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On the critical layer of Alfvén waves in the solar wind

L. M. B. C. CAMPOS and N. L. ISAEVA

Secção de Mecânica Aeroespacial, ISR, Instituto Superior Técnico, 1096 Lisboa Codex, Portugal
(lmbcampos.aero@popsrv.ist.utl.pt)

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Abstract. Alfvén waves are considered in a radial flow and external magnetic field, which relates to some features of the solar wind near the critical point. The Alfvén wave equation for the velocity perturbation is derived, showing that it has in general two singularities (besides the origin and infinity), namely a critical layer (at real distance), where the Alfvén speed equals the mean flow velocity, and a transition level (at imaginary distance), where the spatial derivative of the flow velocity equals the wave frequency. It is shown that in the case of mean flow velocity varying as a power of radial distance the wave field is specified at all distances by a combination of solutions of the Alfvén wave equation around three singularities: a regular singularity at the center, so that ascending power-series solutions exist, some with logarithmic terms; an irregular singularity at infinity, leading to the non-existence of any solution as an ascending Frobenius–Fuchs series, and the existence of two solutions as ascending–descending Laurent series; the region of validity of the preceding solutions is limited by a regular singularity at a finite, non-zero radial distance, which is the critical layer, where the flow velocity and Alfvén speed are equal. The wave field is singular at the critical layer, and has an amplitude jump, which is illustrated by plotting the wave field in the neighborhood of the critical layer, for several values of dimensionless frequency and Alfvén number, combined into a single parameter. When considering Alfvén waves in the solar wind, at least three kinds of boundary conditions could be applied: (i) an initial condition specifying the wave field at the surface of the Sun; (ii) an asymptotic condition excluding wave sources at infinity, by specifying an outward-propagating wave (radiation condition); (iii) a finiteness condition that the wave field be finite at the critical layer. Since the Alfvén wave equation is of second order, only two conditions can in general be applied. It is shown, for example, that (ii) and (iii) are generally incompatible. If the conditions (i) and (iii) are chosen, i.e. an initial wave field is given and the radiation condition of outward propagation at infinity is met, then (ii) will not in general be met; thus the wave field would be singular at the critical layer, in the absence of dissipation, corresponding to the resonance of a linear undamped system. It is shown that in the presence of dissipation, either by fluid viscosity or Ohmic resistivity, the wave field would be finite at the critical layer, corresponding to the resonance of a linear damped system.

1. Introduction

It is well known that the solar wind has a critical point where the flow velocity equals the sound speed (Parker 1959). The point where the mean flow velocity equals the Alfvén speed may be similarly called the critical layer for Alfvén waves propagating with the wind; beyond the critical layer the mean flow velocity exceeds the Alfvén speed, and inward wave propagation is impossible. The presence of a background flow is one of the main effects affecting Alfvén waves in the solar wind (Belcher and Davis 1971; Denskat and Burlaga 1977; Verma and Roberts 1993), together with non-uniform, non-radial magnetic field, non-uniform mass density, and the presence of several ion species. Several combinations of these effects have been studied, with (Belcher 1971; Whang 1973; McKenzie et al. 1979; McKenzie 1994) or without (Heinemann and Olbert 1980; Velli 1993; Campos 1994; Campos and Gil 1995) use of the JWKB approximation. Considering a radial external magnetic field and radially stratified medium, three cases of radial mean flow are of particular interest in the solar wind case: (i) near the Earth the mean flow velocity is uniform, and the convected Alfvén wave equation applies (Heinemann and Olbert 1980; Barkhudarov 1991; Lou 1994; Campos and Gil 2002); (ii) closer to the Sun the mean flow velocity is a linear function of the radial distance, an extended form of the convected Alfvén wave equation applies (Campos and Isaeva 1999), again with a singularity at the critical layer; (iii) for any other mean flow velocity profile there is another singularity, namely a transition level at imaginary distance, where the radial derivative of the mean flow velocity equals the wave frequency (Campos and Isaeva 2003). In the present paper these two singularities, namely the critical layer and transition level, are considered for the mean flow velocity profile $U \sim r^{\nu}$ giving particular attention to the case $\nu = 1/2$, which corresponds to the mean flow velocity of the solar wind $U \sim r^{1/2}$ near the critical point.

There are several cases of atmospheric waves, specified by second-order equations with one critical layer: (i) Alfvén waves propagating in an isothermal atmosphere under a vertical uniform magnetic field in the presence of electrical resistance with constant rate-of-ionization (Campos 1983a, b, 1987); (ii) as before, with fluid viscosity also present, assuming that the viscous and resistive diffusivities are sufficiently small for their product to be negligible (Heyvaerts and Priest 1983; Nocera et al. 1984; Nocera et al. 1986; Campos 1988a, 1993a, b); (iii) as before, allowing for non-uniform horizontal magnetic field, decaying on twice the density scale height (Campos 1990); (iv) in the latter case, a critical layer also occurs for non-dissipative Alfvén waves in the presence of the Hall effect (McKenzie 1979; Campos and Isaeva 1992); (v) non-dissipative magnetosonic–gravity waves in a uniform horizontal magnetic field also have a critical layer (McKenzie 1973; Summers 1976; Nye and Thomas 1976; Adam 1977; Campos 1983c, 1985, 1988b); (vi) vertical acoustic–gravity waves have a critical layer in an isothermal atmosphere, in the presence of viscous dissipation (Yanowitch 1967; Campos 1983a, b, 1986); (vii) non-dissipative acoustic–gravity waves can have a critical layer in the presence of temperature gradients (Campos 1983d). In several of these cases of waves with a critical layer, the wave fields can be represented by Gaussian hypergeometric functions, because the wave equation has only three singularities, all regular: the critical layer, and the deep and high layers of the atmosphere. This is also the case for Alfvén waves in a radial wind which is either uniform (Heinemann and Olbert 1980; Campos and Gil 2002) or where the mean flow velocity is proportional to the

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radius (Campos and Isaeva 1999); the wave equation has three regular singularities, at the center, the critical layer and at infinity, and thus the wave fields are specified exactly in terms of Gaussian hypergeometric functions. In the present problem, in which the mean flow velocity varies as the square root of the radius, there are more than three singularities, e.g. the transition layer is a fourth singularity, and some singularities are irregular (i.e. the singularity of the differential equation at infinity); thus, although power-series solutions exist, they are more complicated than the Gaussian hypergeometric series, in particular in the recurrence formulas for the coefficients.

There are also cases of fourth-order wave equations with two critical layers, such as: (i) acoustic–gravity waves in an isothermal atmosphere, in the presence of viscosity and thermal conduction (Lyons and Yanowitch 1974); (ii) magnetosonic–gravity waves in an isothermal atmosphere, under a uniform horizontal magnetic field, in the presence of more than one of four possible dissipation mechanisms: electrical resistance, fluid viscosity, thermal radiation, and conduction (Alkahby and Yanowitch 1991). The critical layer for Alfvén waves in a radial flow is in some ways analogous to that of sound in a shear flow (Mohring et al. 1963; Campos and Serrão 1999; Campos et al. 1999; Campos and Kobayashi 2000). The critical layer for non-dissipative Alfvén waves is due to the mean flow, and does not occur in the presence of stratification alone (Ferraro and Plumpton 1958, 1965; Leroy 1980; Campos 1983c; An et al. 1989, 1990; Musielak et al. 1992; Musielak and Moore 1995; Rosner et al. 1991; Krogulec et al. 1994), nor when it is combined with: (i) either displacement (Leroy 1983) or Hall (Campos 1992) currents in a uniform magnetic field; or (ii) a non-uniform magnetic field (Oliver et al. 1992; Campos 1994; Campos and Gil 1995). The literature on Alfvén waves in the solar wind (Barkhudarov 1991; Hollweg 1990; Velli 1993; Lou 1994) has not emphasized the existence of a critical layer, with one notable exception (Heinemann and Olbert 1980). Moreover, the methods based on the use of Elsasser (1956) equations in their original form do not allow for the mass density of the background medium to vary on a scale of a wavelength. The present paper discusses the critical layer of Alfvén waves, without restriction on the gradients of background quantities, such as mass density or mean flow velocity. Thus it is limited neither by the WKB approaches nor by low-frequency approximations.

2. Alfvén wave equation for the velocity perturbation in a radial flow

The equations of non-dissipative magnetohydrodynamics are considered for perturbations of a radial flow and external magnetic field, which are transverse along parallels (Sec. 2.1); it is shown that they lead to a second-order Alfvén wave equation for the velocity perturbation (Sec. 2.2), which is discussed in the case of monopole external magnetic field and flow velocity varying as a power of radial distance (Sec. 2.3).

2.1. Linear perturbation of a radially inhomogeneous magnetohydrodynamic state

The equations of ideal (or non-dissipative) magnetohydrodynamics (MHD) are those of induction (1a) and momentum (1b):

$$\partial\vec{H}/\partial t + \nabla \wedge (\vec{H} \wedge \vec{V}) = 0, \tag{1a}$$

$$\partial\vec{V}/\partial t + (\vec{V}.\nabla)\vec{V} + (1/\rho)\nabla p = \vec{g} + (\mu/4\pi\rho)[\vec{H} \wedge (\nabla \wedge \vec{H})], \tag{1b}$$

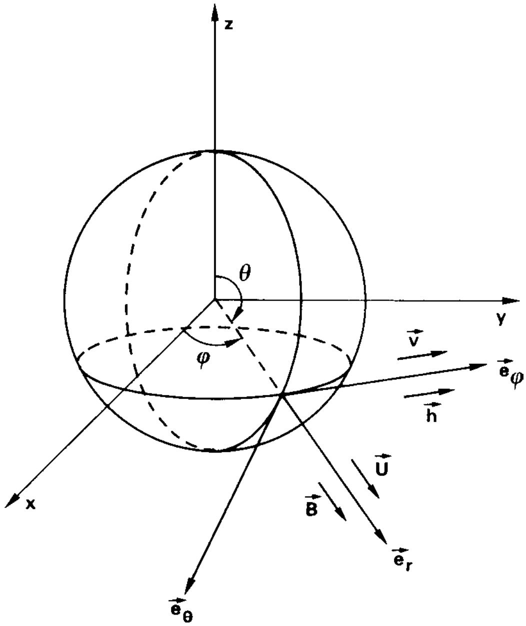

Figure 1. One-dimensional spherical Alfvén wave, propagating radially along the external magnetic field, parallel to the mean flow, with velocity and magnetic field perturbation along parallels.

where $\vec{V}$ is the velocity, $\vec{H}$ is the magnetic field, ρ is the mass density, p is the pressure, $\vec{g}$ is the acceleration of gravity, and μ is the magnetic permeability. For Alfvén waves, which are transversal and hence incompressible, the equation of state and energy are not needed. The total velocity $\vec{V}$ and magnetic field $\vec{H}$ are assumed to consist of a radial non-uniform, steady background component U, H, plus tangential unsteady, non-uniform perturbations along (Fig. 1) the parallels:

$$\vec{V}(\vec{x},t) = U(r)\vec{e}_r + v(r,t)\vec{e}_\varphi, \tag{2a}$$

$$\vec{H}(\vec{x},t) = B(r)\vec{e}_r + h(r,t)\vec{e}_\varphi. \tag{2b}$$

Substitution of (2a, b) in (1a, b) and linearization leads to

$$\frac{\partial h}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}(Brv - hrU) = 0, \tag{3a}$$

$$\frac{\partial v}{\partial t} + \frac{U}{r}\frac{\partial}{\partial r}(rv) = \frac{\mu B}{4\pi\rho}\frac{1}{r}\frac{\partial}{\partial r}(rh). \tag{3b}$$

Since the background medium is assumed to be inhomogeneous but steady, it is convenient to use a Fourier decomposition in time:

$$(r/r_0)v, h(r,t) = \int_{-\infty}^{\infty} F, G(r,\omega) e^{-i\omega t}\, d\omega, \tag{4a, b}$$

where F, G are the spectra, for a wave of frequency ω at radial distance r of respectively the velocity v (4a) and magnetic field h (4b) perturbations, multiplied by the radius r divided by a reference radius r_0 (e.g. the solar radius). Substitution of (4a, b)

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simplifies the induction (3a) and momentum (3b) equations to

$$i\omega G + (BF - UG)' = 0, \tag{5a}$$

$$i\omega F - UF' + (A^2/B)G' = 0, \tag{5b}$$

where prime denotes derivative with regard to the radius, e.g. $F' \equiv dF/dr$, and

$$A^2 \equiv \mu B^2/4\pi\rho \tag{6}$$

is the square of the Alfvén speed. It can be shown that (5a, b) hold equally well for three-dimensional Alfvén waves, with velocity and magnetic field perturbations both along parallels and meridians (Lou 1994), provided that v, h in (4a, b) are replaced by the radial components of respectively the vorticity and electric current (Campos and Isaeva 2002).

