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EXTREMAL CASES OF RAPOPORT–ZINK SPACES

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Abstract We investigate qualitative properties of the underlying scheme of Rapoport–Zink formal moduli spaces of p -divisible groups (resp., shtukas). We single out those cases where the dimension of this underlying scheme is zero (resp., those where the dimension is the maximal possible). The model case for the first alternative is the Lubin–Tate moduli space, and the model case for the second alternative is the Drinfeld moduli space. We exhibit a complete list in both cases.

Keywords and Phrases: Rapoport–Zink spaces; Drinfeld space; Lubin–Tate space; affine Deligne–Lusztig varieties; affine Weyl group

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1. Introduction

Let F be a non-Archimedean local field and let \mathbf{G} be a connected reductive group over F . Let μ be a conjugacy class of cocharacters of \mathbf{G} (over the algebraic closure \overline{F}), and let $b \in \mathbf{G}(\check{F})$, where \check{F} denotes the completion of the maximal unramified extension of F . The main character of this paper is the set

$$X(\mu, b)_K = X^{\mathbf{G}}(\mu, b)_K := \{g\check{K} \in \mathbf{G}(\check{F})/\check{K} \mid g^{-1}b\sigma(g) \in \check{K} \text{Adm}(\mu)\check{K}\}. \quad (1.1)$$

Here K denotes an F -rational parahoric level structure of \mathbf{G} , with corresponding standard parahoric subgroup $\check{K} \subset \mathbf{G}(\check{F})$. Also, $\text{Adm}(\mu)$ denotes the μ -admissible subset of the Iwahori–Weyl group of \mathbf{G} . (See Section 2 for details on this notion and other notation used here.) By [19, Thm. A], $X(\mu, b)_K$ is nonempty if and only if $[b] \in B(\mathbf{G}, \mu)$ (i.e., $[b]$ is neutral acceptable), which we assume from now on.

The set in (1.1) has a geometric structure: if F is a function field, then $X(\mu, b)_K$ is a finite-dimensional closed subscheme of the partial affine flag variety $\mathbf{G}(\check{F})/\check{K}$, locally of finite type over the algebraic closure of the residue field of F . If F is p -adic, then the partial affine flag variety and its finite-dimensional closed subscheme $X(\mu, b)_K$ have to be understood in the sense of Bhatt and Scholze [1, Def. 9.4] and Zhu [41, Thm. 0.1] as a perfect scheme.

The interest in the set in (1.1) comes from the fact that in the case of a p -adic field and when μ is *minuscule*, sets of this form arise as the set of geometric points of the *underlying reduced set* of a Rapoport–Zink formal moduli space of p -divisible groups (cf. [32, §4]). Something analogous holds in the function-field case for formal moduli spaces of *shtukas* (cf. [37]; in this case, the minuscule hypothesis can be dropped). Both classes of formal schemes are very mysterious. In fact, we know explicitly these formal schemes essentially only in two cases: the *Lubin–Tate case* and the *Drinfeld case*. In the Lubin–Tate case, the formal scheme is a disjoint union of formal spectra of formal power series rings with coefficients in $O_{\tilde{F}}$, hence the underlying reduced scheme is just a disjoint union of points. In the Drinfeld case, the formal scheme is π -adic and the underlying reduced set is a disjoint union of special fibres of the Deligne–Drinfeld formal model of the p -adic half-space corresponding to the local field F .

In this paper, we address the question of classifying the cases when $X(\mu, b)_K$ has minimal dimension zero (as in the Lubin–Tate case) or maximal dimension $\langle \mu, 2\rho \rangle$ (as in the Drinfeld case).

Let us first discuss our results pertaining to the case of dimension zero. For the group \mathbf{G} , we denote by $\{\omega_i^\vee\}$ the set of fundamental coweights, where i runs over the index set of the simple roots. Here we use the same labeling as Bourbaki [2, Plate I–X].

Theorem 1.1 (cf. Theorem 4.1). *Assume that \mathbf{G} is quasi-simple over F and that μ is noncentral. Let b be basic and let K be an F -rational parahoric level structure. Then $X(\mu, b)_K$ is zero-dimensional if and only if \mathbf{G}_{ad} is isomorphic to $\text{Res}_{\tilde{F}/F}(\text{PGL}_n)$, for some n and some finite extension \tilde{F} of F , and there exists a unique F -embedding $\varphi_0: \tilde{F} \rightarrow \overline{F}$ such that $\mu_{\text{ad}, \varphi}$ is trivial for $\varphi \neq \varphi_0$ and $\mu_{\text{ad}, \varphi_0} = \omega_1^\vee$.*

Here we write, for any \tilde{F} -group $\tilde{\mathbf{G}}$, a cocharacter μ of $\text{Res}_{\tilde{F}/F}(\tilde{\mathbf{G}})$ as $\mu = (\mu_\varphi)_\varphi$ for cocharacters μ_φ of $\tilde{\mathbf{G}}$, where φ runs over $\text{Hom}_F(\tilde{F}, \overline{F})$.

In particular, if \mathbf{G} is absolutely quasi-simple, then the Lubin–Tate case (Example 2.7) is the only one when the dimension of $X(\mu, b)_K$ is zero. In general, when the dimension of $X(\mu, b)_K$ is zero, then μ is automatically minuscule. Also, the statement that the dimension of $X(\mu, b)_K$ is zero is independent of the choice of K . The case (\mathbf{G}, μ) that appears in Theorem 1.1 is called the *extended Lubin–Tate case* (we use the term *extended* because there is an extension \tilde{F}/F involved).

When we vary K , we obtain the transition morphisms $\pi_{K, K'}: X(\mu, b)_K \rightarrow X(\mu, b)_{K'}$, whenever $K \subset K'$. In the extended Lubin–Tate case, the fibres of $\pi_{K, K'}$ are finite for any $K \subsetneq K'$. For the next statement, let us exclude this case.

Theorem 1.2 (cf. Theorem 4.2). *Assume that \mathbf{G} is quasi-simple over F and that μ is noncentral. Let b be basic. Also, exclude the extended Lubin–Tate case discussed in Theorem 1.1. Fix a pair $K \subsetneq K'$ of F -rational parahoric level structures.*

Then the fibres of $\pi_{K, K'}$ are all finite if and only if \mathbf{G}_{ad} is isomorphic to $\text{Res}_{\tilde{F}/F}(\tilde{\mathbf{G}}_{\text{ad}})$, where \tilde{F} is a finite extension of F and where $\tilde{\mathbf{G}}_{\text{ad}}$ is the adjoint group of a unitary group associated to a split \tilde{F}'/\tilde{F} -Hermitian vector space V for an unramified quadratic extension \tilde{F}'/\tilde{F} , and the following two conditions are satisfied:

- There exists a unique F -embedding $\varphi_0: \tilde{F} \rightarrow \overline{F}$ such that $\mu_{\text{ad},\varphi}$ is trivial for $\varphi \neq \varphi_0$ and $\mu_{\text{ad},\varphi_0} = \omega_1^\vee$.
- The pair (K, K') satisfies the following: let the maximal unramified subextension F_d of \tilde{F}/F have degree d . Correspondingly, write K and K' as $K = (K_1, \dots, K_d)$ and $K' = (K'_1, \dots, K'_d)$, where the entries are parahoric subgroups of $\text{Res}_{\tilde{F}/F_d}(\tilde{\mathbf{G}}_{\text{ad}})$.¹ Then $K'_1 \setminus K_1 \subset \{s_0, s_{\frac{d}{2}}\}$, and if $s_i \in K'_1 \setminus K_1$, then $s_{i+1} \notin K_1$.

Both implications of the theorem are interesting. Indeed, in the case singled out by the theorem, assume for simplicity that $\tilde{F} = F$ and consider a maximal self-dual periodic lattice chain

$$\{\dots \subset \Lambda_{-2} \subset \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \dots\}$$

in V . The case when $K' \setminus K = \{s_0\}$ is given as follows: K' stabilises a subchain Λ_I which contains Λ_1 but not the self-dual lattice Λ_0 , and K stabilises Λ_0 in addition to Λ_I . Under these conditions, the theorem states the following. Let N be an \tilde{F} , vector space of dimension $2\dim V$, equipped with an action of \tilde{F} and an alternating bilinear form \langle, \rangle which is Hermitian with respect to the \tilde{F} -action. Let ϕ be a σ -linear automorphism of N which commutes with the \tilde{F} -action and which is isoclinic of slope $1/2$ and such that $\langle \phi(x), \phi(y) \rangle = \pi \sigma(\langle x, y \rangle)$ for all $x, y \in N$. Here π denotes a uniformiser in F . Let \mathcal{M}_I be a self-dual chain of $O_{\tilde{F}}$ -lattices in N which are invariant under $O_{\tilde{F}}$, of type Λ_I . Assume that $\pi \mathcal{M}_i \subset \phi(\mathcal{M}_i) \subset {}^1 \mathcal{M}_i$ for all $i \in I$. Then there are only finitely many ways of completing the chain \mathcal{M}_I to a self-dual chain by adding a self-dual lattice \mathcal{M}_0 such that $\pi \mathcal{M}_0 \subset \phi(\mathcal{M}_0) \subset {}^1 \mathcal{M}_0$.

The case when $K' \setminus K = \{s_m\}$ when $n = 2m$ is similar (with a self-dual lattice replaced by a lattice which is self-dual up to a scalar); and the case when $K' \setminus K = \{s_0, s_m\}$ when $n = 2m$ is a concatenation of the previous cases.

From a global perspective, i.e., the point of view of Shimura varieties, Theorem 1.1 implies that the only cases where the basic locus is zero-dimensional are those which at the fixed prime p give rise to the extended Lubin–Tate case. This is the situation considered by Harris and Taylor in [14].

Now let us discuss our results pertaining to the case of maximal dimension. First, we have the following well-known upper bound on the dimension of $X(\mu, b)_K$. As usual, ρ denotes the half sum of all positive roots, and by $\langle \mu, 2\rho \rangle$ we mean the value of 2ρ on a dominant representative of μ .

Proposition 1.3 (cf. Corollary 9.6). *The dimension of $X(\mu, b)_K$ is bounded as*

$$\dim X(\mu, b)_K \leq \langle \mu, 2\rho \rangle.$$

If equality holds, then b is basic.

It is thus a natural question to ask in which cases this upper bound is attained. A well-known example is the Drinfeld case, but there are other cases, too.

¹Note that $\text{Res}_{\tilde{F}/F_d}(\tilde{\mathbf{G}}_{\text{ad}})$ has affine Dynkin type \tilde{A}_{n-1} ; we use standard notation for the simple reflections in this case.

Theorem 1.4 (cf. Theorem 10.1). *Assume that \mathbf{G} is quasi-simple over F and that μ is not central. If $\dim X(\mu, b)_K = \langle \mu, 2\rho \rangle$, then b is basic, the σ -centraliser group \mathbf{J}_b is a quasi-split inner form of \mathbf{G} and $\underline{\mu}$ is minuscule (in the échelonnage root system²; see Section 2.2). If $K = \emptyset$ is the Iwahori level, the converse holds.*

For a general parahoric level, $\dim X(\mu, b)_K = \langle \mu, 2\rho \rangle$ if and only if b is basic and $W(\underline{\mu})_{K, \text{fin}} \neq \emptyset$. In this case, the orbits of the action of $\mathbf{J}_b(F)$ on the set of irreducible components of $X(\mu, b)_K$ of dimension $\langle \mu, 2\rho \rangle$ are parametrised by the finite set $W(\underline{\mu})_{K, \text{fin}}$.

We refer to equation (10.1) for the definition of $W(\underline{\mu})_{K, \text{fin}}$, a finite set of translation elements, which is related to Drinfeld’s notion of a critical index (see Proposition 12.1).

The constraints on (\mathbf{G}, μ, b, K) imposed by Theorem 1.4 are in fact quite weak. For instance, if (\mathbf{G}, μ, b) is such that μ is minuscule, b is basic and \mathbf{G} is split over \check{F} , then there always exists an inner form \mathbf{H} of \mathbf{G} such that $\dim X^{\mathbf{H}}(\mu, b)_{\emptyset} = \langle \mu, 2\rho \rangle$.

On the other hand, the condition that $\dim X(\mu, b)_K$ be equidimensional of maximal dimension is much stronger.

Theorem 1.5 (cf. Theorem 10.2). *Assume that \mathbf{G} is quasi-simple over F and that μ is not central. Let $b \in \mathbf{G}(\check{F})$ be a representative of the unique basic element in $B(\mathbf{G}, \mu)$. Then $X(\mu, b)_K$ is equidimensional of dimension equal to $\langle \mu, 2\rho \rangle$ if and only if the triple $(\mathbf{G}_{\text{ad}}, \mu_{\text{ad}}, K)$ is isomorphic to one of the following:*

- (1) $(\text{Res}_{\check{F}/F}(D_{1/n}^{\times})_{\text{ad}}, \omega_1^{\vee}(\varphi_0), \emptyset)$.
- (2) $(\text{Res}_{\check{F}/F} \text{PGL}_2(D_{1/2}), \omega_2^{\vee}(\varphi_0), \emptyset)$.
- (3) $(\text{Res}_{\check{F}/F}(\text{PGL}_n), \mu, \emptyset)$.

Here \check{F} denotes a finite extension of F , and for an adjoint reductive group $\check{\mathbf{G}}$ over \check{F} and a cocharacter $\tilde{\mu}$ of $\check{\mathbf{G}}$ and an embedding $\varphi_0: \check{F} \rightarrow \overline{F}$, we denote by $\tilde{\mu}(\varphi_0)$ the cocharacter μ of $\text{Res}_{\check{F}/F}(\check{\mathbf{G}})$ with $\mu_{\varphi} = 0$ for $\varphi \neq \varphi_0$ and $\mu_{\varphi_0} = \tilde{\mu}$. Furthermore, $D_{1/n}$ denotes the central division algebra over \check{F} with invariant $1/n$, and $D_{1/n}^{\times}$ the algebraic group over \check{F} associated to its multiplicative group. In case (3), there are two embeddings $\varphi_0, \varphi_1: \check{F} \rightarrow \overline{F}$ such that their restrictions to the maximal unramified subextension of \check{F}/F are distinct, and the cocharacter μ is given as follows: $\mu_{\varphi_0} = \omega_1^{\vee}$ and $\mu_{\varphi_1} = \omega_{n-1}^{\vee}$ and $\mu_{\varphi} = 0$ for $\varphi \notin \{\varphi_0, \varphi_1\}$.

Case (1) is the *extended Drinfeld case*. Case (2) is somewhat surprising and was unknown to us before. Case (3) in the case of an unramified quadratic extension \check{F}/F is the *Hilbert–Blumenthal case*. It was discovered by Stamm [35] in the case $\mathbf{G} = \text{Res}_{\check{F}/F} \text{GL}_2$.

It is remarkable that in all three cases, the parahoric level structure K is the Iwahori level. This implies the following characterisation of the Drinfeld case:

Corollary 1.6 (cf. Corollary 15.1). *Assume that \mathbf{G} is quasi-simple over F and that μ is not central. Then $X(\mu, \tau)_K$ is equidimensional of dimension equal to $\langle \mu, 2\rho \rangle$ for every F -rational parahoric level structure K if and only if $(\mathbf{G}_{\text{ad}}, \mu_{\text{ad}})$ is isomorphic to $(\text{Res}_{\check{F}/F}(D_{1/n}^{\times})_{\text{ad}}, \omega_1^{\vee}(\varphi_0))$.*

²The latter condition implies that μ is minuscule but is slightly stronger if \mathbf{G} does not split over \check{F} .

One of our motivations for this paper was to characterise the Drinfeld case. Scholze suggested characterising it through the dimension of its underlying reduced scheme. Theorem 1.5 shows that this is not quite possible, but Corollary 1.6 shows that it is possible when K is varying.

As a consequence of Corollary 1.6, we can characterise the Drinfeld case as the only Rapoport–Zink space which is a π -adic formal scheme. We place ourselves in the context of [21, §4]; in particular, in the rational RZ-data $(F, B, V, (\cdot, \cdot), *, G, \{\mu\}, [b])$, the first entry F is a field. Also, RZ-spaces are modeled on the local models of [21, §2.6]; hence we make a tame ramification hypothesis (cf. [21]).

Theorem 1.7. *Let $\mathcal{D}_{\mathbb{Z}_p}$ be integral RZ-data such that the associated reductive group \mathbf{G} is connected and quasi-simple over \mathbb{Q}_p , and the associated cocharacter μ is noncentral. Let E be its reflex field and let $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ be the associated RZ-space, a formal scheme flat over $\mathrm{Spf} O_{\check{E}}$. Then $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ is a π -adic formal scheme if and only if $\mathcal{D}_{\mathbb{Z}_p}$ is of extended Drinfeld type, in which case $\mathcal{M}_{\mathcal{D}_{\mathbb{Z}_p}}$ is isomorphic to the disjoint sum of copies of $\widehat{\Omega}_E^n \widehat{\otimes}_{O_E} O_{\check{E}}$, where $\widehat{\Omega}_E^n$ is the Deligne–Drinfeld formal model of the Drinfeld half-space attached to E .*

Here the integral RZ-data are said to be of extended Drinfeld type if the rational RZ-data are of type (EL) with $B = D_{1/n}$, $\dim_B(V) = 1$, $\mu = \omega_1^\vee(\varphi_0)$ and b basic, and the integral RZ-data are given by a complete periodic O_B -lattice chain in V .

Through Rapoport–Zink uniformisation, this theorem implies that there is no p -adic uniformisation of Shimura varieties beyond the Drinfeld case. Note that the characterisation of p -adic uniformisation through the fact that the basic Newton stratum makes up the whole special fibre leads to Kottwitz’s determination of all *uniform* pairs (\mathbf{G}, μ) (cf. Section 15.3 and [25, §6]). It appears interesting to us that one can also characterise the Drinfeld case in a *purely local* way, without relating it to a Shimura variety.

This paper consists of three parts. In the first part, we provide the necessary background and introduce the terminology used. The second part is devoted to the case of dimension zero. In Section 4, we discuss the main results of this part; Sections 5 and 6 are devoted to the proofs. In Section 7, we explain in lattice-theoretic terms the minimal cases of Theorems 1.1 and 1.2. In Section 8, we give the proofs of Theorems 1.1 and 1.2. The third part is devoted to the case of maximal dimension. In Section 9, we recall the dimension theory of some subsets of \check{G} and prove Proposition 1.3. In Section 10, we discuss the main results of this part. Section 11 is preparatory for the proof but also contains results on Drinfeld’s critical index set which are of independent interest (in particular, we solve a problem posed 20 years ago in [34, §3]). In Section 12, we give the proof of Theorem 1.4, and in Section 13 the proof of Theorem 1.5. In Section 14, we explain the equi-maximal cases in lattice-theoretic terms. In Section 15, we discuss various ways of singling out the Drinfeld case among the three cases occurring in the classification of Theorem 1.5. Section 16 gives the proofs of the results for the case of maximal dimension.

Notation. For a local field F , we denote by O_F its ring of integers and by k its residue field. We denote by \check{F} the completion of the maximal unramified extension, by $O_{\check{F}}$ or \check{O}_F its ring of integers and by σ its Frobenius generator of $\mathrm{Gal}(\check{F}/F)$.

