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COCHARACTERS, CODIMENSIONS AND HILBERT SERIES OF THE POLYNOMIAL IDENTITIES FOR 2×2 MATRICES WITH INVOLUTION

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ABSTRACT. Let $M_2(K,*)$ be the algebra of 2×2 matrices with involution over a field K of characteristic 0. We obtain the exact values of the cocharacters, codimensions and Hilbert series of the $*$-T-ideal of the polynomial identities for $M_2(K,*)$.

Introduction. Let R be an algebra with involution over a field K of characteristic 0 and let $T(R,*)$ be the $*$-T-ideal of all $*$-polynomial identities of R. In the case of ordinary polynomial identities, a lot of information for the polynomial identities is carried by the S_n-cocharacter sequence, the codimension sequence and of the Hilbert series of the polynomial identities for the algebra. These are also the main objects for quantitative investigation of the polynomial identities for algebras with involution. In this case the characters of S_n are replaced by characters of the wreath product $\mathbf{Z}_2 \wr S_n$ [4].

In this paper we study the $*$-polynomial identities of the 2×2 matrix algebra $M_2(K,*)$ with involution $*$. Two kinds of involution define different $*$-T-ideals $T\big(M_t(K),*\big)$—the transpose and the symplectic involutions. We obtain the exact values of the cocharacters, codimensions and the Hilbert series of the $*$-polynomial identities for the 2×2 matrix algebra. The essentially new results are in the case of transpose involution. Most of the results for the symplectic involution are obtained by Procesi [6] or can be easily derived from there.

Usually the investigation of the matrix polynomial identities involves trace identities and invariant theory. Here we follow another approach which is based on the so called proper (or commutator) polynomial identities. For the ordinary polynomial identities the simplest version of the method can be traced back in the Specht's paper [7]. The further development allowed to obtain explicit results for algebras satisfying an identity of low degree. A selfcontained exposition for the application of the method to the ordinary 2×2 matrices can be found in [2].

1. **Group actions on the polynomial identities with involution.**

Let K be a field of characteristic 0, $X = \{x_1, x_2, \ldots\}$ a countable set of unknowns and let $K\langle X,*\rangle = K\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ be the free unitary algebra with involution $*$. Let us denote by $F_m(*) = K\langle x_1, x_1^*, \ldots, x_m, x_m^*\rangle$ the free subalgebra of rank m.

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Let R be a unitary algebra with involution $*$. We consider involutions of first kind only, *i.e.* $(\alpha r)^* = \alpha r^*$ for $\alpha \in K$, $r \in R$. An element $f(x_1, x_1^*, \ldots, x_m, x_m^*)$ from $K\langle X, *\rangle$ is called a $*$-*polynomial identity* for R if $f(r_1, r_1^*, \ldots, r_m, r_m^*) = 0$ for all substitutions $r_1, \ldots, r_m \in R$. The set $T(R,*)$ of all $*$-polynomial identities of R is a $*$-T-ideal of $K\langle X, *\rangle$, *i.e.* an ideal which is invariant under all endomorphisms of $K\langle X, *\rangle$ commuting with the involution. Then $F(R,*) = K\langle X,*\rangle / T(R,*)$ is the relatively free algebra in the variety of algebras with involution var(R) satisfying all $*$-polynomial identities of R and $F_m(R,*) = F_m(*)/\big(F_m(*) \cap T(R,*)\big)$ is the relatively free algebra of rank m in var(R). Let

$$P_n(*) = \operatorname{span}\{x_{\sigma(1)}^{a_1} \cdots x_{\sigma(n)}^{a_n} \mid \sigma \in S_n, a_i = 1, *\}$$

be the space of multilinear $*$-polynomials in $x_1, \ldots, x_n, x_1^*, \ldots, x_n^*$. We denote by $P_n(R,*) = P_n(*)/\big(P_n(*) \cap T(R,*)\big)$ the set of multilinear elements of degree n in $F(R,*)$. The n-th codimension of R is $c_n(R,*) = \dim P_n(R,*)$, $n = 0, 1, 2, \ldots$. We set $s_i = x_i + x_i^*$ and $k_i = x_i - x_i^*$, $i = 1, 2, \ldots$. Then $F_m(*) = K\langle s_1, \ldots, s_m, k_1, \ldots, k_m\rangle$ and we assume that the same variables $s_1, \ldots, s_m, k_1, \ldots, k_m$ generate the relatively free algebra $F_m(R,*)$. The vector space $F_m(*)$ has a natural multigrading obtained by counting the degree in the symmetric variables $s_1, \ldots, s_m$ and in the skew-symmetric variables $k_1, \ldots, k_m$. Since the ideal $F_m(*) \cap T(R,*)$ is multihomogeneous, $F_m(R,*)$ inherits the multigrading. Let $F_m^{(\mathbf{a},\mathbf{b})}(R,*)$, $(\mathbf{a},\mathbf{b}) = (a_1, \ldots, a_m, b_1, \ldots, b_m)$, be the multihomogeneous component of degree a_i in s_i and of degree b_i in k_i, $i = 1, \ldots, m$. The $*$-Hilbert series of $F_m(R,*)$ is defined as the formal power series

$$H(R,*,y_1, \ldots, y_m, z_1, \ldots, z_m) = \sum_{(\mathbf{a},\mathbf{b})} \dim F_m^{(\mathbf{a},\mathbf{b})}(R,*) y_1^{a_1} \cdots y_m^{a_m} z_1^{b_1} \cdots z_m^{b_m}.$$

Let S_n be the symmetric group acting on $1, \ldots, n$ and let $\mathbf{Z}_2 = \{1, *\}$ be the cyclic group of order 2. The wreath product $\mathbf{Z}_2 \wr S_n$ is defined by

$$\mathbf{Z}_2 \wr S_n = \{(a_1, \ldots, a_n; \sigma) \mid a_i \in \mathbf{Z}_2, \sigma \in S_n\}$$

with multiplication given by

$$(a_1, \ldots, a_n; \sigma)(b_1, \ldots, b_n; \tau) = (a_1 b_{\sigma^{-1}(1)}, \ldots, a_n b_{\sigma^{-1}(n)}; \sigma\tau).$$

The action of the group $\mathbf{Z}_2 \wr S_n$ on $P_n(*)$ defined in [4] can be rewritten in the following way. For $(a_1, \ldots, a_n; \sigma) \in \mathbf{Z}_2 \wr S_n$ and $i = 1, \ldots, n$ we define $(a_1, \ldots, a_n; \sigma)s_i = s_{\sigma(i)}$ and $(a_1, \ldots, a_n; \sigma)k_i = k_{\sigma(i)}^{a_{\sigma(i)}} = \pm k_{\sigma(i)}$. Since

$$P_n(*) = \operatorname{span}\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, w_i = s_i \text{ or } w_i = k_i, i = 1, \ldots, n\},$$

the action of $\mathbf{Z}_2 \wr S_n$ on s_i and k_i can be extended diagonally on $P_n(*)$. It is easily checked that under this action $P_n(*)$ becomes a left $\mathbf{Z}_2 \wr S_n$-module.

Similarly, let $U = \operatorname{span}\{s_1, \ldots, s_m\}$, $V = \operatorname{span}\{k_1, \ldots, k_m\}$. The group $\mathrm{GL}(U) \times \mathrm{GL}(V) \cong \mathrm{GL}_m \times \mathrm{GL}_m$ acts on the left on the space $U \oplus V$ and we extend this action diagonally to get an action on $F_m(*)$. We remark that S_n acts also from the right on the homogeneous component $F_m^{(n)}(*)$ of degree n by place permutation.