In order to eliminate between (5a, b) and obtain an Alfvén wave equation for the velocity perturbation, (5b) is solved for G' and substituted in (5a), namely:

$$i\omega G + (BF)' - U'G = UG' = (UB/A^2)(UF' - i\omega F). \tag{7}$$

Solving (7) for G, and then differentiating and substituting (5b), yields

$$\{[(BF)' - (UB/A^2)(UF' - i\omega F)]/(U' - i\omega)\}' = G' = (B/A^2)(UF' - i\omega F), \tag{8}$$

which involves only F, F', F'' and thus the Alfvén wave equation for the velocity perturbation is of second order.

2.2. *Background magnetic field, mass density, and mean flow velocity*

The Maxwell equation $\nabla \cdot \vec{B} = 0$ requires that the radial external magnetic field decays with the inverse square of the distance (12a):

$$B(r) = b(r_0/r)^2, \tag{9a}$$

$$\rho U r^2 = \text{constant} = \rho_0 u r_0^2, \tag{9b}$$

and conservation of the mass flux requires (9b), where B, ρ, U denote as before the external magnetic field, background mass density, and mean flow velocity, which are generally functions of the radius r, and b, ρ_0, u denote their respective constant values at the reference radius $r = r_0$. From the definition of Alfvén speed (6) it follows that $BU/A^2 \propto \rho U/B$ and $\rho U \sim r^{-2} \sim B$ by (9a, b) implying that $BU/A^2 =$ constant. This condition can be used to re-write (8) the Alfvén wave equation for the velocity perturbation, as

$$\begin{aligned}&(U' - i\omega)(A^2 - U^2)F'' \\ &\quad + [-U(U' - i\omega)^2 + (U' - i\omega)(2A^2B'/B - U'U + i\omega U) - U''(A^2 - U^2)]F' \\ &\quad + [i\omega(U' - i\omega)^2 + (U' - i\omega)A^2B''/B - U''(A^2B'/B + i\omega U)]F = 0,\end{aligned} \tag{10}$$

which simplifies to:

$$U = \text{constant}: \quad (A^2 - U^2)F'' + 2(A^2B'/B + i\omega U)F' + (\omega^2 + A^2B''/B)F = 0, \tag{11}$$

in the case of uniform flow (Heinemann and Olbert 1980; Barkhudarov 1991; Lou 1994; Campos and Gil 2002). The vanishing of the coefficient of F'' indicates that the Alfvén wave equation has two singularities:

$$U(r_1) = A(r_1), \tag{12a}$$

$$U'(r_2) = i\omega, \tag{12b}$$

namely: (i) a critical layer (12a), where the mean flow velocity equals the Alfvén speed, which occurs at a real distance $r = r_1$, and separates an outer region $U > A$, where Alfvén waves can propagate only outward, from an inner region $U < A$, where Alfvén waves can propagate inward and outward; (ii) a transition level (12b), where the radial derivative of the mean flow velocity equals the wave frequency, which occurs at 'imaginary' or complex radius, and thus could affect the wave field only if it limits the radius of convergence of a solution, e.g. a power-series solution around $r = r_1$ can have a radius of convergence not exceeding $R \leq |r_1 - r_2|$.

2.3. Critical layer and transition level for radial Alfvén waves

The mean flow velocity may be specified at will, e.g. if it is a power with exponent ν of the radial distance (13a):

$$U(r) = u(r/r_0)^{\nu}, \tag{13a}$$

$$\rho(r) = \rho_0(r/r_0)^{-2-\nu}, \tag{13b}$$

then it follows from the condition of mass conservation (9b) that the background mass density is given by (13b). For the solar wind between the solar corona and the Earth, the mean flow velocity varies as a power of the radial distance $U \sim r^s$ with $0 < \nu < 1$, suggesting that one proceeds with general exponent ν. The profile of the Alfvén speed (6), which follows from (9a), (13b), is (14a):

$$[A(r)]^2 = a^2(r/r_0)^{\nu-2}, \tag{14a}$$

$$a \equiv B\sqrt{\mu/4\pi\rho_0}, \tag{14b}$$

where (14b) is the value of the Alfvén speed at radius $r = r_0$.

Substitution of (9a), (12a), (14a) in (10), and the introduction of a dimensionless radial distance (15a):

$$s \equiv r/r_0, \tag{15a}$$

$$\Phi(s) = F(r;\omega), \tag{15b}$$

leads, for general exponent ν, to the wave equation

$$\begin{aligned}(1 - N^2 s^{\nu+2})D(s)\Phi'' &+ \{i\Omega N s^2[D(s)]^2 - (4/s + \nu N^2 s^{\nu+1} - i\Omega N s^2)D(s)\\ &- i\nu(\nu-1)(N/\Omega)s^{\nu-2}(1 - N^2 s^{\nu+2})\}\Phi' + \{\Omega^2 s^{2-\nu}[D(s)]^2 + (6/s^2)D(s)\\ &+ \nu(\nu-1)N^2 s^{\nu} + 2i\nu(\nu-1)(N/\Omega)s^{\nu-3}\}\Phi = 0,\end{aligned} \tag{16a}$$

where

$$D(s) \equiv 1 + i\nu(N/\Omega)s^{\nu-1}, \tag{16b}$$

and only two dimensionless parameters appear:

$$\Omega \equiv r_0\omega/a, \tag{17a}$$

$$N \equiv u/a, \tag{17b}$$

namely, the dimensionless frequency (17a) and the initial Alfvén number (17b), i.e. the ratio of flow velocity to Alfvén speed at the reference radius. Introducing the initial wavenumber $k = \omega/a$, which is a wavenumber at the reference radius r_0, and the corresponding wavelength $\lambda = 2\pi/k = 2\pi a/\omega$, it follows that $\Omega \equiv 2\pi r_0/\lambda$. The length scale of variation of mass density (13b) is $1/L = -d(\log\rho)/dr = (\nu+2)/r$,

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and thus $\Omega \sim L/\lambda$. It follows that the literature which uses the JWKB approximation $L^2 \gg \lambda^2$ is restricted to $\Omega^2 \gg 1$; the literature using Elsasser equations assumes $L \gg \lambda$, and thus (Campos et al. 1999) has the stronger restriction to $\Omega \gg 1$. In the present approach to Alfvén waves in an inhomogeneous moving medium, there is no restriction on Ω, and the cases $\Omega \leqslant 1$, which imply a stronger interaction with the background medium, are included.

The factor (16b) corresponds to the transition level (12b), which for the velocity profile (13a) occurs at complex distance:

$$D(s_2) = 0: \quad s_2 = (i\Omega/\nu N)^{1/(\nu-1)}, \quad r_2 = r_0(i\omega r_0/\nu u)^{1/(\nu-1)}. \tag{18a, b}$$

The transition level does not occur in the two cases only: (i) uniform mean flow:

$$\nu = 0: \quad (1 - N^2 s^2)s^2\Phi'' + 2(i\Omega N s^3 - 2)s\Phi' + (6 + \Omega^2 s^4)\Phi = 0, \tag{19}$$

corresponding to the convected Alfvén wave equation (Heinemann and Olbert 1980; Barkhudarov 1991; Lou 1994, Campos and Gil 2002); (ii) mean flow velocity proportional to the radius

$$\nu = 1: \quad (1 - N^2 s^3)s^2\Phi'' + 2(-2 + iNs^3(\Omega + iN))s\Phi' + (6 + \Omega s^3(\Omega + iN))\Phi = 0, \tag{20}$$

which has some extra terms relative to the convected Alfvén wave equation (Campos and Isaeva 1999).

The coefficient of Φ'' in the second curved brackets in (10) specifies the position of the critical layer (12a), for the mean flow velocity (13a) and Alfvén speed (14a) profiles:

$$s_1 = N^{-1/(1+\nu/2)}, \tag{21a}$$

$$r_1 = r_0(a/u)^{1/(1+\nu/2)}. \tag{21b}$$

The critical layer does not occur only if the mean flow velocity (13a) and Alfvén speed (14a) have the same exponent as a function of the radius $\nu = \nu/2 - 1$, i.e.

$$\begin{aligned} \nu = -2: \quad & (1 - N^2)D_1(s)\Phi'' \\ & + \{i\Omega N s^2[D_1(s)]^2 - [2(2 - N^2)/s - i\Omega N s^2]D_1(s) - 6i(N/\Omega)s^{-4}(1 - N^2)\}\Phi' \\ & + \{\Omega^2 s^4[D_1(s)]^2 + (6/s^{-2})D_1(s) + 6N^2/s^2 + 12i(N/\Omega)/s^5\}\Phi = 0, \end{aligned} \tag{22a}$$

where

$$\nu = -2: \quad D_1(s) \equiv 1 - 2i(N/\Omega)s^{-3}. \tag{22b}$$

In the particular case $N = 1$, the Alfvén speed equals the mean flow velocity at all points:

$$\begin{aligned} \nu = -2, N = 1: \quad & [iN\Omega s^2 D_0(s) - 2/s + i\Omega s^2]D_0(s)\Phi' \\ & + \{\Omega^2 s^4[D_0(s)]^2 + (6/s^2)D_0(s) + 6/s^2 + 12(i/\Omega)/s^5\}\Phi = 0, \end{aligned} \tag{23a}$$

where

$$\nu = -2, N = 1: \quad D_0(s) \equiv 1 - 2(i/\Omega)s^{-3}. \tag{23b}$$

In this case the wave equation reduces to first order because Alfvén waves can only propagate in the direction of the flow (Campos and Isaeva 2003).

3. Singularities of the wave equation, including the critical layer

The Alfvén wave equation has a critical layer, where the flow velocity equals the Alfvén speed, and since it is a regular singularity, the wave field may have power-law or logarithmic behaviour in its vicinity (Sec. 3.3). In the case of flow velocity varying as the square root of the radius, the wave field may be obtained for all values of the radius by using series expansions around two more singularities, namely a regular one (Sec. 3.1) at the center $r = 0$ and an irregular one (Sec. 3.2) at infinity $r = \infty$; these solutions are respectively ascending and ascending–descending power series of the radius.

3.1. Regular singularity and ascending power series near center

The velocity profile of the solar wind is not a simple power-law function of the radial distance. It may be approximated by a power law, with different exponents in regions of particular interest: (i) exponent $\nu = 0$ or uniform mean flow velocity near the Earth (Heinemann and Olbert 1980; Barkhudarov 1991; Lou 1994; Campos and Gil 2002): (ii) near the Sun the velocity of the solar wind cannot be considered as uniform, and a better approximation (Campos and Isaeva 1999) is proportional to the radius or exponent $\nu = 1$; (iii) near the critical layer an intermediate value of the exponent $\nu = 1/2$ is more appropriate. The case (iii) is considered in the sequel, and a fuller discussion of the background solar wind model is given in Sec. 6.1. Thus the application to the solar wind near to the critical point suggests that the differential equation (16a) be considered with an exponent $\nu = 1/2$ corresponding to a mean flow increasing as the square root of distance (24a):

$$\nu = 1/2: \quad U(s) = u\sqrt{s}, \quad \rho(s) = \rho_0 s^{-5/2}, \quad A(s) = a s^{-3/4}, \tag{24a, b, c}$$

corresponding to the mass density (13b) is equivalent to (24b) and the Alfvén speed (14a) is equivalent to (24c). The differential equation (16a, b) with $\nu = 1/2$ has radicals in the coefficients, which can be eliminated by the change of variable:

$$z = \sqrt{s}, \tag{25a}$$

$$\Psi(z) \equiv \Phi(s), \tag{25b}$$

leading to the differential equation

$$(z + iN/2\Omega)(1 - N^2 z^5) z^2 \Psi'' - (4iN/\Omega + 97 + iN^3 z^5/2\Omega + N^2 z^6 - 2i\Omega N z^7) z\Psi'$$
$$+ 2(5iN/\Omega + 12z - N^6 z^6 + 2i\Omega\omega z^7 + 2\Omega^2 z^8)\Psi = 0, \tag{26}$$

where the coefficients are polynomials of degree up to eight.