Part 1. Background

2. Preliminaries

2.1. The Iwahori–Weyl group

Let F be a non-Archimedean local field and \check{F} be the completion of the maximal unramified extension F^{un} of F . We denote by σ its Frobenius morphism, and by $\pi \in O_F$ a uniformiser. Let \mathbf{G} be a connected reductive group over F . We fix a σ -stable Iwahori subgroup $\check{\mathcal{I}}$ of $\check{G} = \mathbf{G}(\check{F})$.

We fix a maximal torus T which after extension of scalars is contained in a Borel subgroup of $\mathbf{G} \otimes_F \check{F}$, and such that $\check{\mathcal{I}}$ is the Iwahori subgroup fixing an alcove \mathfrak{a} in the apartment attached to the split part of T . The *Iwahori–Weyl group* is defined by

$$\check{W} = N(\check{F}) / (T(\check{F}) \cap \check{\mathcal{I}})$$

(cf. [13], [36, §1]). Let $W_0 = N(\check{F}) / T(\check{F})$. Then we have

$$\check{W} = X_*(T)_{\Gamma_0} \rtimes W_0, \tag{2.1}$$

where $\Gamma_0 = \text{Gal}(\overline{F}/F^{\text{un}})$. The splitting depends on the choice of a special vertex of the base alcove \mathfrak{a} that we fix in the sequel. When considering an element $\lambda \in X_*(T)_{\Gamma_0}$ as an element of \check{W} , we write t^λ .

Let $\check{\mathbb{S}}$ be the set of simple reflections in \check{W} determined by the base alcove \mathfrak{a} and $\mathbb{S} = \check{\mathbb{S}} \cap W_0$. For any subset K of $\check{\mathbb{S}}$, we denote by W_K the subgroup of \check{W} generated by simple reflections in K . We also denote by ${}^K \check{W}$ the set of representatives of minimal length of the cosets $W_K \backslash \check{W}$. If W_K is a finite group, we denote by \check{K} the corresponding standard parahoric subgroup.

The Iwahori–Weyl group is a quasi-Coxeter group. More precisely,

$$\check{W} = W_a \rtimes \Omega, \tag{2.2}$$

where W_a is the affine Weyl group with set $\check{\mathbb{S}}$ as simple reflections and Ω is the set of elements stabilising the base alcove \mathfrak{a} (cf. [18, §2.2]). The length function on W_a is extended to \check{W} by $\ell(w\tau) = \ell(w)$, for $w \in W_a$ and $\tau \in \Omega$. For $w \in \check{W}$, we denote by $\tau(w)$ its image in Ω .

2.2. Admissible sets and acceptable sets

Let μ be a conjugacy class of cocharacters of \mathbf{G} . We can always choose an \check{F} -rational representative μ_+ in this conjugacy class. We make a definite choice as follows. We identify $X_*(T)_{\Gamma_0, \mathbb{R}}$ with the standard apartment (the apartment attached to the split part of T), using our choice of special vertex of \mathfrak{a} . We then fix the unique Weyl chamber containing \mathfrak{a} , which we declare to be the dominant Weyl chamber. Then μ_+ is to be chosen such that $t^{\mu_+} \mathfrak{a} = \mu_+ + \mathfrak{a}$ is contained in the dominant Weyl chamber. We denote by $\underline{\mu}$ the image in $X_*(T)_{\Gamma_0}$ of μ_+ .

Remark 2.1. The choice of dominant Weyl chamber determines a Borel subgroup B of $\mathbf{G} \otimes_F \check{F}$ containing T . Note that $\underline{\mu}$ is equal to the image in $X_*(T)_{\Gamma_0, \mathbb{R}}$ of the B -antidominant representative of the conjugacy class $\mu \subset X_*(T)$! This phenomenon is

already visible when \mathbf{G} is split and is reflected by the minus sign in [36, Eqn. (5), p. 31]. The minus sign in turn is forced upon us by [36, Eqn. (4)], which could not possibly extend to the noncommutative normaliser if the left-hand side were replaced by $sX_\alpha s^{-1}$. This means that for $\lambda \in X_*(T)$, the element $\lambda(\pi)$ acts on the apartment for T by translation by $-\lambda$, i.e., as the element $t^{-\lambda}$.

It also means that even for a split group, the values $\langle \mu, \alpha \rangle$ and $\langle \underline{\mu}, \alpha \rangle$ for a root α differ by sign.

If $\underline{\mu}$ is minuscule, then μ is minuscule; but the converse does not hold (cf. the table just before [21, Lem. 5.4]). More precisely, we have the following:

Lemma 2.2. *Write $\mathbf{G}_{\text{ad}} = \text{Res}_{\tilde{F}/F}(\tilde{\mathbf{G}}_{\text{ad}})$, where the \tilde{F} -group $\tilde{\mathbf{G}}_{\text{ad}}$ is absolutely simple. Let the maximal unramified subextension F_d of \tilde{F}/F have degree d , and write correspondingly $\underline{\mu} = (\underline{\mu}_1, \dots, \underline{\mu}_d)$, where the entries $\underline{\mu}_i$ correspond to the various embeddings $\iota_i: F_d \rightarrow \tilde{F}$. If $\underline{\mu}$ is minuscule, then for every i there exists an embedding $\varphi_{i,0}: \tilde{F} \rightarrow \bar{F}$ inducing ι_i such that $\mu_\varphi = 0$ for every $\varphi \neq \varphi_{i,0}$ inducing ι_i and with $\mu_{\varphi_{i,0}}$ minuscule.*

Proof. One is immediately reduced to the case where \tilde{F}/F is totally ramified – that is, $d = 1$; therefore, we may drop the index i . Let \tilde{T} be a maximal torus of $\tilde{\mathbf{G}}$ which after extension to \tilde{F} is contained in a Borel subgroup, and let $T = \text{Res}_{\tilde{F}/F}(\tilde{T})$. The sum homomorphism $X_*(T) = \text{Ind}_{\Gamma_0}^{\tilde{\Gamma}_0}(X_*(\tilde{T})) \rightarrow X_*(\tilde{T})$ induces an identification

$$X_*(T)_{\Gamma_0} = X_*(\tilde{T}). \tag{2.3}$$

Here $\tilde{\Gamma}_0 = \text{Gal}(\bar{F}/\tilde{F}^{\text{un}})$. Under the identification of equation (2.3), we have $\underline{\mu} = \sum_{\varphi} \mu_{+, \varphi}$. From this the claim follows easily. \square

Furthermore, we have the following:

Lemma 2.3. *With notation as before, μ is central if and only if $\underline{\mu}$ is central.*

Proof. If μ is central, then clearly $\underline{\mu}$ is central. Conversely, assuming that $\underline{\mu}$ is central, we need to show that $\langle \mu_+, \alpha \rangle = 0$ for every (absolute) root α . Assume by contradiction that $\langle \mu_+, \alpha \rangle < 0$ for some α (cf. Remark 2.1). Let us write $[\underline{\mu}]$ when considering $\underline{\mu}$ as an element of $X_*(T)_{\mathbb{Q}}^{\Gamma_0}$. We want to show that the relative root $\text{res}(\alpha)$ defined by α by restriction to $X_*(T)_{\mathbb{Q}}^{\Gamma_0}$ takes a strictly positive value on $[\underline{\mu}]$. However, with μ_+ , also every Galois translate of μ_+ under an element of Γ_0 is antidominant; and $[\underline{\mu}]$ is the average over the Γ_0 -orbit of μ_+ . But then $\text{res}(\alpha)$ takes a strictly positive value on $[\underline{\mu}]$, and this contradicts the assumption that $\underline{\mu}$ is central. \square

The μ -admissible set is defined by

$$\text{Adm}(\mu) = \{w \in \tilde{W} \mid w \leq t^{x(\underline{\mu})} \text{ for some } x \in W_0\} \tag{2.4}$$

(cf. [31, §3]). For λ a cocharacter (rather than a conjugacy class of cocharacters), we denote by $\text{Adm}(\lambda)$ the admissible set of the conjugacy class of λ . Let $B(\mathbf{G})$ be the set of

σ -conjugacy classes in \check{G} . Kottwitz [24, 25] gave a description of the set $B(\mathbf{G})$. It uses the *Kottwitz map*,

$$\kappa : B(\mathbf{G}) \rightarrow \pi_1(\mathbf{G})_\Gamma, \tag{2.5}$$

where Γ is the Galois group of \bar{F} over F . Any σ -conjugacy class $[b]$ is determined by two invariants:

- the element $\kappa([b]) \in \pi_1(\mathbf{G})_\Gamma$ and
- the Newton point v_b in the dominant chamber of $X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$.

The set of *neutrally acceptable* σ -conjugacy classes is defined by

$$B(\mathbf{G}, \mu) = \{[b] \in B(\mathbf{G}) \mid \kappa([b]) = \kappa(\mu), v_b \leq \mu^\diamond\}, \tag{2.6}$$

where $\mu^\diamond = [\Gamma : \text{Stab}_\Gamma(\mu_+)]^{-1} \sum_{\gamma \in \Gamma/\text{Stab}_\Gamma(\mu_+)} \gamma(\mu_+)$ is the Galois average of μ_+ , an element of $X_*(T)^{\Gamma_0} \otimes \mathbb{Q} \cong X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$.

2.3. Affine Deligne–Lusztig varieties

The *affine Deligne–Lusztig variety* (for the Iwahori subgroup) associated to $w \in \check{W}$ and $b \in \check{G}$ is

$$X_w(b) = \{g\check{I} \in \check{G}/\check{I} \mid g^{-1}b\sigma(g) \in \check{I}w\check{I}\} \tag{2.7}$$

(cf. [31, §4]). Then $X_w(b)$ is a subset of the set of $\bar{\mathbb{F}}_p$ -points of the affine flag variety of \mathbf{G} . If F is of equal characteristic, then by the affine flag variety we mean the ‘usual’ affine flag variety; in the case of mixed characteristic, this notion should be understood in the sense of perfect schemes, as developed by Zhu [41, Thm. 0.1] and by Bhatt and Scholze [1, Def. 9.4]. More precisely, $X_w(b)$ is the set of $\bar{\mathbb{F}}_p$ -points of a locally closed (perfect) subscheme of the affine flag variety, locally (perfectly) of finite type over $\bar{\mathbb{F}}_p$ and of finite dimension, which we denote by the same symbol. This follows from [34, Thm. 1.4]. In fact, the main theorem of that paper implies that $X_w(b)$ is contained in a union $\bigcup_{g \in \mathbf{G}(F)} g\bar{C}$, for some Schubert variety \bar{C} . Since for any $g \in \mathbf{G}(F)$ there are only finitely many $g' \in \mathbf{G}(F)$ such that $g\bar{C}$ and $g'\bar{C}$ have nonempty intersection, this union is a k -scheme locally of finite type, and of finite dimension, and hence so is $X_w(b)$.

Denote by \mathbf{J}_b the σ -centraliser group of b , an algebraic group over F with F -rational points

$$\mathbf{J}_b(F) = \{g \in \mathbf{G}(\check{F}) \mid g^{-1}b\sigma(g) = b\}. \tag{2.8}$$

Then $\mathbf{J}_b(F)$ acts on $X_w(b)$. Let $K \subset \check{S}$ such that W_K is finite, with corresponding standard parahoric subgroup $\check{K} \subset \check{G}$. Here and whenever we consider the space $X(\mu, b)_K$, we assume that $\sigma(K) = K$. We set

$$X(\mu, b)_K = \{g\check{K} \in \check{G}/\check{K} \mid g^{-1}b\sigma(g) \in \check{K} \text{Adm}(\mu)\check{K}\}. \tag{2.9}$$

For $K = \emptyset$, we write simply $X(\mu, b)$ for $X(\mu, b)_K$. Then $X(\mu, b)$ is a union of affine Deligne–Lusztig varieties.

We will need the following result (conjectured in [27, 31]):

Theorem 2.4 ([19]). *Let $K \subset \check{\mathbb{S}}$ such that $\sigma(K) = K$ and W_K is finite. Then $X(\mu, b)_K \neq \emptyset$ if and only if $[b] \in B(\mathbf{G}, \mu)$.*

2.4. Fine affine Deligne–Lusztig varieties

We recall the definition of fine affine Deligne–Lusztig varieties inside the partial affine flag variety \check{G}/\check{K} (cf. [9, §3.4]). For $K \subset \mathbb{S}$, $w \in {}^K\check{W}$ and $b \in \check{G}$, the associated *fine affine Deligne–Lusztig variety* is

$$X_{K,w}(b) = \{g\check{K} \mid g^{-1}b\sigma(g) \in \check{K} \cdot {}_{\sigma}\check{I}w\check{I}\}. \tag{2.10}$$

Note that we have the decomposition of the partial affine flag variety \check{G}/\check{K} into ordinary affine Deligne–Lusztig varieties (for the parahoric subgroup associated to K),

$$\check{G}/\check{K} = \bigsqcup_{x \in W_K \backslash \check{W}/W_K} \{g\check{K} \mid g^{-1}b\sigma(g) \in \check{K}x\check{K}\}.$$

An ordinary affine Deligne–Lusztig variety decomposes in turn into a disjoint sum of fine affine Deligne–Lusztig varieties,

$$\{g\check{K} \mid g^{-1}b\sigma(g) \in \check{K}x\check{K}\} = \bigsqcup_{w \in {}^K\check{W} \cap W_K x W_K} X_{K,w}(b) \tag{2.11}$$

(cf. [9, §3.4]).

2.5. The decomposition of $X(\mu, b)_K$

We set

$${}^K\text{Adm}(\mu) = \text{Adm}(\mu) \cap {}^K\check{W}.$$

It is proved in [19, Thm. 6.1] that ${}^K\text{Adm}(\mu) = W_K \text{Adm}(\mu) W_K \cap {}^K\check{W}$. Hence

$$X(\mu, b)_K = \bigsqcup_{w \in {}^K\text{Adm}(\mu)} X_{K,w}(b). \tag{2.12}$$

We can read definition (2.10) as saying that $X_{K,w}(b)$ is the image of $X_w(b)$ under the projection map $\check{G}/\check{I} \rightarrow \check{G}/\check{K}$. We call this decomposition the *EKOR stratification*, and accordingly call the subsets $X_{K,w}(b)$ the *EKOR strata* inside $X(\mu, b)_K$. If $K = \emptyset$, we speak of the *KR stratification* and *KR strata* instead. These stratifications are the ‘local analogues’ of the stratifications defined in the global context in [22]. But since here we always fix a σ -conjugacy class $[b]$, an EKOR stratum in our context really corresponds to the intersection of a global EKOR stratum with the Newton stratum attached to $[b]$. In [9, §5.1], EKOR strata were called EO strata.

2.6. Tits data

We recall the notion of Tits data and Coxeter data from [21, Def. 5.3]. For an affine Coxeter system $(W_a, \check{\mathbb{S}})$, we denote by W_0 the finite Weyl group, and by \check{W} the associated extended affine Weyl group and by X_* the translation lattice of \check{W} .

Definition 2.5.

- (i) A *Tits datum* (over \check{F}) is a pair $(\tilde{\Delta}, \lambda)$, where $\tilde{\Delta}$ is a local Dynkin diagram and λ is a W_0 -conjugacy class in X_* .
- (ii) A *Coxeter datum* (over \check{F}) is a pair $((W_a, \tilde{S}), \lambda)$, where (W_a, \tilde{S}) is an affine Coxeter system and λ is a W_0 -conjugacy class in X_* .

A Tits datum yields a Coxeter datum by forgetting the arrows in the Dynkin diagram. In general, different Tits data may give rise to the same Coxeter datum. However, in type A and more generally for any simply laced Dynkin diagram, the Coxeter datum determines the Tits datum uniquely.

We need to generalise this notion as follows, to cover also the situation over F . Over \check{F} , simple adjoint groups are classified up to isomorphism by their (absolute) local Dynkin diagram (cf. [36, §4.2]). Over F , we need to take into account the case of groups which are not residually split. In [36, §4.3], Tits gives the classification in terms of the ‘local index’ and ‘relative local Dynkin diagram’. Here we choose to work instead with the absolute local Dynkin diagram (i.e., the affine Dynkin diagram attached to \mathbf{G} over \check{F}), together with the diagram automorphism induced by Frobenius. This datum is determined by \mathbf{G}/F (up to isomorphism), and determines the group \mathbf{G} over F up to isomorphism.

Definition 2.6.

- (i) A *Tits datum over F* is a triple $(\tilde{\Delta}, \delta, \lambda)$, where $\tilde{\Delta}$ is an absolute local Dynkin diagram, δ is a diagram automorphism of $\tilde{\Delta}$ and λ is a W_0 -conjugacy class in the coweight lattice X_* of $\tilde{\Delta}$.
- (ii) A *Coxeter datum over F* is a tuple $((W_a, \tilde{S}), \delta, \lambda)$, where (W_a, \tilde{S}) is an affine Coxeter system, δ is a length-preserving automorphism of W_a and λ is a W_0 -conjugacy class in X_* .

Note that a Tits datum over F gives rise to a Coxeter datum over F . In [21], the notion of *enhanced* Tits and Coxeter data was used, where an enhanced datum in addition specifies a parahoric level structure. Note that for an enhanced Coxeter datum $((W_a, \tilde{S}), \lambda, K)$ in the sense of [21, Def. 5.3], the associated parahoric subgroup is the one generated by the Iwahori and all simple affine reflections which are *not* contained in K , a convention opposite to the one used in this paper.

Next we explain the notion of restriction of scalars of Dynkin types over F (i.e., Dynkin types together with a diagram automorphism) along an unramified field extension. It models the form of the extended affine Weyl group of a group which arises as such a restriction of scalars. Let F_d/F denote the unramified extension of degree d , and let $(\tilde{\Delta}, \delta_d)$ be a local Dynkin diagram with diagram automorphism δ_d . We then define $\text{Res}_{F_d/F}(\tilde{\Delta}, \delta_d)$ as the Dynkin type

$$\tilde{\Delta}_1 \times \cdots \times \tilde{\Delta}_d$$

with diagram automorphism δ , where $\tilde{\Delta}_i = \tilde{\Delta}$ for all i , δ is given by $\text{id}: \tilde{\Delta}_i \rightarrow \tilde{\Delta}_{i+1}$ for $i = 1, \dots, d - 1$ and $\delta_d: \tilde{\Delta}_d \rightarrow \tilde{\Delta}_1$. So δ permutes the components cyclically, and the restriction of δ^d to any component is equal to δ_d .

Specifying a translation element for $\text{Res}_{F_d/F}(\tilde{\Delta}, \delta_d)$ amounts to giving a tuple $(\lambda_1, \dots, \lambda_d)$ consisting of d translation elements for $\tilde{\Delta}$. It is central (resp., minuscule) if and only if all the λ_i are central (resp., minuscule).

Example 2.7 (The Lubin–Tate case). This is the case with Tits datum $(\tilde{A}_{n-1}, \text{id}, \omega_1^\vee)$. The corresponding group is GL_n . This is a fully Hodge–Newton decomposable case (Section 3), and is even of Coxeter type in the sense of [9, §5.1] (and in this case the Coxeter property holds for arbitrary parahoric level). See Section 4.2 for a discussion of this case as a ‘minimal dimension’ case, and Section 7.1 for a ‘lattice description’ of the Lubin–Tate case.

