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PARABOLIC SUBROOT SYSTEMS AND THEIR APPLICATIONS

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Abstract. In this note we consider parabolic subroot systems of a complex simple Lie Algebra. We describe root theoretic data of the subroot systems in terms of that of the root system and we give a selection of applications of our results to the study of generalized flag manifolds.

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1. Introduction. Some years ago R. Carles [**12**, **13**] considered the subroot system of an irreducible (reduced, crystallographic) root system $\Re$ obtained by taking the orthogonal complement of the highest root. The Dynkin diagram of such a subroot system is obtained from the Dynkin diagram of $\Re$ by deletion of one or two (for type A root systems) nodes. We will refer to a subroot system whose Dynkin diagram is obtained by deletion of a subset I of nodes (simple roots) from the Dynkin diagram of $\Re$ as a parabolic subroot system, as it corresponds naturally to a parabolic subgroup P of a complex simple Lie group $G^{\mathbb{C}}$ corresponding to $\Re$. Carles related root theoretic data such as the cardinality and the sum 2ρ of the set of positive roots of this subroot system to those of the root system $\Re$ (see also [**17**] p. 524). We extend these results to all maximal (i.e. I is a singleton) parabolic subroot systems of $\Re$. We also give a selection of geometric applications of our results to the study of flag manifolds. Our results can also be applied to the study of the closely related subroot systems obtained by the same process in the extended Dynkin diagram (see for instance Theorem 3.4). These subroot systems also have many geometric applications such as the study of the compact homogeneous spaces with positive Euler characteristic of [**16**], the centrioles of [**20**, **15**] and the orbits of compact symmetric spaces under the action of the isotropy subgroups of [**24**]. In this note, we will restrict our applications to (generalized) flag manifolds M, that is, homogeneous spaces of the form $M = G^{\mathbb{C}}/P$, where $G^{\mathbb{C}}$ is the complexification of a compact connected semisimple Lie group G and P is a parabolic subgroup of $G^{\mathbb{C}}$. M also has a description of the form $M = G/K$, where K is the centralizer of a torus in G (connected, compact, and semisimple). We denote by $o = eK$ the identity coset of G/K and by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Taking a reductive, orthogonal (w.r.t. the negative of the Killing form B) decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, we have a natural isomorphism between the tangent space T_oM and $\mathfrak{m}$. Also, the isotropy representation of K is equivalent to the adjoint representation of K restricted to $\mathfrak{m}$. G-invariant metrics on G/K are therefore determined by $Ad(K)$-invariant inner products $Q(.,.)$ on $\mathfrak{m}$. Taking a Q-orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ of $\mathfrak{m}$ into its *Ad(K)*-irreducible (inequivalent) submodules, one obtains a family of G-invariant metrics on G/K, given by $\langle , \rangle = x_1 Q_{|\mathfrak{m}_1} + \cdots + x_q Q_{|\mathfrak{m}_q}$ where $(x_1, \ldots, x_q) \in \mathbb{R}^q_+$. Such a metric belongs to the space of

G-invariant, unit volume metrics on M if and only if $\prod_{i=1}^{q} x_i^{d_i} = 1$, where $d_i = \dim \mathfrak{m}_i$, on which variational methods can be applied to the scalar curvature functional (also involving the d_i) to obtain Einstein metrics on M. In Theorem 4.1 we will give uniform formulae for the dimensions d_i when P is a maximal parabolic subgroup of $G^{\mathbb{C}}$ as an alternative to a case-by-case application of Weyl's dimension formula.

A flag manifold $M = G^{\mathbb{C}}/P$ also admits an equivariant holomorphic embedding into a complex projective space. For any smooth projective variety X embedded in projective space via the global sections of a very ample line bundle L, we can consider the codimension of the variety $X' \subset \mathbb{P}^N$ of hyperplanes tangent to X, known as the *dual* or *discriminant* variety of X. Typically X' is a hypersurface and therefore the defect of (X, L), defined to be $def(X, L) = \operatorname{codim} X' - 1$, is typically zero. If the defect is positive, then it is determined by the nef value $\tau(X, L)$ (defined in Section 4), which in our context also determines the first Chern class of X [**21**] and by [**4**] we have that $\{def(X, L) = 2(\tau(X, L) - 1) - \dim X$. In Proposition 4.2 and Theorem 3.3 we give uniform formulae for $\tau(M, L)$ and $\dim M$, respectively, when $M = G^{\mathbb{C}}/P$ and P is a maximal parabolic subgroup of $G^{\mathbb{C}}$ and L is a minimal (very) ample line bundle on M. Most known examples of smooth varieties with positive defect are homogeneous and the flag varieties with positive defect have been classified in [**18**] and [**21**]. The classification in [**18**] is based on invariant theory and considers the different cases corresponding to the type of the group, whereas in [**21**] the relationship between the defect and the nef value is exploited to give a fairly straightforward classification. Even so, the numerical criterion that $d_P := 2(\tau(M, L) - 1) - \dim M$ be positive must be checked for all $M = G^{\mathbb{C}}/P$, where $P = P_{\alpha_i}$ is a maximal parabolic subgroup, in order to arrive at a list of candidates for positive defect flag varieties. In Theorem 4.2 we give a fairly comprehensive description of when d_P is not positive (the norm) in terms of the coefficient $n_i^{\tilde{\alpha}}$ of α_i in the expression of the highest root $\tilde{\alpha}$ w.r.t. the simple roots, the length of the corresponding fundamental weight ω_i and the dual Coxeter number g. Namely we prove:

THEOREM. (i) If $n_i^{\tilde{\alpha}} = 1$, then $d_P = (2 - \langle \omega_i, \omega_i \rangle)g - 2$.
(ii) If $n_i^{\tilde{\alpha}} \geq 2$ and α_i is long, then $d_P < 0$.

