

$$B := Z(G)^{\circ} B_1 \cdots B_n$$

is a Borel subgroup of G with unipotent radical $U := U_1 \cdots U_n$. Setting $\mathfrak{u} := \mathrm{Lie}(U)$ and $\mathfrak{u}_i := \mathrm{Lie}(U_i)$, we have

$$\mathrm{mod}(B; \mathfrak{u}) = \sum_{i=1}^{n} \mathrm{mod}(B_i; \mathfrak{u}_i),$$

so that [7, (1.1)] yields the result. □

Lemma 4.3.2. *Suppose that* $\mathrm{mod}(B; \mathfrak{u}) = 0$. *If* $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))$, *then there is a unique orbit* $\mathcal{O}_C \subseteq \mathrm{pr}_1(C)$ *such that*

(a) $\mathcal{O}_C$ *is dense and open in* $\mathrm{pr}_1(C)$ *and*

(b) $C = \mathfrak{C}(x)$ *for all* $x \in \mathcal{O}_C$.

Proof. Since the component C is B-stable, so is the closed subset $\mathrm{pr}_1(C) \subseteq \mathfrak{u}$, cf. Lemma 1.2. By assumption, B thus acts with finitely many orbits on the irreducible variety $\mathrm{pr}_1(C)$. Hence there is a B-orbit $\mathcal{O}_C \subseteq \mathrm{pr}_1(C)$ such that $\overline{\mathcal{O}}_C = \mathrm{pr}_1(C)$. Consequently, $\mathcal{O}_C$ is open in $\mathrm{pr}_1(C)$. The unicity of $\mathcal{O}_C$ follows from the irreducibility of $\mathrm{pr}_1(C)$.

Let $x \in \mathcal{O}_C$, so that $\mathcal{O}_C = B.x$. Then there is $y \in \mathfrak{u}$ such that $(x, y) \in C$. In particular, $y \in C_{\mathfrak{u}}(x)$, so that $(x, y) \in B.(\{x\} \times C_{\mathfrak{u}}(x)) = \mathrm{pr}_1^{-1}(\mathcal{O}_C)$. Thanks to (a), $\mathrm{pr}^{-1}(\mathcal{O}_C)$ is open in $\mathrm{pr}_1^{-1}(\mathrm{pr}_1(C))$. It follows that $(B.(\{x\} \times C_{\mathfrak{u}}(x))) \cap C$ is a non-empty open subset of C, so that

$$C = \overline{(B.(\{x\} \times C_{\mathfrak{u}}(x))) \cap C} \subseteq \mathfrak{C}(x).$$

Since the latter set is irreducible, while C is a component, we have equality. □

Remarks.

(1) The lemma holds more generally for each $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))$ with $\mathrm{mod}(B; \mathrm{pr}_1(C)) = 0$.

(2) Suppose that $\mathrm{mod}(B; \mathfrak{u}) = 0$. In view of Theorem 4.1.2 and Lemma 4.1.1, each distinguished B-orbit $B.x$ gives rise to an irreducible component $\mathfrak{C}(x)$ of maximal dimension.

Suppose that $\mathrm{mod}(B; \mathfrak{u}) = 0$. Using Lemma 4.3.2, we define

$$\mathrm{msupp}(C) = \mathrm{msupp}(\mathcal{O}_C)$$

for every $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))$.

5. Almost simple groups

The purpose of this technical section is the proof of the following result, which extends [6, § 3] to good characteristics.

Proposition 5.1. *The following statements hold.*

(1) *If G has type $(A_n)_{n\leq 4}$, then $\mathcal{C}_2(\mathfrak{u})$ is equidimensional and*

$$|\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))| = \begin{cases} 5, & n = 4, \\ 2, & n = 3, \\ 1 & \textit{else}. \end{cases}$$

(2) *If $\operatorname{char}(k) \neq 2$ and G has type B_2, then $\mathcal{C}_2(\mathfrak{u})$ is equidimensional and $|\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))| = 2$.*

For G as above, the Borel subgroup $B \subseteq G$ acts on $\mathfrak{u}$ with finitely many orbits. We let $\mathfrak{R} \subseteq \mathfrak{u}$ be a set of orbit representatives, so that

$$\mathcal{C}_2(\mathfrak{u}) = \bigcup_{x \in \mathfrak{R}} \mathfrak{C}(x)$$

is a finite union of closed irreducible subsets. We will determine in each case the set $\{x \in \mathfrak{R};\ \mathfrak{C}(x) \in \operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))\}$. A list of orbit representatives is given in [**6**, § 3] and we will follow the notation established there.

5.1. Special linear groups

Let $G = \mathrm{SL}_{n+1}(k)$ and $\mathfrak{g} = \mathfrak{sl}_{n+1}(k)$, where $1 \leq n \leq 4$. Moreover, B, T, and U denote the standard subgroups of upper triangular, diagonal and upper unitriangular matrices, respectively.

For $i \leq j \in \{1, \ldots, n+1\}$, we let $E_{i,j}$ be the (i,j)-elementary matrix, so that

$$\mathfrak{u} := \bigoplus_{i<j} kE_{i,j}$$

is the Lie algebra of the unipotent radical U of B. We denote the set of simple roots by $\Delta := \{\alpha_1, \ldots, \alpha_n\}$. Let $i < j \leq n+1$. Then $E_{i,j}$ is the root vector corresponding to the root $\alpha_{i,j} := \sum_{\ell=i}^{j-1} \alpha_\ell$. We therefore have $\alpha_i = \alpha_{i,i+1}$ for $1 \leq i \leq n$, and

$$R_T^+ := \{\alpha_{i,j};\ 1 \leq i < j \leq n+1\}$$

is the set of roots of $\mathfrak{u}$ relative to T (the set of positive roots of $\mathfrak{sl}_{n+1}(k)$).

Recall that

$$E_{i,j}E_{r,s} = \delta_{j,r}E_{i,s},$$

as well as

$$[E_{i,j}, E_{r,s}] = \delta_{j,r}E_{i,s} - \delta_{s,i}E_{r,j} \quad \text{for all } i, j, r, s \in \{1, \ldots, n+1\}.$$

Let $\alpha = \alpha_{i,j}$ be a positive root. Then

$$U_\alpha := \{1 + aE_{i,j};\ a \in k\}$$

is the corresponding root subgroup of U, and the formula above implies that

$$\operatorname{Ad}(1 + aE_{i,j})(x) = (1 + aE_{i,j})x(1 - aE_{i,j}) = x + a[E_{i,j}, x]$$

for all $x \in \mathfrak{u}$.

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Note that $A := \{(a_{ij}) \in \mathrm{Mat}_{n+1}(k);\ a_{ij} = 0 \text{ for } i > j\}$ is a subalgebra of the associative algebra $\mathrm{Mat}_{n+1}(k)$. We consider the linear map

$$\zeta : A \longrightarrow A; \qquad E_{i,j} \mapsto E_{n+2-j,n+2-i}.$$

Then we have

(a) $\zeta(ab) = \zeta(b)\zeta(a)$ for all $a, b \in A$ and

(b) $\det(\zeta(a)) = \det(a)$ for all $a \in A$.