Similarly, we have the *extended Lubin–Tate case* $(\text{Res}_{F_d/F}(\tilde{A}_{n-1}, \text{id}), (\omega_1^\vee, 0, \dots, 0))$.

Example 2.8 (The Drinfeld case). Here we consider the Tits datum $(\tilde{A}_{n-1}, \varrho_{n-1}, \omega_1^\vee)$, where ϱ_{n-1} denotes rotation by $n - 1$ steps, $\varrho_{n-1}(s_0) = s_{n-1}$, and so on. The corresponding algebraic group is the group of units of a central division algebra of invariant $1/n$. This is a fully Hodge–Newton decomposable case (Section 3), and even a ‘Coxeter’ case (for arbitrary parahoric level). See Section 14 for a ‘lattice description’ of the Drinfeld case.

Similarly, we have the *extended Drinfeld case* $(\text{Res}_{F_d/F}(\tilde{A}_{n-1}, \varsigma_1), (\omega_1^\vee, 0, \dots, 0))$.

2.7. Reduction to \check{F} -simple groups

Let us recall the construction of [10, §3.4]. Given an F -simple group \mathbf{G} of adjoint type together with a conjugacy class μ of cocharacters, we can decompose

$$\mathbf{G}_{\check{F}} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_d,$$

where the \mathbf{G}_i are simple algebraic groups over \check{F} and where the Frobenius σ induces maps $\mathbf{G}_i \rightarrow \mathbf{G}_{i+1}$ (with indices viewed in \mathbb{Z}/d). Let F_d denote the unramified extension of F of degree d in \check{F} . We denote by \mathbf{G}' the algebraic group over F_d , with $(\mathbf{G}')_{\check{F}} = \mathbf{G}_1$, with Frobenius given by $(\sigma^d)_{|_{G_1}}$. In other words, we write $\mathbf{G} = \text{Res}_{F_d/F}(\mathbf{G}')$ for a quasi-simple group over F_d which stays quasi-simple over \check{F} . Correspondingly, the Tits datum of \mathbf{G} arises by restriction of scalars along F_d/F as defined in Section 2.6.

We also define $\mu' = \sum_{i=1}^d \sigma_0^i(\mu_+)$, where σ_0 denotes the L -action (cf. [10, Def. 2.1]), i.e., the Frobenius action corresponding to the quasi-split inner form of \mathbf{G} .

Now suppose that $K = (K_1, \dots, K_d)$ is an F -rational parahoric level structure for \mathbf{G} . Then K_1 is an F_d -rational parahoric level structure for \mathbf{G}' .

We now consider the special situation that $\mu = (\mu_1, \dots, \mu_d)$ is a conjugacy class of cocharacters of \mathbf{G} where μ_i is central for all $i > 1$. Let $\tau = (\tau_1, \dots, \tau_d)$ be a σ -conjugacy class in $B(\mathbf{G}, \mu)$; we may choose τ_i central for all $i > 1$. Let $\tau' = \Pi \tau_i$ (this is well defined, as only one of the τ_i is noncentral).

Then it is easy to see that projection to the first factor induces an isomorphism $X^{\mathbf{G}}(\mu, \tau)_K \cong X^{\mathbf{G}'}(\mu', \tau')_{K_1}$. Examples of this situation are the extended Lubin–Tate case and the extended Drinfeld case already mentioned in the examples.

Moreover, if $K' = (K'_1, \dots, K'_d)$ is another F -rational parahoric level and $K \subseteq K'$, then we likewise have $X^{\mathbf{G}}(\mu, \tau)_{K'} \cong X^{\mathbf{G}'}(\mu', \tau')_{K'_1}$ and we obtain a commutative diagram

$$\begin{CD} X^{\mathbf{G}}(\mu, \tau)_K @>\cong>> X^{\mathbf{G}'}(\mu', \tau')_{K_1} \\ @VVV @VVV \\ X^{\mathbf{G}}(\mu, \tau)_{K'} @>\cong>> X^{\mathbf{G}'}(\mu', \tau')_{K'_1}, \end{CD}$$

where the vertical maps are the natural projections.

3. Fully Hodge–Newton decomposable case

3.1. The σ -support

For $w \in W_a$, we denote by $\text{supp}(w)$ the support of w , i.e., the set of $i \in \tilde{\mathbb{S}}$ such that s_i appears in some (or equivalently, every) reduced expression of w . For any length-preserving automorphism θ of \tilde{W} , we set

$$\text{supp}_\theta(w\tau) = \bigcup_{n \in \mathbb{Z}} (\text{Ad}(\tau) \circ \theta)^n(\text{supp}(w)). \tag{3.1}$$

This applies in particular to the Frobenius action σ . Then $\text{supp}_\sigma(w\tau)$ is the minimal $\text{Ad}(\tau)\sigma$ -stable subset J of $\tilde{\mathbb{S}}$ such that $w\tau\sigma \in W_J \rtimes \langle \tau\sigma \rangle$.

3.2. Classification of fully Hodge–Newton decomposable pairs (\mathbf{G}, μ)

In [10], the notion of a fully Hodge–Newton decomposable pair (\mathbf{G}, μ) is introduced. We refer to [10, Def. 3.1] for the definition. Here we use the following equivalent characterisations [10, Thm. B, Thm. 3.3]:

Theorem 3.1. *Let (\mathbf{G}, μ) be a pair as before, with \mathbf{G} quasi-simple over F , and let $K \subset \tilde{\mathbb{S}}$ with $\sigma(K) = K$ and W_K finite. The following are equivalent:*

- (1) *The pair (\mathbf{G}, μ) is fully Hodge–Newton decomposable.*
- (2) *For each $w \in \text{Adm}(\mu)$, there exists a unique $[b] \in B(\mathbf{G}, \mu)$ such that $\check{I}w\check{I} \subset [b]$.*
- (3) *For each $w \in {}^K\text{Adm}(\mu)$ with $X_{K,w}(\tau) \neq \emptyset$, the set $W_{\text{supp}_\sigma(w)}$ is finite.*

Here τ denotes a representative of the unique basic element $[\tau]$ in $B(\mathbf{G}, \mu)$.

In particular, condition (3) is independent of K .

In particular, in this case, for any $K \subset \tilde{\mathbb{S}}$ with W_K finite and any $w \in {}^K\text{Adm}(\mu)$, there exists a unique $[b] \in B(\mathbf{G}, \mu)$ such that $\check{K} \cdot_\sigma \check{I}w\check{I} \subset [b]$. This gives us a natural map

$${}^K\text{Adm}(\mu) \rightarrow B(\mathbf{G}, \mu), \quad w \mapsto [w]. \tag{3.2}$$

We will later use the following statement:

Proposition 3.2 ([10, Prop. 5.6, Lem. 5.8]). *Let $x \in \tilde{W}$. The following are equivalent:*

- (1) $\check{K} \cdot_{\sigma} \check{I}x\check{I} \subset [\tau]$.
- (2) $\kappa(x) = \kappa(\tau)$ and $W_{\text{supp}_{\sigma}(x)}$ is finite.
- (3) $\kappa(x) = \kappa(\tau)$ and $\text{Ad}(x) \circ \sigma$ fixes a point in the closure of the base alcove.

In the next two theorems, we give the classification of the fully Hodge–Newton decomposable cases following [10, Thm. 3.5].

Theorem 3.3. *Assume that \mathbf{G} over F is absolutely simple and that μ is not central. Then (\mathbf{G}, μ) is fully Hodge–Newton decomposable if and only if the associated Tits datum is one of the following:*

$(\tilde{A}_{n-1}, \text{id}, \omega_1^{\vee})$	$(\tilde{A}_{n-1}, \varrho_{n-1}, \omega_1^{\vee})$	$(\tilde{A}_{n-1}, \varsigma_0, \omega_1^{\vee})$
$(\tilde{A}_{2m-1}, \varrho_1 \varsigma_0, \omega_1^{\vee})$	$(\tilde{A}_{n-1}, \text{id}, \omega_1^{\vee} + \omega_{n-1}^{\vee})$	
$(\tilde{A}_3, \text{id}, \omega_2^{\vee})$	$(\tilde{A}_3, \varsigma_0, \omega_2^{\vee})$	$(\tilde{A}_3, \varrho_2, \omega_2^{\vee})$
$(\tilde{B}_n, \text{id}, \omega_1^{\vee})$	$(\tilde{B}_n, \text{Ad}(\tau_1), \omega_1^{\vee})$	
$(\tilde{C}_n, \text{id}, \omega_1^{\vee})$	$(\tilde{C}_2, \text{id}, \omega_2^{\vee})$	$(\tilde{C}_2, \text{Ad}(\tau_2), \omega_2^{\vee})$
$(\tilde{D}_n, \text{id}, \omega_1^{\vee})$	$(\tilde{D}_n, \varsigma_0, \omega_1^{\vee})$	

Theorem 3.4. *Assume that \mathbf{G} is quasi-simple over F and that μ is not central. Then the pair (\mathbf{G}, μ) is fully Hodge–Newton decomposable if and only if the associated Tits datum is of type $(\text{Res}_{F_d/F}(\tilde{\Delta}, \delta), (\mu_1, \dots, \mu_d))$, where one of the following two possibilities occur.*

- (1) *There is a unique i such that μ_i is noncentral and $(\tilde{\Delta}, \delta, \mu_i)$ is one of the triples listed in Theorem 3.3.*
- (2) *$(\tilde{\Delta}, \delta) = (\tilde{A}_{n-1}, \text{id})$ and there exist $i \neq i'$ such that $\mu_i = \omega_1^{\vee}$, $\mu_{i'} = \omega_{n-1}^{\vee}$ and μ_j is central for all $j \neq i, i'$.*

Here we use the same labelling of the Coxeter graph as Bourbaki [2, Plate I–X]. If ω_i^{\vee} is minuscule, we denote the element $\tau(t^{\omega_i^{\vee}}) \in \Omega$ by τ_i ; conjugation by τ_i is a length-preserving automorphism of \tilde{W} , which we denote by $\text{Ad}(\tau_i)$. For type A_n , $\text{Ad}(\tau_i)$ is the rotation of the affine Dynkin diagram by i steps (i.e., s_0 is mapped to s_i , s_1 is mapped to s_{i+1} , etc.), and we denote it by ϱ_i instead. Let ς_0 be the unique nontrivial diagram automorphism for the finite Dynkin diagram if W_0 is of type A_n, D_n (with $n \geq 5$) or E_6 . For type D_4 , we also denote by ς_0 the diagram automorphism which interchanges α_3 and α_4 .

If we assume that μ is noncentral in every component of the affine Dynkin diagram, the fully Hodge–Newton decomposable cases are the cases in Theorem 3.3 and the Hilbert–Blumenthal case $(\tilde{A}_{n-1} \times \tilde{A}_{n-1}, {}^1\zeta_0, (\omega_1^\vee, \omega_{n-1}^\vee))$, where the automorphism ${}^1\zeta_0$ on $\tilde{A}_{n-1} \times \tilde{A}_{n-1}$ is the automorphism which exchanges the two factors.

To derive Theorem 3.4 from Theorem 3.3, note that for a group \mathbf{G} which is quasi-simple over F but not over \tilde{F} , we can apply the construction in [10, §3.4] (cf. Section 2.7). We then have that \mathbf{G}' is quasi-simple over \tilde{F} and that μ is minute if and only if μ' is minute (cf. [10, Def. 3.2 and §3.4]). Applying Theorem 3.3 to (\mathbf{G}', μ') , we obtain Theorem 3.4.

3.3. Basic case

Let $\tau = \tau(\mu) \in \Omega$ be the length-0 element in \tilde{W} such that $\text{Adm}(\mu) \subset W_a\tau$. Then $[\tau]$ is the unique basic σ -conjugacy class in $B(\mathbf{G}, \mu)$.

Set

$${}^K\text{Adm}(\mu)_0 = \{w \in {}^K\text{Adm}(\mu) \mid W_{\text{supp}_\sigma(w)} \text{ is finite}\}. \tag{3.3}$$

If (\mathbf{G}, μ) is fully Hodge–Newton decomposable, the set ${}^K\text{Adm}(\mu)_0$ is just the fibre over the unique basic element of $B(\mathbf{G}, \mu)$ of the map in (3.2).

The following result is proved in [10, Thm. B (5)]:

Theorem 3.5. *Suppose that (\mathbf{G}, μ) is a fully Hodge–Newton decomposable pair. Then*

$$X(\mu, \tau)_K = \bigsqcup_{w \in {}^K\text{Adm}(\mu)_0} X_{K,w}(\tau),$$

and $X_{K,w}(\tau) \neq \emptyset$ for all $w \in {}^K\text{Adm}(\mu)_0$.

Part 2. Minimal dimension

In this part we determine those cases when $X(\mu, b)_K$ is zero-dimensional, in case b is basic. When b is basic, we also determine the cases when the transition morphism $X(\mu, b)_K \rightarrow X(\mu, b)_{K'}$ has finite fibres.

4. Statement of results

4.1. Change of parahoric

In this section, we are concerned with the following two theorems.

Theorem 4.1. *Assume that \mathbf{G} is quasi-simple over F and that μ is not central. Let $K \subsetneq \tilde{\mathbb{S}}$ be σ -stable. The following are equivalent:*

- (1) $\dim X(\mu, \tau)_K = 0$.
- (2) (\mathbf{G}, μ) is of extended Lubin–Tate type, i.e., $(\tilde{\Delta}, \sigma, \mu) = (\text{Res}_{F_d/F}(\tilde{A}_{n-1}, \text{id}), (\omega_1^\vee, 0, \dots, 0))$ for a finite unramified extension F_d/F .

See Example 2.7 for a discussion of the (extended) Lubin–Tate case. We will prove a stronger version of this theorem later (see Theorem 4.5).

For any σ -stable subsets $K \subsetneq K' \subset \tilde{S}$, we denote by

$$\pi_{K, K'} : X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'} \tag{4.1}$$

the projection map.

Theorem 4.2. *Assume that \mathbf{G} is quasi-simple over F and that μ is not central. Let $K \subsetneq K' \subsetneq \tilde{S}$ be σ -stable parahoric level structures. Write the Tits datum of (\mathbf{G}, μ) in the form $(\text{Res}_{F_d/F}(\tilde{\Delta}, \sigma), (\mu_1, \dots, \mu_d))$, and correspondingly write the parahoric level structures as $K = (K_1, K_2, \dots, K_d)$, $K' = (K'_1, K'_2, \dots, K'_d)$. Then the following are equivalent:*

- (1) *The projection $X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$ has discrete fibres.*
- (2) *There exists a unique j such that μ_j is noncentral, we have $\mu_j = \omega_1^\vee$ and*
 - *σ acts as id on the affine Dynkin diagram or*
 - *$n \geq 3$ and the action of σ on \tilde{A}_{n-1} preserves s_0 and induces the nontrivial diagram automorphism ζ_0 on A_{n-1} . Furthermore, the pair (K_1, K'_1) satisfies Condition 4.3.*

Here is the Condition 4.3 that appears in Theorem 4.2, case (2):

Condition 4.3. *Every element of $K'_1 \setminus K_1$ is fixed by σ_d , and if $s_i \in K'_1 \setminus K_1$, then $s_{i+1} \notin K_1$.*

Note that K and K' are assumed to be σ -stable, so requiring that the inclusion $K' \subsetneq \tilde{S}$ be strict implies that in each connected component of \tilde{S} there exists a vertex not lying in K' , and similarly for the inclusion $K \subsetneq K'$.

Remark 4.4. Let us enumerate the cases for the second alternative in Theorem 4.2, case (2), when \mathbf{G} is quasi-simple over \check{F} . By assumption K and K' are σ -stable; also, the corresponding algebraic group is a quasi-split unitary group which splits over an unramified quadratic extension.

- *n odd:* In this case, $\sigma(s_0) = s_0$ and $\sigma(s_1) = s_{n-1}$. Then $K \subset \tilde{S} \setminus \{s_0, s_1, s_{n-1}\}$ is σ -stable and $K' = K \cup \{s_0\}$.

Extreme case $n = 3$; then $K = \emptyset$, $K' = \{s_0\}$.

- *$n = 2m$ even:* In this case, $\sigma(s_0) = s_0, \sigma(s_m) = s_m$ and $\sigma(s_{m+1}) = s_{m-1}$. Then the following three possibilities occur:

- (i) $K \subset \tilde{S} \setminus \{s_0, s_1, s_{n-1}\}$ is σ -stable and $K' = K \cup \{s_0\}$.
- (ii) $K \subset \tilde{S} \setminus \{s_{m-1}, s_m, s_{m+1}\}$ is σ -stable and $K' = K \cup \{s_m\}$.
- (iii) $K \subset \tilde{S} \setminus \{s_0, s_1, s_{m-1}, s_m, s_{m+1}, s_{n-1}\}$ is σ -stable and $K' = K \cup \{s_0, s_m\}$.

Extreme case $n = 4, m = 2$; then for (K, K') the following possibilities occur: $(\emptyset, \{s_0\})$ or $(\emptyset, \{s_2\})$ or $(\emptyset, \{s_0, s_2\})$ or $(\{s_2\}, \{s_0, s_2\})$ or $(\{s_0\}, \{s_0, s_2\})$.

The proof of Theorem 4.2 will occupy the next two sections. In the rest of this section, we give more details on the two alternatives of the theorem.

4.2. The Lubin–Tate case

Theorem 4.5. *Assume that \mathbf{G} is quasi-simple over F and that μ is not central. The following are equivalent:*

- (1) *The pair (\mathbf{G}, μ) is of extended Lubin–Tate type (cf. the statement of Theorem 4.1).*
- (2) *$\dim X(\mu, \tau)_K = 0$ for some parahoric K .*
- (3) *$\dim X(\mu, \tau)_K = 0$ for all parahorics K .*
- (4) *The projection $X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$ has finite fibres for all $K \subsetneq K'$.*
- (5) *The projection $X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$ is a bijection for all $K \subsetneq K'$.*

Proof. (3) \Rightarrow (2) and (5) \Rightarrow (4) are obvious.

(1) \Rightarrow (3) & (5): This follows from Remark 4.6.

(2) \Rightarrow (1): This is Theorem 4.1.

(4) \Rightarrow (1): By Theorem 4.2, the Dynkin type is $\text{Res}_{F_d/F}(\tilde{A}_{n-1}, \sigma_d)$, with $\sigma_d = \text{id}$ or $\sigma_d = \zeta_0$ (up to isomorphism). Moreover, as we may take $K = \{s_0\}$, Condition 4.3 implies that σ_d cannot be ζ_0 . Hence $\sigma = \text{id}$. □

Remark 4.6. Properties (3) and (5) in Theorem 4.5 are well known in the Lubin–Tate case, and we explain this in terms of lattices in Section 7. Alternatively, we could apply the methods of [9, §6.3], with Case 1 for $i = 1$ (cf. also [10]). There is only one basic EKOR stratum in this case. (Note that EKOR strata were called EO strata in [9].) Let $\mathbf{J} = \mathbf{J}_\tau$ be the σ -centraliser of τ (cf. equation (2.8)). The index set for the stratification in a single connected component is a quotient of $\mathbf{J}(F)^1$ by a parahoric subgroup (where $\mathbf{J}(F)^1$ is the kernel of the Kottwitz homomorphism). Since $\mathbf{J}(F)^1$ is anisotropic, this quotient is a single point, so the EKOR stratification has a single stratum. This stratum is attached to the length 0 element τ , thus the corresponding classical Deligne–Lusztig variety is just a point. Note that this argument can be applied to arbitrary parahoric level structures, not only maximal parahoric as in the setting of [9]. By either of the two methods, we obtain the more precise statement that $X(\mu, \tau)_K$ has only one point in each connected component of the affine flag variety.