2. Preliminaries.

Let G be a simple, compact, connected Lie group, with Lie Algebra $\mathfrak{g}$. For a fixed Cartan subalgebra $\mathfrak{h}$, let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \Re} \mathfrak{g}^{\alpha}$ be the root space decomposition of the complexification of $\mathfrak{g}$ w.r.t. $\mathfrak{h}^{\mathbb{C}}$. As the restriction of the Killing form $(,)$ on $\mathfrak{g}^{\mathbb{C}}$ to $\mathfrak{h}^{\mathbb{C}}$ is non-degenerate, there corresponds to each root $\alpha \in (\mathfrak{h}^{\mathbb{C}})^*$ an element $h_\alpha \in \mathfrak{h}^{\mathbb{C}}$, with $\alpha(h) = (h, h_\alpha)$ for all $h \in \mathfrak{h}^{\mathbb{C}}$. In this way we obtain a non-degenerate bilinear form on the real linear span E of the roots by defining $(\alpha, \beta) := (h_\alpha, h_\beta)$. We will normalise $(,)$ to an inner product $\langle,\rangle$ so that for the highest root $\tilde{\alpha}$ we have that $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = 2$. The two inner products are related by $(,) = \frac{1}{2g}\langle,\rangle$, where g (called the dual Coxeter number) is the eigenvalue of the Casimir element of $\mathfrak{g}^{\mathbb{C}}$ in its adjoint representation (see [**11**], Proposition 2.1). Choosing a fixed linear functional on E that does not vanish on any of the roots, we can define positive roots $\Re^+$ and simple positive roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, where $r = \dim_{\mathbb{C}} \mathfrak{h}^{\mathbb{C}}$ is the rank of $\mathfrak{g}^{\mathbb{C}}$, and we set $\rho = \frac{1}{2} \sum_{\alpha \in \Re^+} \alpha$. The Coxeter number h is the order of a Coxeter element $\sigma = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_r}$, where s_{α_i} is reflection in the hyperplane orthogonal to α_i in E. We will express the highest root $\tilde{\alpha}$ as a positive integer linear combination $\tilde{\alpha} = \sum_{i=1}^{r} n_i^{\tilde{\alpha}} \alpha_i$, in terms of the simple roots $\alpha_1, \ldots, \alpha_r$, labelled as in [**8**], and in general we will express a root α in the form $\alpha = \sum_{i=1}^{r} n_i^{\alpha} \alpha_i$. For $\alpha \in \Re^+$, we let $ht(\alpha)$ denote the height of α, that is, the sum of the coefficients of α relative to the basis of positive simple roots. It is well known that $ht(\tilde{\alpha}) = h - 1$. Recall that the integers $c_{ij} = \frac{2\langle \alpha_i, \alpha_l \rangle}{\langle \alpha_i, \alpha_i \rangle}$ are called the Cartan integers and the Dynkin diagram $\triangle_{\Re}$ is the (undirected, multi) graph with r vertices (labelled by the positive simple roots), and $c_{ij}c_{ji}$ edges joining α_i to α_j. The extended Dynkin diagram $\tilde{\triangle}_{\Re}$ is the (undirected, multi) graph constructed from $\triangle_{\Re}$ by adding a new vertex $\alpha_0 = -\tilde{\alpha}$ and joining it to any vertex α_i by (the old rule of) $n(\alpha_i, \tilde{\alpha}) \cdot n(\tilde{\alpha}, \alpha_i)$ edges, where for $\alpha, \beta \in \Re$, $n(\alpha, \beta) = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$, We then write the coefficient $n_i^{\tilde{\alpha}}$ over the vertex α_i and $n_0^{\tilde{\alpha}} = 1$ over α_0, The following result from [19] will prove very useful.

PROPOSITION 2.1. *$\sum_{\alpha \in \Re^+} \langle \alpha, \gamma \rangle \alpha = g\gamma$ for all $\gamma \in E$ where g is the dual Coxeter number. If $\mathfrak{g}$ is simply laced, then g is also the Coxeter number of $\mathfrak{g}$.*

Finally, for a fixed node α_i of the Dynkin diagram, we define integers n_k and k_i as follows: $n_k := |\{\alpha \in \Re : n_i^{\alpha} = k\}|$ and $k_i := \frac{\langle \tilde{\alpha}, \tilde{\alpha} \rangle}{\langle \alpha_i, \alpha_i \rangle}$.

3. Cardinality and root-sum formulae. Rather than working with the subroot systems obtained by deletion of a node α_i from the Dynkin diagram we will instead consider the closely related subroot systems obtained by deletion of a node α_i from the extended Dynkin diagram (or two nodes when $n_i^{\tilde{\alpha}} = 1$.) This means that our subroot systems are maximal closed subroot systems of $\Re$, corresponding to maximal rank subgroups K_i, which are also maximal when $n_i^{\tilde{\alpha}}$ is a prime, by the following theorem of Borel and de Siebenthal [6]. Recall that a subroot system $\Re'$ of $\Re$ is said to be closed if for any $\alpha, \beta \in \Re'$ we have that $\alpha + \beta \in \Re'$ whenever $\alpha + \beta$ is a root.

We will denote the root system of K_i (which we may assume contains a maximal torus of G) by $\Re_{K_i}$ and call $\Psi := \Re \setminus \Re_{K_i}$ the set of complementary roots.

THEOREM 3.1. ([6, 25]) *Let G be a compact centerless simple Lie group and let $1 \leq i \leq r$.*

(i) *Suppose that $n_i^{\tilde{\alpha}} = 1$, then the centralizer of the circle group $\{\exp(2\pi i t v_i) : t \in \mathbb{R}\}$ (where $v_1, \ldots, v_r$ satisfy $\alpha_i(v_j) = \frac{1}{n_i}\delta_{ij}$) is a maximal connected subgroup of maximal rank in G with $\Pi_{K_i} = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r\}$ as a system of simple roots.*

(ii) *Suppose that $n_i^{\tilde{\alpha}}$ is a prime $p > 1$, then the centralizer of the element $\exp(2\pi i v_i)$ (of order p) is a maximal connected subgroup of maximal rank in G with $\Pi_{K_i} = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r, -\tilde{\alpha}\}$ as a system of simple roots.*

(iii) *Every maximal connected subgroup of maximal rank in G is conjugate to one of the above groups.*

Our starting point will be to follow [25, p. 282], to give a description of the isotropy representation of K_i on the tangent space to G/K_i, for all i. The isotropy representation of K_i complexifies to a representation of K_i on $\sum_{\alpha \in \Psi} \mathfrak{g}^{\alpha}$ and it comes from the adjoint representation of G. We will denote this representation by ad_{G/K_i} and of course the set of complementary roots $\Psi = \Re \setminus \Re_{K_i}$ are its weights. Denoting the irreducible representation of K_i with highest weight v by π_v we have:

PROPOSITION 3.1. *Let π_v be the irreducible representation of K_i with highest weight v.*

(i) *If $n_i^{\tilde{\alpha}} = 1$, then $ad_{G/K_i} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^*$.*
(ii) *If $n_i^{\tilde{\alpha}} = 2$, then $ad_{G/K_i} = \pi_{-\alpha_i}$.*
(iii) *If $n_i^{\tilde{\alpha}} = 3$, then $ad_{G/K_i} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^*$.*
(iv) *If $n_i^{\tilde{\alpha}} = 4$, then $ad_{G/K_i} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^* + \pi_{-\beta_i}$, where β_i is the lowest height positive root with $n_i^{\beta_i} = 2$.*

(v) *If* $n_i^{\tilde{\alpha}} = 5$, *then* $ad_{G/K_i} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^* + \pi_{-\beta_i} + (\pi_{-\beta_i})^*$, *where* β_i *is the lowest height positive root with* $n_i^{\beta_i} = 2$.

(vi) *If* $n_i^{\tilde{\alpha}} = 6$, *then* $ad_{G/K_i} = \pi_{-\alpha_i} + (\pi_{-\alpha_i})^* + \pi_{-\beta_i} + (\pi_{-\beta_i})^* + \pi_{-\gamma_i}$, *where* β_i *is as above and* γ_i *is the lowest height positive root with* $n_i^{\gamma_i} = 3$.