There results a homomorphism

$$\tau : B \longrightarrow B; \qquad a \mapsto \zeta(a)^{-1}$$

of algebraic groups such that $\tau(U) = U$. We write $\mathfrak{b} := \mathrm{Lie}(B)$ and put $\Upsilon := \mathrm{d}(\tau)|_{\mathfrak{u}}$. As ζ is linear, [**17**, (4.4.12)] implies that

$$\Upsilon(E_{i,j}) = -E_{n+2-j,n+2-i}, \qquad 1 \le i < j \le n+1.$$

Thus, Υ is an automorphism of $\mathfrak{u}$ of order 2 such that

$$\Upsilon(\mathfrak{u}_{\alpha_{ij}}) = \mathfrak{u}_{\alpha_{n+2-j,n+2-i}}.$$

Since Δ is a basis for the root lattice $\mathbb{Z}R_T^+ = \mathbb{Z}R_T$, there is an automorphism $\sigma : \mathbb{Z}R_T^+ \longrightarrow \mathbb{Z}R_T^+$ of order 2 such that

$$\sigma(\alpha_i) = \alpha_{n+1-i} \qquad 1 \le i \le n.$$

Thus, $\sigma(R_T^+) = R_T^+$ and

$$\Upsilon(\mathfrak{u}_\alpha) = \mathfrak{u}_{\sigma(\alpha)} \qquad \forall \alpha \in R_T^+.$$

We denote by $(\mathfrak{u}^n)_{n\in\mathbb{N}}$ the descending series of the nilpotent Lie algebra $\mathfrak{u}$, which is inductively defined via $\mathfrak{u}^1 := \mathfrak{u}$ and $\mathfrak{u}^{n+1} := [\mathfrak{u}, \mathfrak{u}^n]$. Note that $\mathfrak{u}^n = \mathfrak{u}^{(\ge n)}$ for all $n \ge 1$.

Lemma 5.1.1. *Let $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))$. Then we have*

$$\mathrm{msupp}([\Upsilon \times \Upsilon](C)) = \sigma(\mathrm{msupp}(C)).$$

Proof. We put $\mathcal{O}_C = B.x$. In view of $\Upsilon = \mathrm{d}(\tau)|_{\mathfrak{u}}$, we have

$$\Upsilon(b.x) = \tau(b).\Upsilon(x) \qquad \forall b \in B, x \in \mathfrak{u}.$$

Consequently,

$$\Upsilon(\mathcal{O}_C) = \Upsilon(B.x) = B.\Upsilon(x)$$

is an open orbit of $\Upsilon(\mathrm{pr}_1(C)) = \mathrm{pr}_1([\Upsilon \times \Upsilon](C))$, so that

$$\mathcal{O}_{[\Upsilon\times\Upsilon](C)} = \Upsilon(\mathcal{O}_C).$$

Setting $d{:=}\deg(x)$, we have

$$x \equiv \sum_{\alpha\in \operatorname{msupp}(x)} x_\alpha \mod \mathfrak{u}^{(\geq d+1)}.$$

Thus,

$$\Upsilon(x) \equiv \sum_{\alpha\in \operatorname{msupp}(x)} -x_{\sigma(\alpha)} \mod \mathfrak{u}^{(\geq d+1)},$$

whence

$$\operatorname{msupp}([\Upsilon\times\Upsilon](C)) = \operatorname{msupp}(\Upsilon(x)) = \sigma(\operatorname{msupp}(x)) = \sigma(\operatorname{msupp}(C)),$$

as desired. □

Remark. The list of orbit representatives for the case A_4 given in [**6**, (3.4)] contains some typographical errors, which we correct as follows.

(a) In the form stated *loc. cit.*, the element e_3 satisfies $\operatorname{rk}(\mathbb{Z}\operatorname{supp}(e_3)) = 3$, so that it is not distinguished, see Corollary 2.2.4. We write $e_3 = 110101\mathbf{0}000$, so that $e_3 = \Upsilon(e_7)$.

(b) In [**6**, (3.4)], we have $e_4 = e_5$. We put $e_4 := 1101000000$ (the element e_3 of [**6**, (3.4)]), so that $e_4 = \Upsilon(e_8)$.

Lemma 5.1.2. *Let* $G = \mathrm{SL}_5(k)$*. Then* $\mathcal{C}_2(\mathfrak{u})$ *is equidimensional and* $|\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))| = 5$.

Proof. Let $C \in \operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))$ be a component and pick $x \in \mathcal{O}_C$, so that $C = \mathfrak{C}(x)$, cf. Lemma 4.3.2. We consider

$$S_C := \operatorname{msupp}(C) \cup \operatorname{msupp}([\Upsilon\times\Upsilon](C)) = \operatorname{msupp}(x) \cup \operatorname{msupp}(\Upsilon(x)).$$

According to Lemma 5.1.1, S_C is a σ-stable subset of R_T^+.

We will repeatedly apply Lemma 2.1.4 to B-submodules of $\mathfrak{u}$.

(a) We have $x \notin \bigcup_{i=1}^3 kE_{i,i+2}\oplus\mathfrak{u}^3$.

Suppose that $x \in kE_{i,i+2}\oplus\mathfrak{u}^3$ for some $i \in \{1, 2, 3\}$. Since $\mathfrak{u}^3 = kE_{1,4}\oplus kE_{2,5}\oplus kE_{1,5}$, we have $[E_{2,3}, \mathfrak{u}^3] = (0)$. It thus follows from Lemma 2.1.4 that $\deg(x) \leq 2$. Consequently, $\deg(x) = 2$ and $|\operatorname{msupp}(x)| = 1$. If $|S_C| = 1$, then $i = 2$. Since $[E_{2,3}, kE_{2,4} + \mathfrak{u}^3] = (0)$, we may apply Lemma 2.1.4 to $\mathfrak{v}{:=}kE_{2,4} + \mathfrak{u}^3$ to obtain a contradiction. Alternatively, we may assume that $i = 1$. As $[E_{2,4}, kE_{1,3} + \mathfrak{u}^3] = (0)$, another application of Lemma 2.1.4 rules out this case.

(b) We have $\deg(x) = 1$ and $|S_C| = 2, 4$.

Suppose that $\deg(x){\geq}2$. In view of (a), we have $\deg(x) = 2$ and $|\operatorname{msupp}(x)| \geq 2$. If $|\operatorname{msupp}(x)| = 2 = |S_C|$, then $\operatorname{msupp}(x) = S_C$ is σ-stable, so that $\operatorname{msupp}(x) = \{\alpha_{1,3}, \alpha_{3,5}\}$. Thus, $B.x \subseteq \mathfrak{v} := kE_{1,3}\oplus kE_{3,5}\oplus\mathfrak{u}^3$ (see also Lemma 4.2.1). Since $E_{2,4} \in C_{\mathfrak{u}}(\mathfrak{v})$, Lemma 2.1.4 yields a contradiction. If $|\operatorname{msupp}(x)|{=}2$ and $|S_C| = 3$, then $\operatorname{msupp}(x) \cap \operatorname{msupp}(\Upsilon(x))$ contains a fixed point of σ, and we may assume that $\operatorname{msupp}(x) = \{\alpha_{1,3}, \alpha_{2,4}\}$. In view of [**6**, (3.4)], we may assume that $x = e_{48} = E_{1,3} + E_{2,4}$.

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Since $B.x \subseteq \mathfrak{v} := kE_{1,3} \oplus kE_{2,4} \oplus \mathfrak{u}^3$, while $E_{1,2} + E_{3,4} \in C_{\mathfrak{u}}(x)$, Lemma 2.1.4 yields a contradiction.

We thus assume that $|\operatorname{msupp}(x)| = 3$. Then [6, (3.4)] in conjunction with Lemma 4.2.1 gives $x = e_{47} = E_{1,3} + E_{2,4} + E_{3,5}$. Since $E_{1,2} + E_{3,4} \in C_{\mathfrak{u}}(x)$, while $B.x \subseteq \mathfrak{u}^2$, this contradicts Lemma 2.1.4.