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For every $*$-T-ideal $T(R,*)$ the vector spaces $P_n(*) \cap T(R,*)$ and $F_m(*) \cap T(R,*)$ are invariant under the above actions of $\mathbf{Z}_2 \wr S_n$ and $\mathrm{GL}_m \times \mathrm{GL}_m$, respectively. Hence we can view $P_n(R,*)$ and $F_m(R,*)$ respectively as $\mathbf{Z}_2 \wr S_n$- and $\mathrm{GL}_m \times \mathrm{GL}_m$-modules and we want to study their structure.

Now we describe briefly the representation theory of $\mathbf{Z}_2 \wr S_n$ on $P_n(*)$ [4] and that of $\mathrm{GL}_m \times \mathrm{GL}_m$ on $F_m(*)$ [3]. The irreducible modules for both the groups are described by pairs of partitions (λ, μ), where $\lambda \in \mathrm{Part}(r)$, $\mu \in \mathrm{Part}(n-r)$ for all $r = 0, 1, \ldots, n$. We write $M_{\lambda,\mu}$ and $N_{\lambda,\mu}$ for the corresponding $\mathbf{Z}_2 \wr S_n$- and $\mathrm{GL}_m \times \mathrm{GL}_m$-modules, respectively. More precisely let $(K\langle s_1, \ldots, s_m\rangle)^{(r)}$ be the homogeneous component of degree r of $K\langle s_1, \ldots, s_m\rangle$. Let N_λ be an irreducible $\mathrm{GL}(U)$-submodule of $(K\langle s_1, \ldots, s_m\rangle)^{(r)} \cong U^{\otimes r}$ corresponding to λ. It is well known (see *e.g.* [8]) that the highest weight space of N_λ is one-dimensional and there exists an isomorphic copy of N_λ in $(K\langle s_1, \ldots, s_m\rangle)^{(r)}$ such that its highest weight space is spanned on the product of standard polynomials

$$f_\lambda = \prod_{b=1}^{c} \Big(\sum_{\sigma \in S_{\lambda'_b}} (\operatorname{sign} \sigma) s_{\sigma(1)} \cdots s_{\sigma(\lambda'_b)} \Big),$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_c)$ is the conjugate partition of λ. Similarly we define a $\mathrm{GL}(V)$-submodule N_μ of $(K\langle k_1, \ldots, k_m\rangle)^{(n-r)}$ with a generator f_μ. Now the irreducible $\mathrm{GL}_m \times \mathrm{GL}_m$-module $N_{\lambda,\mu}$ is isomorphic to the tensor product $N_\lambda \otimes_K N_\mu$. A generator of an $N_{\lambda,\mu}$ is $f_\lambda(s_1, \ldots, s_p) f_\mu(k_1, \ldots, k_q)$, where $p = \lambda'_1$, $q = \mu'_1$. Any isomorphic copy of $N_{\lambda,\mu}$ in $F_m(*)$ is generated by a non-zero element

$$f_\lambda(s_1, \ldots, s_p) f_\mu(k_1, \ldots, k_q) \sum_{\tau \in S_n} a_\tau \tau, \qquad a_\tau \in K.$$

Similarly, the $\mathbf{Z}_2 \wr S_n$- module $M_{\lambda,\mu}$ can be described as follows [4]: Let t_λ be a λ-Young tableau in the integers $1, \ldots, r$ and t_μ a μ-tableau in the integers $r+1, \ldots, n$. Let e_{t_λ} and e_{t_μ} be the corresponding essential idempotents of KS_r and $KS_{(r+1,\ldots,n)} \equiv KS_{n-r}$, respectively. Then, if Γ is a left transversal of $S_r \times S_{n-r}$ in S_n, we have

$$M_{\lambda,\mu} \cong \Big(\bigoplus_{\gamma \in \Gamma} \gamma K S_r e_{t_\lambda} \otimes_K K S_{n-r} e_{t_\mu} \Big) (s_1 \cdots s_r k_{r+1} \cdots k_n),$$

where S_r and S_{n-r} act on the sets of variables $\{s_1, \ldots, s_r\}$ and $\{k_{r+1}, \ldots, k_n\}$, respectively. Again, every isomorphic copy of $M_{\lambda,\mu}$ in $P_n(*)$ is obtained by multiplication from the right by a suitable element $\sum_{\tau \in S_n} a_\tau \tau$, $a_\tau \in K$. We denote by $\chi_{\lambda,\mu}$ the irreducible $\mathbf{Z}_2 \wr S_n$-character associated with the pair (λ, μ). For any $*$-PI-algebra R the n-th cocharacter $\chi_n(R,*)$, $n = 0, 1, \ldots,$ is defined as the $\mathbf{Z}_2 \wr S_n$-character of $P_n(R,*)$. The irreducible character of the $\mathrm{GL}_m \times \mathrm{GL}_m$-module $N_{\lambda,\mu}$ is the product of Schur functions $S_\lambda(y_1, \ldots, y_m) S_\mu(z_1, \ldots, z_m)$ and the $\mathrm{GL}_m \times \mathrm{GL}_m$-character of $F_m(R,*)$ is the Hilbert series $H(R, *, y_1, \ldots, y_m, z_1, \ldots, z_m)$. The $\mathbf{Z}_2 \wr S_n$-module structure of $P_n(R,*)$ and the $\mathrm{GL}_m \times \mathrm{GL}_m$-module structure of $F_m(R,*)$ are related by the following result.

PROPOSITION 1.1. (THEOREMS 1 AND 2 [3]). *If*

$$\chi_n(R,*) = \sum_{|\lambda|+|\mu|=n} a_{\lambda,\mu}\chi_{\lambda,\mu}$$

and

$$H(R,*,y_1,\ldots,y_m,z_1,\ldots,z_m) = \sum_{n\geq 0}\sum_{|\lambda|+|\mu|=n} b_{\lambda,\mu}S_\lambda(y_1,\ldots,y_m)S_\mu(z_1,\ldots,z_m),$$

then $a_{\lambda,\mu} = b_{\lambda,\mu}$ *for all* λ, μ. ■

Let $R^+ = \{r \in R \mid r^* = r\}$ and $R^- = \{r \in R \mid r^* = -r\}$ and let $h(\lambda)$ denote the height h of the partition $\lambda = (\lambda_1,\ldots,\lambda_h)$, (*i.e.* $\lambda_h \neq 0$). The following assertion is an immediate consequence of Theorem 5.8 [4].

LEMMA 1.2. *Let* $\dim R^+ = p$ *and* $\dim R^- = q$. *Then the partitions* λ *and* μ *in the formula for* $\chi_n(R,*)$ *from Proposition 1.1 satisfy* $h(\lambda) \leq p$ *and* $h(\mu) \leq q$. ■

Let $\{i_1,\ldots,i_r\} \cup \{j_1,\ldots,j_{n-r}\} = \{1,\ldots,n\}$ and let $f \in P_n(*)$. We denote by $g_{\mathbf{i}}(s_{i_1},\ldots,s_{i_r},k_{j_1},\ldots,k_{j_{n-r}})$ the sum of all monomials of f in which only the variables $s_{i_1},\ldots,s_{i_r},k_{j_1},\ldots,k_{j_{n-r}}$ appear. Then we write

$$f = \sum_{\mathbf{i}} g_{\mathbf{i}}(s_{i_1},\ldots,s_{i_r},k_{j_1},\ldots,k_{j_{n-r}}).$$

Since the $*$-T-ideals are multihomogeneous, $f \in T(R,*)$ implies that $g_{\mathbf{i}}(s_{i_1},\ldots,s_{i_r},k_{j_1},\ldots,k_{j_{n-r}}) \in T(R,*)$ for all choices of $\mathbf{i} = (i_1,\ldots,i_r)$. For r fixed we define

$$P_n^{(r)}(*) = \operatorname{span}\{w_{\sigma(1)}\ldots w_{\sigma(n)} \mid \sigma \in S_n, w_i = s_i \text{ for } i = 1,\ldots,r \text{ and } w_i = k_i \text{ for } i = r+1,\ldots,n\}.$$