Using the construction in Section 2.7, the result can be generalised to the extended Lubin–Tate case, where a restriction of scalars is allowed.

4.3. The exotic case

The second alternative in Theorem 4.2, where Condition 4.3 is relevant, will be studied in detail in Section 6.4 in group-theoretic terms and in Section 7.2 in terms of lattices. Using either approach, we will determine the cardinalities of the fibres of the map $\pi_{K, K'}$. If $\#(K'_1 \setminus K_1) = 1$, then the fibre cardinalities are 1, 2 and $q^d + 1$. If $\#(K'_1 \setminus K_1) = 2$, then each fibre is naturally a product of two sets as in the first case, so the cardinalities which occur are 1, 2, 4, $q^d + 1$, $2(q^d + 1)$ and $(q^d + 1)^2$. We give precise criteria in group-theoretic terms as well as in lattice terms for which case occurs when (see Section 6.7 and Proposition 7.9).

5. Proof of (1) \Rightarrow (2) in Theorems 4.1 and 4.2

In this section, we prove the implications (1) \Rightarrow (2) in Theorem 4.1 and Theorem 4.2. We will handle both theorems simultaneously by allowing $K = \mathbb{S}$, with the convention that $X(\mu, \tau)_{\mathbb{S}} = \check{G}/\check{G}$ is a single point. Hence the condition that the map $\pi_{K, \mathbb{S}}$ has discrete fibres is equivalent to the condition that $\dim X(\mu, \tau)_K = 0$.

We assume that μ is not central.

5.1. Preparations

We start with some properties of the admissible set.

Lemma 5.1 ([23, Lem. 6.6]). *For any $s \in \mathbb{S}$, $s\tau \in \text{Adm}(\mu)$.*

Lemma 5.2. *Let W be an irreducible Coxeter group and \mathbb{S} be the set of simple reflections. Let $K \subsetneq \mathbb{S}$; then there exists a Coxeter element $c \in {}^K W$.*

Proof. Let Γ be the Coxeter graph associated to the Coxeter system (W, \mathbb{S}) . The vertices are \mathbb{S} . The two simple reflections s, t are connected in Γ if and only if the order of st in W is at least 3. In this case, the edge is labeled by the order of st .

Let $s \in \mathbb{S} - K$. We reorder the simple reflections of \mathbb{S} in the following way: let $r_1 = s$, and for any $i \leq j$, the distance between r_i and s in the graph Γ is less than or equal to the distance between r_j and s . Let n be the cardinality of \mathbb{S} . Set $c = r_1 r_2 \cdots r_n$. Then it is easy to see that for any $i \neq 1$, $r_i r_1 r_2 \cdots r_n$ is a reduced expression of $r_i c$ and thus $r_i c > c$. So $c \in {}^K W$. □

Proposition 5.3. *Suppose that \mathbf{G} is quasi-simple over \check{F} and that μ is noncentral. If $(\check{\Delta}, \mu) \neq (\check{A}_{n-1}, \omega_1^\vee)$ or $(\check{A}_{n-1}, \omega_{n-1}^\vee)$ for some n , then there exists $w \in \text{Adm}(\mu)$ such that $\text{supp}(w\tau^{-1}) = \check{\mathbb{S}}$.*

Remark 5.4.

- (1) Note that $(\check{A}_{n-1}, \omega_1^\vee)$ and $(\check{A}_{n-1}, \omega_{n-1}^\vee)$ are isomorphic. We often mention only one of these two isomorphic pairs.
- (2) In Theorem 11.1, we will prove a stronger statement by a different method. We decided to keep the present proof, because it is simpler and uses only the combinatorics of the affine Weyl group.

Proof. Let w_0 be the longest element in W_0 and $K = \{s \in \mathbb{S} \mid sw_0(\underline{\mu}) = w_0(\underline{\mu})\}$. By [20, Thm. 2.2], we have $\ell(w_K w_0 t^{w_0(\underline{\mu})}) = \ell(t^{w_0(\underline{\mu})}) - \ell(w_K w_0)$ and $w_K w_0 t^{w_0(\underline{\mu})} \in {}^{\mathbb{S}} \check{W}$. Here w_K denotes the longest element in W_K . Then we have

$$\text{supp}(t^{w_0(\underline{\mu})} \tau^{-1}) = \text{supp}(w_K w_0) \cup \text{supp}(w_K w_0 t^{w_0(\underline{\mu})} \tau^{-1}).$$

Since $\underline{\mu}$ is noncentral, we have $K \subsetneq \mathbb{S}$. By Lemma 5.2, there exists an element $c \in W_0$ such that $\ell(cw_K) = \ell(c) + \ell(w_K)$. In particular, $w_K c^{-1} \leq w_0$ and $c^{-1} \leq w_K w_0$. Hence $\text{supp}(w_K w_0) = \mathbb{S}$.

If $\underline{\mu}$ is nonminuscule, we have $w_K w_0 t^{w_0(\underline{\mu})} \tau^{-1} \neq 1$. Since $w_K w_0 t^{w_0(\underline{\mu})} \in {}^{\mathbb{S}} \check{W}$, we have $\check{\mathbb{S}} \setminus \mathbb{S} \subset \text{supp}(w_K w_0 t^{w_0(\underline{\mu})} \tau^{-1})$. Thus $\text{supp}(t^{w_0(\underline{\mu})} \tau^{-1}) = \check{\mathbb{S}}$.

Now we assume that $\underline{\mu}$ is minuscule. Then $t^{w_0(\underline{\mu})} = w_K w_0 \tau$, where $K = \mathbb{S} \setminus \{s\}$ for certain $s \in \mathbb{S}$. Let s_0 be the unique element in $\tilde{\mathbb{S}} \setminus \mathbb{S}$ and $s' = \tau s_0 \tau^{-1} \in \mathbb{S}$. Then we have

$$t^{s_0 w_0(\underline{\mu})} = s_0 w_K w_0 s' \tau.$$

If \tilde{W} is of type \tilde{A}_{n-1} and $\mu \notin \{\omega_1^\vee, \omega_{n-1}^\vee\}$, then by direct computation, $\text{supp}(w_K w_0 s') = \mathbb{S}$ and thus $\text{supp}(t^{s_0 w_0(\underline{\mu})} \tau^{-1}) = \tilde{\mathbb{S}}$. If \tilde{W} is not of type \tilde{A} , then by the explicit formula for the reduced expressions of $w_K w_0$ given in [16, §1.5], we still have $\text{supp}(w_K w_0 s') = \mathbb{S}$ and $\text{supp}(t^{s_0 w_0(\underline{\mu})} \tau^{-1}) = \tilde{\mathbb{S}}$. □

Lemma 5.5. *Let \tilde{W} be the Iwahori–Weyl group of type \tilde{A}_{n-1} . If μ is noncentral and not equal to ω_1^\vee or ω_{n-1}^\vee , then for any $s, s' \in \tilde{\mathbb{S}}$, $ss'\tau \in \text{Adm}(\mu)$.*

Proof. If s commutes with s' , then by Proposition 5.3 there exists $w \in \text{Adm}(\mu)$ such that $s, s' \in \text{supp}(w\tau^{-1})$ and hence $ss' \leq w\tau^{-1}$. So $ss'\tau \leq w$ and $ss'\tau \in \text{Adm}(\mu)$.

Let τ_1 be the automorphism of \tilde{W} sending s_0 to s_1 , s_1 to s_2 , ..., s_{n-1} to s_0 . Then the conjugation action of τ_1 preserves μ and we have $\tau_1 \text{Adm}(\mu) \tau_1^{-1} = \text{Adm}(\mu)$. Since τ_1 acts transitively on $\tilde{\mathbb{S}}$, it suffices to show that there exists j with $0 \leq j \leq n-1$ such that $s_j s_{j+1} \tau, s_{j+1} s_j \tau \in \text{Adm}(\mu)$. Here, by convention, we set $s_n = s_0$.

Let $\kappa : \tilde{W} \rightarrow \mathbb{Z}/n\mathbb{Z}$ be the Kottwitz map (cf. expression (2.5)). Let $i = \kappa(\mu)$. If $i \notin \{0, 1, n-1\}$, then $\mu_+ \geq \omega_i^\vee$. By direct computation, $s_0 s_1 \tau, s_1 s_0 \tau \leq t^{\omega_i^\vee}$ and hence $s_0 s_1 \tau, s_1 s_0 \tau \in \text{Adm}(\omega_i^\vee) \subset \text{Adm}(\mu)$.

If $i = 0$, then $\mu_+ \geq \omega_1^\vee + \omega_{n-1}^\vee$. By direct computation, $s_1 s_2 \tau, s_2 s_1 \tau \leq t^{\omega_1^\vee + \omega_{n-1}^\vee}$ and hence $s_1 s_2 \tau, s_2 s_1 \tau \in \text{Adm}(\omega_1^\vee + \omega_{n-1}^\vee) \subset \text{Adm}(\mu)$.

If $i = 1$ and $\mu_+ \neq \omega_1^\vee$, then $\mu_+ \geq \omega_2^\vee + \omega_{n-1}^\vee$. By direct computation, $s_0 s_1 \tau, s_1 s_0 \tau \leq t^{\omega_2^\vee + \omega_{n-1}^\vee}$ and hence $s_0 s_1 \tau, s_1 s_0 \tau \in \text{Adm}(\omega_2^\vee + \omega_{n-1}^\vee) \subset \text{Adm}(\mu)$.

If $i = n-1$ and $\mu_+ \neq \omega_{n-1}^\vee$, then $\mu_+ \geq \omega_1^\vee + \omega_{n-2}^\vee$. By direct computation, $s_0 s_1 \tau, s_1 s_0 \tau \leq t^{\omega_1^\vee + \omega_{n-2}^\vee}$ and hence $s_0 s_1 \tau, s_1 s_0 \tau \in \text{Adm}(\omega_1^\vee + \omega_{n-2}^\vee) \subset \text{Adm}(\mu)$. □

Proposition 5.6. *Let $K \subsetneq K' \subseteq \tilde{\mathbb{S}}$ be σ -stable. If $s\tau\sigma(s) \in \text{Adm}(\mu)$ for some $s \in K' \setminus K$, then the projection $\pi_{K, K'} : X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$ has nondiscrete fibres.*

Proof. Let \check{K}_s be the standard parahoric subgroup generated by $\check{\mathcal{I}}$ and s . We then have

$$\check{K}_s \cdot_\sigma \check{\mathcal{I}} \tau \check{\mathcal{I}} \subseteq \check{\mathcal{I}} s \check{\mathcal{I}} \tau \check{\mathcal{I}} \sigma(s) \check{\mathcal{I}} \subseteq \check{\mathcal{I}} \tau \check{\mathcal{I}} \cup \check{\mathcal{I}} s \tau \check{\mathcal{I}} \cup \check{\mathcal{I}} \tau \sigma(s) \check{\mathcal{I}} \cup \check{\mathcal{I}} s \tau \sigma(s) \check{\mathcal{I}} \subseteq \check{\mathcal{K}} \text{Adm}(\mu) \check{\mathcal{K}}.$$

By definition, $\tau \in \text{Adm}(\mu)$. By Lemma 5.1, $s\tau, \tau\sigma(s) \in \text{Adm}(\mu)$. By assumption, $s\tau\sigma(s) \in \text{Adm}(\mu)$. Hence $\check{K}_s \check{\mathcal{K}} / \check{\mathcal{K}} \subseteq X(\mu, \tau)_K$, and this is a subset of dimension 1 which maps to a point in $X(\mu, \tau)_{K'}$. □

5.2. Reduction to the case where \mathbf{G} is quasi-simple over \tilde{F}

From now on we assume that condition (1) in either Theorem 4.1 or Theorem 4.2 holds for $K \subsetneq K' \subseteq \tilde{\mathbb{S}}$. We may assume that \mathbf{G} is adjoint, so we can write $\mathbf{G}_{\tilde{F}} = G_1 \times \cdots \times G_d$ for \tilde{F} -simple groups \mathbf{G}_i .

Correspondingly, \tilde{W} is of the form

$$\tilde{W} = \tilde{W}_1 \times \tilde{W}_2 \times \cdots \times \tilde{W}_d,$$

where $\tilde{W}_1 \cong \tilde{W}_2 \cong \cdots \cong \tilde{W}_m$ are the extended affine Weyl groups with connected Dynkin diagram. Since \mathbf{G} is quasi-simple over F , we have (up to renumbering, if necessary) $\sigma(\tilde{W}_1) = \tilde{W}_2, \dots, \sigma(\tilde{W}_{d-1}) = \tilde{W}_d, \sigma(\tilde{W}_d) = \sigma(\tilde{W}_1)$.

Write $\mu = (\mu_1, \dots, \mu_d)$ and $\tau = (\tau_1, \dots, \tau_d)$. Since by assumption μ is noncentral, at least one of the μ_i is noncentral in \tilde{W}_i . Suppose that there is more than one noncentral μ_i . Without loss of generality, we may assume that μ_1 is noncentral in \tilde{W}_1 and that i is the smallest positive integer > 1 such that μ_i is noncentral in \tilde{W}_i . Then $\text{Ad}(\tau_j)$ is the identity group automorphism on \tilde{W}_j for $1 < j < i$.

Let s be a simple reflection of \tilde{W}_1 that is contained in $K' \setminus K$. Let

$$Z = \{(g, \sigma(g), \dots, \sigma^{i-2}(g), 1, \dots, 1) \mid g \in \check{\mathcal{K}}_s\}.$$

Then $Z \subset \check{\mathcal{K}}'$ and $Z\check{\mathcal{K}}/\check{\mathcal{K}} \subset \check{\mathcal{K}}'/\check{\mathcal{K}}$ is 1-dimensional. By direct computation, $Z \cdot_\sigma \tau \subset \check{\mathcal{I}}s\tau\sigma^{i-1}(s)\check{\mathcal{I}}$. By Lemma 5.1, $s\tau_1 \in \text{Adm}(\mu_1)$ and $\tau_i\sigma^{i-1}(s) \in \text{Adm}(\mu_i)$. Therefore $s\tau\sigma^{i-1}(s) \in \text{Adm}(\mu)$. Hence $Z\check{\mathcal{K}}/\check{\mathcal{K}} \subseteq X(\mu, \tau)_K$, and this is a subset of dimension 1 which maps to a point in $X(\mu, \tau)_{K'}$.

It follows that μ_i is noncentral \tilde{W}_i for a unique i , say $i = 1$. We can thus carry out the construction in Section 2.7 and find an algebraic group \mathbf{G}' over F_d and a commutative diagram

$$\begin{array}{ccc} X^{\mathbf{G}}(\mu, \tau)_K & \xrightarrow{\cong} & X^{\mathbf{G}'}(\mu', \tau')_{K_1} \\ \downarrow \pi_{K, K'} & & \downarrow \pi_{K_1, K'_1} \\ X^{\mathbf{G}}(\mu, \tau)_{K'} & \xrightarrow{\cong} & X^{\mathbf{G}'}(\mu', \tau')_{K'_1}. \end{array}$$

It is then enough to show property (2) in Theorem 4.1 or Theorem 4.2, respectively, for the \check{F} -simple group \mathbf{G}' .

5.3. Reduction to the case $(\tilde{A}_{n-1}, \omega_1^\vee)$

Now we assume that \mathbf{G} is quasi-simple over \check{F} . Let $s \in K' \setminus K$. Suppose that the projection $\pi_{K, K'} : X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$ has discrete fibres. By Proposition 5.6, we then have $s\tau\sigma(s) \notin \text{Adm}(\mu)$. We distinguish cases.

Case (I): s commutes with $\tau\sigma(s)\tau^{-1}$.

By Proposition 5.3, if $(\tilde{\Delta}, \mu) \neq (\tilde{A}_{n-1}, \omega_1^\vee)$ or $(\tilde{A}_{n-1}, \omega_{n-1}^\vee)$ for some n , then there exists $w \in \text{Adm}(\mu)$ with $\text{supp}(w\tau^{-1}) = \check{S}$. Hence $s\tau\sigma(s) \leq w$ and $s\tau\sigma(s) \in \text{Adm}(\mu)$: a contradiction.

Case (II): s does not commute with $\tau\sigma(s)\tau^{-1}$.

Then \tilde{W} is of type $\tilde{A}_n, \tilde{C}_{2n+1}$ or \tilde{D}_{2n+1} . If \tilde{W} is of type \tilde{C}_{2n+1} or \tilde{D}_{2n+1} , then $\{s, \tau\sigma(s)\tau^{-1}\} = \{s_n, s_{n+1}\}$. Then by direct computation, $s_n s_{n+1} \tau, s_{n+1} s_n \tau \in \text{Adm}(\mu)$ for any minuscule or quasi-minuscule coweight μ . For general μ , there exists a minuscule

or quasi-minuscule coweight μ' such that $\mu \geq \mu'$. Hence $\text{Adm}(\mu') \subset \text{Adm}(\mu)$ and $s_n s_{n+1} \tau, s_{n+1} s_n \tau \in \text{Adm}(\mu)$: a contradiction. If \tilde{W} is of type \tilde{A}_{n-1} but μ_+ is not ω_1^\vee or ω_{n-1}^\vee , then by Lemma 5.5, $s\tau\sigma(s) \in \text{Adm}(\mu)$: a contradiction.

In summary, we may now assume that $(\tilde{\Delta}, \mu) = (\tilde{A}_{n-1}, \omega_1^\vee)$.

5.4. The case $(\tilde{A}_{n-1}, \omega_1^\vee)$

If $(\tilde{\Delta}, \mu) = (\tilde{A}_{n-1}, \omega_1^\vee)$, then s does not commute with $\tau\sigma(s)\tau^{-1}$. Indeed, assume that s does commute with $\tau\sigma(s)\tau^{-1}$. The maximal elements in $\text{Adm}(\mu)$ are $\tau s_{n-1} s_{n-2} \cdots s_1, \tau s_{n-2} s_{n-3} \cdots s_0, \dots, \tau s_0 s_{-1} \cdots s_{-(n-2)}$. If $s = \tau\sigma(s)\tau^{-1}$, then $s\tau\sigma(s) = \tau \in \text{Adm}(\mu)$: a contradiction to Proposition 5.6. If $s \neq \tau\sigma(s)\tau^{-1}$, then since s commutes with $\tau\sigma(s)\tau^{-1}$, we have $n \geq 3$ and hence $t_1 t_2 \tau \in \text{Adm}(\mu)$ for any $t_1, t_2 \in \tilde{S}$ with $t_1 t_2 = t_2 t_1$. We again have $s\tau\sigma(s) \in \text{Adm}(\mu)$: a contradiction to Proposition 5.6.