In deriving formulae for the cardinality of the subroot systems described in the introduction, we are lead naturally to formulae for certain sums of roots. Such sums of roots formulae have a long history, starting most notably with [7] where they appear in the calculation of the first Chern class of certain homogeneous spaces (see also [2]). More recently similar formulae appear in [14] to describe the barycentres of the faces of the root polytope corresponding to the root system. For our purposes we now prove the following:

PROPOSITION 3.2. *Let* $n_i^{\tilde{\alpha}} = n$, *and let* $j \in \mathbb{N}$, *with* $j \leq n$. *Then* $\sum_{\{n_i^{\alpha}=j\}} \alpha = s_j \omega_i$ *for some integer* $s_j \in \mathbb{N}$, *and for* $n \geq 2$, *and* $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ *we have*

$$s_j = s_{n-j}.$$

Proof. Let $\Re_{i,j} = \{\alpha \in \Re : n_i^{\alpha} = j\}$. Since

$$s_{\alpha_k}\alpha = \alpha - 2\frac{\langle \alpha,\ \alpha_k \rangle}{\langle \alpha_k,\ \alpha_k \rangle}\alpha_k,$$

we have that s_{α_k} permutes the elements of $\Re_{i,j}$ for $k \neq i$, so that $\langle \sum_{\alpha \in \Re_{i,j}} \alpha,\ \alpha_k \rangle = \langle s_{\alpha_k}(\sum_{\alpha \in \Re_{i,j}} \alpha),\ s_{\alpha_k}\alpha_k \rangle = \langle \sum_{\alpha \in \Re_{i,j}} \alpha,\ s_{\alpha_k}\alpha_k \rangle = -\langle \sum_{\alpha \in \Re_{i,j}} \alpha,\ \alpha_k \rangle$ and $\langle \sum_{\alpha \in \Re_{i,j}} \alpha,\ \alpha_k \rangle = 0$. Therefore, $\sum_{\alpha \in \Re_{i,j}} \alpha = c_j \omega_i$ for some $c_j \in \mathbb{R}$. Recalling that every root can be expressed as an integral linear combination of fundamental weights, we see that $\sum_{\alpha \in \Re_{i,j}} \alpha = s_j \omega_i$ for some $s_j \in \mathbb{Z}$. Finally, the α_i coefficient of $\sum_{\alpha \in \Re_{i,j}} \alpha$ can be alternatively written as jn_j or $s_j k_i \langle \omega_i, \omega_i \rangle$ so that for $j > 0$, $s_j \in \mathbb{N}$.

Let π_v be an irreducible summand of ad_{G/K_i}. As its weights are invariant under the Weyl group of K_i we again have that for $k \neq i$ $\langle \sum_{\lambda \in \pi_v} \lambda,\ \alpha_k \rangle = \langle s_k(\sum_{\lambda \in \pi_v} \lambda),\ s_k\alpha_k \rangle = \langle \sum_{\lambda \in \pi_v} \lambda,\ s_k\alpha_k \rangle = -\langle \sum_{\lambda \in \pi_v} \lambda,\ \alpha_k \rangle$ and therefore $\langle \sum_{\lambda \in \pi_v} \lambda,\ \alpha_k \rangle = 0$. Similarly $\langle \sum_{\lambda \in \pi_v} \lambda,\ \tilde{\alpha} \rangle = 0$ as $s_{\tilde{\alpha}}$ is in the Weyl group of K_i and therefore $\sum_{\lambda \in \pi_v} \lambda = 0$. The result we now show follows from the description of the weights of the irreducible summands of ad_{G/K_i} in the proof of Theorem 3.1. For $m \in \mathbb{Z}$, let $\Lambda_m = \{\alpha \in \Re : n_i^{\alpha} = m\}$.

The weights of $\pi_{-\alpha_i}$, $\pi_{-\beta_i}$, and $\pi_{-\gamma_i}$ for the various values of n are: for $n = 1$, the set of weights of $\pi_{-\alpha_i}$ is Λ_{-1}; for $n \geq 2$, the set of weights of $\pi_{-\alpha_i}$ is $\Lambda_{-1} \cup \Lambda_{n-1}$; for $n \geq 4$, the set of weights of $\pi_{-\beta_i}$ is $\Lambda_{-2} \cup \Lambda_{n-2}$. Finally for $n = 6$, the set of weights of $\pi_{-\gamma_i}$ is $\Lambda_{-3} \cup \Lambda_{n-3}$ which merely says that $s_3 = s_3$.

PROPOSITION 3.3. *Let k be an integer,* $0 < k < n$, *and let* $n_i^{\tilde{\alpha}} = n$ *and* $n \geq 2$ *then:*

$$s_j = s_{n-(j+k)} \text{ for } k + 1 \leq j \leq \left\lfloor \frac{n-k}{2} \right\rfloor.$$

Also

$$s_j + s_{n-(k-j)} - s_{n-(j+k)} = 0 \text{ for } 1 \leq k \leq n \text{ and } j \leq k < \left\lfloor \frac{n-k}{2} \right\rfloor.$$

Proof. Let $\beta \in \Lambda_j$ with $k + 1 \leq j$ and let $\alpha \in \Lambda_{n-k}$ with $k \geq 1$ (we choose α to be long in non-simply laces cases). If $\langle \alpha, \beta \rangle = 0$, then $s_\alpha \beta = \beta$, and if $\langle \alpha, \beta \rangle \neq 0$,

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then either $\alpha+\beta$ is a root (when $\langle\alpha,\beta\rangle<0$) or $\alpha-\beta$ is a root (when $\langle\alpha,\beta\rangle>0$). However, $\alpha+\beta$ would be contained in Λ_{n-k+j}, but $n-k+j>n$. This means that $s_\alpha\beta=-\gamma$ where $n_i^\gamma=n-(j+k)$. The set $\Lambda_j-\Lambda_{n-(j+k)}:=\Lambda_j\cup\Lambda_{(j+k)-n}$ is therefore invariant under s_α so that $\langle\sum_{\lambda\in\Lambda_j-\Lambda_{n-(j+k)}}\lambda,\ \alpha\rangle=\langle s_\alpha(\sum_{\lambda\in\Lambda_j-\Lambda_{n-(j+k)}}\lambda),\ s_\alpha\alpha\rangle=-\langle\sum_{\lambda\in\Lambda_j-\Lambda_{n-(j+k)}}\lambda,\ \alpha\rangle=0$. Using Proposition 3.2. we have $\sum_{\lambda\in\Lambda_j-\Lambda_{n-(j+k)}}\lambda=(s_j-s_{n-(j+k)})\omega_i$ so that $0=\langle\alpha,\sum_{\lambda\in\Lambda_j-\Lambda_{n-(j+k)}}\lambda\rangle=(s_j-s_{n-(j+k)})\langle\alpha,\ \omega_i\rangle$. As $n_i^\alpha=n-k$, we have that $\langle\alpha,\ \omega_i\rangle\neq 0$ and therefore $s_j-s_{n-(j+k)}=0$. □

We now consider the cases when $j\leq k$. For $j\leq k$, either $s_\alpha\beta=-\gamma$ where as above $n_i^\gamma=n-(j+k)$, or $s_\alpha\beta=\psi$ with $n_i^\psi=n-(k-j)$. The set $\Lambda_j-\Lambda_{n-(j+k)}+\Lambda_{n-(k-j)}:=\Lambda_j-\Lambda_{n-(j+k)}\cup\Lambda_{n-(k-j)}$ is therefore invariant under s_α so that $\langle\sum_{\Lambda_j-\Lambda_{n-(j+k)}+\Lambda_{n-(k-j)}}\lambda,\ \alpha\rangle=0$. Again by Proposition 3.2. $\sum_{\Lambda_j-\Lambda_{n-(j+k)}+\Lambda_{n-(k-j)}}\lambda=(s_j-s_{n-(j+k)}+s_{n-k-j)})\omega_i$ and as $\langle\alpha,\ \omega_i\rangle\neq 0$ we have that $s_j-s_{n-(j+k)}+s_{n-(k-j)}=0$.