Although in solar–terrestrial context the model is relevant beyond a solar radius and close to the critical point, the singularity of the differential equation (26) at $r = 0$, affects the wave field up to the nearest singularity, which may be either the critical layer (21b), which is equivalent to (27a), or the transition level (18b), which is equivalent to (27b),

$$r < |r_1|, |r_2|; \quad r_1 = r_0 N^{-4/5}, \quad r_2 = -(N/2\Omega)^2. \tag{27a, b}$$

Thus it is necessary to consider the solution of (26) by expanding in a Frobenius–Fuchs series about the center:

$$\Psi_\sigma(z) = \sum_{n=0}^{\infty} a_n(\sigma) z^{n+\sigma}. \tag{28}$$

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This solution exists, if $z = 0$ is a regular singularity (Forsyth 1929) of the differential equation (26); this is the case because when (26) is written in the form:

$$\sum_{m=0}^{2} z^m f_m(z) \Psi^{(m)}(z) = 0, \tag{29}$$

the functions $f_m(z)$ are analytic at $z = 0$, namely

$$f_2(z) \equiv iN/2\Omega + z - iN^3 z^5/2\Omega - N^2 z^6, \tag{30a}$$

$$f_1(z) \equiv -4iN/\Omega - 9z - iN^3 z^5/2\Omega - N^2 z^6 + 2i\Omega N z^7, \tag{30b}$$

$$f_0(z) \equiv 10iN/\Omega + 24z - 2N^2 z^6 + 4iN\Omega z^7 + 4\Omega^2 z^8, \tag{30c}$$

i.e. they are polynomials of degree not exceeding eight:

$$m = 0, 1, 2: \quad f_m(z) = \sum_{r=0}^{8} f_{m,r} z^r, \tag{31}$$

with coefficients

$$f_{0,r} \equiv \{10iN/\Omega, 24, 0, 0, 0, 0, -2N^2, 4iN\Omega, 4\Omega^2\}, \tag{32a}$$

$$f_{1,r} \equiv \{-5iN/\Omega, -9, 0, 0, 0, iN^3/2\Omega, -N^2, 2i\Omega N\}, \tag{32b}$$

$$f_{2,r} \equiv \{iN/2\Omega, 1, 0, 0, 0, -iN^3/2\Omega, -N^2, 0, 0\}. \tag{32c}$$

Substituting (28), (31) into (29) yields

$$0 = \sum_{n=0}^{\infty} a_n(\omega) \sum_{m=0}^{2} \sum_{r=0}^{8} f_{m,r} (n+\sigma)_m z^{n+\sigma+r} = 0, \tag{33}$$

where the Pochhammer symbol

$$(n+\sigma)_m \equiv (n+\sigma)(n+\sigma-1)\cdots(n+\sigma-m+1) \tag{34}$$

is used.

Re-writing (33) in the form

$$0 = \sum_{n=0}^{\infty} z^{n+\sigma} \left\{ \sum_{r=0}^{5} a_{n-r}(\sigma) \sum_{m=0}^{2} f_{m,r} (n+\sigma-r)_m \right\}, \tag{35}$$

and noting that the coefficients of all powers of z must vanish, leads to the recurrence formula for the coefficients:

$$0 = \sum_{n=0}^{\infty} a_{n-r}(\alpha) \sum_{m=0}^{2} f_{m,r} (n+\sigma-r)(n+\sigma-r-1)\cdots(n+\sigma-r-m+1). \tag{36}$$

Using (32a, b, c), this can be written explicitly:

$$\begin{aligned} &\{(iN/2\Omega)[20 + (n+\sigma)(n+\sigma-9)]\} a_n(\sigma) \\ &\quad = -(24 + (n+\sigma-1)(n+\sigma-11)) a_{n-1}(\sigma) \\ &\qquad + (iN^3/2\Omega)(n+\sigma-5)^2 a_{n-5}(\sigma) + N^2[2 + (n+\sigma-6)^2] a_{n-6}(\sigma) \\ &\qquad - 4iN\Omega(n+\sigma-6) a_{n-7}(\sigma) - 4\Omega^2 a_{n-8}(\sigma). \end{aligned} \tag{37}$$

Setting $n = 0$, and noting that $0 = a_{-1} = a_{-2} = \ldots$, yields the indicial equation:

$$n = 0: \quad a_0(20 + \sigma(\sigma - 9)) = 0, \tag{38}$$

where $a_0 \neq 0$ otherwise a trivial solution would result, since $a_0 = 0$ implies $0 = a_1 = a_2 = \ldots$ by (37), and $\Phi = 0$ from (28). The indicial equation (38) has two roots $\sigma = 4, 5$, each corresponding to one particular integral (28) of the differential equation (26). Setting $\sigma = 4$ in (37) yields

$$\begin{aligned}\sigma = 4 : (iN/2\Omega)n(n-1)a_n(4) &= -(n-1)(n-3)a_{n-1}(4)\\ &+ i(N^3/2\Omega)(n-1)2a_{n-5}(4)\\ &+ N^2(n^2-4n+6)a_{n-6}(4) - 4iN\Omega(n-2)a_{n-7}(4) - 4\Omega^2 a_{n-8}(4). \end{aligned}\tag{39}$$

Setting $n = 1$ yields $0.a_1(4) = 0.a_0(4)$, so that both $a_0(4)$ and $a_1(4)$ are arbitrary and independent.

The coefficients

$$a_0(4) = 1, \quad a_1(4) = 0: \quad a_n(4) \equiv d_n, \tag{40a}$$

specify a solution

$$\Psi_4(z) = \sum_{n=0}^{\infty} d_n z^{n+4}, \tag{40b}$$

starting with the fourth power; this is linearly independent of the solution:

$$\Psi_5(z) = \sum_{n=0}^{\infty} e_n z^{n+5}, \tag{41a}$$

with coefficients

$$a_0(5) = 0, \quad a_1(5) = 1: \quad a_{n+1}(5) \equiv e_n, \tag{41b}$$

which starts with the fifth power. Recalling (25a, b); (15a, b), the wave fields corresponding to (40b), (41a) are, respectively,

$$F_4(r;\omega) = (r/r_0)^2 \sum_{n=0}^{\infty} d_n (r/r_0)^{n/2}, \tag{42a}$$

$$F_5(r,\omega) = (r/r_0)^{5/2} \sum_{n=0}^{\infty} e_n (r/r_0)^{n/2}. \tag{42b}$$

The general integral

$$r < |r_1|, |r_2|: \quad F(r;\omega) = C_4 F_4(r;\omega) + C_5 F_5(r;\omega), \tag{42c}$$

where C_4, C_5 are arbitrary constants, is equivalent to the sum of (40b) and (41a), with $d_0 = C_4$, $e_0 = C_5$, respectively, in (40a), (41b).

Introducing the spectrum of the velocity perturbation

$$v(r,t) = \int_{-\infty}^{+\infty} W(r;\omega) e^{-i\omega t}\, d\omega, \tag{43a}$$

which is related to (4a):

$$W(r;\omega) = (r_0/r) F(r;\omega), \tag{43b}$$

it is clear that the wave field vanishes at the center like r/r_0 for the first term in (42c), and like $(r/r_0)^{3/2}$ for the second; the model is not physically relevant to the interior of the Sun, but the wave field (42c) is specified up to the critical layer (21b) which is equivalent to (44a) and transition level (18b) which is equivalent to (44b):

$$r_1 = r_0(a/u)^{4/5}, \tag{44a}$$

$$r_2 = -r_0(u/2r_0\omega)^2, \tag{44b}$$

by the series solutions (42a, b); thus the nature of the singularity of the model at the center affects the wave fields in the solar wind up to the nearer of these two singularities.

3.2. Irregular singularity and asymptotic ascending–descending series solutions

In order to consider the wave fields at large distance $r > |r_1|, |r_2|$, it is necessary to expand around the point at infinity $s = \infty$, which is mapped to the origin $\zeta = 0$, by the transformation

$$\zeta = 1/s, \tag{45a}$$

$$Q(\zeta) = \Psi(z), \tag{45b}$$

which transforms the wave equation (26) into

$$(1+iN\zeta/2\Omega)(1-N^2\zeta^{-5})\zeta Q''+(11+5iN\zeta/\Omega+iN^3\zeta^{-4}/2\Omega-N^2\zeta^{-5}-2i\Omega N\zeta^{-6})Q'$$
$$+(10iN/\Omega+24\zeta^{-1}-2N^2\zeta^{-6}+4i\Omega N\zeta^{-7}+4\Omega^2\zeta^{-8})Q=0. \tag{46}$$

Again the model does not correspond to the solar wind at large distances, but the nature of the singularity of the differential equation at infinity affects the wave fields in the solar wind beyond the farthest singularity at finite distance.

The differential equation (46) can be written in the standard form, comparable to (29), namely

$$\sum_{m=0}^{2} \zeta^m \gamma_m(\zeta) Q^{(m)}(\zeta) = 0, \tag{47}$$

where the coefficients involve

$$\gamma_2(\zeta) \equiv (1-N^2\zeta^{-5})(1/\zeta+iN/2\Omega) \sim \mathrm{O}(\zeta^{-6}), \tag{48a}$$

$$\gamma_1(\zeta) \equiv 11/\zeta+5iN/\Omega-iN^3\zeta^{-5}/2\Omega-N^2\zeta^{-6}-2i\Omega N\zeta^{-7} \sim \mathrm{O}(\zeta^{-7}), \tag{48b}$$

$$\gamma_0(\zeta) \equiv 10iN/\Omega+24\zeta^{-1}-2N^2\zeta^{-6}+4iN\Omega\zeta^{-7}+4\Omega^2\zeta^{-8} \sim \mathrm{O}(\zeta^{-8}), \tag{48c}$$

which, unlike (30a–c), are not analytic functions at $\zeta = 0$. Thus $\zeta = 0$ (or $z = \infty$) is an irregular singularity of the differential equation (46) (or (26)), and in its neighborhood it is not possible to find two linearly independent particular integrals in the form of a Frobenius–Fuchs expansion:

$$Q_\nu(\zeta) = \sum_{n=0}^{\infty} b_n(\nu)\zeta^{n+\nu} = \sum_{n=0}^{\infty} b_n(\sigma)s^{-n-\nu}, \tag{49}$$

The question of whether no solution of this type (i.e. a series of descending powers of the radius) exists, or one solution exists, remains open.

In order to clarify this, only the lowest powers in the coefficients of the differential equation (46) are needed, namely

$$-N^2\zeta^4[1+O(\zeta)]Q''-2i\Omega N\zeta^{-6}[1+O(\zeta)]Q'+4\Omega^2\zeta^{-8}[1+\mathrm{O}(\zeta)]Q=0. \tag{50}$$

The reason is that substitution of the Frobenius–Fuchs series (49) yields

$$0 = [1 + \mathrm{O}(\zeta)] \sum_{n=0}^{\infty} b_n(\nu)\{-N^2(n+\nu)(n+\nu-1)\zeta^{n+\nu-6} \\ -2i\Omega N(n+\nu)\zeta^{n+\nu-7} + 4\Omega^2\zeta^{n+\nu-8}\}, \quad (51)$$

which can be re-arranged

$$0 = \sum_{n=0}^{\infty} \zeta^{n+\nu}\{4\Omega^2 b_{n+8}(\nu) + \mathrm{O}(b_{n+7}, \ldots, b_n)\}. \quad (52)$$

Equating to zero each power of ζ leads to the recurrence formula for the coefficients

$$\Omega^2 b_{n+8}(\nu) = \mathrm{O}(b_{n+7}, \ldots, b_n). \quad (53)$$

Only the term involving the highest coefficient b_{n+8} has been written explicitly, because it alone specifies the indicial equation

$$n = -8: \quad 4\Omega^2 b_0 = 0. \quad (54)$$

The implication is that the indicial equation is determined only by the lowest powers (50) in the coefficients (48a, b, c) of the differential equation (46). The number of roots of the indicial equation specifies the number of regular integrals (49) of the differential equation. Since the indicial equation (54) does not involve ν, it has no root, and the differential equation (46) has no regular integral, namely (54) implies that $b_0 = 0$, and hence from (52) all $b_n = 0$, leading to a trivial solution $Q(\zeta) = 0$ in (49).