Thus the elements of $P_n^{(r)}(*)$ are polynomials in $s_1,\ldots,s_r,k_{r+1},\ldots,k_n$. It is clear that in order to study $P_n(*)\cap T(R,*)$ it is enough to study $P_n^{(r)}(*)\cap T(R,*)$ for all r. Let us write $P_n^{(r)}(R,*) = P_n^{(r)}(*)/\big(P_n^{(r)}(*)\cap T\big(R,*\big)\big)$. We denote

$$c_n^{(r)}(R,*) = \dim P_n^{(r)}(R,*).$$

Now, if we let S_r act on the symmetric variables $s_1,\ldots,s_r$ and S_{n-r} act on the skew-symmetric variables $k_{r+1},\ldots,k_n$, we obtain an action of $S_r \times S_{n-r}$ on $P_n^{(r)}(*)$. This, in turn, makes $P_n^{(r)}(R,*)$ an $S_r \times S_{n-r}$-module. The irreducible $S_r \times S_{n-r}$-modules are the tensor products $M_\lambda \otimes_K M_\mu$ of the irreducible S_r- and S_{n-r}-modules M_λ and M_μ, where $\lambda \in \operatorname{Part}(r)$ and $\mu \in \operatorname{Part}(n-r)$. We denote by $\zeta_{\lambda,\mu}$ the corresponding $S_r \times S_{n-r}$-character. Now we relate the structure of $P_n(R,*)$ as a $\mathbf{Z}_2 \wr S_n$-module to the structure of $P_n^{(r)}(R,*)$ as an $S_r \times S_{n-r}$-module.

THEOREM 1.3. *If the n-th cocharacter of $T(R,*)$ is*

$$\chi_n(R,*) = \sum_{r=0}^{n}\sum_{\lambda,\mu} a_{\lambda,\mu}\chi_{\lambda,\mu}, \quad \lambda \in \mathrm{Part}(r),\ \mu \in \mathrm{Part}(n-r)$$

and the $S_r \times S_{n-r}$-character of $P_n^{(r)}(R,*)$ is

$$\zeta_n^{(r)}(R,*) = \sum_{\lambda,\mu} b_{\lambda,\mu}\zeta_{\lambda,\mu}, \quad \lambda \in \mathrm{Part}(r),\ \mu \in \mathrm{Part}(n-r),$$

then $a_{\lambda,\mu} = b_{\lambda,\mu}$ for all λ and μ. The codimensions satisfy the relation

$$c_n(R,*) = \sum_{r=0}^{n}\binom{n}{r} c_n^{(r)}(R,*).$$

PROOF. For fixed partitions $\lambda \in \mathrm{Part}(r)$ and $\mu \in \mathrm{Part}(n-r)$, let

$$M_{\lambda,\mu} \cong \bigoplus_{\gamma\in\Gamma} \gamma KS_r e_{t_\lambda} \otimes_K KS_{n-r} e_{t_\mu}(s_1\cdots s_r k_{r+1}\cdots k_n)$$

be the irreducible $\mathbf{Z}_2 \wr S_n$-module defined above. Notice that $M_{\lambda,\mu} \subset P_n(*)$ and each of the summands of $M_{\lambda,\mu}$ consists of polynomials in $s_{i_1},\dots,s_{i_r}$, $k_{j_1},\dots,k_{j_{n-r}}$. The sets $\{i_1,\dots,i_r\}$ are pairwise different for the different elements $\gamma \in \Gamma$. Every summand is isomorphic to the irreducible $S_r \times S_{n-r}$-module $M_\lambda \otimes_K M_\mu$ under the action of S_r and S_{n-r} on $s_{i_1},\dots,s_{i_r}$ and $k_{j_1},\dots,k_{j_{n-r}}$, respectively. Therefore the multiplicities of the irreducible characters in $\chi_n(R,*)$ and $\zeta_n^{(r)}(R,*)$ are the same. Since the degrees of the characters $\chi_{\lambda,\mu}$ and $\zeta_{\lambda,\mu}$ satisfy

$$\deg(\chi_{\lambda,\mu}) = \binom{n}{r}\deg(\zeta_{\lambda,\mu})$$

and $c_n(R,*) = \deg(\chi_{\lambda,\mu})$, $c_n^{(r)}(R,*) = \deg(\zeta_{\lambda,\mu})$, we obtain the relation between the codimensions. ∎

2. **Proper $*$-polynomial identities.** Till the end of the paper we follow the exposition of [2]. Since some of the proofs are similar to those for algebras without involution, we refer to [2] for the missing details. We define (higher) commutators by

$$[u_1,u_2] = u_1(\mathrm{ad}\,u_2) = u_1u_2 - u_2u_1,$$
$$[u_1,\dots,u_{n-1},u_n] = [u_1,\dots,u_{n-1}](\mathrm{ad}\,u_n), \quad n > 2.$$

The free algebra $F_m(*) = K\langle s_1,\dots,s_m,k_1,\dots,k_m\rangle$ is a universal enveloping algebra of the free Lie algebra L_{2m} generated by the free generators $s_1,\dots,s_m,k_1,\dots,k_m$. Let $s_1,\dots,s_m,k_1,\dots,k_m,u_1,u_2,\dots$ be an ordered basis of L_{2m}, where $u_1,u_2,\dots$ are higher commutators. By the Poincaré-Birkhoff-Witt theorem $F_m(*)$ has a basis

$$\{s_1^{p_1}\cdots s_m^{p_m}k_1^{q_1}\cdots k_m^{q_m}u_1^{r_1}\cdots u_n^{r_n} \mid p_h,q_i,r_j \geq 0\}.$$

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LEMMA 2.1. *Let*

$$f(s_1,\dots,s_m,k_1,\dots,k_m)=\sum_{\mathbf{p}}\sum_{\mathbf{q}}\sum_{\mathbf{r}}\alpha_{\mathbf{pqr}}s_1^{p_1}\cdots s_m^{p_m}k_1^{q_1}\cdots k_m^{q_m}u_1^{r_1}\cdots u_n^{r_n}$$

be a $*$-polynomial identity for an algebra R, where $\mathbf{p}=(p_1,\dots,p_m)$, $\mathbf{q}=(q_1,\dots,q_m)$, $\mathbf{r}=(r_1,\dots,r_n)$ and $\alpha_{\mathbf{pqr}}\in K$. Then for every fixed $\mathbf{p}$

$$f_{\mathbf{p}}=\sum_{\mathbf{q}}\sum_{\mathbf{r}}\alpha_{\mathbf{pqr}}k_1^{q_1}\cdots k_m^{q_m}u_1^{r_1}\cdots u_n^{r_n}$$

is also a $*$-polynomial identity for R.