THEOREM 3.2. *Let $\Re$ be an irreducible reduced crystallographic root system and let $n_i^{\tilde\alpha}=n$. Denote by V_{ω_i} the hyperplane perpendicular to ω_i and let $\Re_{\omega_i}=\Re\cap V_{\omega_i}$, then:*
$\Re_{\omega_i}$ is a root system and for $n_i^{\tilde\alpha}\geq 2$, card $\Re^+_{\omega_i}$ =

$$card\ \Re^+-\left\{\frac{2gk_i}{n(n-1)}\left[1+\frac{1}{2}\cdots+\frac{1}{n-1}\right]-s_n\left(1+\frac{1}{n(n-1)}\right)\right\}k_i\langle\omega_i,\ \omega_i\rangle.$$

For $n_i^{\tilde\alpha}=1$, card $\Re^+_{\omega_i}=card\ \Re^+-(2g-s_1)k_i\langle\omega_i,\ \omega_i\rangle=card\ \Re^+-g\langle\omega_i,\ \omega_i\rangle$.

Proof. $\Re_{\omega_i}=\Re\cap V_{\omega_i}$, so $\Re_{\omega_i}$ consists of those roots with α_i coefficient equal to zero, and they constitute the root system (usually not irrreducible) with Dynkin diagram obtained from that of $\mathfrak{g}$ by the deletion of the node labelled α_i. We now count the number of roots in the complement (in $\Re^+$) of $\Re^+_{\omega_i}$, that is, the positive roots with non-zero α_i coefficient. For $0\leq j\leq n$, the α_i coefficient of $\sum_{\{n_i^\alpha=j\}}\alpha$ can be alternatively written as jn_j or $s_jk_i\langle\omega_i,\omega_i\rangle$ so that $n_j=\frac{s_jk_i}{j}\langle\omega_i,\omega_i\rangle$. When $n_i^{\tilde\alpha}\geq 2$ we use the equations in $s_1,\ldots,s_n$ derived from Propositions 3.1 and 3.2, together with the additional equation $s_1+2s_2+\cdots+(n-1)s_{n-1}+ns_n=gk_i$ (from Proposition 2.1). These $n-1$ equations are easily solved in terms of g, n, and s_n using Gaussian elimination (when $n>2$). All rows of the extended matrix with the exception of the last (coming from Proposition 2.1) consist of two or three non-zero entries (equal to ±1) and are essentially in upper echelon form. Killing the entries $1,\ 2,\ \ldots,n-1$ in the last row has the effect of making $1+2+\cdots+n-1=\frac{n(n-1)}{2}$ the coefficient of s_{n-1} and s_n in the last row of the reduced extended matrix to give the equation $\frac{n(n-1)}{2}s_{n-1}+\frac{n(n-1)}{2}s_n=gk_i$. Back substitution using $s_{n-1}=\frac{2}{n(n-1)}gk_i-s_n\ (=s_1)$ then gives $s_2=\cdots=s_{n-2}=\frac{2}{n(n-1)}gk_i$ and recalling that $n_j=\frac{s_jk_i}{j}\langle\omega_i,\omega_i\rangle$ the result follows in these cases. When $n_i^{\tilde\alpha}=1$, the result follows from the equation $s_1+2s_2+\cdots+(n-1)s_{n-1}+ns_n=gk_i$ where $n=1$ and the fact that $k_i=1$ as α_i is always long (because $\tilde\alpha$ is). Also when $n=2$ we have that $s_1=gk_i-2s_2$. □

COROLLARY 3.1. ([**12**, **23**]) *Let $\Re$ be an irreducible reduced crystallographic root system and let $V_{\tilde\alpha}$ denote the hyperplane perpendicular to $\tilde\alpha$ and let $\Re'=\Re\cap V_{\tilde\alpha}$, then $\Re'$ is a root system and*

$$card\ \Re'=card\ \Re-4g+6.$$

Proof. When $\Re$ is not of type A, we have that $\tilde\alpha=c\omega_i$ with $c\in\{1,2\}$ and $n_i^{\tilde\alpha}=2$. When $c=1$, $\langle\tilde\alpha,\tilde\alpha\rangle=2=\langle\tilde\alpha,\omega_i\rangle=\langle 2\alpha_i,\omega_i\rangle$, so that α_i is long and $k_i=1$. The next highest long root $\alpha=s_{\alpha_i}\tilde\alpha$ has $n_i^\alpha=1$, as does the highest short root, so that $s_2=1$. When

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$\tilde{\alpha} = 2\omega_i$ (for C_r), the argument is similar but now $s_2 = 2$ and α_i is short so that $k_i = 2$. By Theorem 3.3 card $\mathfrak{R}'^{+} = \text{card}\,\mathfrak{R}^{+} - 2g + 3$. When $\mathfrak{R}$ is of type A_r, card $\mathfrak{R} = r(r+1)$ and $\tilde{\alpha} = \omega_1 + \omega_r$ so that $\mathfrak{R}'$ is of type A_{r+2} and $g = h = r + 1$. □

COROLLARY 3.2. *In the notation of Theorem 3.1 we have the following formulae for the dimensions of the irreducible components of* ad_{G/K_i}.

(i) *For* $n_i^{\tilde{\alpha}} = 1$, *we have* $\dim \pi_{-\alpha_i} = g\langle\omega_i,\ \omega_i\rangle$.
(ii) *For* $n_i^{\tilde{\alpha}} = 2$, *we have* $\dim \pi_{-\alpha_i} = \{2(gk_i - 2s_n)\}k_i\langle\omega_i,\ \omega_i\rangle$ *and for* $n > 2$, $\dim \pi_{-\alpha_i} = \left\{\frac{n}{n-1}\left(\frac{2gk_i}{n(n-1)} - s_n\right)\right\}k_i\langle\omega_i,\ \omega_i\rangle$.
(iii) *For* $n_i^{\tilde{\alpha}} \geq 4$, *we have* $\dim \pi_{-\beta_i} = \left\{\left(\frac{n}{2(n-2)}\right)\frac{2gk_i}{n(n-1)}\right\}k_i\langle\omega_i,\ \omega_i\rangle$.
(iv) *For* $n_i^{\tilde{\alpha}} = 6$, *we have* $\dim \pi_{-\gamma_i} = \left\{\left(\frac{n}{3(n-3)}\right)\frac{2gk_i}{n(n-1)}\right\}k_i\langle\omega_i,\ \omega_i\rangle$.

Proof. Recall from Proposition 3.1 the descriptions of the weights of $\pi_{-\alpha_i}$, $\pi_{-\beta_i}$, and $\pi_{-\gamma_i}$ for the various values of n. For $n = 1$ the set of weights of $\pi_{-\alpha_i}$ is Λ_{-1} so that $|\Lambda_{-1}| = |\Lambda_1| = n_1 = g\langle\omega_i,\ \omega_i\rangle$ by Theorem 3.2. For $n \geq 2$ the set of weights of $\pi_{-\alpha_i}$ is $\Lambda_{-1} \cup \Lambda_{n-1}$. In the case that $n = 2$ the value of $2n_1$ is determined by the equation $s_1 = gk_i - 2s_2$. When $n > 2$, $|\Lambda_{-1}| + |\Lambda_{n-1}| = n_1 + n_{n-1} = \{s_1 + \frac{1}{n-1}s_{n-1}\}k_i\langle\omega_i,\ \omega_i\rangle = \{(1 + \frac{1}{n-1})(\frac{2}{n(n-1)}gk_i - s_n)\}k_i\langle\omega_i,\ \omega_i\rangle$ by the proof of Theorem 3.2. For $n \geq 4$ the set of weights of $\pi_{-\beta_i}$ is $\Lambda_{-2} \cup \Lambda_{n-2}$, so that $|\Lambda_{-2}| + |\Lambda_{n-2}| = n_2 + n_{n-2} = \{\frac{1}{2}s_2 + \frac{1}{n-2}s_{n-2}\}k_i\langle\omega_i,\ \omega_i\rangle = \{(\frac{1}{2} + \frac{1}{n-2})(\frac{2}{n(n-1)}gk_i)\}k_i\langle\omega_i,\ \omega_i\rangle$ by the proof of Theorem 3.2. Finally for $n = 6$, the argument is similar observing that the set of weights of $\pi_{-\gamma_i}$ is $\Lambda_{-3} \cup \Lambda_{n-3}$.