There must be two linearly independent particular integrals, because the differential equation (46) is of order two. Since they cannot be of regular or Frobenius–Fuchs type, they must have (Forsyth 1902) an essential singularity at infinity, i.e. a Laurent series is an adequate representation:

$$r > |r_1|: \quad Q(\zeta) = \sum_{n=-\infty}^{\infty} b_n(\nu)\zeta^{n+\nu}. \quad (55)$$

Substituting (55) into (46) leads to the same expression (53) as before, which is no longer a recurrence relation, because it concerns a doubly infinite sequence of coefficients $b_0, b_{\pm1}, b_{\pm2}, \ldots$ in the ascending–descending series (55). Now (53) is an infinite system of linear equations, with non-trivial solution for b_n if and only if the determinant vanishes; this is the indicial equation, specifying ν, as its roots $\nu_\pm$. After substitution of each root $\nu_\pm$, (53) becomes an infinite linear inhomogeneous system of equations, whose solution specifies all $b_{\pm1}, b_{\pm2}, \ldots$ in terms of b_0. These infinite systems of equations can be solved approximately by truncation. Further details on finding a complete set of linearly independent integrals of a differential equation in the vicinity of an irregular singularity can be found elsewhere (Forsyth 1902; Ince 1926). The solutions of the Alfvén wave equation within ($r < r_1$ in Sec. 3.1), and outside ($r > r_1$ in Sec. 3.2) the critical layer, both fail to converge at the critical layer, and cannot be used to study the properties of the wave field there. In order to study the properties of Alfvén waves near the critical layer it is necessary to consider the latter as the singularity around which series solutions are sought. This is done next.

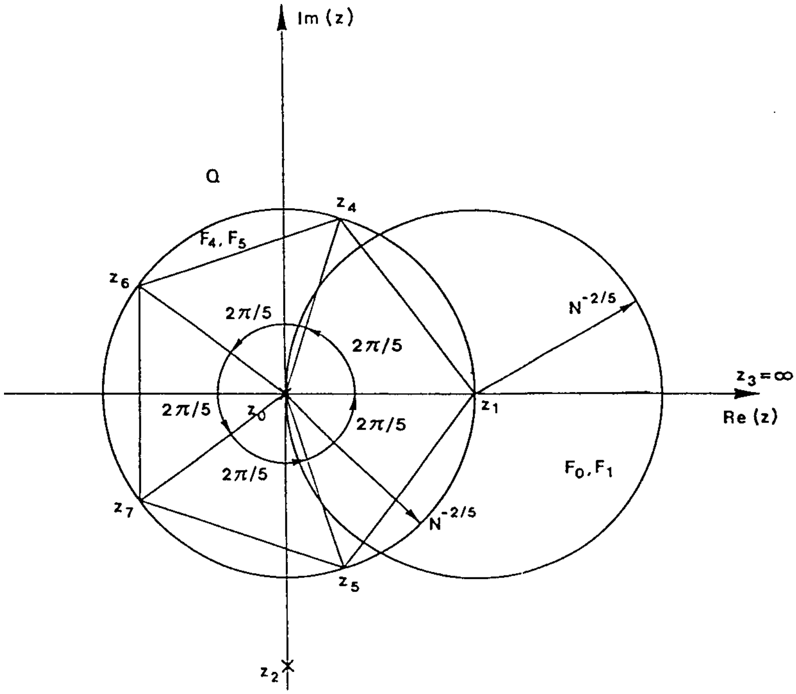

Figure 2. Seven singularities of the differential equation, of which the center z_0, critical layer z_1 and point at infinity z_2 can be used to obtain series solutions covering the whole positive real z axis, which is the physical region.

3.3. Wave fields in the neighborhood of the critical layer

The differential equation (26) has eight singularities, namely (Fig. 2) the origin (56a), the point at infinity (56b), the transition level (56c)

$$z_0 = 0, \tag{56a}$$

$$z_3 = \infty, \tag{56b}$$

$$z_2 = -iN/(2\Omega) = -iu/(2\omega r_0), \tag{56c}$$

and the five roots

$$z_{1,4,5,6,7} = \sqrt[5]{1}\, N^{-2/5} = N^{-2/5}\{1, e^{\pm i2\pi/5}, e^{\pm i4\pi/5}\}, \tag{57a, b, c, d, e}$$

of which the one on the real axis z_1 is the critical layer. The roots (57a, b, c, d, e) lie (Fig. 2) on a circle of radius $N^{-2/5}$ and center at the origin, at the vertices of a regular pentagon, with one vertex on the real axis. The length of the side of the inscribed pentagon is larger than the radius of the circle, because it is larger than the side of the inscribed hexagon, which is equal to the radius. Thus the singularities z_2, z_3, z_4, z_5 are farther from z_1 than the origin, and the series expansion about the critical layer has a radius of convergence up to the origin. The solution around the origin (Sec. 3.1) has region of validity (42c); the solution around the point at infinity (Sec. 3.2) is valid beyond the critical layer. It follows (Fig. 2) that the solution around the critical layer completes the coverage of the physical region and the remaining five singularities need not be considered.

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The critical layer is placed at the origin by the change of variable

$$\xi \equiv 1 - N^{2/5}z = 1 - \sqrt{r/r_1}, \tag{58a}$$

$$R(\xi) \equiv \Psi(z), \tag{58b}$$

which transforms the differential equation (26) to the standard form

$$\sum_{m=0}^{2} \xi^m q_m(\xi) R^{(m)}(\xi) = 0, \tag{59}$$

where the critical layer $\xi = 0$ is a regular singularity, since the coefficients $q_m(\xi)$ are all analytic at $\xi = 0$, namely

$$q_2(\xi) \equiv (1-\xi)^2[1-\xi+i/(2\alpha)](5-10\xi+10\xi^2-5\xi^3+\xi^4), \tag{60a}$$

$$q_1(\xi) \equiv (1-\xi)[4i/\alpha + 9(1-\xi) + (i/2\alpha)(1-\xi)^5 + (1-\xi)^6 - 2i\alpha(1-\xi)^7], \tag{60b}$$

$$q_0(\xi) \equiv \xi[10i/\alpha + 24(1-\xi) - 2(1-\xi)^6 + 4i\alpha(1-\xi)^7 + 4\alpha^2(1-\xi)^8], \tag{60c}$$

and involve only one dimensionless parameter

$$\alpha \equiv \Omega N^{-7/5} = (\omega r_0/a)(u/a)^{-7/5} = \omega r_0 a^{2/5} u^{-7/5}. \tag{61}$$

Thus the dimensionless frequency (17a) and Alfvén number (17b) appear only in the combination (61), and this is the only parameter in our problem.

All coefficients of (60a, b, c) of the differential equation (59) are polynomials of degree not exceeding nine:

$$m = 0, 1, 2, 3: \quad q_m(\xi) = \sum_{r=0}^{9} \beta_{m,r}\xi^r, \tag{62}$$

where the coefficients $\beta_{m,r}$ are given in the Appendix. Since the critical layer $\xi = 0$ is a regular singularity, in its vicinity two linearly independent Frobenius–Fuchs expansions must exist:

$$R_\vartheta(\xi) = \sum_{n=0}^{\infty} c_n(\vartheta)\xi^{n+\vartheta}. \tag{63}$$

Substituting (62), (63) in (59) leads as before (33), (35) to a recurrence formula analogous to (36), replacing a, α by c, β and $r = 0, \dots, 8$ by $r = 0, \dots, 9$:

$$0 = \sum_{r=0}^{9} c_{n-r}(\vartheta) \sum_{m=0}^{2} \beta_{m,r}(n+\vartheta-r)(n+\vartheta-r-1)\cdots(n+\vartheta-r-m+1). \tag{64}$$

The recurrence formula is written explicitly in the Appendix since what is needed next is only the indicial equation, which is obtained setting by $n = 0$:

$$0 = c_0(\vartheta)\{\beta_{0,0} + \vartheta[\beta_{1,0} + (\vartheta-1)\beta_{2,0}]\}, \tag{65}$$

from (60a–c) it follows that

$$\beta_{0,0} \equiv q_0(0) = 0, \tag{66a}$$

$$\beta_{1,0} = q_1(0) = 10 - 2i\alpha + 9i/2\alpha, \tag{66b}$$

$$\beta_{2,0} \equiv q_2(0) = 5 + 5i/2\alpha, \tag{66c}$$

so that the indicial equation is

$$0 = \vartheta\{5\vartheta[1 + i/(2\alpha)] + 5 - 2i\alpha + 2i/\alpha\}, \tag{67}$$

and it has a root zero and a complex root.

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The first root $\vartheta = 0$, from (63), is given by

$$R_0(\xi) = \sum_{n=0}^{\infty} c_n(0)\xi^n, \tag{68}$$

which corresponds to a wave field (58a, b); (25a, b); (15a, b); (43b):

$$W_0(r;\omega) = (r_0/r)\sum_{n=0}^{\infty} c_n(0)(1-\sqrt{r/r_1})^n, \tag{69}$$

which is finite at the critical layer

$$c_0(1) \equiv 1: \quad F_0(r_1,\omega) = r_0/r_1 = N^{-4/5} = (a/u)^{4/5}. \tag{70}$$

The constant $c_0 = 1$ has been incorporated in the arbitrary constants of integration in the general integral

$$W(r;\omega) = C_0 W_0(r;\omega) + C_1 W_1(r;\omega), \tag{71}$$

where the second linearly independent particular integral corresponds (67) to the index

$$\vartheta_1 = (2i\alpha - 5 - 2i/\alpha)/(5 + 5i/2\alpha) = 2(i\alpha - 2)/5, \tag{72}$$

where the real part $\mathrm{Re}(\sigma_1)$ is negative, and hence the wave field is singular at the critical layer

$$c_0(\vartheta_1) \equiv 1: \quad W_1(r;\omega) = (r_0/r)\sum_{n=0}^{\infty} c_n(\sigma_1)(1-\sqrt{r/r_1})^{n-4/5+2i\alpha/5}. \tag{73}$$

The leading term of the wave field (73) as the critical layer is approached is

$$F_1(r,\omega) \propto (1-\sqrt{r/r_1})^{-4/5+2i\alpha/5} = (1-\sqrt{r/r_1})^{-4/5}\exp[(2i\alpha/5)\log(1-\sqrt{r/r_1})], \tag{74}$$

where the first factor confirms that the amplitude is singular at the critical layer since $\mathrm{Re}(\sigma_1) = -4/5 < 0$; the second factor in (74) has a discontinuity across the critical layer:

$$\exp\{(2i\alpha/5)\log\xi\} = \exp\{(2i\alpha/5)\log|\xi|\} \times \begin{cases} 1 & \text{if } r_1 > r, \quad (75\text{a}) \\ \exp(-2\pi\alpha/5) & \text{if } r_1 < r, \quad (75\text{b}) \end{cases}$$

where the sign in (75b) is decided by giving the Alfvén speed a small positive imaginary part in (76a):

$$a = \bar{a} + i\varepsilon, \tag{76a}$$

$$\exp(i\omega r/a) = \exp[i\omega r/\bar{a} + \varepsilon\omega r/\bar{a}^2 + \mathrm{O}(\varepsilon^2)], \tag{76b}$$

so that for an outward-propagating wave (4a) the spatial phase term (76b) has a slow growth. From (58a), (17b), (76a) it follows that

$$\xi = 1 - N^{2/5}z = 1 - z(u/a)^{2/5} = 1 - z(u/\bar{a})^{2/5} + iz(u/\bar{a})^{2/5}(2\varepsilon/5\bar{a}) + \mathrm{O}(\varepsilon^2), \tag{77}$$

so that $\mathrm{Im}(\xi) > 0$ as $\varepsilon \to 0+$, and the sign should be chosen in $\log\xi = \log|\xi| + i\pi$ in (75b).