Consequently, $\deg(x) = 1$, so that $\operatorname{msupp}(x) \subseteq \Delta$. Since σ acts without fixed points on Δ, every σ-orbit of Δ has two elements. As $S_C \subseteq \Delta$ is a disjoint union of σ-orbits, we obtain $|S_C| = 2, 4$.

(c) We have $|\operatorname{msupp}(x)| \geq 2$.
Alternatively, (b) provides $i \in \{1, \ldots, 4\}$ such that $B.x \subseteq \mathfrak{v} := kE_{i,i+1} + \mathfrak{u}^2$. Applying Υ, if necessary, we may assume that $i \in \{1, 2\}$.

Suppose that $i = 1$. Then Lemma 4.2.1 in conjunction with [6, (3.4)] implies that we have to consider the following cases:

$$x = e_{16} = E_{1,2} + E_{2,4} + E_{3,5};\; x = e_{17} = E_{1,2} + E_{2,4};$$
$$x = e_{18} = E_{1,2} + E_{3,5} + E_{2,5};$$
$$x = e_{19} = E_{1,2} + E_{3,5};\; x = e_{20} = E_{1,2} + E_{2,5};\; x = e_{21} = E_{1,2}.$$

Consequently, $E_{3,4} \in C_{\mathfrak{u}}(x) \smallsetminus \mathfrak{v}$, which contradicts Lemma 2.1.4.

Suppose that $i = 2$. Then [6, (3.4)] implies

$$x = e_{29} = E_{2,3} + E_{3,5} + E_{1,4};\; x = e_{30} = E_{2,3} + E_{3,5};$$
$$x = e_{31} = E_{2,3} + E_{1,4};$$
$$x = e_{32} = E_{2,3} + E_{1,5};\; x = e_{33} = E_{2,3}.$$

Since $E_{4,5} \in [C_{\mathfrak{u}}(e_{30}) \cap C_{\mathfrak{u}}(e_{32}) \cap C_{u}(e_{33})] \smallsetminus \mathfrak{v}$, Lemma 2.1.4 rules out these possibilities. In view of $E_{4,5} + E_{1,3} \in C_{\mathfrak{u}}(e_{29}) \smallsetminus \mathfrak{v}$, it remains to discuss the case where $x = e_{31}$.

We consider the morphism

$$\mathfrak{x} : k \longrightarrow \mathfrak{u}; \qquad \alpha \mapsto e_{31} + \alpha E_{3,5}.$$

Then we have $\mathfrak{x}(\alpha) \in B.e_{29}$ for all $\alpha \in k^{\times}$, while $\mathfrak{x}(0) = e_{31}$. Direct computation shows that

$$C_{\mathfrak{u}}(e_{31}) = kE_{2,3} \oplus kE_{1,3} \oplus kE_{2,4} \oplus \mathfrak{u}^3.$$

For $y = aE_{2,3} + bE_{1,3} + cE_{2,4} + z \in C_{\mathfrak{u}}(e_{31})$, where $z \in \mathfrak{u}^3$, we consider the morphism

$$\mathfrak{y} : k \longrightarrow \mathfrak{u}; \qquad \alpha \mapsto y + b\alpha E_{4,5} + a\alpha E_{3,5}.$$

Since $[\mathfrak{x}(\alpha), \mathfrak{y}(\alpha)] = 0$ for all $\alpha \in k^{\times}$, Lemma 2.1.2 yields

$$(e_{31}, y) = (\mathfrak{x}(0), \mathfrak{y}(0)) \in \mathfrak{C}(\mathfrak{x}(1)) = \mathfrak{C}(e_{29}).$$

Consequently, $\mathfrak{C}(e_{31}) \subseteq \mathfrak{C}(e_{29})$. Since $\mathfrak{C}(e_{29}) \notin \operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))$, we again arrive at a contradiction.

(d) We have $|S_C| = 4$.
Suppose that $|S_C| \neq 4$. Then (b) implies $|S_C| = 2$ and (c) shows that $\operatorname{msupp}(x) \subseteq \Delta$ is σ-stable with two elements. Consequently, $\operatorname{msupp}(x) = \{\alpha_1, \alpha_4\}$ or $\operatorname{msupp}(x) = \{\alpha_2, \alpha_3\}$.

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If $x = E_{2,3} + E_{3,4} + y$, where $y \in \mathfrak{u}^2$, then [6, (3.4)] yields $x \in B.e_{23} \cup B.e_{24}$, where $e_{23} := E_{2,3} + E_{3,4} + E_{1,5}$ and $e_{24} := E_{2,3} + E_{3,4}$. We may invoke Lemma 2.1.3 to see that $\mathfrak{C}(e_{24}) \subseteq \mathfrak{C}(e_{23})$. It was shown in [6, (3.4)] that $\mathfrak{C}(e_{23}) \subseteq \mathfrak{C}(e_1)$. Hence $\mathfrak{C}(x)$ is not a component, a contradiction.

It follows that $\operatorname{msupp}(x) = \{\alpha_1, \alpha_4\}$, so that [6, (3.4)] implies

$$x \in B.e_{13} \cup B.e_{14} \cup B.e_{15},$$

where $e_{15} := E_{1,2} + E_{4,5}$, $e_{14} := e_{15} + E_{2,5}$ and $e_{13} := e_{15} + E_{2,4}$. In view of $C_{\mathfrak{u}}(e_{15}) \subseteq kE_{1,2} \oplus kE_{4,5} \oplus \mathfrak{u}^2$, we have $[C_{\mathfrak{u}}(e_{15}), E_{2,5}] \subseteq k[E_{1,2}, E_{2,5}] = k[e_{15}, E_{2,5}]$. Lemma 2.1.3 thus shows that $\mathfrak{C}(e_{15}) \subseteq \mathfrak{C}(e_{14})$. In [6, (3.4)] it is shown that $\mathfrak{C}(e_{14}) \subseteq \mathfrak{C}(e_3)$. According to (b), the latter set is not a component, so neither is $\mathfrak{C}(e_{14})$.

It remains to dispose of the case $x = e_{13}$. For $(\alpha, \beta) \in k^2$, we consider the elements

$$e_1(\alpha, \beta) := E_{1,2} + \alpha E_{2,3} + \beta E_{3,4} + E_{4,5} \quad \text{and} \quad e_{13}(\alpha, \beta) := e_1(\alpha, \beta) + E_{2,4}$$

of $\mathfrak{u}$. Let $u_{i,j}(t) := 1 + tE_{i,j} \in U$ $(t \in k)$, so that $u_{i,j}(t).x = x + t[E_{i,j}, x]$ for all $x \in \mathfrak{u}$. We thus obtain $e_{13}(\alpha, \beta) = u_{2,3}(\beta^{-1})u_{1,2}(\alpha^{-1}\beta^{-1}).e_1(\alpha, \beta)$ for $\alpha\beta \neq 0$. As a result,

$$e_{13}(\alpha, \beta) \in B.e_1 \qquad \text{for } \alpha\beta \neq 0,$$

where $e_1 = e_1(1, 1)$.