REMARKS: (i) As the number of complementary roots with $n_i^{\alpha} = n$ (or equivalently s_n) is relatively small, it is easily computed or can be read off the extended Dynkin diagram in many cases. When the node α_i lies in the extended Dynkin diagram path from α_0 up to and including the branch node (when $\mathfrak{R}$ is simply laced), or the node nearest to α_0 with a multiple connection (when $\mathfrak{R}$ is non-simply laced), n_n is equal to the number of nodes in the extended Dynkin diagram in the path joining α_0 to α_i not counting α_0. This is because we can reflect $\tilde{\alpha}$ by the compositions of the simple reflections in the extended Dynkin diagram path above without changing the α_i coefficient.

(ii) In [5] the quantity $m(G/K_i)$ is defined for an Hermitian symmetric space, to be the number of positive roots $\alpha \in \Psi$, $\alpha \neq \alpha_i$ for which $\alpha - \alpha_i$ is a root, and it is related to the scalar curvature of the space. When $n_i^{\tilde{\alpha}} = 1$, this is the same as the number of positive roots $\alpha \in \Psi$, $\alpha \neq \alpha_i$ for which m_i^{α} is positive, where $\alpha = \sum_{j=1}^{r} m_j^{\alpha}\omega_j$. Taking the latter as the definition of $m(G/K_i)$, it was proved in [9] that when $n = 2$, $s_2 = g - m(G/K_i) - 2$ when α_i is long and $s_2 = h - m(G/K_i) - 2$, when α_i is short. For $n > 2$, it is still the case that $s_n = g - m(G/K_i) - 2$, when α_i is long.

EXAMPLE: For $\mathfrak{R}$ of type F_4, choosing $\alpha_i = \alpha_2$, we have $n_2^{\tilde{\alpha}} = n = 3$, $\omega_2 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4$, [8, p. 273] so that $\langle\omega_2,\ \omega_2\rangle = \langle 6\alpha_2,\ \omega_2\rangle = 3\langle\alpha_2,\ \alpha_2\rangle = 6$ (as α_2 is long, i.e. $k_2 = 1$). By Corollary 3.2 $\dim \pi_{-\alpha_2} = \{\frac{n}{n-1}(\frac{2gk_2}{n(n-1)} - s_n)\}k_2\langle\omega_2,\ \omega_2\rangle$. As the number of nodes in the extended Dynkin diagram path joining α_0 to α_2, not counting α_0, is two, we have that $s_n = 1$. Finally as $g = 9$ we have that $\dim \pi_{-\alpha_2} = 18$. From Theorem 3.2 we calculate card $\mathfrak{R}^{+}_{\omega_2}$ to be $24 - \{\frac{9}{2} - \frac{7}{6}\}6 = 4$.

We next turn our attention to the sum of the positive roots of the related root systems $\mathfrak{R}_{\omega_i}$ and $\mathfrak{R}_{K_i}$, which we will denote by $2\rho_{\omega_i}$ and $2\rho_{K_i}$ respectively, and we relate them to the sum of the positive roots 2ρ of $\Re$. Note that by Theorem 3.1 $2\rho_{\omega_i} = 2\rho_{K_i}$ when K_i is semisimple, that is, when $n_i^{\tilde{\alpha}} = 1$, but otherwise they are different.

THEOREM 3.3. *Let $\Re$ be an irreducible reduced crystallographic root system and let $n_i^{\tilde{\alpha}} = n \geq 2$ and $k_i = \frac{(\tilde{\alpha},\tilde{\alpha})}{(\alpha_i,\alpha_i)}$, then:*

$$2\rho_{K_i} = 2\rho - \frac{2k_i g}{n}\,\omega_i$$

and for $n_i^{\tilde{\alpha}} = n \geq 1$

$$2\rho_{\omega_i} = 2\rho - \left(\frac{2k_i g}{n} - s_n\right)\,\omega_i.$$

In particular for $n_i^{\tilde{\alpha}} = 1$ we have that $\quad 2\rho_{\omega_i} = 2\rho - g\,\omega_i$.

Proof. By definition $\Re_{\omega_i} = \{\alpha \in \Re : n_i^{\alpha} = 0\}$ and therefore

$$2\rho_{\omega_i} = 2\rho - \left(\sum_{n_i^{\alpha}=1} \alpha + \cdots + \sum_{n_i^{\alpha}=n} \alpha\right).$$

Recall that $\sum_{\{n_i^{\alpha}=j\}} \alpha = s_j\omega_i$, $\langle\alpha_i,\,\omega_i\rangle = 1/k_i$ and by Proposition 2.1,

$$\sum_{\alpha\in\Re^+} \langle\alpha,\ \omega_i\rangle\alpha = \sum_{n_i^{\alpha}=0} \langle\alpha,\ \omega_i\rangle\alpha + \sum_{n_i^{\alpha}=1} \langle\alpha,\ \omega_i\rangle\alpha + \cdots + \sum_{n_i^{\alpha}=n} \langle\alpha,\ \omega_i\rangle\alpha = g\,\omega_i,$$

so that $s_1\omega_i + 2s_2\omega_i + \cdots + ns_n\omega_i = gk_i\omega_i$. When n is odd therefore

$$s_1 + 2s_2 + \cdots + \left(\frac{n-1}{2}\right) s_{\frac{n-1}{2}} + \left(\frac{n-1}{2} + 1\right) s_{\frac{n-1}{2}+1} + \cdots + (n-1)s_{n-1} + ns_n = gk_i,$$

and by Proposition 3.2 $s_1 = s_{n-1}, \ldots, s_{\frac{n-1}{2}} = s_{\frac{n-1}{2}+1}$, so that $ns_n + ns_{n-1} + \cdots + ns_{\frac{n-1}{2}+1} = gk_i$ and $2s_n + 2s_{n-1} + \cdots + 2s_{\frac{n-1}{2}+1} = \frac{2gk_i}{n}$. Applying Proposition 3.2 again, we have that $s_1 + s_2 + \cdots + s_{n-1} + 2s_n = \frac{2gk_i}{n}$ and

$$s_1 + s_2 + \cdots + s_n = \frac{2k_i g}{n} - s_n.$$

Similarly when n is even, by Proposition 3.1 $s_{\frac{n}{2}-1} = s_{\frac{n}{2}+1}$, therefore $ns_n + ns_{n-1} + \cdots + ns_{\frac{n}{2}+1} + \frac{n}{2}s_{\frac{n}{2}} = gk_i$, and again