From (75a, b) it follows that the wave decreases in amplitude, as it crosses the critical layer in the outward direction, by a factor

$$T = \exp(-2\pi\alpha/5) = \exp\left\{-(2\pi\alpha/5)\omega r_0 a^{2/5} u^{-7/5}\right\}, \tag{78}$$

where α is given by (61), and is proportional to the frequency

$$T = e^{-\omega/\omega_*}, \tag{79a}$$

$$\omega_* = (u/r_0)(5/2\pi)(u/a)^{2/5}; \tag{79b}$$

thus the transmission factor (79a) depends on the ratio of wave frequency ω to the filtering frequency (79b). For $\omega \ll \omega_*$ there is a total transmission $T \sim 1$, for $\omega \gg \omega_*$ no transmission $T \ll 1$, and for $\omega \sim \omega_*$ partial transmission. Note that the significance of the transmission coefficient is reduced because: (i) it does not apply to the wave field (69) which is finite at the critical layer (70); (ii) it applies only to the wave field (73), which is singular at the critical layer (74), but in this case the transmission coefficient is actually determined by the decay of the wave field on either side. This suggests plotting the wave field near to the critical layer. Before proceeding to do so we consider the radiation condition at infinity.

4. Asymptotic wave fields and radiation condition

The asymptotic wave fields are written explicitly (Sec. 4.1) to show that they consist of inward- and outward-propagating waves (Sec. 4.2). It follows that the radiation condition (Sec. 4.3) is generally incompatible with the condition of finite wave field at the critical layer.

4.1. Wave fields in the JWKB approximation

The JWKB approximation (Jeffreys 1924; Wentzel 1926; Kramers 1926; Brillouin 1926) assumes that the wave frequency is sufficiently high for the medium to be uniform on the scale of a wavelength; this is true at large distances $r \to \infty$, when the mean flow velocity (24a) is much larger than the Alfvén speed (24c), and thus $N = U/A \to \infty$. The wave equation (26) is approximated by:

$$\Omega \gg N \to \infty: \quad -N^2 z^8 \Psi'' - 2i\Omega N z^8 \Psi' + 4\Omega^2 z^8 \Psi = 0, \tag{80}$$

which has constant coefficients

$$\Psi'' - 2i(\Omega/N)\Psi' - (2\Omega/N)^2 \Psi = 0, \tag{81}$$

and hence exponential solutions

$$\Psi(z) = e^{\vartheta z}, \tag{82a}$$

$$\vartheta^2 - 2i(\Omega/N)\vartheta + (2i\Omega/N)^2 = 0. \tag{82b}$$

The two roots specify the wave fields

$$\vartheta_\pm = i\Omega/N \pm 2\sqrt{3}\Omega/N: \quad \Psi_\pm \sim \exp\{1(\Omega/N)z\}\exp\{\pm 2\sqrt{3}(\Omega/N)z\}, \tag{83a, b}$$

which correspond (15a, b); (25a, b); (17a, b); (43b) to velocity perturbations

$$W_\pm(r;\omega) \sim \exp\{i(r_0\omega/u)\sqrt{r/r_0}\}(r/r_0)\exp\{\pm 2\sqrt{3}(r_0\omega/u)\sqrt{r/r_0}\}. \tag{84a, b}$$

Note that both wave fields propagate outward, because the mean flow is super-alfvénic; in fact, since the mean flow velocity $U(r) = u\sqrt{r/r_0} \gg A(r)$ is much larger than the Alfvén speed, the former appears in the wave fields:

$$U(r) = u\sqrt{r/r_0}: \quad W_\pm(r;\omega) \sim (r_0/r)\exp\{i[r_0\omega/U(r)](r/r_0)\}, \tag{85}$$

as the wave speed. In order to apply the radiation condition, it is necessary to consider the wave speed relative to the mean flow, i.e. go beyond the approximation

(85). Note that since the JWKB solution is valid only at large distance it cannot be matched to the initial wave field.

4.2. Exact solution as superposition of inward- and outward-propagating waves

It was shown (Sec. 3.2) that the point at infinity $r = \infty$ is an irregular singularity of the Alfvén wave equation (46), and thus the wave field has an essential singularity there. The nature of this essential singularity was identified (Sec. 4.1) by noting that the leading term of the wave field at infinity (84a, b) corresponds to the convection of Alfvén waves by the mean flow. Noting (83a, b); (45a), the leading term of the wave field at infinity is used in the change of dependent variable:

$$Q(\zeta) = \exp(\vartheta_\pm/\zeta)P(\zeta), \tag{86}$$

which transforms the differential equation (46) to:

$$\begin{aligned}(1 + iN\zeta/2\Omega)&(1 - N^2/\zeta^s)\zeta P'' + [siN\zeta/\Omega + 12 \pm i2\sqrt{3} - 2\vartheta_\pm\zeta^{-1} - iN^3\zeta^{-4}/2\Omega\\ &+ (-1 \pm i2\sqrt{3} - N^2)\zeta^{-5} \mp 4\sqrt{3}\Omega N\zeta^{-6}]P' + \{10iN/\Omega + (19 \mp i2\sqrt{3})\zeta^{-1}\\ &+ (-23/2 \pm i\sqrt{3})\vartheta_\pm\zeta^{-2} + (\vartheta_\pm)^2\zeta^{-3} - (5/2 \pm i\sqrt{3})N^2\zeta^{-6}\\ &+ [(5 \pm 2\sqrt{3})\Omega N - iN^3(\vartheta_\pm)^2/2\Omega]\zeta^{-7}\}P = 0.\end{aligned} \tag{87}$$

This differential equation (87) is simpler than (46) because, although the coefficient of the highest derivative Q'' or P'' is the same, the coefficient of P' is $\mathrm{O}(\zeta^{-6})$ of the same order as $\mathrm{O}(\zeta^{-6})$ for Q', and the coefficient of P is $\mathrm{O}(\zeta^{-7})$ instead of $\mathrm{O}(\zeta^{-8})$ for Q. It was shown in (50)–(54) that the differential equation (46) had no solution as a Frobenius–Fuchs series (49), because all solutions have an essential singularity. It will be proven next that the exponential in (86) specifies partially the essential singularity of the wave field at infinity, by showing that the differential equation (87) does have one solution as a power series of Frobenius–Fuchs type:

$$P_\chi(\zeta) = \sum_{n=0}^{\infty} d_n(\chi)\zeta^{n+\chi}. \tag{88}$$

Substitution of (88) into (87) leads to the recurrence formula for the coefficients

$$\begin{aligned}&[(n+\chi)(n+\chi+9)+20]d_n(\chi)\\ &\quad = i(2\Omega/N)[(n+\chi+1)(n+\chi+12 \pm i2\sqrt{3}) + 19 \mp i2\sqrt{3}]d_{n+1}(\chi)\\ &\qquad + 2(\Omega/N)^2[2(n+\chi+2) + 23/2 \pm i\sqrt{3}](1 \mp i2\sqrt{3})\, d_{n+2}(\chi)\\ &\qquad + 2i(\Omega/N)^3(i \pm 2\sqrt{3})^2 d_{n+3}(\chi) + N^2(n+\chi+5)^2 d_{n+5}(\chi)\\ &\qquad + 2i(\Omega/N)\{(n+\chi+6)[-1 \pm i2\sqrt{3} - N^2(n+\chi+6)] - 5/2 \pm i\sqrt{3}\}d_{n+6}(\chi)\\ &\qquad + i2\Omega^2[\mp 4\sqrt{3}(n+\chi+7) + 5 \pm 2\sqrt{3} - (i/2)(i \pm 2\sqrt{3})^2]d_{n+7}(\chi).\end{aligned} \tag{89}$$

Setting $n = -7$ leads to the indicial equation (90a):

$$n = -7: \quad 0 = d_0(\chi)(\mp 4\sqrt{3}\chi + 5 - 11i/2 \mp 4\sqrt{3}), \quad \chi_\pm = 1 \pm 5/4\sqrt{3} \mp i11/8\sqrt{3}, \tag{90a, b}$$

which has one root (90b) for each value of $\vartheta_\pm$ in (83a), corresponding (88) to the two exact explicit solutions

$$|\zeta| < 1: \quad P_\pm(\zeta) = \zeta^{\chi_\pm} \sum_{n=0}^{\infty} d_n(\chi_\pm)\zeta^n, \tag{91}$$

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around the point at infinity which are valid up to the critical layer. These two solutions correspond to inward- and outward-propagating waves, as shown next.

4.3. *Incompatibility with finite wave field at the critical layer*

Substituting (91) in (86); (45a, b); (25a, b); (15a, b); (43b) it follows that the velocity perturbation spectrum is specified beyond the critical layer by

$$r > |r_1|, |r_2| : \quad \omega(r;\omega) = C_+ W_+(r;\omega) + C_- W_-(r;\omega), \tag{92}$$

where

$$\begin{aligned} W_\pm(r;\omega) &= (r_0/r) F_\pm(r;\omega) = (r_0/r)\Phi_\pm(r/r_0) = (r_0/r)\Psi_\pm(\sqrt{r/r_0}) \\ &= (r_0/r) Q_\pm(\sqrt{r_0/r}) = (r_0/r)\exp(\vartheta_\pm \sqrt{r_0/r}) P_\pm(\sqrt{r_0/r}) \end{aligned} \tag{93a,b}$$

are given explicitly (91) by

$$W_\pm(r,\omega) = (r_0/r)^{-1+\chi_\pm} \exp[\vartheta_\pm \sqrt{r_0/r}] \sum_{n=0}^{\infty} d_n(\chi_\pm)(r_0/r)^{n/2}, \tag{94}$$

where $\vartheta_\pm$, $\chi_\pm$ are given by (83a), (90b). Apart from the factor (84), the leading term as $r \to \infty$, namely

$$(r_0/r)^{-1+\chi_\pm} = (r_0/r)^{\pm 5/4\sqrt{3}} \exp[\mp i(11/8\sqrt{3})\log(r/r_0)], \tag{95}$$

shows that for increasing r the phase increases (decreases) for W_+, (a) (for W_-, (b)) and thus it corresponds to an outward- (inward-) propagating wave.

The radiation condition, selecting the outward-propagating wave P_+, corresponds to setting to zero ($C_- = 0$) the amplitude of the inward-propagating wave P_- in (91) or W_+ in (92):

$$C_- = 0 : \quad W(r;\omega) = C_+ W_+(r;\omega). \tag{96}$$

Analytic continuation across the critical layer (71) yields

$$W_\pm(r;\omega) = D_0^\pm W_0(r;\omega) + D_1^\pm W_1(r;\omega), \tag{97}$$

where in general $D_0^\pm, D_1^\pm$ are non-zero constants. Substitution of (97) in (96) shows that the wave field is singular at the critical layer. In general

$$W(r;\omega) = (C_- D_0^- + C_+ D_0^+) W_0(r;\omega) + (C_- D_1^- + C_+ D_1^+) W_1(r;\omega). \tag{98}$$

A finite wave field at the critical layer requires

$$C_- D_1^- + C_+ D_1^+ = 0; \tag{99}$$

if a radiation condition (96) is imposed ($C_- = 0$), it follows that $C_+ D_1^+ = 0$; since in general $D_1^+ \neq 0$, it follows that $C_+ = 0$, and hence a null or zero wave field results. Thus the radiation condition together with a condition of finite wave field at the critical layer generally leads to a trivial solution. This point could not be addressed in the preceding literature (Heinemann and Olbert 1980; Barkhudarov 1991; Velli 1993; Lou 1994); which did not include the exact solution of the Alfvén wave equation both near the critical layer and near the point at infinity. It is confirmed by exact solutions in the cases of constant mean flow velocity (Campos and Isaeva 1999) and mean flow velocity proportional to the radius (Campos and Gil 2002). The conclusion is predictable, since the Alfvén wave equation is of the second order, and thus it is not generally possible to impose *three* conditions: (i) given initial