Direct computation shows that

$$C_{\mathfrak{u}}(e_{13}) = ke_{13} \oplus kE_{1,3} \oplus kE_{3,5} \oplus k(E_{1,4} + E_{2,5}) \oplus kE_{1,5}.$$

Let $y = ae_{13} + bE_{1,3} + cE_{3,5} + d(E_{1,4} + E_{2,5}) + eE_{1,5} \in C_{\mathfrak{u}}(e_{13})$ be such that $b, c \neq 0$. We consider the morphisms

$$\mathfrak{x} : k \longrightarrow \mathfrak{u}; \qquad \alpha \mapsto e_{13}(\alpha, \alpha cb^{-1}) \;\text{ and }\; \mathfrak{y} : k \longrightarrow \mathfrak{u};$$
$$\alpha \mapsto y + \alpha a E_{2,3} + \alpha acb^{-1} E_{3,4} + \alpha c E_{2,4}$$

and observe that

(a) $\mathfrak{x}(\alpha) \in B.\mathfrak{x}(1)$ for all $\alpha \in k^\times$ and

(b) $[\mathfrak{x}(\alpha), \mathfrak{y}(\alpha)] = 0$ for all $\alpha \in k$.

Thus, Lemma 2.1.2 implies that $(e_{13}, y) = (\mathfrak{x}(0), \mathfrak{y}(0)) \in \mathfrak{C}(\mathfrak{x}(1)) = \mathfrak{C}(e_1)$. Since the set of those y with $bc \neq 0$ lies dense in $C_{\mathfrak{u}}(e_{13})$, it follows that $\mathfrak{C}(e_{13}) \subseteq \mathfrak{C}(e_1)$, a contradiction. This completes the proof of (d).

If $\operatorname{msupp}(x) = S_C$, (d) shows that $\deg(x) = 1$ and $|\operatorname{msupp}(x)| = 4$. Hence x is regular and $\mathfrak{C}(x) = \mathfrak{C}(e_1)$ is an irreducible component.

If $|\operatorname{msupp}(x)| = 2$, then $S_C = \operatorname{msupp}(x) \sqcup \sigma(\operatorname{msupp}(x))$ and we only need to consider the cases

$$\operatorname{msupp}(x) = \{\alpha_1, \alpha_2\}; \; \{\alpha_1, \alpha_3\}.$$

If $\operatorname{msupp}(x) = \{\alpha_1, \alpha_2\}$, then Lemma 4.2.1 yields $B.x \subseteq \mathfrak{v} := kE_{1,2} + kE_{2,3} + \mathfrak{u}^2$, while [6, (3.4)] implies

$$x = e_5 = E_{1,2} + E_{2,3} + E_{3,5}; \; x = e_6 = E_{1,2} + E_{2,3}.$$

Consequently, $E_{4,5} \in C_{\mathfrak{u}}(x) \smallsetminus \mathfrak{v}$, a contradiction.

If $\operatorname{msupp}(x) = \{\alpha_1, \alpha_3\}$, then $B.x \subseteq \mathfrak{v} := kE_{1,2} + kE_{3,4} + \mathfrak{u}^2$ and [6, (3.4)] implies

$$x = e_9 = E_{1,2} + E_{3,4} + E_{2,4} + E_{2,5};\ x = e_{10} = E_{1,2} + E_{3,4} + E_{2,4};$$
$$x = e_{11} = E_{1,2} + E_{3,4} + E_{2,5};\ x = e_{12} = E_{1,2} + E_{3,4}.$$

Given $(\alpha, \beta) \in k^2$, we put

$$x(\alpha, \beta) = E_{1,2} + E_{3,4} + \alpha E_{2,4} + \beta E_{2,5}.$$

Note that

$$x(\alpha, \beta) \in B.x(1, 1) = B.e_9 \qquad \text{for } \alpha, \beta \neq 0.$$

We put $\mathfrak{w} := kE_{3,4} \oplus k(E_{1,3} + E_{2,4}) \oplus kE_{3,5} \oplus kE_{1,4} \oplus kE_{1,5}$. Direct computation shows that

$$C_{\mathfrak{u}}(x(\alpha, \beta)) = k(E_{1,2} + \alpha E_{2,4} + \beta E_{2,5}) \oplus \mathfrak{w}$$

for all $(\alpha, \beta) \in k^2$. We have $e_i = x(\delta_{i,10}, \delta_{i,11})$ for $i \in \{10, 11, 12\}$. Thus, if $y = a(E_{1,2} + \delta_{i,10}E_{2,4} + \delta_{i,11}E_{2,5}) + w \in C_{\mathfrak{u}}(e_i)$, where $a \in k$ and $w \in \mathfrak{w}$, then

$$y(\alpha, \beta) = y + (a\alpha - a)\delta_{i,10}E_{2,4} + (a\beta - a)\delta_{i,11}E_{2,5} \in C_{\mathfrak{u}}(x(\alpha, \beta)).$$

Let $i \in \{10, 11, 12\}$. Then the morphisms

$$\mathfrak{x}_i : k \longrightarrow \mathfrak{u}; \qquad \alpha \mapsto x(\alpha(\delta_{i,11} + \delta_{i,12}) + \delta_{i,10}, \alpha(\delta_{i,10} + \delta_{i,12}) + \delta_{i,11})$$

and

$$\mathfrak{y}_i : k \longrightarrow \mathfrak{u}; \qquad \alpha \mapsto y(\alpha(\delta_{i,11} + \delta_{i,12}) + \delta_{i,10}, \alpha(\delta_{i,10} + \delta_{i,12}) + \delta_{i,11})$$

fulfil the conditions of Lemma 2.1.2, so that

$$(e_i, y) = (\mathfrak{x}_i(0), \mathfrak{y}_i(0)) \in \mathfrak{C}(\mathfrak{x}_i(1)) = \mathfrak{C}(e_9).$$

As a result, $\mathfrak{C}(e_i) \subseteq \mathfrak{C}(e_9)$ for $10 \leq i \leq 12$.

We have $\dim_k \operatorname{im}(\operatorname{ad} e_9)(\mathfrak{b}) = \dim_k \operatorname{im}(\operatorname{ad} e_9) + 4$, so that $C_{\mathfrak{u}}(e_9) = C_{\mathfrak{b}}(e_9)$. Thus, Proposition 3.2 implies

$$\dim C_B(e_9) \leq \dim_k C_{\mathfrak{b}}(e_9) = \dim_k C_{\mathfrak{u}}(e_9) = \dim C_U(e_9),$$

so that $C_B(e_9)^\circ = C_U(e_9)^\circ$. Consequently, the element e_9 is distinguished and $\mathfrak{C}(e_9)$ is a component. Hence $\Upsilon(e_9)$ is also distinguished and [6, (3.4)] in conjunction with Corollary 2.2.4 implies that $\mathfrak{C}(e_{25})$ is also a component.

It remains to consider the case where $|\operatorname{msupp}(x)| = 3$. Then $\operatorname{msupp}(x) \cap \sigma(\operatorname{msupp}(x))$ is a σ-stable subset of Δ of cardinality 2, so that

$$\operatorname{msupp}(x) \cap \sigma(\operatorname{msupp}(x)) = \{\alpha_1, \alpha_4\};\ \{\alpha_2, \alpha_3\}.$$

Suppose that $\operatorname{msupp}(x) \cap \sigma(\operatorname{msupp}(x)) = \{\alpha_1, \alpha_4\}$. Then we may assume that $\operatorname{msupp}(x) = \{\alpha_1, \alpha_2, \alpha_4\}$. Thanks to [6, (3.4)], this yields $x = e_3, e_4$. The above methods show that $\mathfrak{C}(e_4) \subseteq \mathfrak{C}(e_3)$, while e_3 is a distinguished element. Hence $\mathfrak{C}(e_3)$ and $\Upsilon(\mathfrak{C}(e_3)) = \mathfrak{C}(e_7)$ are components of $\mathcal{C}_2(\mathfrak{u})$.