$$\frac{2gk_i}{n} = 2s_n + 2s_{n-1} + \cdots + 2s_{\frac{n}{2}+1} + s_{\frac{n}{2}},$$

so that

$$2\rho_{\omega_i} = 2\rho - \left(\frac{2k_i g}{n} - s_n\right)\,\omega_i$$

in all cases. By Theorem 3.1, $\Pi_{K_i} = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r, -\tilde{\alpha}\}$ is a system of positive simple roots of K_i so that $\Re_{K_i}^+ = \Re_{\omega_i}^+ \cup \{-\alpha \in \Re : n_i^{\alpha} = n\}$ and $2\rho_{K_i} = 2\rho_{\omega_i} - s_n\omega_i = 2\rho - \frac{2k_i g}{n}\,\omega_i$. When $n_i^{\tilde{\alpha}} = 1$, we note that α_i is necessarily long (because $\tilde{\alpha}$ is) so that $s_1\omega_i = \sum_{\{n_i^{\alpha}=1\}} \alpha = \sum_{\alpha\in\Re^+} \langle\alpha,\ \omega_i\rangle\alpha = g\omega_i$ by Proposition 2.1. and the result follows. As we have

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proved that $s_1 = g$ and $k_i = 1$ when $n_i^{\tilde{\alpha}} = n = 1$ we have that $2\rho_{\omega_i} = 2\rho - (\frac{2k_i g}{n} - s_n)\,\omega_i$ in all cases. $\square$

4. Applications to flag manifolds. The background material for this section can be found in [**1**, **2**, **3**, **21**].

A flag manifold M is a homogeneous space G/K, where G is a compact connected Lie group and $K = C(S)$ is the centralizer of a torus $S \subseteq G$, or equivalently they are the orbits of the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Flag manifolds have an alternative description of the form $G^{\mathbb{C}}/P$, where $G^{\mathbb{C}}$ is the complexification of G and P is a parabolic subgroup of $G^{\mathbb{C}}$, the definition of which we now recall.

The subalgebra $\mathfrak{b} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha<0} \mathfrak{g}_\alpha$ is a maximal solvable subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let B be the closed connected solvable subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{b}$, then any conjugate of B is called a *Borel* subgroup.

DEFINITION 4.1. *A connected subgroup P of $G^{\mathbb{C}}$ containing a Borel subgroup is called a parabolic subgroup.*

The Lie algebra of P is given by

$$\mathfrak{p} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha>0} \mathfrak{g}_{-\alpha} + \sum_{\alpha\in\Re_P^+} \mathfrak{g}_\alpha,$$

where $\Re_P^+$ is a closed (under addition) subset of positive roots. As $\Re_P := \Re^- \cup \Re_P^+$ is also a closed set of roots containing all negative simple roots, it follows that $\Re_P^+$ is generated by a set of positive simple roots $\{\alpha_i : i \in I\}$, where $I \subseteq \{1, \ldots, r\}$.

The set of positive complementary roots $\Re^+ \setminus \Re_P^+$ is denoted $\Re_M$ and is called the set of roots of M. In particular if $|I| = r - 1$, we call P a maximal parabolic subgroup and it is this case that we will focus on in this section. When $I = \{1, \ldots, \hat{i}, \ldots, r\}$ with $n_i^{\tilde{\alpha}} = n$, we note that $\Re_M = \Psi^+ \cup \{\alpha \in \Re : n_i^\alpha = n\}$. We now adapt the results of Section 3 to study the isotropy representation of K on T_oM, where $o = eK$ the identity coset of G/K. This representation is equivalent to the adjoint representation of K on $\mathfrak{m}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a reductive, orthogonal (w.r.t. the negative of the Killing form B) decomposition of $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Re^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)$, where $A_\alpha = E_\alpha + E_{-\alpha}$, $B_\alpha = i(E_\alpha - E_{-\alpha})$ and $\mathfrak{g}_\alpha = \mathbb{C}E_\alpha$. The intersection $\mathfrak{k} = \mathfrak{p} \cap \mathfrak{g}$ is the Lie algebra of K. In particular we will derive formulae for the dimensions $d_j = \dim \mathfrak{m}_j$, where $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ is the decomposition of $\mathfrak{m}$ into its *Ad(K)*-irreducible (inequivalent) real submodules, because of their importance in finding Einstein metric on M.

THEOREM 4.1. *Let $M = G/K = G^{\mathbb{C}}/P$, where $P = P_{\alpha_i}$ is a maximal parabolic subgroup corresponding to $I = \{1, \ldots, \hat{i}, \ldots, r\}$ with $n_i^{\tilde{\alpha}} = n$, then $\mathfrak{m}$ decomposes as a sum $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n$ of Ad(K)-irreducible (inequivalent) real submodules of dimensions $d_1, \ldots, d_n$ given as follows:*

(i) $d_j = \frac{4gk_i}{n(n-1)} \frac{k_i\langle\omega_i,\omega_i\rangle}{j}$, *for* $2 \le j \le n-2$.
(ii) $d_1 = (n-1)d_{n-1} = \left(\frac{4gk_i}{n(n-1)} - 2s_n\right) k_i\langle\omega_i, \omega_i\rangle$.
(iii) $d_1 + nd_n = jd_j$, *for* $2 \le j \le n-2$.

Proof. Minor adjustments to the arguments of Theorem 3.2 and Corollary 3.2, to take account of the fact that $\tilde{\alpha}$ is not a simple root of K, yield that $\mathfrak{m}^{\mathbb{C}}$ decomposes into complex

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irreducible $ad(\mathfrak{k}^{\mathbb{C}})$-submodules $\mathfrak{m}'_l$, one for each $l \in \{-n, -n+1, \dots, n\}$. Recalling the notation $\Lambda_m = \{\alpha \in \mathfrak{R} : n_i^\alpha = m\}$ for $m \in \mathbb{Z}$, we have that $\mathfrak{m}'_l = \sum_{\alpha \in \Lambda_l} \mathbb{C}E_\alpha$. Accordingly $\mathfrak{m}$ decomposes as $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n$ of real irreducible $ad(\mathfrak{k})$-submodules $\mathfrak{m}_j$, $1 \le j \le n$, where $\mathfrak{m}_j = \sum_{\alpha \in \Lambda_j} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)$. Alternatively the decomposition can be obtained using the theory of *T-roots* [2, 3]. To complete the proof we note that $d_j = \dim \mathfrak{m}_j = 2|\Lambda_j| = \frac{2s_jk_i}{j}\langle \omega_i, \omega_i \rangle$ and use the values for s_j obtained by solving the system of linear equations in Theorem 3.3. □