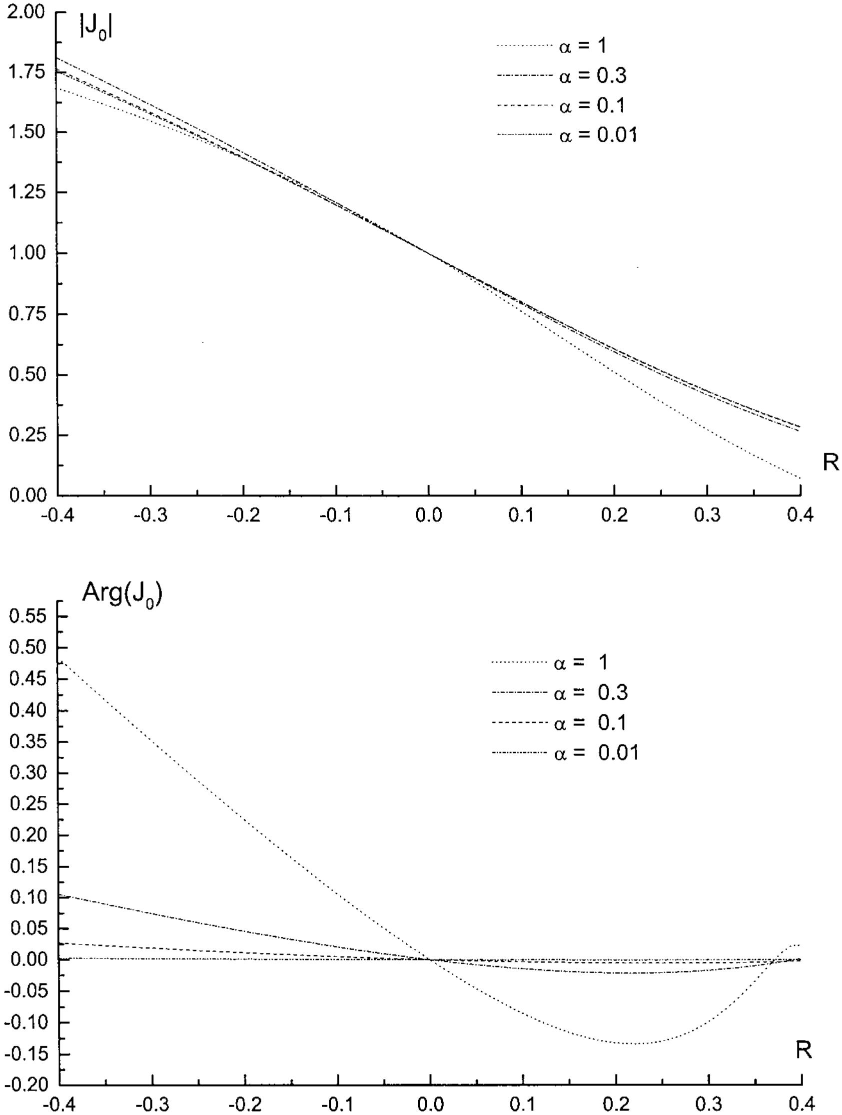
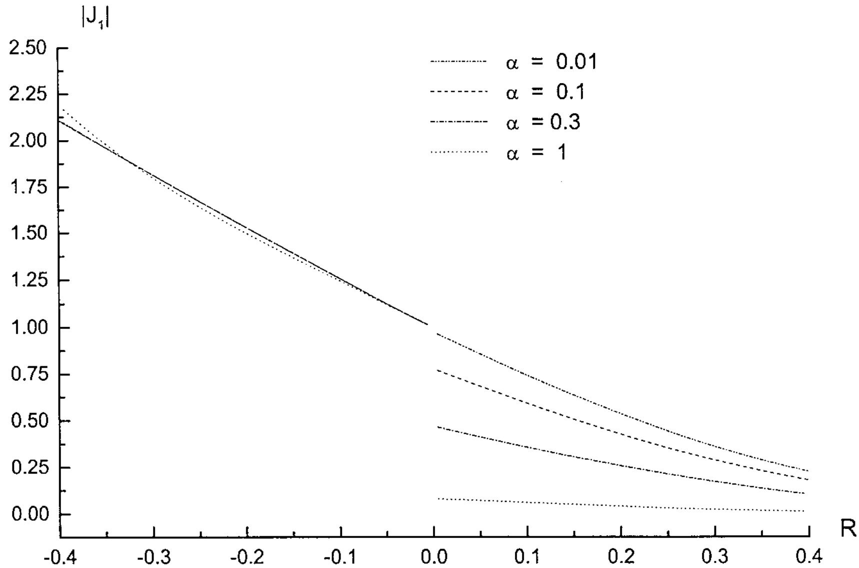


Figure 3. Amplitude (a) and phase (b) for the wave field component finite at the critical layer (82a), versus dimensionless distance from the critical layer (83b), for four values (83a) of dimensionless parameter (61).

wave field at the Sun; (ii) radiation condition at infinity; (iii) finiteness condition at the critical layer. The latter two are incompatible, as shown before.

5. Properties of Alfvén waves in the vicinity of the critical layer

Since the radiation condition of outward propagation at infinity is generally incompatible with the condition of finite amplitude at the critical layer (Sec. 5.1), both the finite and singular components of the wave field are plotted (Figs 3 and 4) across the critical layer (Sec. 5.2). Also, the physical mechanism giving rise to the appearance of the critical layer is identified (Sec. 5.3).



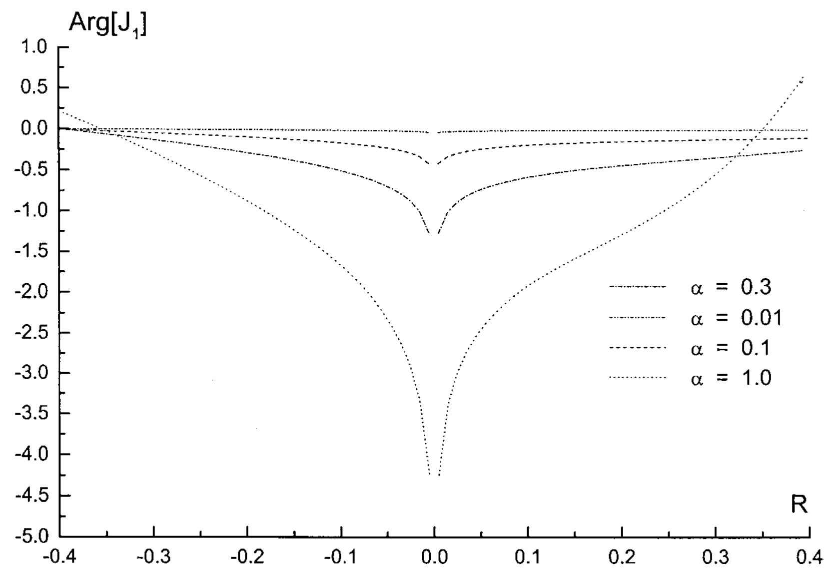


Figure 4. As Fig. 3 for the component of the wave field (82b), which is singular at the critical layer.

5.1. Incompatibility of finiteness and radiation conditions

Before proceeding to plot the wave fields near the critical layer, it should be noted that W_1 is omitted in most of the literature on Alfvén waves in the solar wind, by setting $C_1 = 0$ in (71), so as to ensure that the wave field remains finite at the critical layer. This practice is at variance with the much broader and older literature on waves with critical layers, where examples with singularities abound. The very first study of a critical layer (Bretherton 1966; Booker and Bretherton

1967), for internal waves in a shear flow, leads to a wave amplitude which is singular at the critical layer. There are many other examples of hydrodynamic and hydromagnetic waves in atmospheres which have a singular amplitude at the critical layer(s) (Lyons and Yanowitch 1974; Yanowitch 1967; McKenzie 1973, 1979; Adam 1977; Campos 1983a, 1983c, 1985, 1987, 1988b; Alkahby and Yanowitch 1993; Campos and Isaeva 1999; Campos and Gil 2002). It should be understood that a critical layer is a resonance of a linear, undamped system, and in such conditions an infinite amplitude is possible. Of course the amplitude will become finite if either dissipation or nonlinear effects are taken into account. It will be shown in the sequel that, in the present problem, if either fluid viscosity (Sec. 6.3) or electrical resistance (Sec. 6.2) are included, the wave field is finite at all finite distances, including the critical layer.

The practice adopted in the literature on Alfvén waves in the solar wind, of setting $C_1 = 0$ in (71) to exclude the singular wave field, implies that there is only one constant of integration left C_0, which is determined from the initial wave field at some radius r_*, e.g.

$$C_0 = W(r_*, \omega)/W_0(r_*, \omega). \tag{100a}$$

The wave field

$$W(r; \omega) = C_0 W_0(r; \omega) \tag{100b}$$

will not in general satisfy a radiation condition at infinity. This means that, in order to be able to specify the initial wave field at some radius (100a), and to enforce a finite wave amplitude at the critical layer $C_1 = 0$, it is necessary to have wave sources at infinity. In the case of the solar wind, it is usually accepted that the only source of Alfvén waves is the Sun. The issue of whether (a) to satisfy the radiation condition at infinity or (b) to impose a finiteness condition at the critical layer may remain open to debate. Even though the present model may not apply to the solar wind at large distance, the meeting of the radiation condition may be seen as a matter of self-consistency. In order to be able to specify the initial wave field (100a) and exclude waves coming from infinity (radiation condition), then in general $C_1 \neq 0$, and the wave field is singular at the critical layer, due to the presence of $W_1(r; \omega)$. The latter would become finite at the critical layer, if dissipation and/or nonlinearity were taken into account. The consideration of linear non-dissipative waves should not make one of the wave components disappear. It is prudent, therefore, to consider both components of the wave field W_0 and W_1 in the plots which follow.

Since the question of the properties of Alfvén waves at the critical layer is a major point, it is worth noting that the conclusions in this paper are based on: (i) a derivation of the Alfvén wave equation in the form appropriate to the problem at hand; (ii) an identification of all singularities of that wave equation; (iii) exact solutions around all singularities, thus covering the whole physical region. This provides a self-consistent basis for analysis and conclusions. The solution at infinity consists of inward- and outward-propagating waves (Sec. 4.2). Applying a radiation condition to exclude inward-propagating waves sets one constant of integration equal to zero. The remaining constant of integration is determined by the initial wave field at a given radius. Thus it is generally *not* possible to impose a third condition requiring the wave field to be finite at the critical layer. Another way to reach the same conclusion is to note that the solution which represents an outward-propagating wave at infinity can be extended analytically to a linear combination of solutions around the critical layer. Since one of the solutions is singular at the critical layer, the total wave field will also be in general singular at the critical layer.

5.2. Amplitude and phase as functions of distance across the critical layer

In what follows both components of the wave field are plotted, using the general solutions, which hold for all frequencies. The component of the wave field which is finite at the critical layer (69) is plotted (Fig. 3) with first coefficient unity (101a):

$$J_0(R) \equiv W_0(r;\omega), \tag{101a}$$

$$J_1(R) \equiv R^{-4/5} W_1(r;\omega), \tag{101b}$$

whereas for the singular component of the wave field (73), the singular amplitude in (74) is suppressed (101b), where R is a dimensionless distance from the critical layer:

$$R \equiv 1 - \sqrt{r/r_1}: \quad J_0(R) = (1-R)^{-2}\left\{1 + \sum_{n=0}^{\infty} c_n(0) R^n\right\}, \tag{102a}$$

$$J_1(R) = R^{i2\alpha/5}(1-R)^{-2}\left\{1 + \sum_{n=1}^{\infty} c_n(-4/5 + 2i\alpha/5) R^n\right\}. \tag{102b}$$

The index is zero for the finite solution (102a) and is specified by (72) for the singular solution (102b), where the factor $R^{-1-4ia/5}$ is given by (75a, b). The only parameter is (61), which appears in the recurrence formulas for the coefficients given in the Appendix.

