

THE MINIMUM MODULUS OF GAP POWER SERIES

by P. C. FENTON
(Received 12th October 1976)

1. Introduction

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \tag{1.1}$$

be an entire function, where (λ_n) is a strictly increasing sequence of non-negative integers. The *maximum modulus*, $M(r)$, the *minimum modulus*, $m(r)$, and the *maximum term*, $\mu(r)$, of f are defined by

$$M(r) = \max_{|z|=r} |f(z)|, \quad m(r) = \min_{|z|=r} |f(z)|, \quad \mu(r) = \max_{n \geq 0} |a_n| r^{\lambda_n}.$$

The largest number λ_n for which $\mu(r) = |a_n| r^{\lambda_n}$ is called the *central index*, $N(r)$, of f .

We recall briefly the notions of logarithmic measure and logarithmic density of linear sets. Given a set E of non-negative numbers, the *logarithmic measure* of E is defined by

$$\log \text{meas } E = \int_{E \cap [1, \infty)} \frac{dt}{t}.$$

The *upper logarithmic density*, $\overline{\log \text{dens } E}$, and the *lower logarithmic density*, $\underline{\log \text{dens } E}$, of E are defined by

$$\overline{\log \text{dens } E} = \overline{\lim}_{r \rightarrow \infty} (\log r)^{-1} \int_{E \cap (1, r)} \frac{dt}{t}. \tag{1.2}$$

Should the two limits of (1.2) prove to be equal we speak simply of the *logarithmic density* of E .

In (1), P. Erdős and A. J. Macintyre established conditions under which

$$\overline{\lim}_{r \rightarrow \infty} \frac{m(r)}{M(r)} = \overline{\lim}_{r \rightarrow \infty} \frac{\mu(r)}{M(r)} = 1 \tag{1.3}$$

for the entire function (1.1). Their results ((1) Theorems 1 and 4) may be gathered as follows:

(a) If $\sum_{k=0}^{\infty} \frac{1}{\lambda_{k+1} - \lambda_k} < \infty$; or

(b) if $\sum_{k=0}^{\infty} \frac{1}{\lambda_{k+1} - \lambda_k} = o(\log \lambda_n)$ and f is of finite order; or

(c) if $\sum_{k=0}^n \frac{1}{\lambda_{k+1} - \lambda_k} = O(\log \lambda_n)$ and f is of zero order;

then (1.3) holds.

It is the purpose of this note to prove a sharpened version of this result both by establishing an estimate of the size of the set on which, as $r \rightarrow \infty$,

$$\mu(r) = (1 + o(1))M(r), \quad m(r) = (1 + o(1))M(r) \tag{1.4}$$

hold together and also by relaxing slightly the conditions (b) and (c).

2. The result

We will prove

(i) If $\sum_{k=0}^{\infty} (1/\lambda_{k+1} - \lambda_k) < \infty$, then (1.4) hold together as r tends to infinity outside a set of finite logarithmic measure.

(ii) If $\overline{\lim}_{n \rightarrow \infty} (\log \lambda_n)^{-1} \sum_{k=0}^n (1/\lambda_{k+1} - \lambda_k) = 0$ and f is of finite order; or

(iii) if $\overline{\lim}_{n \rightarrow \infty} (\log \lambda_n)^{-1} \sum_{k=0}^n (1/\lambda_{k+1} - \lambda_k) < \infty$ and f is of zero order; then (1.4) hold together as r tends to infinity outside a set of zero lower logarithmic density.

Erdős and Macintyre show by counterexamples that if any one of the hypotheses concerning the indices λ_n in (i), (ii) and (iii) is false then there is an entire function of the form (1.1) of the appropriate order (that is, of finite order in case (ii) and of zero order in case (iii)) for which both

$$\overline{\lim}_{r \rightarrow \infty} \frac{m(r)}{M(r)} \leq 1/2, \quad \overline{\lim}_{r \rightarrow \infty} \frac{\mu(r)}{M(r)} \leq 1/2.$$

In this sense, then, the result is best possible.

3. Preliminaries and a lemma

Let

$$\epsilon_n = \frac{1}{\lambda_{n+1} - \lambda_n}, \quad n = 0, 1, 2, \dots \tag{3.1}$$

Given any non-negative integer n , let $n_0 = n_0(n)$ be such that

$$\lambda_{n_0} \leq 2\lambda_n < \lambda_{n_0+1}, \tag{3.2}$$

and let us define a number δ_n by

$$\delta_n = \max_{i \leq n \leq j \leq n_0} (j - i + 1)^{-3/2} \sum_{\nu=i}^j \epsilon_{\nu}. \tag{3.3}$$

This δ_n is a slight variation of an ingenious function introduced in (1), the difference being that in (1) there is no upper restriction on j . We have, for every non-negative n ,

$$\sum_{k=0}^n \delta_k \leq C \sum_{k=0}^{n_0} \epsilon_k, \tag{3.4}$$

where C is an absolute constant. For, for some i_k, j_k , with $i_k \leq k \leq j_k \leq n_0$, $k =$