We deduce that $\sigma = \text{id}, \sigma = \zeta_0$ (for $n \geq 3$) or $\sigma = \text{Ad}(\tau_{n-2})$. Now $\text{Ad}(\tau_{n-2})$ acts on the affine Dynkin diagram by sending s_2 to s_0, s_3 to s_1, \dots, s_1 to s_{n-1} . By direct computation, if $\sigma = \text{Ad}(\tau_{n-2})$, then $s\tau\sigma(s) \in \text{Adm}(\mu)$: a contradiction to Proposition 5.6.

If $\sigma = \zeta_0$, then $s\tau\sigma(s) \notin \text{Adm}(\mu)$ if and only if $s = s_0$ for n odd and $s = s_0$ or $s = s_m$ for $n = 2m$ even. Now assume that $K' \setminus K \subset \{s_0, s_{\frac{n}{2}}\}$, and let us check Condition 4.3 on (K, K') . We argue by contradiction.

If $s_0 \in K' \setminus K$ and $s_1 \in K$, then $\check{K}_{s_0, s_1} \subset \check{K}'$, where \check{K}_{s_0, s_1} is the standard parahoric subgroup generated by \check{I} and s_0, s_1 . We have

$$\check{I} s_0 \tau \check{I} \subset \check{K}_{s_0, s_1} \cdot_\sigma \tau \subset \check{K}' \cdot_\sigma \tau.$$

Since $s_0 \tau \in \text{Adm}(\mu)$, the set

$$\{g \in \check{K}' / \check{K} \mid g^{-1} \tau \sigma(g) \in \check{K}' \cdot_\sigma \check{I} s_0 \tau \check{I}\}$$

is a one-dimensional subvariety of $X(\mu, \tau)_K$ in the fibre over $\check{K}' / \check{K}' \in X(\mu, \tau)_{K'}$: a contradiction.

If $n = 2m$ is even, $s_m \in K' \setminus K$ and $s_{m+1} \in K$, then $\check{K}_{s_m, s_{m+1}} \subset \check{K}'$ and

$$\check{I} s_m \tau \check{I} \subset \check{K}_{s_m, s_{m+1}} \cdot_\sigma \tau \subset \check{K}' \cdot_\sigma \tau.$$

Since $s_m \tau \in \text{Adm}(\mu)$, the set

$$\{g \in \check{K}' / \check{K} \mid g^{-1} \tau \sigma(g) \in \check{K}' \cdot_\sigma \check{I} s_m \tau \check{I}\}$$

is a one-dimensional subvariety of $X(\mu, \tau)_K$ in the fibre over $\check{K}' / \check{K}' \in X(\mu, \tau)_{K'}$: a contradiction.

6. Proof of (2) \Rightarrow (1) in Theorem 4.2

Similarly as before, we may assume that \mathbf{G} is quasi-simple over \check{F} .

6.1. Compatibility of the map $p_{K, \tau}$

Assume that we are in the following situation:

Situation 6.1. Let (\mathbf{G}, μ) and $K \subsetneq K' \subsetneq \tilde{\mathbb{S}}$ be σ -stable and such that we are in either of the following two cases:

- (The Lubin–Tate case) The associated Coxeter datum is isomorphic to $(\tilde{A}_{n-1}, \text{id}, \omega_1^\vee)$.
- (The exotic case) The associated Coxeter datum is isomorphic to $(\tilde{A}_{n-1}, \zeta_0, \omega_1^\vee)$, $n \geq 3$ and Condition 4.3 is satisfied.

Then by Theorem 3.3, the pair (G, μ) is fully Hodge–Newton decomposable. By Theorem 3.5,

$$X(\mu, \tau)_K = \bigsqcup_{w \in {}^K\text{Adm}(\mu)_0} X_{K,w}(\tau),$$

and we define the map $p_{K,\tau}: X(\mu, \tau)_K \rightarrow {}^K\text{Adm}(\mu)_0$ by mapping all points in $X_{K,w}(\tau)$ to w . We prove the following compatibility result for the maps $p_{K,\tau}$ when K varies:

Theorem 6.2. *Let $(\mathbf{G}, \mu, K \subsetneq K')$ be as in Situation 6.1.*

There exists a unique map $\pi'_{K,K'}: {}^K\text{Adm}(\mu)_0 \rightarrow {}^{K'}\text{Adm}(\mu)_0$ such that the following diagram commutes:

$$\begin{array}{ccc} X(\mu, \tau)_K & \xrightarrow{p_{K,\tau}} & {}^K\text{Adm}(\mu)_0 \\ \pi_{K,K'} \downarrow & & \downarrow \pi'_{K,K'} \\ X(\mu, \tau)_{K'} & \xrightarrow{p_{K',\tau}} & {}^{K'}\text{Adm}(\mu)_0. \end{array}$$

That is, for each EKOR stratum in $X(\mu, \tau)_K$, the projection to $X(\mu, \tau)_{K'}$ is a single EKOR stratum. Moreover, the projection map $\pi_{K,K'}: X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$ has finite fibres.

6.2. Partial conjugation

To give the definition of $\pi'_{K,K'}$, we use the partial conjugation method.

Let $w, w' \in \tilde{W}$ and $s \in \tilde{\mathbb{S}}$. We write $w \xrightarrow{s} w'$ if $w' = sw\sigma(s)$ and $\ell(w') \leq \ell(w)$. Let $K \subset \tilde{\mathbb{S}}$. We write $w \rightarrow_{K,\sigma} w'$ if there exists a sequence $w = w_1, w_2, \dots, w_n = w'$ such that for any k , $w_k \xrightarrow{s} w_{k+1}$ for some $s \in K$. We write $w \approx_{K,\sigma} w'$ if $w \rightarrow_{K,\sigma} w'$ and $w' \rightarrow_{K,\sigma} w$.

Proposition 6.3. *Let $(\mathbf{G}, \mu, K \subsetneq K')$ be as in Situation 6.1. For any $w \in {}^K\text{Adm}(\mu)_0$, there exists a unique $w' \in {}^{K'}\text{Adm}(\mu)_0$ such that $w \approx_{K',\sigma} w'$.*

Proof. The uniqueness of w' follows from [15, Cor. 2.5]. Now we prove the existence.

If σ acts as id on the affine Dynkin diagram, for any $s \in \tilde{\mathbb{S}}$, $\text{supp}_\sigma(s\tau) = \tilde{\mathbb{S}}$. Thus ${}^K\text{Adm}(\mu)_0 = \{\tau\}$ for any K . Now we consider the case where $\sigma = \zeta_0$. Note that the maximal elements in $\text{Adm}(\mu)$ are

$$s_0 s_{n-1} s_{n-2} \cdots s_2 \tau, s_1 s_0 s_{n-1} \cdots s_3 \tau, \dots, s_{n-1} s_{n-2} \cdots s_1 \tau.$$

Therefore,

- (1) if $w \in \text{Adm}(\mu)$, then each simple reflection appears at most once in a reduced expression of $w\tau^{-1}$;
- (2) for any $0 \leq i \leq n-1$, $s_i s_{i+1} \tau \notin \text{Adm}(\mu)$. Here, by convention, we set $s_n = s_0$.

We consider here the case where $n = 2m$ for some $m \geq 2$ and $K' \setminus K = \{s_0, s_m\}$; the other cases follow from a similar (but simpler) argument. Let $w \in {}^K \text{Adm}(\mu)_0$.

If $s_0 w > w$ and $s_m w > w$, then $w \in {}^{K'} \text{Adm}(\mu)_0$ and $w' := w$ is the desired element. If $s_0 w < w$ and $s_m w > w$, then s_0 commutes with s_m and $s_m(s_0 w) > s_0 w$. So $s_0 w \in {}^{K'} \tilde{W}$. Since $s_0 w < w$ and $w \in \text{Adm}(\mu)$, s_1 does not occur in any reduced expression of $w\tau^{-1}$. Thus

$$s_0 w \sigma(s_0) = s_0 w s_0 = s_0(w\tau^{-1})s_1 \tau \in {}^{K'} \tilde{W}$$

and has the same length as w . Moreover, by [12, Lem. 4.5], $s_0 w s_0 \in \text{Adm}(\mu)$. So $w' := s_0 w s_0$ is the desired element.

If $s_0 w > w$ and $s_m w < w$, then by a similar argument $s_m w \in {}^{K'} \tilde{W}$, and $w' := s_m w s_m \in {}^{K'} \text{Adm}(\mu)_0$ is the desired element. If $s_0 w < w$ and $s_m w < w$, then by a similar argument $s_0 s_m w \in {}^{K'} \tilde{W}$, and $w' := s_0 s_m w s_m s_0 \in {}^{K'} \text{Adm}(\mu)_0$ is the desired element. \square

Proof of Theorem 6.2 (existence and uniqueness of $\pi'_{K, K'}$). By Theorem 3.5, we have

$$X(\mu, \tau)_K = \bigsqcup_{w \in {}^K \text{Adm}(\mu)_0} X_{K, w}(\tau),$$

and all $X_{K, w}(\tau)$ in the union of the right-hand side are nonempty. The latter fact says that the map $p_{K, \tau}$ is surjective, so $\pi'_{K, K'}$ is unique, if it exists. We define the map $\pi'_{K, K'} : {}^K \text{Adm}(\mu)_0 \rightarrow {}^{K'} \text{Adm}(\mu)_0$ by $w \mapsto w'$, where w' is the unique element in ${}^{K'} \text{Adm}(\mu)_0$ with $w \approx_{K', \sigma} w'$ (cf. Proposition 6.3). Now for any $g\check{K} \in X_{K, w}(\tau)$, we have $g^{-1}\tau\sigma(g) \in \check{K} \cdot_{\sigma} \check{I} w \check{I} \subset \check{K}' \cdot_{\sigma} \check{I} w' \check{I}$. Therefore $\pi_{K, K'}(g\check{K}) \in X_{K', w'}(\tau)$. This proves the commutativity of the diagram and thus shows the existence of $\pi'_{K, K'}$. \square

6.3. The fibres of the map $\pi'_{K, K'}$

Assume that our Tits datum is $(\tilde{\Delta}, \sigma, \mu) = (\tilde{A}_{n-1}, \zeta_0, \omega_1^\vee)$ for $n \geq 3$, and $K' \setminus K \subset \{s_0, s_{\frac{n}{2}}\}$, and if $s_i \in K' \setminus K$, then $s_{i+1} \notin K$. By the proof of Proposition 6.3, if $K' \setminus K = \{s_j\}$ for $j \in \{0, \frac{n}{2}\}$, then for $w' \in {}^{K'} \text{Adm}(\mu)_0$,

$$(\pi'_{K, K'})^{-1}(w') = \begin{cases} \{w', s_j w' s_j\}, & \text{if } w' s_j < w', \\ \{w'\}, & \text{if } w' s_j > w'. \end{cases}$$

If $n = 2m$ and $K' \setminus K = \{s_0, s_m\}$, then for $w' \in {}^{K'}\text{Adm}(\mu)_0$,

$$(\pi'_{K, K'})^{-1}(w') = \begin{cases} \{w', s_0 w' s_0, s_m w' s_m, s_0 s_m w' s_m s_0\}, & \text{if } w' s_0 < w', w' s_m < w', \\ \{w', s_0 w' s_0\}, & \text{if } w' s_0 < w', w' s_m > w', \\ \{w', s_m w' s_m\}, & \text{if } w' s_0 > w', w' s_m < w', \\ \{w'\}, & \text{if } w' s_0 > w', w' s_m > w'. \end{cases}$$

6.4. The fibres of the map $\pi_{K, K'}$

Next we study the fibres of the map $\pi_{K, K'} : X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$. This will also finish the proof of Theorem 6.2.

Theorem 6.4. *Let $b \in \check{G}$. Let $K \subseteq K' \subsetneq \check{S}$. Let $w \in {}^K \check{W}$ and $w' \in {}^{K'} \check{W}$. If $w \approx_{K', \sigma} w'$, then the natural projection map $X_{K, w}(b) \rightarrow X_{K', w'}(b)$ has finite fibres.*

We first recall the following result, which relates a fine affine Deligne–Lusztig variety in the partial affine flag variety \check{G}/\check{K} to an ordinary affine Deligne–Lusztig variety in another partial affine flag variety:

Theorem 6.5 ([9, Thm. 4.1.2]). *Let $K \subsetneq \check{S}$ and $w \in {}^K \check{W}$. Set*

$$K_1 = I(K, w, \sigma) = \max\{K' \subset K \mid \text{Ad}(w) \circ \sigma(K') = K'\}.$$

Let \check{K}_1 be the associated parahoric subgroup. Then the natural projection map $\check{G}/\check{K}_1 \rightarrow \check{G}/\check{K}$ induces an isomorphism

$$X_{K_1, w}(b) \xrightarrow{\cong} X_{K, w}(b).$$

Note that for $s \in K$, the element $w\sigma(s)w^{-1} \in \check{W}$ is not in general a simple reflection; it is part of the condition in the definition of K_1 that this is the case.

Remark 6.6. Since $\text{Ad}(w) \circ \sigma(K_1) = K_1$, we have $\check{K}_1 \cdot_\sigma \check{I} w \check{I} = \check{K}_1 w \sigma(\check{K}_1)$, and thus $X_{K_1, w}(b) = \{g\check{K}_1 \mid g^{-1}b\sigma(g) = \check{K}_1 w \sigma(\check{K}_1)\}$ is an ordinary affine Deligne–Lusztig variety in \check{G}/\check{K}_1 .

Proposition 6.7. *Let $K \subset \check{S}$ and $w \in {}^K \check{W}$ with $\text{Ad}(w) \circ \sigma(K) = K$. Let $b \in \check{G}$ with $X_w(b) \neq \emptyset$. Then each fibre of the projection map $X_w(b) \rightarrow X_{K, w}(b)$ consists of $\sharp(\check{K}/\check{I})^{\text{Ad}(w) \circ \sigma}$ elements.*

Remark 6.8. Note that \check{K}/\check{I} is the flag variety of the reductive quotient of \check{K} , and $\text{Ad}(w) \circ \sigma$ induces a Frobenius morphism on the reductive quotient of \check{K} . Hence $(\check{K}/\check{I})^{\text{Ad}(w) \circ \sigma}$ is the set of rational points of a full flag variety over the finite field k .

Proof. Let $U_{\check{K}}$ be the pronipotent radical of \check{K} and $\check{K} \cong \check{K}/U_{\check{K}}$ be the reductive quotient of \check{K} . Let \underline{B} be the image of \check{I} in \check{K} . Then \underline{B} is a Borel subgroup of \check{K} . Since $\text{Ad}(w) \circ \sigma(K) = K$, the action of $\text{Ad}(w) \circ \sigma$ stabilises \check{K} and hence is a Frobenius morphism on \check{K} .

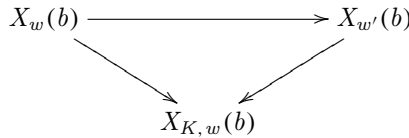
By Lang’s theorem, any element in $\check{K} w \check{K} = \check{K} w$ is of the form $kw\sigma(k)^{-1}$ for some $k \in \check{K}$. Let $g\check{I} \in X_w(b)$. Then the elements in the same fibre as $g\check{I}$ are $gk\check{I}$ for $k^{-1}g^{-1}b\sigma(g)\sigma(k) \in \check{I} w \check{I}$. Note that $g\check{I} \in X_w(b)$. So $g^{-1}b\sigma(g) = uk_1^{-1}w\sigma(k_1)u'$ for some

$k_1 \in \check{K}$ and $u, w' \in U_{\check{K}}$. Thus the condition $k^{-1}g^{-1}b\sigma(g)\sigma(k) \in \check{\mathcal{I}}w\check{\mathcal{I}}$ is equivalent to $\underline{k}^{-1}k_1^{-1}w\sigma(k_1)\sigma(\underline{k}) \in \underline{B}w\sigma(\underline{B})$, where $\underline{k} \in \check{K}$ such that $k \in \underline{k}U_{\check{K}}$. Note that

$$\left\{ \underline{k}B \in \check{K}/B \mid \underline{k}^{-1}k_1^{-1}w\sigma(k_1)\sigma(\underline{k}) \in \underline{B}w\sigma(\underline{B}) \right\} \cong \left\{ \underline{k}B \in \check{K}/B \mid \underline{k}^{-1}w\sigma(\underline{k})w^{-1} \in \underline{B} \right\}.$$

The statement is proved. □

Proposition 6.9. *Let $w, w' \in \check{W}$ and $K \subset \check{S}$ such that $w \approx_{K, \sigma} w'$ and $w \in {}^K \check{W}$. Then there is a commutative diagram*



in which the horizontal arrow is a homeomorphism.

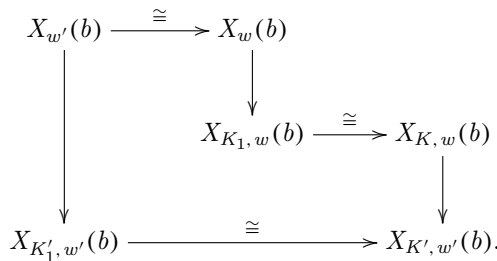
Proof. By definition, there exists a sequence $w = w_1, \dots, w_n = w'$ and $s_1, \dots, s_{n-1} \in K$ such that $\ell(w_1) = \ell(w_2) = \dots = \ell(w_n)$ and $w_{k+1} = sw_k\sigma(s)$ for $1 \leq k \leq n - 1$.

So it suffices to consider the case when $\ell(w') = \ell(w)$ and $w' = sw\sigma(s)$ for some $s \in K$. Without loss of generality, we may assume furthermore that $sw < w$.

By case 1 in the proof of [5, Thm. 1.6] (see also the generalisation to the affine case [8, Proof of Cor. 2.5.3]), for any $g\check{\mathcal{I}}/\check{\mathcal{I}} \in X_w(b)$ there exists a unique element $g'\check{\mathcal{I}}/\check{\mathcal{I}} \in g\check{K}_s/\check{\mathcal{I}}$ such that $g'\check{\mathcal{I}} \in X_{w'}(b)$. Moreover, the map $g\check{\mathcal{I}} \rightarrow g'\check{\mathcal{I}}$ induces a homeomorphism $X_w(b) \rightarrow X_{w'}(b)$. As $g^{-1}g' \in \check{K}_s \subset \check{K}$, the diagram in the statement of the proposition is commutative. □

6.5. Proof of Theorem 6.4

Let $K_1 = I(K, w, \sigma)$ and $K'_1 = I(K', w', \sigma)$. Then we have the following commutative diagram:



Here the vertical maps are the projection maps. The isomorphisms $X_{K_1, w}(b) \cong X_{K, w}(b)$ and $X_{K'_1, w'}(b) \cong X_{K', w'}(b)$ follow from Theorem 6.5. The homeomorphism $X_{w'}(b) \cong X_w(b)$ and the commutativity of the diagram follow from Proposition 6.9. By Proposition 6.7, the maps $X_{w'}(b) \rightarrow X_{K'_1, w'}(b)$ and $X_w(b) \rightarrow X_{K_1, w}(b)$ have finite fibres. Hence the map

$X_{K, w}(b) \rightarrow X_{K', w'}(b)$ has finite fibres. Moreover, each fibre consists of $\frac{\#\check{K}'_1/\check{\mathcal{I}}^{\text{Ad}(w')\circ\sigma}}{\#\check{K}_1/\check{\mathcal{I}}^{\text{Ad}(w)\circ\sigma}}$ elements.