We finally consider $\operatorname{msupp}(x)\cap\sigma(\operatorname{msupp}(x)) = \{\alpha_2,\alpha_3\}$ and assume that $\operatorname{msupp}(x) = \{\alpha_1,\alpha_2,\alpha_3\}$. By [6, (3.4)], this implies

$$x = e_2 = E_{1,2} + E_{2,3} + E_{3,4}.$$

As $\mathfrak{C}(e_2) \subseteq \mathfrak{C}(e_1)$, this case yields no additional components. It follows that

$$\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u})) = \{\mathfrak{C}(e_1), \mathfrak{C}(e_3), \mathfrak{C}(e_7), \mathfrak{C}(e_9), \mathfrak{C}(e_{25})\},$$

so that $|\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))| = 5$. □

Lemma 5.1.3. *Let $G = \mathrm{SL}_4(k)$. Then $\mathcal{C}_2(\mathfrak{u})$ is equidimensional with $|\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))| = 2$.*

Proof. We consider $\mathrm{GL}_n(k) = \mathrm{SL}_n(k)Z(\mathrm{GL}_n)$ along with its standard Borel subgroup $B_n = U_n \rtimes T_n$ of upper triangular matrices, where U_n and T_n are the group's unitriangular and diagonal matrices, respectively. The B-orbits of $\mathfrak{u}_n := \operatorname{Lie}(U_n)$ coincide with those of the standard Borel subgroup $B_n \cap \mathrm{SL}_n(k)$ of $\mathrm{SL}_n(k)$.

We consider $G' := \mathrm{GL}_5(k)$ along with its commuting variety $\mathcal{C}_2(\mathfrak{u}')$. In view of Lemma 5.1.2, we have

$$\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}')) = \{\mathfrak{C}(e'_1), \mathfrak{C}(e'_3), \mathfrak{C}(e'_7), \mathfrak{C}(e'_9), \mathfrak{C}(e'_{25})\}.$$

Let A' and A be the associative algebras of upper triangular (5×5)-matrices and upper triangular (4×4)-matrices, respectively. Then

$$\pi : A' \longrightarrow A; \qquad (a_{ij}) \mapsto (a_{ij})_{1\le i\le j\le 4}$$

are homomorphisms of k-algebras. Thus, if we identify $G := \mathrm{GL}_4(k)$ with a subgroup of the Levi subgroup of G', given by $\Delta_4 := \{\alpha'_1, \alpha'_2, \alpha'_3\}$, then the restriction

$$\pi : B' \longrightarrow B$$

is a homomorphism of groups such that $\pi|_B = \mathrm{id}_B$. It follows that the differential

$$\mathrm{d}(\pi) : \mathfrak{u}' \longrightarrow \mathfrak{u}$$

of the restriction $\pi|_{U'} : U' \longrightarrow U$ is split surjective such that

$$\mathrm{d}(\pi)(b'.x') = \pi(b').\mathrm{d}(\pi)(x') \quad \text{for all } b' \in B',\ x' \in \mathfrak{u}'.$$

As a result, the morphism

$$[\mathrm{d}(\pi)\times\mathrm{d}(\pi)] : \mathcal{C}_2(\mathfrak{u}') \longrightarrow \mathcal{C}_2(\mathfrak{u})$$

is surjective and such that

$$[\mathrm{d}(\pi)\times\mathrm{d}(\pi)](B'.(\{x'\}\times C_{\mathfrak{u}'}(x'))) \subseteq B.(\{\mathrm{d}(\pi)(x')\}\times C_{\mathfrak{u}}(\mathrm{d}(\pi)(x'))),$$

whence

$$[\mathrm{d}(\pi)\times\mathrm{d}(\pi)](\mathfrak{C}(x')) \subseteq \mathfrak{C}(\mathrm{d}(\pi)(x')) \quad \text{for all } x' \in \mathfrak{u}'.$$

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Consequently,

$$\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u})) \subseteq \{\mathfrak{C}(\mathrm{d}(\pi)(e_1')), \mathfrak{C}(\mathrm{d}(\pi)(e_3')), \mathfrak{C}(\mathrm{d}(\pi)(e_7')), \mathfrak{C}(\mathrm{d}(\pi)(e_9')), \mathfrak{C}(\mathrm{d}(\pi)(e_{25}'))\}.$$

Thanks to [**6**, (3.3), (3.4)], we obtain

$$\mathrm{d}(\pi)(e_1') = e_1;\ \mathrm{d}(\pi)(e_3') \in B.e_2;\ \mathrm{d}(\pi)(e_7') = e_3;\ \mathrm{d}(\pi)(e_9') = e_3;\ \mathrm{d}(\pi)(e_{25}') = e_8.$$

In [**6**, (3.3)], the authors show that $\mathfrak{C}(e_8) \subseteq \mathfrak{C}(e_1)$. By applying Lemma 2.1.2 to the morphism

$$\mathfrak{x} : k \longrightarrow \mathfrak{u}; \qquad \alpha \mapsto E_{1,2} + E_{2,3} + \alpha E_{3,4}$$

we obtain $\mathfrak{C}(e_2) \subseteq \mathfrak{C}(e_1)$.

Since the element e_1 is regular, it is distinguished. As $\dim_k(\mathrm{ad}\, e_3)(\mathfrak{b}) = \dim_k(\mathrm{ad}\, e_3)(\mathfrak{u}) + 3 = 5$, we obtain, observing Proposition 3.2,

$$\dim C_B(e_3) \leq \dim_k C_{\mathfrak{b}}(e_3) = \dim_k C_{\mathfrak{u}}(e_3) = \dim C_U(e_3),$$

so that $C_B(e_3)^\circ = C_U(e_3)^\circ$. Hence e_3 is distinguished for B, and $\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u})) = \{\mathfrak{C}(e_1), \mathfrak{C}(e_3)\}$. □

The same method readily shows the following.

Lemma 5.1.4. *Let $G = \mathrm{SL}_n(k)$, where $n = 2, 3$. Then $\mathcal{C}_2(\mathfrak{u})$ is irreducible.*

5.2. Symplectic groups

The following result disposes of the remaining case.

Lemma 5.2.1. *Suppose that $\mathrm{char}(k) \neq 2$. Let $G = \mathrm{Sp}(4)$ be of type $B_2 = C_2$. Then $\mathfrak{C}_2(\mathfrak{u})$ is equidimensional with $|\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))| = 2$.*

Proof. Recall that $R_T^+ := \{\alpha, \beta, \alpha+\beta, \alpha+2\beta\}$ is a system of positive roots, where $\Delta = \{\alpha, \beta\}$. Suppose that $\mathfrak{C}(x)$ is a component. Since $[\mathfrak{u}_\alpha, \mathfrak{u}^{(\geq 2)}] = (0)$, Lemma 2.1.4 implies $\deg(x) = 1$.

Suppose that $|\mathrm{msupp}(x)| = 1$. If $\mathrm{msupp}(x) = \{\alpha\}$, then [**6**, (3.5)] yields $x \in B.x_\alpha \cup B.(x_\alpha + x_{\alpha+2\beta})$, while Lemma 2.1.3 gives $\mathfrak{C}(x_\alpha) \subseteq \mathfrak{C}(x_\alpha + x_{\alpha+2\beta})$.