0, 1, 2, . . . ,

$$\begin{aligned} \sum_{k=0}^n \delta_k &= \sum_{k=0}^n (j_k - i_k + 1)^{-3/2} \sum_{\nu=i_k}^{j_k} \epsilon_\nu \\ &\leq \sum_{k=0}^n \sum_{\nu=i_k}^{j_k} \epsilon_\nu (|\nu - k| + 1)^{-3/2} \\ &\leq \sum_{\nu=0}^{n_0} \epsilon_\nu \sum_{k=0}^n (|\nu - k| + 1)^{-3/2} \\ &\leq C \sum_{\nu=0}^{n_0} \epsilon_\nu, \end{aligned}$$

where $C = 1 + 2 \sum_{k=2}^\infty k^{-3/2}$.

With these definitions we have

Lemma. *For any positive r let $p = p(N(r))$ be such that $N(r) = \lambda_p$. In each of the cases (i), (ii) and (iii) there is a sequence (c_n) , with $c_n \rightarrow \infty$ as $n \rightarrow \infty$, for which*

$$N(re^{\pm c_n \delta_\nu}) = N(r) \tag{3.5}$$

for r outside a set E , where E has finite logarithmic measure in case (i) and zero lower logarithmic density in cases (ii) and (iii).

As the proof in all three cases is basically one we will prove only case (iii), which involves slightly more complications than the others.

Since f is of zero order,

$$\frac{\log N(r)}{\log r} = \eta(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty. \tag{3.6}$$

Moreover

$$\lim_{n \rightarrow \infty} (\log \lambda_n)^{-1} \sum_{k=0}^n \delta_k < \infty. \tag{3.7}$$

For we know that there is a sequence (m_n) of positive integers such that

$$(\log \lambda_{m_n})^{-1} \sum_{k=0}^{m_n} \epsilon_k = O(1) \quad \text{as } n \rightarrow \infty.$$

Let q_n be the largest positive integer such that $\lambda_{q_n-1} \leq \frac{1}{2} \lambda_{m_n}$. Then, from the definition of n_0 given by (3.2), we have $(q_n - 1)_0 \leq m_n$, and so from (3.4)

$$\sum_{k=0}^{q_n-1} \delta_k \leq C \sum_{k=0}^{m_n} \epsilon_k.$$

Also it follows from the definition of δ_k that $0 \leq \delta_k < 1$, $k = 0, 1, 2, \dots$, so

$$\sum_{k=0}^{q_n} \delta_k \leq 1 + C \sum_{k=0}^{m_n} \epsilon_k. \tag{3.8}$$

But $\lambda_{q_n} > \frac{1}{2}\lambda_{m_n}$, so from this together with (3.8) we obtain

$$(\log \lambda_{q_n})^{-1} \sum_{k=0}^{q_n} \delta_k \leq (\log \lambda_{m_n} - \log 2)^{-1} \left\{ 1 + C \sum_{k=0}^{m_n} \epsilon_k \right\} = O(1) \text{ as } n \rightarrow \infty.$$

Thus (3.7) holds and there is an increasing sequence (q_n) of positive integers and a constant K such that

$$\sum_{k=0}^{q_n} \delta_k \leq K \log \lambda_{q_n}, \quad n = 1, 2, 3, \dots$$

Let λ_{M_n} be the smallest value of the central index such that $\lambda_{q_n} \leq \lambda_{M_n}$ and let R_n be such that

$$N(R_n) = \lambda_{M_n} \text{ and } N(r) < \lambda_{M_n} \text{ for } r < R_n.$$

With $\eta(t)$ as given by (3.6), we can find a sequence (c_k) with $c_k \rightarrow \infty$ as $k \rightarrow \infty$ for which, at least for a subsequence,

$$\sum_{k=0}^{q_n} c_k \delta_k \leq K \eta(R_n)^{-1/2} \log \lambda_{q_n}, \quad n = 1, 2, 3, \dots \tag{3.9}$$

and since $\log \lambda_{q_n} \leq \log \lambda_{M_n} = \log N(R_n) = \eta(R_n) \log R_n$, we obtain from (3.9)

$$\sum_{k=0}^{q_n} c_k \delta_k \leq K \{\eta(R_n)\}^{1/2} \log R_n. \tag{3.10}$$

We seek to determine the size of the subset of $[1, R_n]$ at the points of which

$$N(re^{c_p \delta_p}) = N(r), \tag{3.11}$$

where p is such that $N(r) = \lambda_p$, $p = p(N)$.