Finally, we determine explicitly, in each of the two cases of Theorem 6.2, the fibres of the map $\pi_{K,K'}: X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$.

6.6. The case $(\tilde{A}_{n-1}, \text{id}, \omega_1^\vee)$

In this case, $\mathbf{G} = \text{PGL}_n$. Note that $\text{Ad}(\tau) \circ \sigma$ acts transitively on $\tilde{\mathbb{S}}$. For any $w \in W_a \tau$, $\text{supp}_\sigma(w) \neq \tilde{\mathbb{S}}$ if and only if $w = \tau$. Thus by Theorem 3.5, $X(\mu, \tau)_K = X_{K,\tau}(\tau)$. We have $X_\tau(\tau) = \Omega\tilde{I}/\tilde{I} \subset \check{G}/\check{I}$ a finite subset consisting of n points. For any parahoric K , $X_{K,\tau}(\tau)$ is the image of $X_\tau(\tau)$ under the natural projection map $\check{G}/\check{I} \rightarrow \check{G}/\check{K}$. Hence $X(\mu, \tau)_K = X_{K,\tau} = \Omega\check{K}/\check{K} \subset \check{G}/\check{K}$ consists of n points. More precisely, in each connected component of \check{G}/\check{K} there is precisely one point of $X(\mu, \tau)_K$. Moreover, for any $K \subsetneq K' \subsetneq \tilde{\mathbb{S}}$, the projection map $X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$ is bijective.

6.7. The case $(\tilde{A}_{n-1}, \zeta_0, \omega_1^\vee)$

We first discuss the case where $K' \setminus K = \{s_0\}$. By assumption, $s_1, s_{n-1} \notin K$ (recall that K is σ -stable). Recall the explicit description of $\text{Adm}(\mu)$ obtained in the proof of Proposition 6.3: the elements of $\text{Adm}(\mu)$ are τ and the elements of the form

$$s_i s_{i-1} \cdots s_{i-i_k} \tau$$

for $0 < i_1 < \cdots < i_r \leq n-2$ (all indices are understood in $\mathbb{Z}/n\mathbb{Z}$, and r could be 0). An element $w\tau \in \text{Adm}(\mu)$ lies in $\text{Adm}(\mu)_0$ if there exists j , $0 \leq j \leq n-1$, such that $j, n-j+1 \notin \text{supp}(w)$.

Let $w \in {}^K\text{Adm}(\mu)_0$ and $w' = \pi'_{K,K'}(w) \in {}^{K'}\text{Adm}(\mu)_0$. The proof of Proposition 6.3 also shows that we have $w' = w$ or $w' = s_0 w s_0$. Hence at most two K -EKOR strata lie above the K' -EKOR stratum attached to w' , and we have two K -EKOR strata above the K' -EKOR stratum attached to w' if and only if $w' \neq s_0 w s_0 \in {}^K\text{Adm}(\mu)$ and $\pi'_{K,K'}(s_0 w s_0) = w'$. Using elementary properties of the Bruhat order and [12, Lem. 4.5], we check that this is equivalent to $w' s_0 < w'$:

$$\pi_{K,K'}^{-1}(X_{K',w'}(\tau)) = \begin{cases} X_{K,w'}(\tau) \sqcup X_{K,s_0 w' s_0}(\tau), & \text{if } w' s_0 < w, \\ X_{K,w'}(\tau), & \text{if } w' s_0 > w. \end{cases}$$

From the explicit description we obtain that $I(K', w', \sigma) = I(K, w, \sigma)$ or $I(K', w', \sigma) = I(K, w, \sigma) \sqcup \{s_0\}$, and that $s_0 \in I(K', w', \sigma)$ if and only if $w' s_0 = s_0 w'$. Since $s_0 w' > w'$ by assumption, in this case we have $w' s_0 > w'$, and the foregoing shows that there is a single K -EKOR stratum above the K' -EKOR stratum for w' .

By the proof of Theorem 6.4, for $g \in X_{K,w'}(\tau)$ we now obtain

$$\sharp \pi_{K,K'}^{-1}(g) = \begin{cases} q+1, & \text{if } I(K', w', \sigma) = I(K, w, \sigma) \sqcup \{s_0\}, \\ 2, & \text{if } w' s_0 < w', \\ 1, & \text{if } w' s_0 > w' \text{ and } I(K', w', \sigma) = I(K, w, \sigma). \end{cases}$$

Here q denotes the cardinality of the residue class field of F .

Let us express the condition $w' s_0 = s_0 w'$ more explicitly, using once again the explicit description of the admissible set in this case.

Claim. $w's_0 = s_0w'$ if and only if $w' \notin W_0\tau$, and in this case $w's_0 > w'$.

To prove the claim, note that for $w' \notin W_0\tau$, the explicit description (and the assumption that $s_0w' > w'$) shows that w' has the form $\cdots s_1s_0\cdots\tau$, whence $s_0w's_0 = \cdots s_0s_1s_0s_1\cdots\tau = w'$. Since $s_0w' > w'$ by assumption, it is also clear that $w's_0 > w'$ in this case. On the other hand, if $w' \in W_0\tau$, then $s_0\tau \leq s_0w'$ but $s_0\tau \not\leq w'\tau^{-1}s_1\tau = w's_0$.

Altogether we have proved the following:

Proposition 6.10. For $w' \in {}^{K'}\text{Adm}(\mu)_0$ and $g \in X_{K,w'}(\tau)$,

$$\sharp\pi_{K,K'}^{-1}(g) = \begin{cases} q + 1, & \text{if and only if } w' \notin W_0\tau, \\ 2, & \text{if and only if } w's_0 < w', \\ 1, & \text{if and only if } w' \in W_0\tau \text{ and } w's_0 > w'. \end{cases}$$

See Proposition 7.9 for a proof of this proposition in terms of lattices.

The case $n = 2m$, $K' \setminus K = \{s_m\}$ is completely analogous to the case $K' \setminus K = \{s_0\}$ we discussed before. Similarly, if $n = 2m$ for $m \geq 2$ and $K' \setminus K = \{s_0, s_m\}$, then for $w' \in {}^{K'}\text{Adm}(\mu)_0$ and $g \in X_{K',w'}(\tau)$, the fibre $\pi_{K,K'}^{-1}(g)$ has $1, 2, 4, q + 1, 2(q + 1)$ or $(q + 1)^2$, depending on which of the conditions $w's_0 > w'$, $w's_m > w'$, $\ell(s_0s_1w') = \ell(w') - 2$ and $\ell(s_ms_{m+1}w') = \ell(w') - 2$ are satisfied.

Example 6.11. Here we consider the case where $(\tilde{\Delta}, \sigma, \mu) = (\tilde{A}_2, \zeta_0, \omega_1^\vee)$. In this case,

$$\text{Adm}(\mu) = \{\tau, s_0\tau, s_1\tau, s_2\tau, s_0s_2\tau, s_1s_0\tau, s_2s_1\tau\}.$$

Let $K = \emptyset$ and $K' = \{s_0\}$. Then

$$\begin{aligned} {}^K\text{Adm}(\mu)_0 &= \{\tau, s_0\tau, s_1\tau, s_2\tau, s_1s_0\tau\}, \\ {}^{K'}\text{Adm}(\mu)_0 &= \{\tau, s_1\tau, s_2\tau, s_1s_0\tau\}. \end{aligned}$$

The map $\pi'_{K,K'}$ sends τ to τ , $s_2\tau$ to $s_2\tau$, both $s_0\tau$ and $s_1\tau$ to $s_1\tau$ and $s_1s_0\tau$ to $s_1s_0\tau$.

Note that $I(K, w, \sigma) = \emptyset$ for $w \in {}^K\text{Adm}(\mu)_0$ and $I(K', w, \sigma) = \emptyset$ for $w = \tau, s_1\tau, s_2\tau$, and $I(K', s_1s_0\tau, \sigma) = K'$. Hence the natural projection map $\pi_{K,K'}$ induces isomorphisms

$$X_{K,\tau}(\tau) \cong X_{K',\tau}(\tau), X_{K,s_2\tau}(\tau) \cong X_{K',s_2\tau}(\tau), X_{K,s_1\tau}(\tau) \cong X_{K',s_1\tau}(\tau), X_{K,s_0\tau}(\tau) \cong X_{K',s_1\tau}(\tau),$$

and the projection map $X_{K,s_1s_0\tau}(\tau) \rightarrow X_{K',s_1s_0\tau}(\tau)$ is a $(q + 1)$ -to-1 map, where $q + 1$ is the cardinality of $(\tilde{K}'/\tilde{K})^{\text{Ad}(s_1s_0\tau) \circ \sigma}$.

In summary, the fibres of the map $\pi_{K,K'} : X^G(\mu, \tau)_K \rightarrow X^G(\mu, \tau)_{K'}$ are as follows:

- (1) over points in $X_{K',\tau}(\tau)$, each fibre consists of one point;
- (2) over points in $X_{K',s_2\tau}(\tau)$, each fibre consists of one point;
- (3) over points in $X_{K',s_1\tau}(\tau)$, each fibre consists of two points;
- (4) over points in $X_{K',s_1s_0\tau}(\tau)$, each fibre consists of $q + 1$ points.

7. Lattice interpretation of the minimal cases

In this section, we give explicit descriptions in terms of lattices for the Lubin–Tate case and the exotic case in which discrete fibres occur. To avoid too-heavy notation, we do not include cases arising by restriction of scalars, but discuss only the nonextended cases.

7.1. The Lubin–Tate case

In this subsection, we explain what $X(\mu, \tau)_K$ looks like in terms of a lattice description in the Lubin–Tate case (Example 2.7), as described in Theorem 4.5. Let us consider first the case where K is a hyperspecial maximal parahoric subgroup. In this case, we have the following description.

Let (N, ϕ) be an isocrystal of dimension n , where ϕ is a σ -linear automorphism isoclinic of slope $1/n$. Then we have (for $\mathbf{G} = \mathrm{GL}_n$)

$$X(\mu, \tau)_K = \bigsqcup_{v \in \mathbb{Z}} \{M \mid M \supset \phi(M), \mathrm{vol}(M) = v\}. \tag{7.1}$$

The decomposition indexed by v corresponds to the decomposition of the affine Grassmannian, or correspondingly the space of all lattices in N , into connected components. Note that after passing to lattices, there is no dependence on K anymore. More precisely, denote by \mathcal{Latt} the set of all lattices in N . Viewing K as the stabiliser of a lattice Λ , we have an identification $\mathrm{GL}_n(\check{F})/K \cong \mathcal{Latt}$ mapping $g \mapsto g\Lambda$. Using this identification, we view $X(\mu, \tau)_K$ as a subset of \mathcal{Latt} . Likewise, we have an identification $\mathrm{GL}_n(\check{F})/\tau K \tau^{-1} \cong \mathcal{Latt}$, now mapping $g \mapsto g\tau\Lambda$, and this is the identification we use when we want to view $X(\mu, \tau)_{\tau K \tau^{-1}}$ as a subset of \mathcal{Latt} . Since the bijection $\mathrm{GL}_n(\check{F})/K \rightarrow \mathrm{GL}_n(\check{F})/\tau K \tau^{-1}$, $g \mapsto g\tau^{-1}$, maps $X(\mu, \tau)_K$ onto $X(\mu, \tau)_{\tau K \tau^{-1}}$, as subsets of \mathcal{Latt} we have $X(\mu, \tau)_K = X(\mu, \tau)_{\tau K \tau^{-1}}$. By iterating this, we can identify the affine Deligne–Lusztig varieties $X(\mu, \tau)_K$ for all standard hyperspecial parahorics K .

Note that for M in $X(\mu, \tau)_K$ the index of $\phi(M)$ in M is equal to 1.

Lemma 7.1. *The chain of lattices*

$$M \supset \phi(M) \supset \phi^2(M) \supset \dots \supset \phi^{n-1}(M) \supset \phi^n(M) = pM$$

determines the unique fixed point under ϕ in $\mathcal{B}(\mathrm{PGL}_n, \check{\mathbb{Q}}_p)$, i.e., the unique point in $\mathcal{B}(\mathbf{J}_{\tau, \mathrm{ad}}, \mathbb{Q}_p)$. In particular, each connected component of $X(\mu, \tau)_K$ consists of a single point.

Proof. All we have to show is that $\phi^n(M) = pM$: after this, the lattice chain determines an alcove in $\mathcal{B}(\mathrm{PGL}_n, \check{\mathbb{Q}}_p)$ which is obviously fixed by ϕ , i.e., lies in $\mathcal{B}(\mathbf{J}_{\tau, \mathrm{ad}}, \mathbb{Q}_p)$. Since $\mathbf{J}_{\tau, \mathrm{ad}}$ is anisotropic, the latter building consists of only one point.

We consider the chain of lattices

$$M \supset \phi(M) \supset \phi^2(M) + pM \supset \phi^3(M) + pM \supset \dots \supset \phi^{n-1}(M) + pM \supset \phi^n(M) + pM.$$

Claim. *All inclusions are strict.*

Once the claim is proved, we conclude as follows. Since obviously all indices in this chain are ≤ 1 , the claim implies that $[M : (\phi^n(M) + pM)] = n = [M : pM]$. Hence $\phi^n(M) + pM = pM$, i.e., $\phi^n(M) = pM$ (both have index n in M).

Proof of claim. Assume that $\phi^r(M) + pM = \phi^{r+1}(M) + pM$. Then $\phi^{r+1}(M) + p\phi(M) = \phi^{r+2}(M) + p\phi(M)$. Hence

$$\phi^{r+1}(M) + pM = \phi^{r+2}(M) + p\phi(M) + pM = \phi^{r+2}(M) + pM.$$

We conclude that $\phi^r(M) + pM = \phi^j(M) + pM$, for any $j \geq r$. But ϕ is topologically nilpotent, hence $\phi^j(M) \subset pM$ for large j . But this implies $\phi^r(M) \subset pM$, which is absurd for $r \leq n - 1$. □

The lemma implies immediately that $X(\mu, \tau)_K$ has only one element when K is an arbitrary parahoric.

7.2. The exotic case

For the setup, we follow [28] (cf. also [3]). The case of hyperspecial level structure (which corresponds, in terms of the notation used in the following, to the case $r = 0$) was analysed in detail by Vollaard [38].

7.2.1. The isocrystal. Let \tilde{F}/F be the unramified quadratic extension contained in \check{F} . We fix $n \geq 1$ and $1 \leq s \leq n - 1$. We also fix the following data:

- (1) N is an \check{F} -vector space of dimension $2n$ together with an alternating \check{F} -bilinear pairing $\langle \cdot, \cdot \rangle : N \times N \rightarrow \check{F}$.
- (2) There is an \tilde{F} -action on N such that

$$\langle a \cdot x, y \rangle = \langle x, \sigma(a) \cdot y \rangle \quad \text{for all } x, y \in N, a \in \tilde{F}. \tag{7.2}$$

- (3) We have a σ -linear operator $\phi : N \rightarrow N$ which commutes with the \tilde{F} -action and such that all slopes of ϕ are equal to $\frac{1}{2}$, and which satisfies

$$\langle \phi(x), \phi(y) \rangle = \pi \cdot \sigma(\langle x, y \rangle) \quad \text{for all } x, y \in N, \tag{7.3}$$

where π is a fixed uniformiser of F .

Via the \tilde{F} -action, N is a module over $\tilde{F} \otimes_F \check{F} = \check{F} \times \check{F}$, i.e., it decomposes as $N = N^0 \oplus N^1$, where \tilde{F} acts on N^0 via the inclusion $\tilde{F} \subset \check{F}$ and on N^1 via $\sigma : \tilde{F} \rightarrow \check{F}$. We then have $\phi(N^0) = N^1$, $\phi(N^1) = N^0$. The \tilde{F} -action on an element $x = (x^0, x^1)$ is given by $a(x^0, x^1) = (ax^0, \sigma(a)x^1)$. By equation (7.2) (and using the fact that the pairing is alternating), we obtain that N^0 and N^1 are totally isotropic subspaces.

We will consider $O_{\tilde{F}}$ -invariant $O_{\tilde{F}}$ -lattices M . For them we obtain an analogous decomposition $M = M^0 \oplus M^1$. We will impose the *signature condition* for s , i.e., $\pi M \subset \phi(M) \subset M$ with

$$\pi M^0 \subset^{n-s} \phi(M^1) \subset^s M^0. \tag{7.4}$$

Here the upper indices indicate the length as $O_{\bar{F}}$ -modules of the corresponding factor modules.

For a lattice $M \subset N$, we denote by M^\vee its dual with respect to the form $\langle \cdot, \cdot \rangle$, i.e., $M^\vee = \{x \in N \mid \langle x, M \rangle \subseteq O_{\bar{F}}\}$.

We will impose the following condition:

- *There exists an $O_{\bar{F}}$ -stable self-dual lattice $\mathbb{M} \subset N$ such that $\pi\mathbb{M} \subset \phi(\mathbb{M}) \subset \mathbb{M}$ and satisfying the signature condition for s .*

In the setting of the following remark, this condition means that the data arise from a p -divisible group (with an $O_{\bar{F}}$ -action and a p -principal polarisation), as in [33]. See Remark 7.6 for a discussion of this assumption in terms of group theory.

Remark 7.2. Let $F = \mathbb{Q}_p$. Then the tuple $(N, \langle \cdot, \cdot \rangle, \phi)$ is the isocrystal of a supersingular p -divisible group of height $2n$ over $\bar{\mathbb{F}}_p$ with \mathbb{Z}_{p^2} -action which satisfies the determinant condition for signature $(s, n - s)$, with a quasi-polarisation compatible with the \mathbb{Z}_{p^2} -action (cf. [38, Def. 1.1]). In [38], p -divisible groups are considered which admit a p -principal polarisation. These correspond to self-dual lattices, i.e., $M^\vee = M$. Here we will consider more general parahoric level structures. In the case of a maximal but nonhyperspecial level structure, the level structure can be seen as a (non- p -principal) polarisation.

7.2.2. The space of lattices. Now let us fix an integer r , $0 \leq r \leq n/2$. We will see how this corresponds to a choice of maximal rational parahoric level structure.

Consider the following set of pairs of lattices in N :

$$\mathcal{F}^{(2r)} = \{(\pi M_2 \subseteq M_1 \subseteq M_2) \mid M_i \text{ stable under } O_{\bar{F}}, M_1^0 \subseteq^{2r} M_2^0, M_1^1 \subseteq^{2r} M_2^1, M_2 = \pi^c M_1^\vee \text{ for some } c \in \mathbb{Z}\}. \tag{7.5}$$

By mapping $(M_1 \subseteq M_2) \in \mathcal{F}^{(2r)}$ to $(M_1^0 \subseteq M_2^0, c)$, we obtain a bijection between $\mathcal{F}^{(2r)}$ and the set

$$\mathcal{F}^{(2r),0} := \{(\pi A \subseteq B \subseteq^{2r} A, c) \mid B, A \subset N^0 \text{ lattices, } c \in \mathbb{Z}\}. \tag{7.6}$$

This set of lattices will be identified later with the set of \bar{k} -points of the corresponding partial affine flag variety.