EXAMPLE: In [3] the invariant Einstein metrics on flag manifolds with four isotropy summands were calculated. This involved using the Weyl dimension formula to calculate the dimensions d_j of the isotropy summands $\mathfrak{m}_j$ above. There are two types of such flag manifolds, with Type I corresponding to a maximal parabolic subgroup P_{α_i} with $n_i^{\tilde{\alpha}} = n = 4$. We will apply Theorem 4.1 to recalculate these dimensions for the two spaces of Type I with $\mathfrak{R}$ of type E_8. For the case of $n_6^{\tilde{\alpha}} = 4$, we have $\langle \omega_6, \omega_6 \rangle = 12$ [8, p. 269], and as $\mathfrak{R}$ is simply laced $k_i = 1$ and $g = h$ $(= 30)$. By Theorem 4.1 (i) therefore $d_2 = \frac{4.30}{4.3}\frac{12}{2} = 60$. As the number of nodes in the extended Dynkin diagram path joining α_0 to α_6, not counting α_0, is three, we have that $n_4 = 3$ (equivalently $s_4 = 1$), and therefore $d_4 = 6$. Using part (ii) of Theorem 4.1 we get $d_1 = (\frac{4.30}{4.3} - 2)12 = 96$ and $d_3 = 32$. For the flag manifold corresponding to the maximal parabolic subgroup P_{α_3} with $n_3^{\tilde{\alpha}} = n = 4$, the calculation is similar. In this case, however, the number of nodes in the extended Dynkin diagram path joining α_0 to α_3, not counting α_0, is six but it passes through the branch node α_4 so that the simple reflection s_{α_2} (α_2 is connected to the branch node) yields an additional root with $n_3^{\tilde{\alpha}} = n = 4$, giving $n_4 = 7$. As $\langle \omega_3, \omega_3 \rangle = 14$, $s_4 = 2$ and $d_4 = 14$. Now by Theorem 4.1 (ii) $d_1 = 84$, $d_2 = 70$, and $d_3 = 28$.

We next use Theorem 3.4 to study the first Chern class of the tangent bundle of $M = G^{\mathbb{C}}/P$ (or simply the first Chern class of M). The Chern classes can be expressed in terms of the roots of M [1, 7, 21]. The total Chern class $c(M)$ has a description as

$$c(M) = \prod_{\alpha \in \mathfrak{R}_M} (t + \alpha) = \sum_{q=0}^{\dim M} c_q t^{\dim M - q},$$

so that the first Chern class $c_1(M) = \sum_{\alpha \in \mathfrak{R}_M} \alpha$. Theorem 3.4 now has a geometric interpretation in terms of the first Chern class of $M = G^{\mathbb{C}}/P_{\alpha_i}$, where P_{α_i} in the maximal parabolic subgroup and the space of full flags $M = G^{\mathbb{C}}/B$, where B is a Borel subgroup. Noting that for $M = G^{\mathbb{C}}/B$, we have $c_1(M) = \sum_{\alpha \in \mathfrak{R}_M} \alpha = \sum_{\alpha \in \mathfrak{R}^+} \alpha = 2\rho$ we obtain the following:

PROPOSITION 4.1. *Let $M = G^{\mathbb{C}}/P$, where $P = P_{\alpha_i}$ is a maximal parabolic subgroup corresponding to $I = \{1, \dots, \hat{i}, \dots, r\}$ with $n_i^{\tilde{\alpha}} = n$, then*

$$c_1(G^{\mathbb{C}}/B) = 2\rho_{\omega_i} + c_1(G^{\mathbb{C}}/P_{\alpha_i}),$$

and

$$c_1(G^{\mathbb{C}}/P_{\alpha_i}) = \left(\frac{2k_i g}{n_i^{\tilde{\alpha}}} - s_{n_i^{\tilde{\alpha}}}\right)\omega_i.$$

REMARK: The first Chern class $c_1(G^{\mathbb{C}}/P)$ of any parabolic subgroup P can be obtained from knowledge of all the $c_1(G^{\mathbb{C}}/P_{\alpha_i})$, where P_{α_i} are the maximal parabolic subgroups by

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an algorithm described in [22]. Also we note for later that if $\tau := 2\rho - 2\rho_{\omega_i}$, then using the natural isomorphism between $\mathfrak{h}^*$ and $\Omega^2(X)^G$ (the G-invariant 2 forms on M) we see that

$$\tau \longleftrightarrow \frac{i}{2\pi} d\tau = \frac{i}{4\pi} \sum_{\alpha \in \Re_M^+} \langle \tau, \alpha \rangle dx_\alpha \wedge d\bar{x}_\alpha$$

represents the first Chern class of M [2].

We now consider another interpretation of $c_1(M)$ when M is considered as a projective manifold. Let P be a parabolic subgroup defined by a subset I with corresponding roots $\Re_P$ and let

$$\Lambda_M = \{\lambda \in \Lambda : \langle \lambda, \alpha \rangle = 0, \quad \forall \alpha \in \Re_P\},$$

which is generated by $\{\omega_i : i \notin I\}$. Any line bundle on M is homogeneous and is determined by a character $\lambda \in \Lambda_M$ which gives a character $\tilde{\lambda} : P \to \mathbb{C}^*$ so that

$$L = G^{\mathbb{C}} \times_P \mathbb{C}^{\tilde{\lambda}} = G^{\mathbb{C}} \times \mathbb{C}^{\tilde{\lambda}} / (g, z) \sim (gp^{-1}, \tilde{\lambda}(p)z) \quad \forall p \in P.$$

As above, the first Chern class of $L = L_\lambda$ is $c_1(L_\lambda) = \frac{i}{4\pi} \sum_{\alpha \in \Re_\lambda} \langle \lambda, \alpha \rangle dx \wedge d\bar{x}_\alpha$, and we say that L_λ is nef (numerically effective) if $\int_c c_1(L_\lambda) \geq 0$ for all (effective) curves c in M, which in our context means that L_λ is nef $\Leftrightarrow$ all $n_i \geq 0$, where $\lambda = \sum_{i=1}^r n_i \omega_i$. We denote the (holomorphic) sections of L_λ by:

$$\Gamma := H^0(M, \mathbb{C}^{\tilde{\lambda}}) := \{s : G^{\mathbb{C}} \to \mathbb{C}^{\tilde{\lambda}} : s(gp^{-1}) = \tilde{\lambda}(p)s(g) \quad \forall p \in P\}.$$

Γ is a vector space on which $G^{\mathbb{C}}$ acts via $g.s(g\prime) = s(g^{-1}g\prime)$.

Definition 4.2. *A line bundle L_λ on M is said to be ample if some power L_λ^m embeds M via its sections in $\mathbb{P}(\Gamma^*)$.*

In this setting, ampleness is equivalent to the condition that $\lambda = \sum_{i \notin I} n_i \omega_i$, with all $n_i > 0$.

An important line bundle on M is the canonical bundle $K_M = \wedge^{\dim(M)} TM^*$, where TM^* is the cotangent bundle of M and $c_1(K_M) = -\sum_{\alpha \in \Re_M} \alpha$, so that it is never nef.

Definition 4.3. *Let M be a projective manifold whose canonical bundle is not nef and let L be an ample line bundle on M. The nef value of L denoted*

$$\tau(M, L) = inf \left\{ p/q \in \mathbb{Q} : K_M^q \otimes L^p \quad is \quad nef \right\}.$$

In the case that $M = G^{\mathbb{C}}/P$, where P is the maximal parabolic subgroup corresponding to $I = \{1, \ldots, \hat{i}, \ldots, r\}$, it is proved in [21] that $L = L_{\omega_i}$ is the minimal (very) ample line bundle on M and $\tau(M, L)\omega_i = c_1(X)$ so that by Theorem 3.4 we have:

Proposition 4.2. *Let $M = G^{\mathbb{C}}/P$, where $P = P_{\alpha_i}$ is a maximal parabolic subgroup corresponding to $I = \{1, \ldots, \hat{i}, \ldots, r\}$ with $n_i^{\tilde{\alpha}} = n$, and let $L = L_{\omega_i}$ be the minimal ample line bundle on M, then*

$$\tau(M, L) = \frac{2k_i g}{n} - s_n.$$

When M is embedded in $\mathbb{P}(\Gamma^*)$ via the global sections of $L = L_{\omega_i}$, there is a connection between the nef value $\tau(M, L)$ and the codimension of the dual variety $M' \subset \mathbb{P}^{\mathbb{N}}$. Recall that the *defect* of (M, L) is defined to be $def(M, L) = \operatorname{codim} M' - 1$. If $def(M, L) > 0$, then $def(M, L) = 2(\tau(M, L) - 1) - \dim M$ [4]. We denote the quantity $2(\tau(M, L) - 1) - \dim M$ by d_P, so that when $def(M, L) > 0$ we have $def(M, L) = d_P$.