Alternatively, $x \in B.x_\beta$. Since $C_{\mathfrak{u}}(x_\beta) = kx_\beta \oplus kx_{\alpha+2\beta}$, we have $[x_\alpha, C_{\mathfrak{u}}(x_\beta)] = k[x_\alpha, x_\beta]$, and Lemma 2.1.3 implies $\mathfrak{C}(x_\beta) \subseteq \mathfrak{C}(x_\alpha + x_\beta)$. As a result,

$$\mathcal{C}_2(\mathfrak{u}) = \mathfrak{C}(x_\alpha + x_\beta) \cup \mathfrak{C}(x_\alpha + x_{\alpha+2\beta}).$$

Since $\mathrm{char}(k) \neq 2$, the arguments of Lemma 5.1.3 show that these elements are distinguished. Consequently, $\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u})) = \{\mathfrak{C}(x_\alpha + x_\beta), \mathfrak{C}(x_\alpha + x_{\alpha+2\beta})\}$. □

5.3. Proof of Proposition 5.1

Proof. (1) Let us first consider an almost simple group G of type A_n for $n \in \{1, \ldots, 4\}$. In view of [**9**, (II.1.13), (II.1.14)], there is a covering $\pi : \mathrm{SL}_{n+1}(k) \longrightarrow G$. Hence π is surjective and $\ker \pi \subseteq Z(G)$ is diagonalizable. Let $B_{n+1} \subseteq \mathrm{SL}_{n+1}(k)$ be a Borel subgroup,

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and let $U_{n+1} \trianglelefteq B_{n+1}$ be its unipotent radical with Lie algebra $\mathfrak{u}_{n+1}$. Then $B := \pi(B_{n+1})$ is a Borel subgroup of G with unipotent radical $U := \pi(U_{n+1})$. Since $\ker \pi \cap U_{n+1} = \{1\}$, it follows that $\pi|_{U_{n+1}}$ is a closed embedding, so that $\pi|_{U_{n+1}} : U_{n+1} \longrightarrow U$ is an isomorphism. Consequently, its differential

$$\mathrm{d}(\pi) : \mathfrak{u}_{n+1} \longrightarrow \mathfrak{u}$$

is an isomorphism of Lie algebras such that

$$\pi(b).\mathrm{d}(\pi)(x) = \mathrm{d}(\pi)(b.x) \quad \text{for all } x \in \mathfrak{u}_{n+1}, b \in B_{n+1}.$$

Thanks to § 5.1, the variety $\mathcal{C}_2(\mathfrak{u}_{n+1}) \cong \mathcal{C}_2(\mathfrak{u})$ is equidimensional with $|\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))| = |\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}_{n+1}))|$.

(2) Since $\mathrm{Sp}(4)$ is simply connected, we may use the foregoing arguments in conjunction with Lemma 5.2.1. □

5.4. Irreducibility and equidimensionality of $\mathcal{C}_2(\mathfrak{u})$

We record the following direct consequence of Proposition 5.1.

Corollary 5.4.1. *Let G be connected and reductive such that $\mathrm{char}(k)$ is good for G. Suppose that $B \subseteq G$ is a Borel subgroup with unipotent radical U, whose Lie algebra is denoted $\mathfrak{u}$.*

(1) *If $\mathrm{mod}(B; \mathfrak{u}) = 0$, then $\mathcal{C}_2(\mathfrak{u})$ is equidimensional.*

(2) *$\mathcal{C}_2(\mathfrak{u})$ is irreducible if and only if every almost simple component of (G, G) is of type A_1 or A_2.*

Proof. Let $G_1, \ldots, G_n$ be the almost simple components of G. As before, we may write

$$B = Z(G)^{\circ} B_1 \cdots B_n,$$

where $B_i \subseteq G_i$ is a Borel subgroup. Letting U_i be the unipotent radical of B_i and setting $\mathfrak{u}_i := \mathrm{Lie}(U_i)$, we have $\mathcal{C}_2(\mathfrak{u}) \cong \prod_{i=1}^{n} \mathcal{C}_2(\mathfrak{u}_i)$. This shows that

$$\operatorname{Irr}(\mathcal{C}_2(\mathfrak{u})) = \left\{ \prod_{i=1}^{n} C_i;\ C_i \in \operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}_i)) \ \ 1 \leq i \leq n \right\}.$$

(1) The theorem of Hille–Röhrle shows that each G_i is of type $(A_n)_{n \leq 4}$ or B_2. Thanks to Proposition 5.1, each $\mathcal{C}_2(\mathfrak{u}_i)$ is equidimensional. Hence $\mathcal{C}_2(\mathfrak{u})$ enjoys the same property.

(2) If $\mathcal{C}_2(\mathfrak{u})$ is irreducible, then so is each $\mathcal{C}_2(\mathfrak{u}_i)$, and a consecutive application of Theorem 4.1.2, Lemma 4.1.3, [**7**, (1.1)] and Proposition 5.1 ensures that each almost simple group G_i is of type A_1 or A_2. The reverse direction follows directly from Proposition 5.1. □

Remark. Suppose that G is almost simple of type A–D. If $p \geq h(G)$ is good for G, then [**19**, (1.7), (1.8)] in conjunction with the foregoing result implies that the variety $V(U_2)$ of infinitesimal one-parameter subgroups of the second Frobenius kernel U_2 of U is irreducible if and only if G is of type A_1 or A_2.

6. The variety $\mathbb{A}(2,\mathfrak{u})$

Let $\mathfrak{u} := \mathrm{Lie}(U)$ be the Lie algebra of the unipotent radical U of a Borel subgroup B of a connected reductive group G. In this section, we are interested in the projective variety

$$\mathbb{A}(2,\mathfrak{u}) := \{\mathfrak{a} \in \mathrm{Gr}_2(\mathfrak{u});\ [\mathfrak{a},\mathfrak{a}] = (0)\}$$

of two-dimensional abelian subalgebras of $\mathfrak{u}$. Recall that

$$\mathcal{O}_2(\mathfrak{u}) := \{(x,y) \in \mathcal{C}_2(\mathfrak{u});\ \dim_k kx + ky = 2\}$$

is an open, $\mathrm{GL}_2(k)$-stable subset of $\mathcal{C}_2(\mathfrak{u})$, while the map

$$\varphi : \mathcal{O}_2(\mathfrak{u}) \longrightarrow \mathbb{A}(2,\mathfrak{u}); \qquad (x,y) \mapsto kx + ky$$

is a surjective morphism such that $\varphi^{-1}(\varphi(x,y)) = \mathrm{GL}_2(k).(x,y)$ for all $(x,y) \in \mathcal{O}_2(\mathfrak{u})$. Note that $\mathrm{GL}_2(k)$ acts simply on $\mathcal{O}_2(\mathfrak{u})$, so that each fibre of φ is four-dimensional.

The Borel subgroup B acts on $\mathbb{A}(2,\mathfrak{u})$ via

$$b.\mathfrak{a} := \mathrm{Ad}(b)(\mathfrak{a}) \qquad \forall b \in B, \mathfrak{a} \in \mathbb{A}(2,\mathfrak{u}).$$

Moreover, the set $\mathcal{O}_2(\mathfrak{u})$ is B-stable and $\varphi : \mathcal{O}_2(\mathfrak{u}) \longrightarrow \mathbb{A}(2,\mathfrak{u})$ is B-equivariant.

Lemma 6.1. *Suppose that* $\mathrm{rk}_{\mathrm{ss}}(G) \geq 2$. *Then the following statements hold.*

(1) *Given* $x \in \mathfrak{u} \setminus \{0\}$, *there is* $y \in \mathfrak{u}$ *such that* $(x,y) \in \mathcal{O}_2(\mathfrak{u})$.

(2) $\mathcal{O}_2(\mathfrak{u})$ *lies dense in* $\mathcal{C}_2(\mathfrak{u})$.