7.2.3. The action of Frobenius. The operator ϕ on N induces an action on the set $\mathcal{F}^{(2r)}$. In fact, for $(M_1 \subseteq M_2) \in \mathcal{F}^{(2r)}$ with $M_2 = \pi^c M_1^\vee$, we have $\phi(M_2) = \phi(\pi^c M_1^\vee) = \pi^{c+1} \phi(M_1)^\vee$. To describe this action in terms of the bijection $\mathcal{F}^{(2r)} \xrightarrow{\sim} \mathcal{F}^{(2r),0}$, we introduce the following notation.

Let $\tau = \pi^{-1} \phi^2$ be a σ^2 -linear automorphism of N^0 which has all slopes zero. Let $C = (N^0)^{(\tau)}$. Also, let

$$h(x, y) = \delta^{-1} \pi^{-1} \langle x, \phi y \rangle,$$

where $\delta \in O_{\bar{F}}^{\times}$ is such that $\sigma(\delta) = -\delta$. Then the restriction of h to C is a Hermitian form on C . On N^0 , the Hermitian nature of h is given by

$$h(x, y) = \sigma(h(y, \tau^{-1}(x))). \tag{7.7}$$

Definition 7.3. For a lattice $L \subset N^0$, we denote by

$$L^{\sharp} = \{x \in N^0 \mid \pi^{-1}\langle x, \phi(L) \rangle \subseteq O_{\bar{F}}\}$$

the dual of L with respect to the form h , which is again a lattice in N^0 .

Note that

$$(L^{\sharp})^{\sharp} = \tau(L). \tag{7.8}$$

Lemma 7.4. For $(M_1 \subseteq M_2) \in \mathcal{F}^{\{2r\}}$ corresponding to $(B \subseteq A, c) \in \mathcal{F}^{\{2r\}, 0}$, the chain $(\phi(M_1) \subseteq \phi(M_2))$ corresponds to $((\pi^{-c}A)^{\sharp} \subseteq (\pi^{-c}B)^{\sharp}, c+1)$.

Proof. We need to check $\phi(M_1)^0 = (\pi^{-c}M_2^0)^{\sharp}$ and $\phi(M_2)^0 = (\pi^{-c}M_1^0)^{\sharp}$. Now $\phi(M_1)^0 = \phi(M_1^1)$, and

$$\langle \phi(M_1^1), \pi^{-c-1}\phi(M_2^0) \rangle = \sigma(\langle M_1^1, \pi^{-c}M_2^0 \rangle) = \sigma(\langle M_1^1, (M_1^{\vee})^0 \rangle) = O_{\bar{F}}$$

by equation (7.3), so $\phi(M_1)^0 = (\pi^{-c}M_2^0)^{\sharp}$. The computation for $\phi(M_2)^0$ is similar. \square

7.2.4. The parahoric RZ-space. The \bar{k} -valued points of the (relative) RZ-space which we want to describe correspond to those points in $\mathcal{F}^{\{2r\}}$ (or equivalently in $\mathcal{F}^{\{2r\}, 0}$) which are Dieudonné modules of signature $(s, n-s)$:

$$\mathcal{N} = \mathcal{N}^{\{2r\}} = \{(M_1 \subseteq M_2) \in \mathcal{F}^{\{2r\}} \mid \pi M_i \subseteq \phi(M_i) \subseteq M_i, i = 1, 2\}. \tag{7.9}$$

Here $\phi(M_i)^0 \subseteq M_i^0$ has colength s and $\phi(M_i)^1 \subseteq M_i^1$ has colength $n-s$. By Lemma 7.4, we can identify \mathcal{N} with a subset of $\mathcal{F}^{\{2r\}, 0}$, as follows:

$$\mathcal{N} = \{(B \subseteq A, c) \in \mathcal{F}^{\{2r\}, 0} \mid \pi B \subseteq \pi^c A^{\sharp} \subseteq^s B, \pi A \subseteq \pi^c B^{\sharp} \subseteq^s A\}. \tag{7.10}$$

7.2.5. Reduction to the case $c = 0$. We have

$$\mathcal{N} = \bigsqcup_{c \in \mathbb{Z}} \mathcal{N}_c,$$

where for $c \in \mathbb{Z}$ we write

$$\mathcal{N}_c = \{(B \subseteq A \subset N) \mid (B \subseteq A, c) \in \mathcal{F}^{\{2r\}, 0}, \pi B \subseteq \pi^c A^{\sharp} \subseteq B, \pi A \subseteq \pi^c B^{\sharp} \subseteq A\}.$$

Lemma 7.5.

- (1) If nc is odd, then $\mathcal{N}_c = \emptyset$.
- (2) If nc is even, then there exists an automorphism j of N compatible with ϕ and the pairing \langle, \rangle (and hence with the pairing h and the $-^{\sharp}$ construction) such that the map $(B \subseteq A) \mapsto (jB \subseteq jA)$ is an isomorphism $\mathcal{N}_c \cong \mathcal{N}_0$.

Proof. Part (1) follows by a comparison of indices between A, B, A^\sharp, B^\sharp and \mathbb{M} , similarly as in [38, Lem. 1.7]. Part (2) is proved in [38, Lem. 1.17]. \square

From now on we assume $c = 0$, so we consider the set

$$\mathcal{N}_0 = \mathcal{N}_0^{\{2r\}} = \{\pi A \subseteq B \subseteq^{2r} A \subset N^0 \mid \pi B \subseteq A^\sharp \subseteq^s B, \pi A \subseteq B^\sharp \subseteq^s A\}. \tag{7.11}$$

This is the description given in [28] (cf. [3]).³ Note that the Hasse invariant of C is given by $\text{inv}(C) = (-1)^s$.

7.2.6. Nonmaximal level structure. Combining the foregoing data for more than one r , we get analogous descriptions of the RZ-spaces $\mathcal{N}^R, \mathcal{N}_0^R$ with more general parahoric level structure $R \subseteq \{0, \dots, [n/2]\}$. For instance, combining the cases $r = 0$ and $r = 1$, we obtain a nonmaximal parahoric case, given as the set of diagrams

$$\begin{array}{ccccc} B_1 & \subset & B_0 & \subset & A_1 \\ \cup & & \cup & & \cup \\ A_1^\sharp & \subset & B_0^\sharp & \subset & B_1^\sharp. \end{array} \tag{7.12}$$

Here all horizontal inclusions have index 1 and it is understood that $\pi A_1 \subseteq B_1$. The index of the vertical inclusions in this diagram is equal to s .

7.2.7. Description of fibres. From now on we restrict to the case $s = 1$, i.e., to signature $(1, n - 1)$. Let us describe explicitly, in terms of lattices, the projection

$$\mathcal{N}_0^{R \cup \{0\}} \rightarrow \mathcal{N}_0^R$$

for a level structure $R \subseteq \{1, \dots, [n/2]\}$ (i.e., $0 \notin R$) such that $1 \in R$, between spaces with parahoric level structures, which is given by forgetting the lattice at position 0. In terms of the group-theoretic description to be discussed later, this case corresponds to $K' \setminus K = \{s_0\}$. In other words, we need to describe, for a diagram

$$\begin{array}{ccccc} B_1 & \subset & B_0 & \subset & A_1 \\ \cup & & \cup & & \cup \\ A_1^\sharp & \subset & B_0^\sharp & \subset & B_1^\sharp \end{array} \tag{7.13}$$

of lattices in N^0 with all inclusions of index 1 and $\pi A_1 \subseteq B_1$, how many choices there are for B_0 when A_1 and B_1 are fixed. (All the other positions which might be present in R are irrelevant for determining the fibre.)

We distinguish cases, depending on whether $B_1 \subseteq B_1^\sharp$ or not.

First case: $B_1 \not\subseteq B_1^\sharp$. In this case, we have $A_1^\sharp = B_1 \cap B_1^\sharp \supseteq \pi A_1$. Thus A_1/A_1^\sharp is a \bar{k} -vector space with a ‘Hermitian’ form, and $B_0^\sharp/A_1^\sharp \subset B_1^\sharp/A_1^\sharp$ is an isotropic line.

Claim. *There are exactly $q + 1$ such lines.*

³In [3], pairs $M_1 \subset M_2$ are also considered, where $M_1^0 \subset M_2^0$ has *odd* colength.

Proof of claim. By assumption, $A_1/A_1^\sharp = B_1/A_1^\sharp \oplus B_1^\sharp/A_1^\sharp$, and the restriction of the pairing to $B_1^\sharp/A_1^\sharp \times B_1^\sharp/A_1^\sharp$ is nondegenerate. The entirety of all nontrivial subspaces of B_1^\sharp/A_1^\sharp is a projective line. Mapping a line L to $L^\sharp \subset B_1^\sharp/A_1^\sharp$ defines a twisted Frobenius on this projective line over \bar{k} , i.e., a k -structure on this projective line (cf. [38, Lem. 2.12]). The isotropic lines correspond to the rational points with respect to this k -structure. Over a finite field, every form of \mathbb{P}^1 is \mathbb{P}^1 , so there are $q + 1$ points.

Second case: $B_1 \subseteq B_1^\sharp$. In this case, the only possibilities for B_0 are $B_0 = B_1^\sharp$ or $B_0 = \tau^{-1}B_1^\sharp$ (which can equivalently be expressed as $B_0^\sharp = B_1$). In fact, if $B_0 \neq B_1^\sharp$, then $B_0 + B_1^\sharp = A_1$, and similarly, if $B_1 \neq B_0^\sharp$, then $B_1 + B_0^\sharp = B_0$, so from both inequalities together we obtain $B_1^\sharp = B_1 + B_0^\sharp + B_1^\sharp = A_1$, an obvious contradiction.

Depending on whether $B_1 = \tau(B_1)$ or not, we have one or two points in the fibre.

7.2.8. Description of fibres: General case. If n is odd, then the case considered in the previous section is the only possible case. If $n = 2m$ is even, the case of forgetting L_m is completely analogous to the case of forgetting L_0 .

Finally, if n is even, there is the case of forgetting L_0 and L_m . This case corresponds to the case $K' \setminus K = \{s_0, s_m\}$. Since forgetting L_0 and forgetting L_m are independent of each other, the fibres in this case are just products of fibres arising in the case of forgetting one lattice of the chain. In particular, we see that the possible cardinalities of fibres are 1, 2, 4, $q + 1$, $2(q + 1)$ and $(q + 1)^2$.

7.2.9. Connection with group theory. For this subsection, the condition $s = 1$ plays no role. Let V be an n -dimensional \check{F} -vector space with an alternating bilinear form $\langle, \rangle : V \times V \rightarrow F$ such that $\langle av, w \rangle = \langle v, \sigma(a)w \rangle$ for all $a \in \check{F}$, $v, w, \in V$, and let \mathbf{G} be the associated group of similitudes of this pairing (cf. [39, §2.1]). As before, we write $\check{G} = \mathbf{G}(\check{\mathbb{Q}}_p)$. Setting $N = V \otimes_F \check{F}$ and extending the pairing, we obtain a $2n$ -dimensional \check{F} -vector space N with an action of \check{F} and a pairing which satisfy properties (1) and (2) in Section 7.2.1. Conversely, starting with N and a pairing satisfying (1) and (2) and choosing a \check{F} -subvector space $V \subset N$ such that $V \otimes_F \check{F} = N$ and the pairing restricted to $V \times V$ takes values in F , we obtain data as before.

We assume that V contains a self-dual $O_{\check{F}}$ -lattice L_0 , and we fix a self-dual ‘standard lattice chain’ of $O_{\check{F}}$ -lattices in V containing L_0 . This gives us a standard Iwahori subgroup. As in the previous sections, we have the extended affine Weyl group \check{W} , the set \check{S} of simple affine reflections, and so on.

By restricting to part of the standard lattice chain, we can identify each $\mathcal{F}^{(2r)}$ as a quotient of \check{G} by the standard parahoric subgroup of type $K = K^{(r)} = \{0, \dots, n - 1\} \setminus \{r, n - r\}$ if $r > 0$, or $K = K^{(0)} = \{1, \dots, n - 1\}$ if $r = 0$. We obtain analogous identifications for a nonmaximal parahoric level structure.

Now suppose that $N = V \otimes_F \check{F}$ comes equipped with an operator ϕ , as in property (3) of Section 7.2.1. We write $F = b\sigma$, where $b \in \text{GL}(N)$ and $\sigma = \text{id} \otimes \sigma$. Then equation (7.3) amounts to saying that $b \in \check{G}$ with multiplier $c(b) = \pi$. The condition that ϕ be isoclinic is equivalent to requiring that b is basic. Conversely, starting with a basic element $b \in \check{G}$ with multiplier π , we can define $\phi = b\sigma$.

According to the choice of the integer s , $1 \leq s \leq n - 1$, which defines the signature condition, we define the cocharacter $\mu_+ = \omega_s^\vee$. We denote by μ its conjugacy class.

Remark 7.6. To explain the connection with the setup discussed before, we mention the following more specific facts.

- (i) Given the vector space V with the pairing $\langle \cdot, \cdot \rangle$, the existence of a self-dual lattice is equivalent to the existence of a *hyperspecial* parahoric subgroup in \check{G} defined over F . This in turn is equivalent to \mathbf{G} being quasi-split (over F).
- (ii) We have $[b] \in B(\mathbf{G}, \mu)$ if and only if $X(\mu, b)_K \neq \emptyset$ (for any/every K ; see [40]). Since there is a unique basic element in $B(\mathbf{G}, \mu)$, we see that the σ -conjugacy class $[b]$ is uniquely determined by s under the condition $X(\mu, b)_K \neq \emptyset$.
- (iii) The following proposition says that $X(\mu, b)_K \neq \emptyset$ if and only if there exists a self-dual Dieudonné module satisfying the signature condition corresponding to μ_+ . The latter condition is the condition which we imposed in Section 7.2.1.

The map $g = (g^0, g^1) \mapsto (g^0, c(g))$ gives an isomorphism $\mathbf{G}_{\check{F}} \xrightarrow{\sim} \mathrm{GL}(N^0) \times \mathbb{G}_{m, \check{F}}$ of algebraic groups over \check{F} . Via this isomorphism, we can also view $\mathcal{F}^{(2r)}$ as a partial affine flag variety for the group $\mathrm{GL}(N^0) \times \mathbb{G}_{m, \check{F}}$. This corresponds to the identification $\mathcal{F}^{(2r)} = \mathcal{F}^{(2r), 0}$.

Consider the space $\mathcal{N} \subset \mathcal{F}$ as defined previously, for a level structure corresponding to $K \subset \check{\mathbb{S}}$.

Proposition 7.7. *In the setting already outlined,*

$$\mathcal{N}^{(2r)} = X(\mu, b)_{K^{(r)}}$$

as subsets of the corresponding partial affine flag variety \mathcal{F} over \check{F} .

Proof. Inside the partial flag variety, for both these sets, their definition can be expressed by imposing conditions on the relative position between the partial lattice chain and its image under Frobenius. For $\mathcal{N}^{(2r)}$, the condition is that this relative position be μ -permissible in the sense of [26]. For $X(\mu, b)_{K^{(r)}}$, the condition is that it must be μ -admissible. By [26] the two conditions coincide. (Note that because of the identification $\mathbf{G}_{\check{F}} \xrightarrow{\sim} \mathrm{GL}(N^0) \times \mathbb{G}_{m, \check{F}}$, it is enough to know this for GL_n .) □

By analogy with the decomposition $\mathcal{N}^{(2r)} = \sqcup_c \mathcal{N}_c^{(2r)}$, the space $X(\mu, b)_{K^{(r)}}$ decomposes as a union of spaces of the form $X(\mu, b)_{K^{(r)}}$ for a unitary group, rather than a group of unitary similitudes.

The group \mathbf{J}_b , the σ -centraliser of b , can be identified in this context with the unitary similitude group of the Hermitian space C .

7.2.10. Description of fibres and the EKOR stratification. Let us discuss the case of ‘forgetting L_0 ’ with the connection to group theory in mind. As before, we assume $s = 1$. (The other cases can be handled similarly.) Again as before, fix a level structure $R \subseteq \{1, \dots, \lfloor n/2 \rfloor\}$ such that $1 \in R$.

Recall our terminology of KR and EKOR strata (see Section 2.5). In terms of lattices, the KR stratification on the Iwahori level space $\mathcal{N}^{\text{Iw}} \cong X(\mu, b)$ is given by the relative position of L_\bullet and L_\bullet^\sharp . The EKOR stratification on $X(\mu, b)_K$ likewise induces a stratification on the corresponding \mathcal{N} space, which we can describe as the coarsest stratification such that the projection of every KR stratum is a union of EKOR strata (cf. [22, §6.2]). For $w \in {}^K\text{Adm}(\mu)$, the index set for the EKOR stratification, the projection of the KR stratum for w is equal to the EKOR stratum for w , i.e., the partial lattice chains in the EKOR stratum for w are precisely those chains which can be extended to a full lattice chain L_\bullet such that the relative position of L_\bullet and L_\bullet^\sharp is equal to w .

As the standard lattice chain we choose

$$\Lambda_\bullet = \cdots \subset \text{diag}(p, 1, \dots, 1) \subset \text{diag}(1, \dots, 1) \subset \text{diag}(1, \dots, 1, p^{-1}) \subset \cdots,$$

where $\text{diag}()$ denotes a diagonal matrix and a matrix is understood as a lattice by taking the lattice generated by its column vectors.

Let τ be the matrix

$$\begin{pmatrix} & & p \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix},$$

so that $\tau \Lambda_i = \Lambda_{i+1}$. We can also view τ as a length 0 element of the Iwahori–Weyl group of \check{G} .

The simple reflections are given as follows:

$$s_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}, s_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}, \dots, s_0 = \begin{pmatrix} & & & p \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ p^{-1} & & & \end{pmatrix}.$$

Proposition 7.8. *With notation as in diagram (7.13), each of the following conditions describes a union of EKOR strata:*

- (1) $B_1 \subseteq B_1^\sharp, B_1 = \tau(B_1)$.
- (2) $B_1 \subseteq B_1^\sharp, B_1 \neq \tau(B_1)$.
- (3) $B_1 \not\subseteq B_1^\sharp$.

The fibres of the projection $\pi_{K, K'}$ have cardinality 1 in case (1), cardinality 2 in case (2) and cardinality $q + 1$ in case (3).

As before, q denotes the cardinality of the residue class field of F .

Proof. Via our choice of standard lattice chain, the alcove expressions for the identity elements of \tilde{W} and of τ are, respectively,

$$\begin{aligned} \text{alc}(\text{id})_{\bullet} &: \dots, (1^{(2)}, 0^{(n-2)}), (1, 0^{(n-1)}), (0^{(n)}), (0^{(n-1)}, -1), \dots, \\ \text{alc}(\tau)_{\bullet} &: \dots, (1^{(3)}, 0^{(n-3)}), (1^{(2)}, 0^{(n-2)}), (1, 0^{(n-1)}), (0^{(n)}), \dots \end{aligned}$$

Here we use the ‘alcove notation’ of [26]. Similarly, any $w' \in {}^K\text{Adm}(\mu)$ gives rise to such an alcove expression $(\text{alc}(w')_i)$ with each $\text{alc}(w')_i \in \mathbb{Z}^n$, and w' is determined by this datum. The fact that $w' \in \text{Adm}(\mu)$ translates to the condition $\text{alc}(\text{id}) \leq \text{alc}(w') \leq \text{alc}(\text{id}) + (1^{(n)})$, where \leq means that for each index, the respective entries are \leq . The condition $B_1 \subset B_1^\sharp$ translates to $(1, 0^{(n-1)}) \geq \text{alc}(w')_{-1}$, which together with the admissibility implies that $\text{alc}(w')_{-1} = (0^{(n)})$ or $\text{alc}(w')_{-1} = (1, 0^{(n-2)}, -1)$. The latter case is not possible because $w' \in {}^K\tilde{W}$.