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We now evaluate the quantity $d_P = 2(\tau(M, L) - 1) - \dim M$ and we obtain the following theorem:

THEOREM 4.2. (i) *If* $n_i^{\tilde{\alpha}} = 1$, *then* $d_P = (2 - \langle\omega_i, \omega_i\rangle)g - 2$.
(ii) *If* $n_i^{\tilde{\alpha}} \geq 2$ *and* α_i *is long, then* $d_P < 0$.

Proof. We first consider those cases where $n_i^{\tilde{\alpha}} \geq 4$. By Proposition 4.2 the quantity $d_P = 2(\frac{2g}{n_i^{\tilde{\alpha}}} - s_{n_i^{\tilde{\alpha}}}) - \dim M - 2 \leq 2(\frac{2g}{4} - s_{n_i^{\tilde{\alpha}}}) - \dim M - 2 = g - 2s_{n_i^{\tilde{\alpha}}} - \dim M - 2$. Recalling that the quantity $m(G/K_i)$ is defined to be the number of positive roots $\alpha \in \Psi$, $\alpha \neq \alpha_i$ for which m_i^α is positive, where $\alpha = \sum_{j=1}^r m_j^\alpha \omega_j$ and that $m(G/K_i) + 2 = g - s_{n_i^{\tilde{\alpha}}}$ when α_i is long, we have that $d_P = m(G/K_i) - \dim M - s_{n_i^{\tilde{\alpha}}} < 0$. Similarly for $n_i^{\tilde{\alpha}} = 3$ we have $d_P = 2(\frac{2g}{3} - s_3) - \dim M - 2 = m(G/K_i) + \frac{g}{3} - s_3 - \dim M = m(G/K_i) + s_2 - \dim M$ (see the proof of Theorem 3.3). Using the definition of $m(G/K_i)$ we have that $d_P < s_2 - card\,\{\alpha \in \Re_M | m_i^\alpha \leq 0\} \leq card\,\{\alpha \in \Re_M | n_i^\alpha = 2 \text{ and } m_i^\alpha > 0\} - card\,\{\alpha \in \Re_M | m_i^\alpha \leq 0\} = card\,\{\alpha \in \Re_M | n_i^\alpha = 1 \text{ and } m_i^\alpha < 0\} - card\,\{\alpha \in \Re_M | m_i^\alpha \leq 0\} < 0$. Finally when $n_i^{\tilde{\alpha}} = 2$ (and α_i is long) we can lookup the α_i coefficient of ω_i in [8] to establish that $\langle\omega_i, \omega_i\rangle \geq 3$, unless $\langle\omega_i, \omega_i\rangle = 2$ in which case $\tilde{\alpha} = c\omega_i$, and $s_2 = 1$ by Corollary 2.2, so that $d_P = 2(\frac{2g}{2} - s_2) - \dim M - 2 = 2g - 4 - (2g - 3) < 0$. We next deal with the cases where $\langle\omega_i, \omega_i\rangle \geq 4$, then $d_P = 2g - 2s_2 - \dim M - 2 = 2s_1 + 2s_2 - n_1 - n_2 - 2 = \frac{2n_1 + 4n_2}{\langle\omega_i,\omega_i\rangle} - n_1 - n_2 - 2 < 0$. Similarly when $\langle\omega_i, \omega_i\rangle \geq 3$, we have $d_P \leq -\frac{n_1}{3} + \frac{n_2}{3} - 2$, which we now show is negative for $3 \leq \langle\omega_i, \omega_i\rangle \leq 4$. Observing that $m_i^\alpha \in \{0, 1\}$ for $\alpha \in \{\alpha \in \Re_M | n_i^\alpha = 2\}$, we obtain $s_2 = \frac{2n_2}{\langle\omega_i,\omega_i\rangle}$ roots in $\{\alpha \in \Re_M | n_i^\alpha = 1\}$ with $m_i^\alpha = -1$ and there are therefore at least $\frac{4n_2}{\langle\omega_i,\omega_i\rangle}$ roots in $\{\alpha \in \Re_M | n_i^\alpha = 1\}$, so that $n_1 \geq \frac{4n_2}{\langle\omega_i,\omega_i\rangle} \geq n_2$ and $d_P < 0$.

REMARK: We observe that the quantity $k_i\langle\omega_i, \omega_i\rangle$ is important in determining the sign of d_P. It follows from Proposition 2.1 in [10] that $\langle\omega_i, \omega_i\rangle$ increases (and $n_i^{\tilde{\alpha}}$ cannot decrease) as the corresponding node α_i in the extended Dynkin diagram is further along the path from a pendant node to the first node with a branch or multiple connection. Also for root systems of type other than A_r or C_r we have that $\tilde{\alpha} = \omega_j$ so that $\langle\omega_j, \omega_j\rangle = 2$, and the corresponding Dynkin diagram node α_j either is a pendant node or is connected to a pendant node. So in general we expect d_P to be positive when $n_i^{\tilde{\alpha}} \leq 2$ only if the corresponding α_i node in the Dynkin diagram is a pendant node or is adjacent to a pendant node. Checking the few cases not covered already we see that this indeed turns out to be the case and we have the following [22]:

COROLLARY 4.1. *The following infinite families of Flag manifolds* G/P_i *have* $d_P > 0$ *:*

(a) *If* $n_i^{\tilde{\alpha}} = 1$,

(i) $\Re = A_r, \quad i \in \{1, 2, r-1, r\}$.
(ii) $\Re = B_r, \quad i = 1$.
(iii) $\Re = D_r, \quad i = 1$.

(b) *If* $n_i^{\tilde{\alpha}} = 2$, $\Re = C_r, \quad i \in \{1, 2\}$.

COROLLARY 4.2. *The following finite collection completes the list of Flag manifolds* G/P_i *with* $d_P > 0$ *:*

(a) *If* $n_i^{\tilde{\alpha}} = 1$,

(i) $\Re = A_r, \quad r = 5, \quad i = 3$.
(ii) $\Re = D_r, \quad 5 \leq r \leq 7, \quad i \in \{r, r-1\}$.
(iv) $\Re = E_6, \quad i \in \{1, 6\}$.
(iv) $\Re = E_7, \quad i = 1$.

(b) *If* $n_i^{\tilde{\alpha}} = 2$,

(i) $\mathfrak{R} = B_r,\ \ 2 \leq r \leq 6,\ \ i = r.$

(ii) $\mathfrak{R} = F_4,\ \ i = 4.$

(c) *If* $n_i^{\tilde{\alpha}} = 3$, $\mathfrak{R} = G_2,\ \ i = 1.$

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