Proof. (1) Let $z \in C(\mathfrak{u}) \setminus \{0\}$. If $x \in \mathfrak{u} \setminus kz$, then $(x,z) \in \mathcal{O}_2(\mathfrak{g})$. Alternatively, $x \in kz \setminus \{0\}$. Since $\mathrm{rk}_{\mathrm{ss}}(G) \geq 2$, we have $\dim_k \mathfrak{u} > 1$, so that there is $y \in \mathfrak{u} \setminus kx$. It follows that $(x,y) \in \mathcal{O}_2(\mathfrak{u})$.

(2) Let $x \in \mathfrak{u} \setminus \{0\}$. By (1), there is $y \in \mathfrak{u}$ such that $(x,y) \in \mathcal{O}_2(\mathfrak{u})$. Given $\beta \in k$, we consider the morphism

$$f_\beta : k \longrightarrow \mathcal{C}_2(\mathfrak{u}); \qquad \alpha \mapsto (x, \beta x + \alpha y).$$

Then we have $f_\beta(k^\times) \subseteq \mathcal{O}_2(\mathfrak{u})$, so that $f(k) \subseteq \overline{\mathcal{O}_2(\mathfrak{u})}$. In particular, $(x,\beta x) = f(0) \in \overline{\mathcal{O}_2(\mathfrak{u})}$. Setting $\beta = 0$, we obtain $(x,0) \in \overline{\mathcal{O}_2(\mathfrak{u})}$. Using the $\mathrm{GL}_2(k)$-action, we conclude that $(0,x) \in \overline{\mathcal{O}_2(\mathfrak{u})}$. Since

$$g : k \longrightarrow \mathcal{C}_2(\mathfrak{u}); \qquad \alpha \mapsto (\alpha x, 0)$$

is a morphism such that $g(k^\times) \subseteq \overline{\mathcal{O}_2(\mathfrak{u})}$, we conclude that $(0,0) \in \overline{\mathcal{O}_2(\mathfrak{u})}$. As a result, $\mathcal{C}_2(\mathfrak{u}) = \overline{\mathcal{O}_2(\mathfrak{u})}$. □

Lemma 6.2. *Suppose that* $\operatorname{char}(k)$ *is good for* G *and that* $\operatorname{rk}_{ss}(G) \geq 2$. *Let* $\mathcal{O} \subseteq \mathfrak{u} \setminus \{0\}$ *be a* B*-orbit,* $x \in \mathcal{O}$.

(1) *We have* $\varphi(B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u})) = \{\mathfrak{a} \in \mathbb{A}(2, \mathfrak{u});\ \mathfrak{a} \cap \mathcal{O} \neq \emptyset\}$.

(2) *If* $\mathcal{O} = \mathcal{O}_{\mathrm{reg}} \cap \mathfrak{u}$, *then* $\overline{\varphi(B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u}))}$ *is an irreducible component of* $\mathbb{A}(2, \mathfrak{u})$ *of dimension* $\dim B - \dim Z(G) - 4$.

Proof. (1) We put $\mathbb{A}(2, \mathfrak{u})_{\mathcal{O}} := \{\mathfrak{a} \in \mathbb{A}(2, \mathfrak{u});\ \mathfrak{a} \cap \mathcal{O} \neq \emptyset\}$. Let $y \in C_{\mathfrak{u}}(x)$ be such that $(x, y) \in \mathcal{O}_2(\mathfrak{u})$. Then $x \in \varphi(x, y) \cap \mathcal{O}$, so that $\varphi(x, y) \in \mathbb{A}(2, \mathfrak{u})_{\mathcal{O}}$. Since $\mathbb{A}(2, \mathfrak{u})_{\mathcal{O}}$ is B-stable, it follows that $\varphi(B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u})) = B.\varphi((\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u})) \subseteq \mathbb{A}(2, \mathfrak{u})_{\mathcal{O}}$.

Now suppose that $\mathfrak{a} \in \mathbb{A}(2, \mathfrak{u})_{\mathcal{O}}$, and write $\mathfrak{a} = ky \oplus kz$, where $y \in \mathcal{O}$. Then there is $b \in B$ such that $x = b.y$, so that $b.\mathfrak{a} \in \varphi((\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u}))$. As a result, $\mathfrak{a} \in \varphi(B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u}))$.

(2) General theory tells us that $\mathcal{O} = \mathcal{O}_{\mathrm{reg}} \cap \mathfrak{u}$ is an open B-orbit of $\mathfrak{u}$. Note that $\mathcal{O}_{\mathrm{reg}}$ is a conical subset of $\mathfrak{g}$, so that $\mathcal{O}_{\mathrm{reg}} \cap \mathfrak{u}$ is a conical subset of $\mathfrak{u}$. It now follows from (1) and [**2**, (3.2)] that $\varphi(B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u}))$ is an open subset of $\mathbb{A}(2, \mathfrak{u})$. In view of Lemma 6.1, the irreducible set $\{x\} \times C_{\mathfrak{u}}(x)$ meets $\mathcal{O}_2(\mathfrak{u})$, so that $B.((\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u})) = B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u})$ is irreducible. Hence $\varphi(B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u}))$ is a non-empty, irreducible, open subset of $\mathbb{A}(2, \mathfrak{u})$. Let $C \supseteq \varphi(B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u}))$ be an irreducible component of $\mathbb{A}(2, \mathfrak{u})$. Then $\varphi(B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u}))$ lies dense in C, so that $C = \overline{\varphi(B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u}))}$. Observing Lemma 4.1.1, we thus obtain

$$\begin{aligned}\dim C = \dim B.(\{x\} \times C_{\mathfrak{u}}(x)) \cap \mathcal{O}_2(\mathfrak{u}) - 4 &= \dim B.(\{x\} \times C_{\mathfrak{u}}(x)) - 4 \\ &= \dim B - \dim Z(G) - 4,\end{aligned}$$

as desired. □

Given $x \in \mathfrak{u}$, we put

$$\mathbb{A}(2, \mathfrak{u}, x) := \{\mathfrak{a} \in \mathbb{A}(2, \mathfrak{u});\ x \in \mathfrak{a}\}.$$

Proposition 6.3. *Suppose that* $\operatorname{char}(k)$ *is good for* G *and that* $\operatorname{rk}_{ss}(G) \geq 2$.

(1) $\dim \mathbb{A}(2, \mathfrak{u}) = \dim B - \dim Z(G) + \operatorname{mod}(B; \mathfrak{u}) - 4$.

(2) *The variety* $\mathbb{A}(2, \mathfrak{u})$ *is equidimensional if and only if every almost simple component of* (G, G) *is of type* $(A_n)_{n \leq 4}$ *or* B_2. *In that case, every irreducible component* $C \in \operatorname{Irr}(\mathbb{A}(2, \mathfrak{u}))$ *is of the form* $C = \overline{B.\mathbb{A}(2, \mathfrak{u}, x)}$ *for some* B*-distinguished element* $x \in \mathfrak{u}$.

(3) *The variety* $\mathbb{A}(2, \mathfrak{u})$ *is irreducible if and only if every almost simple component of* (G, G) *is of type* A_1 *or* A_2.