PROOF. Without loss of generality we assume that the polynomial f is multihomogeneous. Since $u_j=u_j(s_1,\dots,s_m,k_1,\dots,k_m)$ are higher commutators and the constants from K are symmetric elements, we obtain

$$u_j(s_1+\beta,s_2,\dots,s_m,k_1,\dots,k_m)=u_j(s_1,s_2,\dots,s_m,k_1,\dots,k_m),\quad \beta\in K,$$
$$f(s_1+\beta,s_2,\dots,s_m,k_1,\dots,k_m)=\sum_{\mathbf{p}}\sum_{\mathbf{q}}\sum_{\mathbf{r}}\alpha_{\mathbf{pqr}}(s_1+\beta)^{p_1}s_2^{p_2}\cdots s_m^{p_m}k_1^{q_1}\cdots k_m^{q_m}u_1^{r_1}\cdots u_n^{r_n}.$$

Standard Vandermonde arguments and similar considerations for the other symmetric variables $s_2,\dots,s_m$ give that $\sum_{\mathbf{p}}f_{\mathbf{p}}$ and its multihomogeneous components $f_{\mathbf{p}}$ are also a $*$-polynomial identities for R. ∎

We denote by $B_m(*)$ the vector subspace of $F_m(*)$ spanned by all products

$$k_1^{q_1}\cdots k_m^{q_m}u_1^{r_1}\cdots u_n^{r_n},\quad q_i,r_j\geq 0$$

and call the elements of $B_m(*)$ proper polynomials. Let $\Gamma_n(*)=P_n(*)\cap B_n(*)$ be the set of all multilinear proper polynomials. Lemma 2.1 gives that all $*$-polynomial identities of an algebra R follow from the proper ones. We denote

$$B_m(R,*)=B_m(*)/\big(B_m(*)\cap T(R,*)\big),\quad \Gamma_n(R,*)=\Gamma_n(*)/\big(\Gamma_n(*)\cap T(R,*)\big).$$

Clearly, there is an analogue of Proposition 1.1 for the proper $*$-polynomial identities and the $\mathrm{GL}_m\times\mathrm{GL}_m$-module $B_m(R,*)$ and the $\mathbf{Z}_2\wr S_n$-module $\Gamma_n(R,*)$ have the same structure. Lemma 1.2 is restated in the following way.

LEMMA 2.2. *Let C be the centre of R and let $\dim R^+/(R^+\cap C)=p$, $\dim R^-=q$. Then the partitions λ and μ in the $\mathbf{Z}_2\wr S_n$-character $\chi\big(\Gamma_n(R,*)\big)=\sum b_{\lambda,\mu}\chi_{\lambda,\mu}$ satisfy $h(\lambda)\leq p$ and $h(\mu)\leq q$.* ∎

The following result gives a simple relation between the proper and all the polynomial identities of a $*$-PI-algebra R. The proof repeats verbatim the arguments from the ordinary case in Theorems 2.2 and 2.3 in [2].

THEOREM 2.3. *(i) The Hilbert series of $F_m(R,*)$ and $B_m(R,*)$ satisfy*

$$H(R,*,y_1,\dots,y_m,z_1,\dots,z_m)=\prod_{i=1}^{m}\frac{1}{1-y_i}H\big(B_m(R,*),y_1,\dots,y_m,z_1,\dots,z_m\big).$$

(ii) The codimension sequence $c_n(R,*)$ and the proper codimension sequence $\gamma_n(R,*)=\dim\Gamma_n(R,*)$, $n=0,1,2,\dots$, are related by the equality

$$c_n(R,*)=\sum_{i=0}^{n}\binom{n}{i}\gamma_i(R,*).$$

(iii) The codimension series $c(R,*,t)=\sum c_n(R,*)t^n$ and the proper codimension series $\gamma(R,*,t)=\sum\gamma_n(R,*)t^n$ satisfy the equation

$$c(R,*,t)=\frac{1}{1-t}\gamma\Big(R,*,\frac{t}{1-t}\Big).$$

(iv) The following $\mathrm{GL}_m\times\mathrm{GL}_m$-module isomorphism holds

$$F_m(R,*)\cong K[s_1,\dots,s_m]\otimes_K B_m(R,*),$$

where $K[s_1,\dots,s_m]$ is the algebra of the polynomials in commuting variables with the canonical $\mathrm{GL}(U)$- and the trivial $\mathrm{GL}(V)$-action.

(v) If $\chi_n(R,*)=\sum a_{\lambda,\mu}\chi_{\lambda,\mu}$ and $\chi\big(\Gamma_n(R,*)\big)=\sum b_{\nu,\mu}\chi_{\nu,\mu}$, $n=0,1,\dots$, are respectively the cocharacter and the proper cocharacter sequences of R, then $a_{\lambda,\mu}=\sum b_{\nu,\mu}$, where for fixed $\lambda=(\lambda_1,\dots,\lambda_m)$ and μ the summation runs over all partitions $\nu=(\nu_1,\dots,\nu_m)$, such that $\lambda_1\geq\nu_1\geq\cdots\geq\lambda_m\geq\nu_m$. Equivalently, if

$$H(R,*,y_1,\dots,y_m,z_1,\dots,z_m)=\sum a_{\lambda,\mu}S_\lambda(y_1,\dots,y_m)S_\mu(z_1,\dots,z_m),$$
$$H\big(B_m(R,*),y_1,\dots,y_m,z_1,\dots,z_m\big)=\sum b_{\nu,\mu}S_\nu(y_1,\dots,y_m)S_\mu(z_1,\dots,z_m),$$

then $a_{\lambda,\mu}=\sum b_{\nu,\mu}$, under the same relations between λ and ν. ∎

3. **Proper $*$-identities for 2×2 matrices.** Let us recall that the transpose involution acts on $M_2(K,*)$ by

$$\begin{pmatrix}\alpha_{11}&\alpha_{12}\\ \alpha_{21}&\alpha_{22}\end{pmatrix}^*=\begin{pmatrix}\alpha_{11}&\alpha_{21}\\ \alpha_{12}&\alpha_{22}\end{pmatrix}.$$

Hence the vector space $M_2(K,*)^+$ is spanned on the matrices

$$e=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad a_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad a_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

while the space $M_2(K,*)^-$ of the skew-symmetric matrices is spanned on the matrix

$$a_3=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

The symplectic involution is defined by

$$\begin{pmatrix}\alpha_{11}&\alpha_{12}\\ \alpha_{21}&\alpha_{22}\end{pmatrix}^*=\begin{pmatrix}\alpha_{22}&-\alpha_{12}\\ -\alpha_{21}&\alpha_{11}\end{pmatrix},$$
$$M_2(K,*)^+=\mathrm{span}\{e\},\quad M_2(K,*)^-=\mathrm{span}\{a_1,a_2,a_3\}.$$

Since the centre of $M_2(K,*)$ consists of scalar matrices only, Lemma 2.2 immediately gives:

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LEMMA 3.1. *The partitions λ and μ from the proper cocharacter sequence*

$$\chi\Big(\Gamma_n\big(M_2(K),*\big)\Big)=\sum b_{\lambda,\mu}\chi_{\lambda,\mu}$$

satisfy $h(\lambda)\leq 2$ and $h(\mu)\leq 1$ in the transpose case and $h(\lambda)=0$ and $h(\mu)\leq 3$ in the symplectic case. ■

Now we use a generic construction which is similar to the ordinary generic matrices and works for matrices of any size (for the symplectic involution we require the size to be even). Let $M_2(K[\xi],*)$ be the 2×2 matrix algebra with entries from the polynomial algebra

$$K[\xi]=K[\xi_{ij}^{(h)}\mid i,j=1,2,\ h=1,\dots,m]$$

and equipped with either the transpose or the symplectic involution. The generic matrix algebra $G_m(*)$ is $*$-generated by the generic matrices

$$\bar{x}_h=\begin{pmatrix}\xi_{11}^{(h)} & \xi_{12}^{(h)}\\ \xi_{21}^{(h)} & \xi_{22}^{(h)}\end{pmatrix},\quad h=1,\dots,m.$$