The parameter α involves the dimensionless frequency (17a) and the initial Alfvén number (17b). Assuming that Alfvén waves emerge at the solar radius $r_0 = 7 \times 10^{10}$ cm, where the Alfvén speed is $a = 3 \times 10^7$ cm s^{-1}, for a wave period $\tau = 1$ day $= 24$ h $= 8.6 \times 10^4$ s, the dimensionless frequency is $\Omega = r_0\omega/a = 2\pi r_0/a\tau = 0.16$. The Alfvén number is less than unity ($N < 1$) for a wave starting below the critical layer, so $\alpha < \Omega$, suggesting that $\alpha < 0.2$, if the present model is taken as reference. However the model represents the solar wind only near the critical layer, so α could be calculated there, at $r = r_1$. In this case $u = a$, and (61) simplifies to $\alpha = \omega r_1/a_1$. Retaining a frequency $\omega = 2\pi/(24 \times 3600$ s$) = 7.3 \times 10^{-5}$ s^{-1} corresponding to a period of one day, assuming that the critical layer lies at a distance of 10 solar radii ($r_1 = 10 r_0 = 7 \times 10^{11}$ cm), and using $a_1 = 500$ km s^{-1} $= 5 \times 10^7$ cm s^{-1} for the Alfvén speed, leads to $\alpha = 1.0$. Thus the plots concern four values of α, namely

$$\alpha = 0.01, 0.1, 0.3, 1, \tag{103a}$$

spanning two orders of magnitude up to unity. Since the model applies only near the critical layer at $r_1 = 10r_0$, the range of radial distances which is taken ($r_1/3 < r < 2r_1$) corresponds to $3r_0 < r < 20r_0$, so that the plots concern (102a) the range of dimensionless radial distances

$$0.4 = 1 - 1/\sqrt{3} > R > 1 - \sqrt{2} = -0.4, \tag{103b}$$

for both wave fields, i.e. regular (102a, Fig. 3) and singular (102b, Fig. 4) at the critical layer.

The amplitude (Fig. 3(a)) of the wave field component J_0 which is finite at the critical layer (102a) $\equiv$ (101a) decays with distance, and shows a weak dependence on wave frequency; a somewhat faster decay with distance is noticeable for the largest α, corresponding to the higher frequency. The phase variation is larger for larger α or higher frequency, both for the wave field component which is finite at the critical layer (Fig. 3(b)), and that which singular at the critical layer (Fig. 4(b)). For the latter the phase is singular at the critical layer, and wave propagation is away from the critical layer, i.e. inward below and outward above. The explanation is that: (i) waves are partially reflected at the critical layer, and thus can propagate inward inside the critical layer, against the mean flow, because the Alfvén speed is larger than the mean flow velocity, (ii) outside the critical layer the Alfvén speed is smaller than the mean flow velocity, and thus all waves are convected outward, regardless of whether they propagate inward or outward relative to the mean flow. The amplitude (Fig. 4(a)) of the wave field component which is singular at the critical layer, when the singularity is removed, is weakly dependent on frequency before the critical layer; across the critical layer the amplitude has a larger jump and thus becomes smaller for higher-frequency waves, in agreement with (79a); this corresponds to a steepening of the spectrum, which is a feature observed in the solar wind.

5.3. Wave reflection and counter-flow of critical layer

One type of critical layer which is well known to lead to singular amplitudes is the 'tangential' type, where the wave approaches from one side, and then propagates along the critical layer; since the wave energy is 'trapped' in the critical layer, the amplitude becomes singular. This is not the type of critical layer relevant to the present problem. Another type, which is the one relevant here, is the 'counter-flow' critical layer. A wave propagates against a stream of increasing velocity, up to a point where the wave speed equals the mean flow velocity. At that point the group velocity is zero, and the wave energy accumulates, leading to a singular amplitude. The latter is the type of critical layer relevant to Alfvén waves in the solar wind, and in this connection it is important to distinguish outward- and inward-propagating waves. The outward-propagating wave has no critical layer in a homogeneous medium, because the phase speed adds to the mean flow velocity. However, in an inhomogeneous medium, it gives rise, by reflection, to inward-propagating waves, which have a critical layer, as the Alfvén speed faces an increasing flow velocity. Since the solar wind is inhomogeneous, an outward-propagating wave is gradually reflected as an inward-propagating wave, and thus a critical layer with singular amplitude can exist generally.

Note that the solution presented is not a steady solution, but rather a solution harmonic in time, represented by a Fourier integral, as appropriate for a boundary-value problem. In the case of an initial value problem, a Laplace transform would be used instead, leading to a distinct inversion formula in the time domain. However, apart from replacing $-i\omega$ by $-s$, the spatial dependence is the same in both problems. Thus the nature of the critical layer as a spatial singularity of the wave equation is unchanged in a boundary or initial-value problem. The propagation of Alfvén waves in the solar wind is specified by a wave equation with variable coefficients, which has a singularity at the critical layer. This equation cannot be replaced by one with constant coefficients or without singularities except in particular cases such as the JWKB limit of wavelength short compared with the length scale of change of properties of the medium (Sec. 4.1). For low-frequency waves this condition applies only asymptotically at large distance, and thus the

JWKB 'solution' cannot be matched to the initial wave field across the critical layer.

The JWKB approximation cannot indicate singular wave behaviour at the critical layer because it neglects reflection, and it is precisely the reflected wave which is singular at the critical layer. Besides, the JWKB approximation applies to the solar wind only at large distance, far beyond the critical layer. The JWKB solution (Sec. 4.1) can be confirmed (Sec. 4) as the leading term of the exact, explicit solution of the Alfvén wave equation around the point at infinity (Sec. 4.2). This solution consists (Sec. 4.2) of inward- and outward-propagating waves. Applying the radiation condition (Sec. 4.3) to select the outward-propagating wave leads in general to a wave field which is singular at the critical layer. Thus, in order to have a finite wave field at the critical layer, it is necessary to have waves incoming from infinity, of exactly the right amplitude to cancel the resonance at the critical layer. Thus the radiation condition is incompatible with a finite wave field at the critical layer, opening the question of which of the two should be used.

6. Discussion

The discussion concerns the mean state assumed (Sec. 6.1) in the present theory of non-dissipative Alfvén waves in the solar wind, which leads to a singularity at the critical layer. It is also shown that the amplitude becomes finite at the critical layer in the presence of dissipation, either by Ohmic electrical resistance (Sec. 6.2) or fluid viscosity (Sec. 6.3).

6.1. Self-consistent background mean state for wave propagation

The external magnetic field (9a) is force-free, and thus the momentum equation for the mean flow is

$$dp/dr = -\rho U U' - \rho g, \tag{104}$$

where the acceleration of gravity decays as the inverse square of the radius:

$$g(r) = g_0 (r_0/r)^2 \tag{105}$$

substituting (24a, b); (105) in (104)

$$dp/dr = -(\rho_0 u^2/2r_0)(r_0/r)^{5/2} - \rho_0 g_0 (r_0/r)^{9/2}, \tag{106}$$

and taking the gas pressure to be zero at infinity, leads to

$$p(r) = -(\rho_0 u^2/3)(r_0/r)^{3/2} + (2\rho_0 g_0 r_0/7)(r_0/r)^{7/2}. \tag{107}$$

The background state assumed in the present model of Alfvén wave propagation corresponds to a double-polytropic:

$$p(r)/\rho(r) = (u^2/3)(r/r_0) + (2g_0 r_0/7)(r_0/r), \tag{108}$$

with exponents ± 1. Note that gravity $\vec{g}$ in (1b) drops out of the Alfvén wave equations (3a, b) because the velocity perturbations (2a, b) are transversal. However, gravity affects the propagation of Alfvén waves, by specifying the profiles of the background medium.

The preceding background model (9a); (24a, b, c); (105); (107); (108) is self-consistent (9b); (104), but does not mimic closely the solar wind. The most desirable approach would be to start from an existing solar wind model (Brandt 1970;

Hundhausen 1972), preferably with dissipation (Scarf and Noble 1965), to perform a linear perturbation, and eliminate to form an Alfvén wave equation. However, even the simplest solar wind model, from a coronal hole (Kopp and Holzer 1976), has a profile consisting of the product of a power and an exponential of the radial distance; for such a background, the Alfvén wave equation (10) would be difficult to solve analytically, a numerical solution would not be suited to the application of a radiation condition at infinity, and might not represent adequately the singularity at the critical layer. It should be borne in mind that the published exact solutions of the Alfvén wave equation in a radial 'monopole' magnetic field assume uniform mean flow (Heinemann and Olbert 1980; Barkhudarov 1991; Campos and Gil 2002), with two exceptions, of which the present paper is the second (the first was Campos and Isaeva 1999). These three cases are among the most complex exact solutions of the Alfvén wave equation in the literature (Alfvén 1942, 1948; Alfvén and Falthammar 1962; Ferraro and Plumpton 1965; Lighthill 1978; Priest 1982), and form the available basis for discussion. In all three cases, of mean flow velocity a power law of the radius $U(r) \sim r^{\nu}$, with exponents $\nu = 0, 1, 1/2$, it was found that a radiation condition of outward wave propagation at infinity is incompatible with a finite wave field at the critical layer, suggesting that this is a robust result, not associated with a particular background or mean flow. Each of these problems was solved using consistently the same mean flow velocity profile $U(r) \sim r^{\nu}$ with a given $\nu = 0, \frac{1}{2}, 1$ for all distances outside the solar radius $r_0 \leqslant r < \infty$. In the case of a compound velocity profile, consisting of three regions (I) $\nu = 0$ up to a distance before the critical layer $r_0 < r \leqslant r_1 < r_*$, (II) $\nu = \frac{1}{2}$ around the critical layer $r_1 \leqslant r \leqslant r_2$, and (III) $\nu = 1$ beyond the critical layer $r_* < r_2 \leqslant r < \infty$, the same conclusion would hold: the radiation condition at infinity would imply, by matching from region III to region II, a singular wave field at the critical layer.

It can be argued that those three exact solutions assume that the mean flow velocity is a power law of the radius over an infinite radial distance outside the Sun $r_0 \leqslant r < \infty$, whereas the velocity profile in the solar wind is more complex. Nevertheless the velocity profile of the solar wind can be approximated by a power law, with different exponents near the critical layer and at large distance. The asymptotic solution at large radius will consist of inward- and outward-propagating waves; the solutions near the critical layer will consist of finite and singular modes. The matching coefficients between the two pairs of solutions will depend on the mean flow velocity profile $U(r)$ at and beyond the critical layer $r_1 \leqslant r < \infty$. In this sense, the three exact solutions available specify the matching coefficients for three particular velocity profiles. It would be rather exceptional that the outward-propagating wave at infinity would match exactly to the finite wave field at the critical layer; such a coincidence could occur only for a peculiar mean flow velocity profile $U(r)$, and particular values of the parameters of the problem, like (61). Thus it should be expected that, in general, an outward-propagating wave, meeting the radiation condition at infinity, would involve the singular wave component at the critical layer. This is the conclusion consistent with the three available exact solutions of the Alfvén wave equation in a radial external magnetic field and mean flow. For general profiles of the mean flow velocity $U(r)$ and Alfvén speed $A(r)$ the critical layers are the roots of $U(r_*) = A(r_*)$. For example, if the Alfvén speed is monotonic decreasing $dA/dr < 0$ for all $r < r_0$ due to the density $\rho(r)$ decreasing more slowly than the external magnetic field squared $[B(r)]^2$, and if the mean flow velocity is monotonically increasing or non-decreasing,

$dU/dr \geqslant 0$, then $A(r) = U(r)$ can have only one root, and there is only one critical layer.

6.2. *Effect of Ohmic electric resistivity on convected Alfvén waves*

The singularity of the Alfvén wave field at the critical layer is typical of a linear, undamped resonant system. This analogy suggests that the amplitude would become finite at the critical layer in the presence of damping, e.g. by Ohmic electrical resistivity. In order to check this prediction, the Ohmic diffusivity χ, assumed to be constant, is added on the right-hand side of the induction equation (1a), namely

$$\partial \bar{H}/\partial t + \nabla \wedge (\vec{H} \wedge \vec{V}) = \chi \nabla^2 \vec{H}, \tag{109}$$

so that (3a) is replaced by

$$\frac{\partial h}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}(Bru - hrU) = \frac{\chi}{r}\frac{\partial^2}{\partial r^2}(hr), \tag{110}$$

and hence (5a) is replaced by

$$i\omega G + (BF - UG)' = -\chi G''. \tag{111}$$

The elimination with (5b) is similar, namely, (111) is expanded

$$i\omega G + (BF)' - U'G = UG' - \chi G'', \tag{112}$$

and G' is substituted from (5b)

$$i\omega G + (BF)' - U'G = (UB/A^2)(UF' - i\omega F) - \chi[(B/A^2)(UF' - i\omega F)]'. \tag{113}$$

This equation is solved for G, then differentiated, and G' replaced from (5b):

$$\begin{aligned} &[\{(BF)' - (UB/A^2)(UF' - i\omega F) + \chi[(B/A^2)(UF' - i\omega F)]'\}/(U' - i\omega)]' \\ &\qquad = G' = (B/A^2)(UF' - i\omega F). \end{aligned} \tag{114}$$

Equation (114) is the Alfvén wave equation, and in the presence of Ohmic resistivity it is of third order.