Let r_1 be the first number no less than 1 for which (3.11) is false and let $r'_1 = r_1 e^{c_{p_1} \delta_{p_1}}$, where $p_1 = p(N(r_1))$. Let r_2 be the smallest number no less than r'_1 at which (3.11) is false and let $r'_2 = r_2 e^{c_{p_2} \delta_{p_2}}$, where $p_2 = p(N(r_2))$. We continue in this way until reaching r_s where s is such that $r_s < R_n$ and $r_{s+1} \geq R_n$.

We note that $N(r_i) \geq N(r'_{i-1}) > N(r_{i-1})$, so the sequence $(p_i)_{i=1}^s$ is strictly increasing. Also, since $r_s < R_n$, $N(r_s) < \lambda_{M_n}$ and so, from the definition of M_n , $p(N(r_s)) = p_s < q_n$. Thus (3.11) holds in $[1, R_n]$ outside a subset of logarithmic measure at most

$$\sum_{i=1}^s c_{p_i} \delta_{p_i} \leq \sum_{k=0}^{q_n} c_k \delta_k \leq K \{\eta(R_n)\}^{1/2} \log R_n. \tag{3.12}$$

In a precisely similar way, but defining the sequences (r_i) , (r'_i) to be decreasing from R_n , we see that

$$N(re^{-c_p \delta_p}) = N(r), \tag{3.13}$$

where p is such that $N(r) = \lambda_p$, also holds in $[1, R_n]$ outside a subset of logarithmic measure at most $K \{\eta(R_n)\}^{1/2} \log R_n$. On combining this with (3.11) and (3.12) we see that (3.11) and (3.13) hold together in $[1, R_n]$ outside a subset of logarithmic measure at $2K \{\eta(R_n)\}^{1/2} \log R_n$. Since $R_n \rightarrow \infty$ as $n \rightarrow \infty$ and $\eta(t) \rightarrow 0$ as $t \rightarrow \infty$, the case (iii) of the Lemma is proved.

4. Proof of the result

With p such that $N(r) = \lambda_p$, let us consider

$$S(r) = \sum_{n=0}^{p-1} |a_n| r^{\lambda_n} + \sum_{n=p+1}^{\infty} |a_n| r^{\lambda_n}.$$

It is known (see for example Lemma 2 of (2)) that, for r outside a set F of finite logarithmic measure and for n_0 as given by (3.2),

$$\sum_{n=p_0+1}^{\infty} |a_n| r^{\lambda_n} = o(\mu(r)) \text{ as } r \rightarrow \infty.$$

Hence, as r tends to infinity outside F ,

$$S(r) = \sum_{n=0}^{p-1} |a_n| r^{\lambda_n} + \sum_{n=p+1}^{p_0} |a_n| r^{\lambda_n} + o(\mu(r)).$$

Let r be a number outside $F \cup E$, where E is the exceptional set of the Lemma. For $p + 1 \leq n \leq p_0$, we obtain from one half of the Lemma,

$$|a_n| \{r e^{c_p \delta_p}\}^{\lambda_n} \leq |a_p| \{r e^{c_p \delta_p}\}^{\lambda_p}$$

so that

$$|a_n| r^{\lambda_n} \leq \mu(r) e^{-(\lambda_n - \lambda_p) c_p \delta_p}.$$

Also, for $p + 1 \leq n \leq p_0$ we obtain, with two applications of the arithmetic-geometric means inequality (following (1)),

$$\begin{aligned} \delta_p &\geq (n - p)^{-3/2} \sum_{\nu=p}^{n-1} \frac{1}{\lambda_{\nu+1} - \lambda_{\nu}} \\ &\geq (n - p)^{-1/2} \left\{ \prod_{\nu=p}^{n-1} \frac{1}{\lambda_{\nu+1} - \lambda_{\nu}} \right\}^{1/(n-p)} \\ &\geq (n - p)^{1/2} (\lambda_n - \lambda_p)^{-1}. \end{aligned}$$

Hence, for $p + 1 \leq n \leq p_0$,

$$|a_n| r^{\lambda_n} \leq \mu(r) e^{-c_p(n-p)^{1/2}}.$$

For $0 \leq n \leq p - 1$ we employ the remaining half of the Lemma to obtain, quite analogously,

$$|a_n| r^{\lambda_n} \leq \mu(r) e^{-c_p(p-n)^{1/2}}.$$

Thus, as r tends to infinity outside $F \cup E$,

$$\begin{aligned} S(r) &\leq 2\mu(r) \sum_{n=1}^{\infty} e^{-c_p n^{1/2}} + o(\mu(r)) \\ &= o(\mu(r)). \end{aligned}$$

The result follows.

REFERENCES

(1) P. ERDŐS and A. J. MACINTYRE, Integral functions with gap power series. *Proc. Edinburgh Math. Soc.* (2) **10** (1953), 62–70.

(2) W. K. HAYMAN, The local growth of power series: a survey of the Wiman–Valiron method by, *Canad. Math. Bull.* **17** (1974), 317–358.

UNIVERSITY OF OTAGO
BOX 56, DUNEDIN
NEW ZEALAND