Proof. (1) We write

$$\mathcal{C}_2(\mathfrak{u}) = \bigcup_{C \in \operatorname{Irr}(\mathcal{C}_2(\mathfrak{u}))} C$$

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as the union of its irreducible components. Since $\mathrm{rk}_{\mathrm{ss}}(G)\geq 2$, Lemma 6.1 shows that $\mathcal{O}_2(\mathfrak{u})$ is a dense open subset of $\mathcal{C}_2(\mathfrak{u})$. As a result, every irreducible component $C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))$ meets $\mathcal{O}_2(\mathfrak{u})$. In view of Theorem 4.1.2, we obtain

$$\dim\mathcal{O}_2(\mathfrak{u})=\dim\mathcal{C}_2(\mathfrak{u})=\dim B-\dim Z(G)+\mathrm{mod}(B;\mathfrak{u}).$$

Let $C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))$. Then $C\cap\mathcal{O}_2(\mathfrak{u})$ is a $\mathrm{GL}_2(k)$-stable, irreducible variety of dimension $\dim C$, so that

$$\dim\overline{\varphi(C\cap\mathcal{O}_2(\mathfrak{u}))}=\dim C\cap\mathcal{O}_2(\mathfrak{u})-4=\dim C-4.$$

Consequently,

$$\begin{aligned}\dim\mathbb{A}(2,\mathfrak{u})&=\max_{C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))}\overline{\varphi(C\cap\mathcal{O}_2(\mathfrak{u}))}=\dim\mathcal{C}_2(\mathfrak{u})-4\\&=\dim B-\dim Z(G)+\mathrm{mod}(B;\mathfrak{u})-4.\end{aligned}$$

(2) Suppose that $\mathbb{A}(2,\mathfrak{u})$ is equidimensional. As Lemma 6.2 provides $C\in\mathrm{Irr}(\mathbb{A}(2,\mathfrak{u}))$ such that $\dim C=\dim B-\dim Z(G)-4$, it follows from (1) that $\mathrm{mod}(B;\mathfrak{u})=0$. The theorem of Hille–Röhrle (see Proposition 4.3.1) ensures that every almost simple component of (G,G) is of the asserted type. Assuming this to be the case, Corollary 5.4.1 implies that $\mathcal{C}_2(\mathfrak{u})$ is equidimensional. In view of [**2**, (2.5.1)], $\mathcal{O}_2(\mathfrak{u})$ is equidimensional as well. We may thus apply [**2**, (2.5.2)] to the canonical surjection $\mathcal{O}_2(\mathfrak{u})\twoheadrightarrow\mathbb{A}(2,\mathfrak{u})$ and the $\mathrm{GL}_2(k)$-action on $\mathcal{O}_2(\mathfrak{u})$ to conclude that $\mathbb{A}(2,\mathfrak{u})$ is equidimensional.

Given $C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))$, Lemma 4.3.2 provides $x_C\in\mathfrak{u}$ such that $C=\mathfrak{C}(x_C)$. In view of Lemma 4.1.1, our current assumption shows that x_C is distinguished for B. According to Lemma 6.1, we have $(\{x_C\}\times C_{\mathfrak{u}}(x_C))\cap\mathcal{O}_2(\mathfrak{u})\neq\emptyset$, while Lemma 6.2 yields $\varphi(B.(\{x_C\}\times C_{\mathfrak{u}}(x_C))\cap\mathcal{O}_2(\mathfrak{u}))=B.\mathbb{A}(2,\mathfrak{u},x_C)$.

Let $a\in\mathfrak{C}(x_C)\cap\mathcal{O}_2(\mathfrak{u})$. If $\mathcal{U}\subseteq\mathcal{C}_2(\mathfrak{u})$ is an open subset containing a, then $\mathcal{U}\cap(B.(\{x_C\}\times C_{\mathfrak{u}}(x_C)))$ is a non-empty open subset of the irreducible set $B.(\{x_C\}\times C_{\mathfrak{u}}(x_C))$. Since this also holds for $B.(\{x_C\}\times C_{\mathfrak{u}}(x_C))\cap\mathcal{O}_2(\mathfrak{u})$, we conclude that $\mathcal{U}\cap B.(\{x_C\}\times C_{\mathfrak{u}}(x_C))\cap\mathcal{O}_2(\mathfrak{u})\neq\emptyset$. This shows that $a\in\overline{B.(\{x_C\}\times C_{\mathfrak{u}}(x_C))\cap\mathcal{O}_2(\mathfrak{u})}$. Consequently,

$$\begin{aligned}\mathbb{A}(2,\mathfrak{u})&=\bigcup_{C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))}\varphi(\mathfrak{C}(x_C)\cap\mathcal{O}_2(\mathfrak{u}))\\&\subseteq\bigcup_{C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))}\varphi(\overline{B.(\{x_C\}\times C_{\mathfrak{u}}(x_C))\cap\mathcal{O}_2(\mathfrak{u})})\\&\subseteq\bigcup_{C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))}\overline{\varphi(B.(\{x_C\}\times C_{\mathfrak{u}}(x_C))\cap\mathcal{O}_2(\mathfrak{u}))}\\&\subseteq\bigcup_{C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))}\overline{\varphi(B.[(\{x_C\}\times C_{\mathfrak{u}}(x_C))\cap\mathcal{O}_2(\mathfrak{u})])}\\&=\bigcup_{C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))}\overline{B.\varphi((\{x_C\}\times C_{\mathfrak{u}}(x_C))\cap\mathcal{O}_2(\mathfrak{u}))}\\&=\bigcup_{C\in\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))}\overline{B.\mathbb{A}(2,\mathfrak{u},x_C)}\subseteq\mathbb{A}(2,\mathfrak{u}),\end{aligned}$$

so that $\mathbb{A}(2,\mathfrak{u}) = \bigcup_{C\in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))} \overline{B.\mathbb{A}(2,\mathfrak{u},x_C)}$ is a finite union of closed irreducible subsets. It follows that every irreducible component of $\mathbb{A}(2,\mathfrak{u})$ is of the form $\overline{B.\mathbb{A}(2,\mathfrak{u},x_C)}$ for some $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))$.

(3) Suppose that $\mathbb{A}(2,\mathfrak{u})$ is irreducible. Then (2), Proposition 4.3.1 and Corollary 5.4.1 show that the variety $\mathcal{C}_2(\mathfrak{u})$ is equidimensional. Using [**2**, (2.5.2)], we conclude that $\mathcal{C}_2(\mathfrak{u})$ is irreducible, and Corollary 5.4.1 implies that G has the asserted type. The reverse implication is a direct consequence of Corollary 5.4.1. □

Remark. The arguments of (2) can actually be used to show that $\mathcal{C}_2(\mathfrak{u})$ and $\mathbb{A}(2,\mathfrak{u})$ have the same number of components in the case where one (and hence both) of these spaces is (are) equidimensional. Let $C \in \mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))$. Returning to the proof of Proposition 1.3(3), we find a subset $X_C \subseteq \mathfrak{u}$ such that

$$C = \overline{\mathrm{pr}_1^{-1}(X_C)}.$$

Since C is $\mathrm{GL}_2(k)$-stable, we conclude that $X_C \not\subseteq \{0\}$. Let $x \in X_C \smallsetminus \{0\}$. Then $\{x\} \times C_{\mathfrak{u}}(x) \subseteq C$. The assumption $C_{\mathfrak{u}}(x) = kx$ implies $x \in C(\mathfrak{u})$ and hence $\dim_k \mathfrak{u} = 1$, a contradiction. As a result, $C \cap \mathcal{O}_2(\mathfrak{u}) \neq \emptyset$. In view of [**2**, (2.5.1)], the variety $\mathcal{O}_2(\mathfrak{u})$ is therefore equidimensional with $|\mathrm{Irr}(\mathcal{O}_2(\mathfrak{u}))| = |\mathrm{Irr}(\mathcal{C}_2(\mathfrak{u}))|$. By virtue of [**2**, (2.5.2)], we obtain $|\mathrm{Irr}(\mathcal{O}_2(\mathfrak{u}))| = |\mathrm{Irr}(\mathbb{A}(2,\mathfrak{u}))|$.

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