Bearing in mind that for the radial external magnetic field decaying as the inverse square of distance (9a), the factor UB/A^2 is constant, the Alfvén wave equation with Ohmic resistivity (114), namely

$$\begin{aligned} &(U' - i\omega)^2(UF' - i\omega F) \\ &\quad = (U' - i\omega)[(A^2/B)(BF)'' - U(UF' - i\omega F)' + \chi U(F' - i\omega F/U)''] \\ &\qquad - U''[(A^2/B)(BF)' - U(UF' - i\omega F) + \chi U(F' - i\omega F/U)'], \end{aligned} \tag{115}$$

can be written explicitly for an arbitrary mean flow velocity profile $U(r)$:

$$\begin{aligned} &\chi U(U' - i\omega)F''' + [(U' - i\omega)(A^2 - U^2 - i\omega\chi) - \chi U''U]F'' + \{-U(U' - i\omega)^2 \\ &\quad + (U' - i\omega)[2A^2B'/B - U'U + i\omega(U + 2\chi U'/U)] - U''(A^2 - U^2 - i\omega\chi)\}F' \\ &\quad + \{i\omega(U' - i\omega) + (U' - i\omega)[A^2B''/B + i\omega\chi(U''/U - 2U'^2/U^2)] \\ &\quad - U''[A^2B'/B + i\omega(U + \chi U'/U)]\}F = 0, \end{aligned} \tag{116}$$

and simplifies to

$$U = \text{constant}: \ \chi U F''' + (A^2 - U^2 - i\omega\chi)F'' + 2(A^2 B'/B + i\omega U)F' + (A^2 B''/B + \omega^2)F = 0 \quad (117)$$

for uniform mean flow.

Note that in the case of non-uniform mean flow (116), the only singularity in the coefficient of F'' is the transition layer (12b), which occurs for complex 'altitude'. Thus there is no singularity for real altitude, and the wave field is finite everywhere (for finite r), including at the critical layer. The transition layer does not occur in the case (117) of uniform mean flow velocity. In the absence of Ohmic diffusivity ($\chi = 0$) the wave equation (116) would drop to second order, and coincide with (10). Both for non-uniform (116) and uniform (117) mean flow velocity, the coefficient of F'', is $U^2 - A^2 - i\omega\chi$, which would vanish at the critical layer $A = \pm U$ in the absence of dissipation $\chi = 0$; in the presence of dissipation $\chi \neq 0$ it is complex, and cannot vanish for real altitude. It has been shown that, in the presence of dissipation by Ohmic electrical resistivity, the Alfvén wave equation has finite amplitude at the critical layer, for any mean flow velocity profile. In order to show that this result is not a feature of a particular dissipation mechanism, it is proved next for another dissipation process, namely fluid viscosity.

6.3. *Dissipation by fluid viscosity and finite amplitude at critical layer*

In the presence of constant shear viscosity ν, the momentum equation (1b) has an extra term

$$\partial\vec{V}/\partial t + (\vec{V}.\nabla)\vec{V} + (1/\rho)\nabla p = \vec{g} - (\mu/4\pi\rho)[\vec{H} \wedge (\nabla \wedge \vec{H})] + (\nu/\rho)\nabla^2\vec{V}, \quad (118)$$

and thus (3b) is replaced by

$$\frac{\partial U}{\partial t} + \frac{U}{r}\frac{\partial}{\partial r}(rv) = \frac{\mu B}{4\pi\rho}\frac{1}{r}\frac{\partial}{\partial r}(rh) + \frac{\eta}{r}\frac{\partial^2}{\partial r^2}(rv), \quad (119)$$

where η is the kinematic viscosity (120a), which is not constant because it depends on the mass density:

$$\eta(r) \equiv \nu/\rho(r), \quad (120a)$$

$$i\omega F - UF' + (A^2/B)G' = -\eta F''; \quad (120b)$$

this needs to be taken into account when eliminating between (120b), which replaces (5b), and (5a), which is unchanged. Solving (5a) for G' and substituting (120b) yields

$$i\omega G + (BF)' - U'G = UG' = (UB/A^2)(UF' - i\omega F - \eta F''). \quad (121)$$

Solving (121) for G, differentiating and using (5a) again yields

$$\{[(BF)' - (UB/A^2)(UF' - i\omega F - \eta F'')]/(U' - i\omega)\}' = G' = (B/A^2)(UF' - i\omega F - \eta F''), \quad (122)$$

which is the third-order Alfvén wave equation with viscous dissipation.

Bearing in mind that UB/A^2 is constant but the kinematic diffusivity η is not, the Alfvén wave equation with viscous dissipation can be written explicitly

$$\begin{aligned}(U' - i\omega)^2(B/A^2)(UF' - i\omega F - \eta F'') &= (U' - i\omega)[(BF)'' - (UB/A^2)(UF' - i\omega F - \eta F'')'] \\ &\quad - U''[(BF)' - (UB/A^2)(UF' - i\omega F - \eta F'')],\end{aligned} \quad (123)$$

for arbitrary mean flow velocity profile $U(r)$:

$$\eta U(U'-i\omega)F''' + [\eta(U'-i\omega)^2 + (U'-i\omega)(A^2-U^2-\eta' U) + \eta U'' U]F'' + [-U(U'-i\omega)^2 + (U'-i\omega)(2A^2B'/B - U'U + i\omega U) - U''(A^2-U^2)]F' + [i\omega(U'-i\omega)^2 + (U'-i\omega)A^2B''/B - U''(A^2B'/B + i\omega U)]F = 0, \qquad (124)$$

and simplifies to

$$U = \text{constant}: \ \eta U F'''(A^2-U^2-i\omega\eta+\eta' U)F'' + 2(A^2B'/B + i\omega U)F' + (A^2B''/B+\omega^2)F = 0, \qquad (125)$$

for uniform mean flow.

The coefficient of F'' is the same in (124) and (116), substituting the Ohmic diffusivity χ by the kinematic viscosity η. In both cases, for a non-uniform mean flow velocity, the only singularity of the dissipative Alfvén wave equation is the transition layer (12b), which occurs at complex altitude. In the case of uniform mean flow (125) or (117), there is no transition layer. Thus, both for uniform (125) and non-uniform (124) mean flow, the Alfvén wave equation with viscous dissipation has no singularity at real altitude; it follows that the Alfvén wave fields are finite everywhere (for finite r), including at the critical layer, for any mean flow velocity profile. The coefficient of F'' in the case of uniform flow is $A^2-U^2+\eta\prime U-i\omega\eta$, which would vanish at the critical layer $A=\pm U$, in the absence of viscosity $\eta=0$, when the wave equations (124), (125) reduce to second order. In the presence of kinematic viscosity, this coefficient is complex, and does not vanish at the critical layer, namely it is equal to $-\eta\prime U-i\omega\eta$. The difference from the coefficient $A^2-U^2-i\omega\chi$ of F'' in (116), (117) is that the Ohmic diffusivity is constant ($\chi\prime=0$). We have considered the Alfvén wave equation in the presence of either Ohmic resistivity (116) or shear viscosity (124) to show that the finite wave amplitude at the critical layer results from different dissipation mechanisms, and just one need be present. The third-order dissipative Alfvén wave equation will have three solutions, but all of them will be finite at the critical layer. Thus a finiteness condition is redundant, because it is automatically met by dissipative waves. It goes beyond the scope of the present paper to discuss in more detail dissipative Alfvén waves; the purpose of the present discussion was to show that the singularity of the Alfvén wave at the critical layer disappears, in the presence of dissipation, for any mean flow velocity profile.

Appendix A. Calculation of the wave field near the critical layer

The wave fields near the critical layer are specified by the solution of the differential equation (58), whose coefficients involve the polynomials (60a, b, c) having degrees not exceeding nine, and the coefficients (62) are given by

$$\beta_{0,r} = \{0, 22+4i\alpha+4\alpha^2, -12+28i\alpha-32\alpha^2, -30+84i\alpha+112\alpha^2, 40-140i\alpha-224\alpha^2, -30+140i\alpha+280\alpha^2, 12-84i\alpha-224\alpha^2, -2+28i\alpha+112\alpha^2, -4i\alpha-32\alpha^2, 4\alpha^2\}, \qquad (126a)$$

$$\beta_{1,r} = \{10+9i/\alpha-2i\alpha, -27-7i/\alpha+16i\alpha, 30+15i/\alpha-56i\alpha, -35-10i/\alpha+112i\alpha, 35+15i/\alpha-140i\alpha, -21-3i/\alpha+112i\alpha, 7+i/2\alpha-56i\alpha, -1+16i\alpha, -2i\alpha, 0\}, \qquad (126b)$$

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$$\beta_{2,r} = \{5 + 5i/2\alpha, -25 - 10i/\alpha, 55 + 35i/2\alpha, -70 - 35i/2\alpha,\\ 56 + 8i/\alpha, -28 - 7i/2\alpha, 8 + i/2\alpha, -1, 0, 0\}, \tag{126c}$$

for $r = 1, 2, \ldots, 9$. These coefficients appear in the recurrence formula (64), which can be written explicitly:

$$\begin{aligned} 0 = {} & (n+\vartheta)[10 + 9i/\alpha - 2i\alpha + 5(n+\vartheta-1)(1+i/2\alpha)]c_n(\vartheta) \\ & + \{22 + 4i\alpha + 4\alpha^2 + (n+\vartheta-1)[-27 - 7i/\alpha + 16i\alpha \\ & - 5(n+\vartheta-2)(5+2i/\alpha)\}]c_{n-1}(\vartheta) \\ & + \{-12 + 28i\alpha + 32\alpha^2 + (n+\vartheta-2)[30 + 15i/\alpha - 56i\alpha \\ & + 5(n+\vartheta-3)(11+7i/2\alpha)]\}c_{n-2}(\vartheta) \\ & + \{-30 + 84i\alpha + 112\alpha^2 + (n+\vartheta-3)[-35 - 10i/\alpha + 112i\alpha \\ & - 5(n+\vartheta-4)(14+7i/2\alpha)]\}c_{n-3}(\vartheta) \\ & + \{40 - 140i\alpha - 224\alpha^2 + (n+\vartheta-4)[35 + 15i/\alpha - 140i\alpha \\ & + 8(n+\vartheta-5)(7+i/\alpha)]\}c_{n-4}(\vartheta) \\ & + \{-30 + 140i\alpha + 280\alpha^2 + (n+\vartheta-5)[-21 - 3i/\alpha + 112i\alpha \\ & - 7(n+\vartheta-6)(4+i/2\alpha)]\}c_{n-5}(\vartheta) \\ & + \{12 - 84i\alpha - 224\alpha^2 + (n+\vartheta-6)[7 + i/2\alpha \\ & - 56i\alpha + (n+\vartheta-7)(8+i/2\alpha)]\}c_{n-6}(\vartheta) \\ & + [-2 + 28i\alpha + 112\alpha^2 + (n+\vartheta-7)(7 + 16i\alpha - n - \vartheta)]c_{n-7}(\vartheta) \\ & + [-4i\alpha - 32\alpha^2 - 2i\alpha(n+\vartheta-8)]c_{n-8}(\vartheta) + 4\alpha^2 c_{n-9}(\vartheta). \end{aligned} \tag{127}$$

This recurrence formula starting with $c_0(\sigma) = 1$ specifies the wave field which is finite at the critical layer (102a) for $\sigma = 0$ as plotted in Fig. 3(a, b) and the wave field singular at the critical layer (102b) for $\sigma = \sigma_1$ in (72) as plotted in Fig. 4(a, b).

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