Since the field K is infinite, a polynomial $f(x_1,\dots,x_m,x_1^*,\dots,x_m^*)\in F_m(*)$ vanishes on $\bar{x}_1,\dots,\bar{x}_m,\bar{x}_1^*,\dots,\bar{x}_m^*$ if and only if $f(b_1,\dots,b_m,b_1^*,\dots,b_m^*)=0$ for all matrices $b_1,\dots,b_m$ with entries from K. Therefore the generic matrix algebra $G_m(*)$ is isomorphic to the relatively free algebra $F_m\big(M_2(K),*\big)$. As in the case of the free algebra with involution we prefer to change the set of generators and denote the free symmetric and skew-symmetric generic matrices respectively by $\bar{s}_h=(\eta_{ij}^{(h)})$ and $\bar{k}_h=(\zeta_{ij}^{(h)})$, where

$$K[\eta_{ij}^{(h)},\zeta_{ij}^{(h)}\mid i,j=1,2,\ h=1,\dots,m]$$

is the polynomial ring in commuting variables satisfying the relations obtained by the equalities $\bar{s}_h^*=\bar{s}_h$ and $\bar{k}_h^*=-\bar{k}_h$. In the transpose case this is equivalent to

$$\eta_{12}^{(h)}=\eta_{21}^{(h)},\quad \zeta_{11}^{(h)}=\zeta_{22}^{(h)}=0,\quad \zeta_{21}^{(h)}=-\zeta_{12}^{(h)},$$

while in the symplectic case we have

$$\eta_{11}^{(h)}=\eta_{22}^{(h)},\quad \eta_{12}^{(h)}=\eta_{21}^{(h)}=0,\quad \zeta_{22}^{(h)}=-\zeta_{11}^{(h)}.$$

In the ordinary 2×2 matrix algebra case there is a trick which allows to consider two of the generic matrices of a special form (the first one diagonal and the second one symmetric; see Lemma 3.1 [2]). We repeat the arguments from [2]. Let $*$ be the transpose involution and let

$$K[\Xi']=K[\eta_1,\eta_2,\eta_1^{(h)},\eta_2^{(h)},\eta^{(h)},\zeta^{(i)}\mid h=2,\dots,m,\ i=1,\dots,m],$$
$$s_1'=\begin{pmatrix}\eta_1 & 0\\ 0 & \eta_2\end{pmatrix},\quad s_h'=\begin{pmatrix}\eta_1^{(h)} & \eta^{(h)}\\ \eta^{(h)} & \eta_2^{(h)}\end{pmatrix},\quad h=2,\dots,m,$$
$$k_i'=\begin{pmatrix}0 & \zeta^{(i)}\\ -\zeta^{(i)} & 0\end{pmatrix},\quad i=1,\dots,m.$$

For the symplectic involution we introduce

$$K[\Xi''] = K[\eta^{(h)}, \zeta^{(1)}, \zeta_1^{(2)}, \zeta^{(2)}, \zeta_{11}^{(i)}, \zeta_{12}^{(i)}, \zeta_{21}^{(i)} \mid h = 1, \ldots, m,\ i = 3, \ldots, m],$$

$$s_h'' = \begin{pmatrix} \eta^{(h)} & 0 \\ 0 & \eta^{(h)} \end{pmatrix}, \quad h = 1, \ldots, m,$$

$$k_1'' = \begin{pmatrix} \zeta^{(1)} & 0 \\ 0 & -\zeta^{(1)} \end{pmatrix}, \ k_2'' = \begin{pmatrix} \zeta_1^{(2)} & \zeta^{(2)} \\ \zeta^{(2)} & -\zeta_1^{(2)} \end{pmatrix}, \ k_i'' = \begin{pmatrix} \zeta_{11}^{(i)} & \zeta_{12}^{(i)} \\ \zeta_{21}^{(i)} & -\zeta_{11}^{(i)} \end{pmatrix}, \quad i = 3, \ldots, m.$$

LEMMA 3.2. *(i) The subalgebra of $M_2(K[\Xi'], *)$ generated by the matrices s_1', s_h', k_i', $h = 2, \ldots, m$, $i = 1, \ldots, m$, is isomorphic to the generic algebra $G_m(*)$ with transpose involution.*

(ii) The subalgebra of $M_2(K[\Xi''], *)$ generated by the matrices $s_h'', k_1'', k_2'', k_i''$, $h = 1, \ldots, m$, $i = 3, \ldots, m$, is isomorphic to the generic algebra $G_m(*)$ with symplectic involution.

PROOF. (i) Let Ω be the algebraic closure of the field of fractions of $K[\Xi']$. We assume that the matrix algebra $M_2(\Omega)$ acts on the two-dimensional vector space Ω^2 equipped with a non-degenerate symmetric bilinear form $\langle\,,\,\rangle$. For every $a \in M_2(\Omega)$ there exists a unique $b \in M_2(\Omega)$ such that $\langle a(v_1), v_2\rangle = \langle v_1, b(v_2)\rangle$ for all $v_1, v_2 \in \Omega^2$. If we define the form by

$$\langle(\alpha_1, \alpha_2), (\beta_1, \beta_2)\rangle = \alpha_1\beta_1 + \alpha_2\beta_2, \quad (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \Omega^2,$$

we obtain that $b = a^*$ is the transpose of a. Every orthogonal matrix $g \in M_2(\Omega)$ satisfies $gg^* = 1$ and for every $a \in M_2(\Omega)$

$$\big(g^{-1}(a + a^*)g\big)^* = g^{-1}(a + a^*)g, \quad \big(g^{-1}(a - a^*)g\big)^* = -g^{-1}(a - a^*)g.$$

Hence $g^{-1}M_2(\Omega)^{\pm}g = M_2(\Omega)^{\pm}$. The matrix $\bar{s}_1$ has different eigenvalues and there exists an orthogonal matrix g such that $g^{-1}\bar{s}_1 g$ is diagonal. Clearly the matrices $g^{-1}\bar{s}_h g$, $g^{-1}\bar{k}_i g$, $h, i = 1, \ldots, m$, freely generate an isomorphic copy of $G_m(*)$. Since they are specializations of the matrices s_h', k_i', $h, i = 1, \ldots, m$, we obtain that s_h', k_i' also generate an isomorphic copy of $G_m(*)$.

(ii) Again, let Ω be the algebraic closure of $K[\Xi'']$. Now we consider Ω^2 with a non-degenerate skew-symmetric bilinear form

$$\langle(\alpha_1, \alpha_2), (\beta_1, \beta_2)\rangle = \alpha_1\beta_2 - \alpha_2\beta_1, \quad (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \Omega^2.$$

For $a \in M_2(\Omega)$ and $*$ being the symplectic involution, a^* satisfies $\langle a(v_1), v_2\rangle = \langle v_1, a^*(v_2)\rangle$, $v_1, v_2 \in \Omega^2$. Since $\{g \in M_2(\Omega) \mid gg^* = 1\} = \mathrm{GL}_2(\Omega)$, there exists a matrix $g_1 \in \mathrm{GL}_2(\Omega)$ such that $g_1^{-1}\bar{k}_1 g_1$ is diagonal. Now we can find another matrix $g_2 \in \mathrm{GL}_2(\Omega)$ satisfying

$$g_2^{-1}(g_1^{-1}\bar{k}_1 g_1)g_2 = g_1^{-1}\bar{k}_1 g_1, \quad g_2^{-1}(g_1^{-1}\bar{k}_2 g_1)g_2 = (\gamma_{ij}) \quad \text{and} \quad \gamma_{12} = \gamma_{21}.$$

For example we can choose g_2 to be with non-zero entries on the second diagonal only. Again the matrices $g_2^{-1}g_1^{-1}\bar{s}_h g_1 g_2$, $g_2^{-1}g_1^{-1}\bar{k}_i g_1 g_2$, $h, i = 1, \ldots, m$, are specializations of s_h'', k_i'', $h, i = 1, \ldots, m$, and s_h', k_i' generate an isomorphic copy of $G_m(*)$. ■

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THEOREM 3.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\mu = (\mu_1, \ldots, \mu_q)$ be partitions such that $\lambda_p \neq 0$, $\mu_q \neq 0$ and let*

$$f_{\lambda,\mu}(s_1, \ldots, s_p, k_1, \ldots, k_q) = f_\lambda(s_1, \ldots, s_p) f_\mu(k_1, \ldots, k_q) \sum_{\tau \in S_n} a_\tau \tau, \quad a_\tau \in K,$$

be a proper polynomial generating a $\mathrm{GL}_m \times \mathrm{GL}_m$-submodule $N_{\lambda,\mu}$ of $B_m(*)$.

(i) Let $*$ be the transpose involution of $M_2(K)$. If $p > 2$ or $q > 1$, then $f_{\lambda,\mu}$ is a polynomial identity for $M_2(K,*)$. If $p \leq 2$ and $q \leq 1$, $f_{\lambda,\mu}(s_1, s_2, k_1)$ is a polynomial identity for $M_2(K,*)$ if and only if $f_{\lambda,\mu}(a_1, a_2, a_3) = 0$ for the matrices a_1, a_2, a_3 defined in the beginning of the section.

(ii) For the symplectic involution and $p > 0$ or $q > 3$, $f_{\lambda,\mu}$ is a polynomial identity for $M_2(K,*)$. If $p = 0$ and $q \leq 3$, $f_{\lambda,\mu} = f_\mu(k_1, k_2, k_3)$ is a polynomial identity for $M_2(K,*)$ if and only if $f_\mu(a_1, a_2, a_3) = 0$.

PROOF. (i) The case $p > 2$ or $q > 1$ follows immediately from Lemma 3.1. Now let $p \leq 2$ and $q \leq 1$. Clearly $f_{\lambda,\mu}(s_1, s_2, k_1)$ is a $*$-polynomial identity for $M_2(K,*)$ if and only if $f_{\lambda,\mu}(s_1', s_2', k_1') = 0$. Obviously

$$s_1' = \phi_0 e + \phi a_1, \quad s_2' = \psi_0 e + \psi_1 a_1 + \psi a_2, \quad k_1' = \zeta a_3$$

for some algebraically independent commuting variables. By the definition of the proper polynomial, s_1', s_2' appear only in commutators. Every variable s_2 participates in $f_{\lambda,\mu}(s_1, s_2, k_1)$ in a skew-symmetric combination with a s_1. Hence

$$f_{\lambda,\mu}(s_1', s_2', k_1') = \phi^{\lambda_1} \psi^{\lambda_2} \zeta^{\mu_1} f_{\lambda,\mu}(a_1, a_2, a_3).$$

(ii) Since $M_2(K)^- = sl_2(K)$, we can repeat verbatim the arguments from Theorem 3.2 [2]. We write

$$k_1'' = \zeta a_1, \quad k_2'' = \phi_1 a_1 + \phi a_2, \quad k_3'' = \psi_1 a_1 + \psi_2 a_2 + \psi a_3$$

with algebraically independent ζ, ϕ, ψ and

$$f_\mu(k_1'', k_2'', k_3'') = \zeta^{\mu_1} \phi^{\mu_2} \psi^{\mu_3} f_\mu(a_1, a_2, a_3). \qquad \blacksquare$$

THEOREM 3.4. *(i) For the transpose involution*

$$B_m\big(M_2(K), *\big) \cong \bigoplus N_{\lambda,\mu},$$

where the summation is over all partitions $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1)$ and $\lambda_2 \neq 0$ when $\lambda_1 \neq 0$ and $\mu_1 = 0$.

(ii) For the symplectic involution

$$B_m\big(M_2(K), *\big) \cong \bigoplus N_{0,\mu},$$

where $\mu = (\mu_1, \mu_2, \mu_3)$.

PROOF. (i) The matrices a_1, a_2, a_3 satisfy the following relations

$$a_1^2 = a_2^2 = -a_3^2 = e, \quad a_1 a_2 = -a_2 a_1 = a_3,$$
$$a_2 a_3 = -a_3 a_2 = -a_1, \quad a_3 a_1 = -a_1 a_3 = -a_2.$$

As a consequence, for every multihomogeneous polynomial $f(s_1, s_2, k_1)$

$$f(a_1, a_2, a_3) = \alpha a_1^{\delta_1} a_2^{\delta_2} a_3^{\delta_3},$$

where $\alpha \in K$ and $\delta_i = 0, 1$, $i = 1, 2, 3$, is the parity of the degree of f in s_1, s_2, k_1, respectively. By Lemma 3.1

$$B_m\big(M_2(K), *\big) \cong \bigoplus b_{\lambda,\mu} N_{\lambda,\mu}, \quad \lambda = (\lambda_1, \lambda_2),\ \mu = (\mu_1).$$

Let $b_{\lambda,\mu} > 1$ for some pair (λ, μ) and let $f'_{\lambda,\mu}(s_1, s_2, k_1), f''_{\lambda,\mu}(s_1, s_2, k_1)$ be the generators of two isomorphic copies of $N_{\lambda,\mu}$ in $B_m\big(M_2(K), *\big)$. Obviously the polynomials $f'_{\lambda,\mu}(s_1, s_2, k_1)$ and $f''_{\lambda,\mu}(s_1, s_2, k_1)$ are linearly independent in $B_m\big(M_2(K), *\big)$. There exist non-zero constants $\alpha', \alpha'' \in K$ such that

$$f'_{\lambda,\mu}(a_1, a_2, a_3) = \alpha' a_1^{\delta_1} a_2^{\delta_2} a_3^{\delta_3}, \quad f''_{\lambda,\mu}(a_1, a_2, a_3) = \alpha'' a_1^{\delta_1} a_2^{\delta_2} a_3^{\delta_3}.$$

Therefore, $\alpha'' f'_{\lambda,\mu}(a_1, a_2, a_3) - \alpha' f''_{\lambda,\mu}(a_1, a_2, a_3) = 0$ and this contradicts the linear independence of $f'_{\lambda,\mu}(s_1, s_2, k_1)$ and $f''_{\lambda,\mu}(s_1, s_2, k_1)$. Hence $b_{\lambda,\mu} \leq 1$. The proof will be completed if we show that $b_{\lambda,\mu} = 0$ for $\lambda_1 \neq 0$ and $\lambda_2 = \mu_1 = 0$ and if we construct non-zero proper polynomials which generate $N_{\lambda,\mu} \subset B_m\big(M_2(K), *\big)$ for all other pairs $\big((\lambda_1, \lambda_2), (\mu_1)\big)$. First, let $\lambda_2 = \mu_1 = 0$. Then $f_{\lambda,\mu}$ is a homogeneous polynomial depending on s_1 only and $f_{\lambda,\mu} = s_1^{\lambda_1}$ up to a multiplicative constant. Since $s_1^{\lambda_1}$ is not a proper polynomial, we obtain that $b_{\lambda,\mu} = 0$. If $\lambda_2 \neq 0$, we can choose

$$f_{\lambda,\mu} = k_1^{\mu_1}\Big((s_1 s_2 - s_2 s_1)(\mathrm{ad}^{\lambda_1 - \lambda_2} s_1)\Big)(s_1 s_2 - s_2 s_1)^{\lambda_2 - 1}$$

and for $\lambda_2 = 0$, $\mu_1 \neq 0$

$$f_{\lambda,\mu} = k_1^{\mu_1 - 1}\big(k_1(\mathrm{ad}^{\lambda_1} s_1)\big).$$

Direct verification shows that $f_{\lambda,\mu}(a_1, a_2, a_3) \neq 0$.

(ii) The proof is similar to the proof of (i). We can choose

$$f_\mu = S_3^{\mu_3}(k_1, k_2, k_3) S_2^{\mu_2 - \mu_3}(k_1, k_2) k_1^{\mu_1 - \mu_2},$$

where $S_p(k_1, \ldots, k_p)$ is the standard polynomial. ∎

4. **Cocharacters, Hilbert series and codimensions.** In this section we compute explicitly the cocharacters, the Hilbert series and the codimensions of the $*$-polynomial identities for the 2×2 matrix algebra. Since $\dim M_2(K)^{\pm} \leq 3$, the Hilbert series of $F_3\big(M_2(K), *\big)$ determines completely the $\mathrm{GL}_m \times \mathrm{GL}_m$-module structure of the $*$-polynomial identities of $M_2(K, *)$ for all m and we give also a closed formula for this case.

THEOREM 4.1. *The* $\mathbf{Z}_2 \wr S_n$*-cocharacter of* $M_2(K,*)$ *is* $\chi_n\big(M_2(K),*\big) = \sum a_{\lambda,\mu}\chi_{\lambda,\mu}$, *where*

(i) *For the transpose involution* $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1)$ *and*

$a_{\lambda,\mu} = 1$, *when* $\lambda_2 = \mu_1 = 0$;

$a_{\lambda,\mu} = (\lambda_1 - \lambda_2 + 1)\lambda_2$, *when* $\lambda_2 \neq 0$, $\lambda_3 = \mu_1 = 0$;

$a_{\lambda,\mu} = (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)$, *for other pairs of partitions.*

(ii) *For the symplectic involution* $\lambda = (\lambda_1)$, $\mu = (\mu_1, \mu_2, \mu_3)$ *and* $a_{\lambda,\mu} = 1$.

PROOF. We apply Theorem 2.3 (v) to the decomposition of the $\mathrm{GL}_m \times \mathrm{GL}_m$-module $B_m\big(M_2(K),*\big)$ from Theorem 3.4. Then, by Proposition 1.1, we state the result for the $\mathbf{Z}_2 \wr S_n$-cocharacters. For the transpose involution

$$B_m\big(M_2(K),*\big) \cong \Big(\bigoplus N_{(\lambda_1,\lambda_2),(\mu_1)}\Big) \Big/ \Big(\bigoplus_{\lambda_1>0} N_{(\lambda_1),0}\Big),$$

where the first direct sum is on all pairs of partitions (λ_1, λ_2), (μ_1). Arguments similar to those in the proof of Theorem 4.1 [2] give that

$$K[s_1,\ldots,s_m] \otimes_K \Big(\bigoplus N_{(\lambda_1,\lambda_2),(\mu_1)}\Big) \cong \bigoplus (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1) N_{(\lambda_1,\lambda_2,\lambda_3),(\mu_1)},$$

$$K[s_1,\ldots,s_m] \otimes_K \Big(\bigoplus_{\lambda_1>0} N_{(\lambda_1),0}\Big) \cong \Big(\bigoplus (\lambda_1 - \lambda_2 + 1) N_{(\lambda_1,\lambda_2),0}\Big) \Big/ \Big(\bigoplus N_{(\lambda_1),0}\Big)$$

and this gives the decomposition for $F_m\big(M_2(K),*\big)$ and, hence for $\chi_n\big(M_2(K),*\big)$. The proof for the symplectic involution is an immediate consequence of Theorems 2.3 (v) and 3.4(ii). ∎

THEOREM 4.2. *(i) For the transpose involution*

$$\begin{aligned}
H\big(M_2(K),*,y_1,\ldots,y_m,z_1,\ldots,z_m\big)
&= \prod_{i=1}^{m} \frac{1}{1-y_i} \prod_{i=1}^{m} \frac{1}{1-z_i} \sum_{(\lambda_1,\lambda_2)} S_{(\lambda_1,\lambda_2)}(y_1,\ldots,y_m) \\
&\quad - \prod_{i=1}^{m} \frac{1}{(1-y_i)^2} + \prod_{i=1}^{m} \frac{1}{1-y_i} \\
&= \prod_{i=1}^{m} \frac{1}{1-y_i} \prod_{i=1}^{m} \frac{1}{1-z_i} \sum_{k\geq 0} \Big(h_k^2(y_1,\ldots,y_m) + h_k(y_1,\ldots,y_m)h_{k+1}(y_1,\ldots,y_m)\Big) \\
&\quad - \prod_{i=1}^{m} \frac{1}{(1-y_i)^2} + \prod_{i=1}^{m} \frac{1}{1-y_i},
\end{aligned}$$

where $S_{(\lambda_1,\lambda_2)}(y_1,\ldots,y_m)$ *is the Schur function corresponding to the partition* $\lambda = (\lambda_1, \lambda_2)$ *and* $h_k(y_1,\ldots,y_m) = S_{(k)}(y_1,\ldots,y_m)$ *denotes the k-th complete symmetric function;*

$$\begin{aligned}
H\big(M_2(K),*,y_1,y_2,y_3,z_1,z_2,z_3\big) &= (1 - y_1y_2y_3) \prod_{i=1}^{3} \frac{1}{(1-y_i)^2} \prod_{i<j} \frac{1}{1-y_iy_j} \prod_{i=1}^{3} \frac{1}{1-z_i} \\
&\quad - \prod_{i=1}^{3} \frac{1}{(1-y_i)^2} + \prod_{i=1}^{3} \frac{1}{1-y_i}.
\end{aligned}$$

(ii) For the symplectic involution

$$H\big(M_2(K),*,y_1,\dots,y_m,z_1,\dots,z_m\big)=\prod_{i=1}^{m}\frac{1}{1-y_i}\sum_{\mu}S_{(\mu_1,\mu_2,\mu_3)}(z_1,\dots,z_m)$$
$$=\prod_{i=1}^{m}\frac{1}{1-y_i}\prod_{i=1}^{m}\frac{1}{1-z_i}\sum_{\mu_1\geq 0}S_{(\mu_1,\mu_1)}(z_1,\dots,z_m),$$
$$H\big(M_2(K),*,y_1,y_2,y_3,z_1,z_2,z_3\big)=\prod_{i=1}^{3}\frac{1}{1-y_i}\prod_{i=1}^{3}\frac{1}{1-z_i}\prod_{i<j}\frac{1}{1-z_iz_j}.$$

PROOF. (i) The first expression for the Hilbert series of $F_m\big(M_2(K),*\big)$ follows immediately from Theorem 3.4(i) bearing in mind that the $\mathrm{GL}(U)$-modules $\bigoplus_{\lambda_1\geq 0}N_{(\lambda_1),0}$ and $K[s_1,\dots,s_m]$ are isomorphic and their Hilbert series is equal to $\prod_{i=1}^{m}\frac{1}{1-y_i}$. For the second expression we apply the combinatorial identity

$$h_k(y_1,\dots,y_m)h_l(y_1,\dots,y_m)=\sum S_{(\lambda_1,\lambda_2)}(y_1,\dots,y_m),$$

where the summation runs over all (λ_1,λ_2) such that $\lambda_1+\lambda_2=k+l$, $\lambda_1\geq k\geq\lambda_2$. It can be obtained as a special case of (5.16), p. 42 [5]. Applying once again (5.16), p. 42 [5] we obtain that

$$h_k(y_1,\dots,y_m)S_{(\lambda_1,\lambda_1)}(y_1,\dots,y_m)=\sum S_{(\nu_1,\nu_2,\nu_3)}(y_1,\dots,y_m),$$

where $(\nu_1,\nu_2,\nu_3)\in\mathrm{Part}(2\lambda_1+k)$ and $\nu_1\geq\lambda_1=\nu_2\geq\nu_3$. Now we use the following identity which is a consequence of (5.17), p. 42 [5]:

$$S_{(\lambda_1,\lambda_2,\lambda_3)}(y_1,y_2,y_3)=e_3^{\lambda_3}(y_1,y_2,y_3)S_{(\lambda_1-\lambda_3,\lambda_2-\lambda_3)}(y_1,y_2,y_3)$$
$$=(y_1y_2y_3)^{\lambda_3}S_{(\lambda_1-\lambda_3,\lambda_2-\lambda_3)}(y_1,y_2,y_3).$$

Here $e_p=e_p(y_1,\dots,y_m)=S_{(1^p)}(y_1,\dots,y_m)$ is the p-th elementary symmetric function. Hence

$$\prod_{i=1}^{3}\frac{1}{1-y_i}\sum_{\lambda_1\geq 0}S_{(\lambda_1,\lambda_1)}(y_1,y_2,y_3)=\sum_{k\geq 0}\sum_{\lambda_1\geq 0}h_k(y_1,y_2,y_3)S_{(\lambda_1,\lambda_1)}(y_1,y_2,y_3)$$
$$=\sum_{\lambda}S_{(\lambda_1,\lambda_2,\lambda_3)}(y_1,y_2,y_3)$$
$$=\sum_{k\geq 0}(y_1y_2y_3)^k\sum_{\lambda}S_{(\lambda_1,\lambda_2)}(y_1,y_2,y_3)$$
$$=\frac{1}{1-y_1y_2y_3}\sum_{\lambda}S_{(\lambda_1,\lambda_2)}(y_1,y_2,y_3).$$

Therefore, we have to calculate $\sum S_{(\lambda_1,\lambda_1)}(y_1,y_2,y_3)$. We use the combinatorial identity

$$\sum_{\lambda}S_{(\lambda_1^2,\dots,\lambda_r^2)}=\sum_{k\geq 0}h_k\odot e_2=\prod_{i<j}\frac{1}{1-y_iy_j}$$

from Exercise 5, p. 45 [5], where $h_k \odot e_2$ is the plethysm of the symmetric functions h_k and e_2. Since $m = 3$, the only non-trivial partitions $(\lambda_1^2, \ldots, \lambda_r^2)$ are (λ_1, λ_1) and we establish the formula for the Hilbert series of $F_3\big(M_2(K), *\big)$.

(ii) Again, we make use of (5.16), p. 42 [5] and obtain that

$$\sum_{\mu} S_{(\mu_1,\mu_2,\mu_3)}(z_1, \ldots, z_m) = \prod_{i=1}^{m} \frac{1}{1 - z_i} \sum_{\mu_1 \geq 0} S_{(\mu_1,\mu_1)}(z_1, \ldots, z_m)$$

and for $m = 3$ we obtain from the expression for the plethysm

$$\sum_{\mu_1 \geq 0} S_{(\mu_1,\mu_1)}(z_1, z_2, z_3) = \prod_{i<j} \frac{1}{1 - z_i z_j}.$$

A similar formula

$$\sum_{\mu_1 \geq 0} S_{(\mu_1,\mu_1)}(z_1, z_2, z_3, z_4) = (1 - z_1 z_2 z_3 z_4) \prod_{i<j} \frac{1}{1 - z_i z_j}$$

can be obtained also in the case of four variables (see the proof of Theorem 4.3 [2]). ∎

THEOREM 4.3. *(i) For the transpose involution the codimension series and the codimension sequence are equal respectively to*

$$c\big(M_2(K), *, t\big) = \frac{1}{2t}\left(-1 + \sqrt{\frac{1}{1 - 4t}}\right) - \frac{1}{1 - 2t} + \frac{1}{1 - t},$$

$$c_n\big(M_2(K), *\big) = \frac{1}{2}\binom{2n + 2}{n + 1} - 2^n + 1.$$

(ii) For the symplectic involution

$$c\big(M_2(K), *, t\big) = \frac{1}{t^2}\left(1 - 2t - \sqrt{1 - 4t}\right),$$

$$c_n\big(M_2(K), *\big) = \frac{1}{n + 2}\binom{2n + 2}{n + 1}.$$

PROOF. (i) By virtue of Theorem 4.1 the cocharacters of $M_2(K, *)$ are

$$\chi_n\big(M_2(K), *\big) = \sum\big((\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1) - \eta(\lambda, \mu_1)\big)\chi_{\lambda,(\mu_1)},$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Part}(n - \mu_1)$ and $\eta(\lambda, \mu_1)$ is the correction to count the case $\lambda_3 = \mu_1 = 0$. Now we apply Theorem 1.3. Since the subgroup $1 \times S_{n-r}$ of $S_r \times S_{n-r}$ acts trivially on the modules $M_\lambda \otimes_K M_{(n-r)}$, $\lambda \in \mathrm{Part}(r)$ and $\dim M_{(n-r)} = 1$, we obtain that

$$\dim M_{\lambda,(n-r)} = \binom{n}{r} \dim M_\lambda, \quad \lambda \in \mathrm{Part}(r).$$

Hence the sum

$$c_n^{(r)} = c^{(r)} = \sum(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1) \dim M_{\lambda,(n-r)}$$

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depends on r only and

$$c_n = \sum_{r=0}^{n} \binom{n}{r} c^{(r)}.$$

It is easy to see that the formal power series $c(t) = \sum c_n t^n$ and $d(t) = \sum c^{(r)} t^r$ are related by

$$c(t) = \frac{1}{1-t} d\left(\frac{t}{1-t}\right)$$

(we used this in the proof of Theorem 2.3(iii) [2] and hence in Theorem 2.3(iii) as well). In this way we reduce the problem to the computing of the series

$$d(t) = \sum (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\dim M_\lambda) t^r, \quad \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Part}(r).$$

Again as in Theorem 2.3 (iii) we obtain that

$$d(t) = \frac{1}{1-t} \delta\left(\frac{t}{1-t}\right), \quad \text{where} \quad \delta(t) = \sum (\dim M_{(\lambda_1,\lambda_2)}) t^{\lambda_1+\lambda_2}.$$

A closed formula of this series

$$\delta(t) = \frac{2t - 1 + \sqrt{1-4t^2}}{2t(1-2t)}$$

is given *e.g.* in Lemma 2(b) [1] and we calculate consequently

$$d(t) = \frac{1}{2t}\left(-1 + \sqrt{\frac{1+t}{1-3t}}\right), \quad c(t) = \frac{1}{2t}\left(-1 + \sqrt{\frac{1}{1-4t}}\right).$$

The only non-zero values of the correction $\eta(\lambda, \mu_1)$ are

$$\eta\big((\lambda_1), 0\big) = \lambda_1,\ \lambda_1 > 0, \text{ and } \eta\big((\lambda_1, \lambda_2), 0\big) = \lambda_1 - \lambda_2 + 1, \quad \lambda_2 > 0.$$

With similar considerations (see also the proof of Theorem 4.4 [2]) we obtain that

$$\sum \eta\big((\lambda_1, \lambda_2), 0\big) \dim M_{(\lambda_1,\lambda_2)} t^{\lambda_1+\lambda_2} = \frac{1}{1-t}\delta_1\left(\frac{t}{1-t}\right) = \frac{1}{1-2t} - \frac{1}{1-t},$$

where

$$\delta_1(t) = \sum_{k>0} \dim M_{(k)} t^k = \sum_{k>0} t^k = \frac{t}{1-t}.$$

Hence

$$c\big(M_2(K), *, t\big) = \frac{1}{2t}\left(-1 + \sqrt{\frac{1}{1-4t}}\right) - \frac{1}{1-2t} + \frac{1}{1-t}.$$

The formula for the codimension sequence follows immediately because

$$\frac{1}{2t}\left(-1 + \sqrt{\frac{1}{1-4t}}\right) = \frac{1}{2t}\left(-1 + \sum_{n\geq 0} \binom{2n}{n} t^n\right) = \sum_{n\geq 0} \frac{1}{2} \binom{2n+2}{n+1} t^n.$$

(ii) We can repeat verbatim the arguments from the proof of Theorem 4.4 [2], because the proper codimension series of the ordinary polynomial identities of $M_2(K)$ is "almost" equal to

$$\gamma\big(M_2(K), *, t\big) = \sum (\dim M_\lambda) t^n, \quad \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathrm{Part}(n)$$

and the evaluation of this series was the most difficult part in the considerations there. For the original proof of the result see [6]. ∎

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