Now assume that $B_1 \not\subset B_1^\sharp$; then $\text{alc}(w')_{-1}$ has the form $(0, 0^{(i)}, 1, 0^{(n-i-3)}, -1)$ for some $i \geq 0$. Since these conditions are constant on each KR stratum, and are phrased in terms of the indices 1, -1 of the lattice chain only, they describe unions of EKOR strata.

Now assume that $B_1 \subset B_1^\sharp$, so $\text{alc}(w')_{-1} = (0^{(n)})$. Then $B_1^\sharp = B_0$, so the condition $B_1 = \tau(B_1)$ becomes $B_1 = B_0^\sharp$, which is equivalent to $\text{alc}(w')_0 = (1, 0^{(n-1)})$. Again, this clearly describes a union of EKOR strata. (Note that at this point $B_1 \subset B_1^\sharp$ implies $B_1^\sharp = B_0$, i.e., we do not see the possibility $B_0 = \tau^{-1}B_1^\sharp$ in the second case of Section 7.2.7. This is because we are not considering the full fibre here, only the EKOR strata for w' , for level K and K' .) □

We now recover the characterisation of the loci of different fibre cardinalities as unions of EKOR strata, which we proved group-theoretically as Proposition 6.10. (But note that in the lattice context we did not re-prove Theorem 6.2, because we did not separate the unions of EKOR strata where the fibre cardinality is constant into individual EKOR strata.)

Proposition 7.9. *Fix a point in a parahoric RZ-space \mathcal{N}_0 given by a diagram*

$$\begin{array}{ccccccc} \dots & \subset & B_1 & \subset & A_1 & \subset & \dots \\ & & \cup & & \cup & & \\ \dots & \subset & A_1^\sharp & \subset & B_1^\sharp & \subset & \dots \end{array} \tag{7.14}$$

which lies in the EKOR stratum for $w' \in {}^K\text{Adm}(\mu)$. Then

- $B_1 \subseteq B_1^\sharp$, $B_1 = \tau(B_1)$ if and only if $w' \in W_0\tau$ and $w's_0 > w'$, if and only if the fibre cardinality is 1.
- $B_1 \subseteq B_1^\sharp$, $B_1 \neq \tau(B_1)$ if and only if $w's_0 < w'$, if and only if the fibre cardinality is 2.
- $B_1 \not\subseteq B_1^\sharp$ if and only if $w' \notin W_0\tau$, if and only if the fibre cardinality is $q + 1$.

Proof. First note that by the proof of Proposition 7.8, $B_1 \subseteq B_1^\sharp$ is equivalent to $\text{alc}(w')_{-1} = (0^{(n)})$ or, in other words, $w' \in W_0\tau$. This already proves the third statement. Now if $w' \in W_0\tau$, then $\ell(w')$ is the number of inversions of the permutation $v := w'\tau^{-1}$. We have $B_1 = \tau(B_1)$ if and only if $\text{alc}(w')_0 = (1, 0^{(n-1)})$, if and only if $v(1) = 1$. In this case,

$w's_0 = vs_1\tau$ has length $\ell(w') + 1$. On the other hand, if $v(1) \neq 1$, then by the admissibility of w' , $v(1) = 2$, so $w's_0 = vs_1\tau$ has length $\ell(w') - 1$.

It remains to prove that $w' \notin W_0\tau$ implies $w's_0 > w'$. As we have already found, $w' \notin W_0\tau$ means that $\text{alc}(w')_0 = (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in position $i > 1$. We have $\text{alc}(w')_i = \text{alc}(w's_0)_i$ for all $i \neq 0$, and hence $\text{alc}(w')_1 \leq \text{alc}(w's_0)_0 \leq \text{alc}(w')_{-1}$ and $\text{alc}(w's_0)_0 \neq \text{alc}(w')_{-1}$. Thus the only possibility for $\text{alc}(w's_0)_0$ is $(1, 0, \dots, 0, 1, 0, \dots, 0, -1)$, whence $w's_0 \notin \text{Adm}(\mu)$. This is only possible if $w's_0 > w'$. \square

7.2.11. The EKOR stratification in the case of signature (1,2). In the case $n = 3$, we can describe explicitly all the KR and EKOR strata (cf. Example 6.11). As a preparation, we write down explicitly the KR strata in terms of lattices. In this section, we consider the full affine flag variety for GL_3 over \check{F} . The set of \bar{k} -valued points is the set of full periodic lattice chains L_\bullet . Since all lattice chains are periodic, we usually only consider degrees 1, 0 and -1 .

Lemma 7.10. *Let L_\bullet, L'_\bullet be lattice chains and denote by $\text{inv}(L_\bullet, L'_\bullet) \in \check{W}$ their relative position.*

- (1) $\text{inv}(L_\bullet, L'_\bullet) = \tau$ if and only if $L'_i = L_{i+1}$ for $i = 1, 0, -1$ (equivalently: for all i).
- (2) $\text{inv}(L_\bullet, L'_\bullet) \in \{s_0\tau, \tau\}$ if and only if $L'_1 = L_2 (= \pi L_{-1})$ and $L'_0 = L_1$.
- (3) $\text{inv}(L_\bullet, L'_\bullet) \in \{s_1\tau, \tau\}$ if and only if $L'_1 = L_2 (= \pi L_{-1})$ and $L'_{-1} = L_0$.
- (4) $\text{inv}(L_\bullet, L'_\bullet) \in \{s_2\tau, \tau\}$ if and only if $L'_{-1} = L_0$ and $L'_0 = L_1$.
- (5) $\text{inv}(L_\bullet, L'_\bullet) \in \{s_1s_0\tau, s_0\tau, s_1\tau, \tau\}$ if and only if $L'_1 = L_2 (= \pi L_{-1})$.

The lemma describes all KR strata for $w \in \text{Adm}(\mu)_0$. We omit the easy proof. As a consequence, we obtain the following description of the EKOR strata in $\mathcal{N}_0^{(2)}$. (It is possible to characterise the EKOR strata by other conditions, in the style of the original definition of the EO stratification in the Siegel case – see, for instance, [30]; we have made a choice which is close to the criteria we found earlier for the cardinality of the fibres of the projection from the Iwahori space.)

Proposition 7.11. *A point in $\mathcal{N}_0^{(2)}$, given by a diagram*

$$\begin{array}{ccc}
 B_1 & \subset & A_1 \\
 \cup & & \cup \\
 A_1^\sharp & \subset & B_1^\sharp,
 \end{array} \tag{7.15}$$

lies in the EKOR stratum attached to

- (1) τ if and only if $pA_1 = A_1^\sharp, B_1 \subseteq B_1^\sharp, B_1 = \tau(B_1)$,
- (2) $s_1\tau$ if and only if $B_1 \subseteq B_1^\sharp, B_1 = \tau(B_1)$ (and on this stratum $\pi A_1 = A_1^\sharp$),
- (3) $s_2\tau$ if and only if $\pi A_1 \neq A_1^\sharp$ (and on this stratum $B_1 \subseteq B_1^\sharp, B_1 = \tau(B_1)$),
- (4) $s_1s_0\tau$ if and only if $B_1 \not\subseteq B_1^\sharp$ (and on this stratum $\pi A_1 = A_1^\sharp$).

8. Proof of Theorems 1.1 and 1.2

In this section, we deduce Theorems 1.1 and 1.2 from Theorems 4.1 and 4.2, respectively. Let (\mathbf{G}, μ) be such that \mathbf{G} is quasi-simple and μ noncentral. Write $\mathbf{G} = \text{Res}_{\tilde{F}/F} \tilde{\mathbf{G}}$, for a finite field extension \tilde{F} and an absolutely quasi-simple group $\tilde{\mathbf{G}}$ over \tilde{F} . We also write $\mu = (\mu_\varphi)$, where μ_φ are cocharacters of $\tilde{\mathbf{G}}$. Here φ runs over $\text{Hom}_F(\tilde{F}, \overline{F})$. Let F_d be the maximal unramified subextension of \tilde{F} , $d = [F_d : F]$, and fix an embedding of F_d into \overline{F} . Let $\mathbf{G}_d = \text{Res}_{\tilde{F}/F_d} \tilde{\mathbf{G}}$. Then $\mathbf{G} = \text{Res}_{F_d/F}(\mathbf{G}_d)$, and the Tits datum over F of (\mathbf{G}, μ) is equal to $(\text{Res}_{F_d/F}(\tilde{\Delta}_{\mathbf{G}_d}, \sigma_d), (\underline{\mu}_{d,i})_i)$, where $\tilde{\Delta}_{\mathbf{G}_d}$ is the absolute Dynkin diagram of $\mathbf{G}_d \otimes_{F_d} \tilde{F}$ with its action σ_d of the Frobenius over F_d and where, for $i = 0, \dots, d - 1$, we denote by $\underline{\mu}_{d,i}$ the element in the translation lattice corresponding to $\mu_{d,i} = (\mu_\varphi)_\varphi$. Here φ runs over those elements of $\text{Hom}_F(\tilde{F}, \overline{F})$ whose restriction to F_d is equal to σ^i . Note that $\tilde{\Delta}_{\mathbf{G}_d}$ coincides with the absolute local Dynkin diagram $\tilde{\Delta}_{\tilde{\mathbf{G}}}$ of $\tilde{\mathbf{G}} \otimes_{\tilde{F}} \tilde{F}$, where $\tilde{F} = \tilde{F} \otimes_{F_d} \tilde{F}$ is the completion of the maximal unramified extension of \tilde{F} (cf. [36, §1.13]).

Now let (\mathbf{G}, μ) satisfy the conclusions of Theorems 4.1 and 4.2. In the case of Theorem 4.1, it follows that $(\tilde{\Delta}_{\mathbf{G}_d}, \sigma_d) = (\tilde{A}_{n-1}, \text{id})$. Furthermore, by changing the embedding of F_d into \overline{F} , we deduce from $\underline{\mu}_d = (\omega_1^\vee, 0, \dots, 0)$ that for $i \neq 0$, $\underline{\mu}_{d,i}$ is central and then that $\mu_{d,i}$ is central (cf. Lemma 2.3). From $\underline{\mu}_{d,0} = \omega_1^\vee$, we similarly deduce that there exists a unique $\varphi_0 \in \text{Hom}_{F_d}(\tilde{F}, \overline{F})$ such that $\mu_{\varphi_0} = \omega_1^\vee$ and μ_φ is central for all $\varphi \in \text{Hom}_{F_d}(\tilde{F}, \overline{F}) \setminus \{\varphi_0\}$ (cf. Lemma 2.2 and the table right before [21, Lem. 5.4]). It also follows that $\tilde{\mathbf{G}}_{\text{ad}} = \text{PGL}_n$, and Theorem 1.1 follows.

In the case of Theorem 4.2, and excluding the case treated in Theorem 4.1, it follows that $(\tilde{\Delta}_{\mathbf{G}_d}, \sigma_d) = (\tilde{A}_{n-1}, \zeta_0)$. Analogously to the case just treated, we obtain that there exists a unique $\varphi_0 \in \text{Hom}_F(\tilde{F}, \overline{F})$ such that $\mu_{\varphi_0} = \omega_1^\vee$ and μ_φ is central for all $\varphi \neq \varphi_0$ (cf. Lemma 2.2). It follows that $\tilde{\mathbf{G}}_{\text{ad}}$ is an outer twist of PGL_n by an unramified quadratic extension \tilde{F}' of \tilde{F} . Hence $\tilde{\mathbf{G}}_{\text{ad}} = \text{U}(V)_{\text{ad}}$, for an \tilde{F}'/\tilde{F} -Hermitian vector space V . The condition on (K, K') in Theorem 1.2 follows directly from Theorem 4.1, and implies that the Hermitian space V is split (existence of a lattice which is self-dual or self-dual up to a scalar). Theorem 1.2 is proved.

Part 3. Maximal dimension

In this part, we consider the problem opposite to the one of the previous part: When is $X(\mu, b)_K$ of maximal dimension?

9. Dimension of affine Deligne-Lusztig varieties

9.1. Admissible sets

In this subsection, we introduce a dimension notion for certain subsets of \check{G} . We follow [18, §2.5]. We view \check{G} as the set of \bar{k} -valued points of the loop group of \mathbf{G} and equip it with the ind-topology. Then the closure $\overline{\check{I}x\check{I}}$ is equal to the (perfect) scheme $\bigcup_{x' \leq x} \check{I}x'\check{I}$,

and a subset V is closed if and only if its intersection with $\overline{\check{I}x\check{I}}$ is closed for the Zariski topology, for all $x \in \check{W}$.

A subset V of \check{G} is called *admissible*⁴ if for any $w \in \check{W}$, the set $V \cap \check{I}w\check{I}$ is stable under the right action of an open compact subgroup \check{K}_w which contains a congruence subgroup \check{I}_n of \check{G} . This is equivalent to asking that for any $w \in \check{W}$, the set $V \cap \overline{\check{I}w\check{I}}$ be stable under the right action of an open compact subgroup \check{K}_w which contains a congruence subgroup \check{I}_n of \check{G} . We say that V is *bounded* if $V \cap \check{I}w\check{I} = \emptyset$ for all but finitely many $w \in \check{W}$.

For any compact open subgroup \check{K} of \check{G} , we define

$$\dim_{\check{K}} V = \sup_w \dim((V \cap \overline{\check{I}w\check{I}})/\check{K}_w) - \dim(\check{K}/\check{K}_w),$$

where \check{K}_w is chosen as in the foregoing and such that $\check{K}_w \subseteq \check{K}$.

This definition is applicable in our case because of the following fact:

Theorem 9.1 ([19, Thm. A.1]). *Any σ -conjugacy class in \check{G} is an admissible subset.*

We also recall the following fact. Note that in [18] the notation $X_{K,w}(b)$ has a different meaning than here.

Theorem 9.2 ([18, Thm. 2.23]). *Let $[b] \in B(\mathbf{G})$. Then for every $w \in \text{Adm}(\mu)$,*

$$\dim_{\check{I}}(\check{I}w\check{I} \cap [b]) = \dim X_w(b) + \langle v_b, 2\rho \rangle.$$

Furthermore, for a σ -stable parahoric subgroup \check{K} of \check{G} ,

$$\dim_{\check{K}}(\check{K} \text{Adm}(\mu)\check{K} \cap [b]) = \dim X(\mu, b)_K + \langle v_b, 2\rho \rangle.$$

9.2. Closure relations of fine affine Deligne–Lusztig varieties

We recall from [15, §4] the partial order on ${}^K\check{W}$. Let $w, w' \in {}^K\check{W}$. Then $w' \leq_{K,\sigma} w$ if there exists $x \in W_K$ such that $xw'\sigma(x)^{-1} \leq w$. The relation to the closure relation is given by the following fact:

Theorem 9.3 ([17, Prop. 2.5], [18, Thm. 2.11]). *For $w \in {}^K\check{W}$, the closure of $\check{K} \cdot_{\sigma} \check{I}w\check{I}$ is given as follows:*

$$\overline{\check{K} \cdot_{\sigma} \check{I}w\check{I}} = \bigsqcup_{\{w' \in {}^K\check{W} \mid w' \leq_{K,\sigma} w\}} \check{K} \cdot_{\sigma} \check{I}w'\check{I}.$$

We also need the following fact:

Theorem 9.4 ([18, Thm. 2.5]). *There is a disjoint sum decomposition into locally closed subsets*

$$\check{K} \text{Adm}(\mu)\check{K} = \bigsqcup_{x \in {}^K\text{Adm}(\mu)} \check{K} \cdot_{\sigma} \check{I}x\check{I}.$$

Furthermore, $\dim_{\check{K}}(\check{K} \cdot_{\sigma} \check{I}x\check{I}) = \ell(x)$, for any $x \in {}^K\text{Adm}(\mu)$.

⁴This notion of admissibility is not related to the μ -admissible set.

From these facts we can now deduce the following statement:

Proposition 9.5. *The admissible set $\check{K} \text{Adm}(\mu)\check{K}$ is equidimensional with*

$$\dim_{\check{K}}(\check{K} \text{Adm}(\mu)\check{K}) = \langle \mu, 2\rho \rangle.$$

The irreducible components of $\check{K} \text{Adm}(\mu)\check{K}$ are the $\overline{\check{K}t^\lambda\check{K}} = \overline{\check{K} \cdot_\sigma \check{I}t^\lambda\check{I}}$ for $\lambda \in W_0(\underline{\mu})$ with $t^\lambda \in {}^K\tilde{W}$.

Proof. If $t^\lambda \in {}^K\tilde{W}$, then the maximal element in $W_K t^\lambda W_K$ is $w_K t^\lambda$, where w_K is the longest element in W_K . In this case, $\overline{\check{K}t^\lambda\check{K}} = \overline{\check{I}w_K t^\lambda\check{I}}$ and $\ell(w_K t^\lambda) = \ell(w_K) + \ell(t^\lambda) = \ell(w_K) + \ell(t^\pm)$. Hence $\dim_{\check{K}}(\check{K}t^\lambda\check{K}) = \ell(t^\pm) = \langle \mu, 2\rho \rangle$. Moreover, $\check{K} \cdot_\sigma \check{I}t^\lambda\check{I} \subset \check{K}t^\lambda\check{K}$ and $\dim_{\check{K}}(\check{K} \cdot_\sigma \check{I}t^\lambda\check{I}) = \ell(t^\lambda) = \ell(t^\pm)$. Thus $\overline{\check{K}t^\lambda\check{K}} = \overline{\check{K} \cdot_\sigma \check{I}t^\lambda\check{I}}$.

We have $\check{K} \text{Adm}(\mu)\check{K} = \cup_{\lambda \in W_0(\underline{\mu})} \overline{\check{K}t^\lambda\check{K}}$, and each $\overline{\check{K}t^\lambda\check{K}}$ is irreducible. If $\lambda' \in W_K(\lambda)$, then $\check{K}t^\lambda\check{K} = \check{K}t^{\lambda'}\check{K}$. It remains to show that for any λ , there exists $\lambda' \in W_K(\lambda)$ with $t^{\lambda'} \in {}^K\tilde{W}$.

Let $w \in W_K$ such that $wt^\lambda \in {}^K\tilde{W}$. Then by definition, for any simple root α in K we have that $(wt^\lambda)^{-1}(\alpha) = (t^\lambda)^{-1}w^{-1}(\alpha)$ is a positive root in the affine root system. Hence $\langle -\lambda, w^{-1}(\alpha) \rangle > 0$. This is equivalent to saying that $\langle w(\lambda), \alpha \rangle < 0$. Hence $(t^{w(\lambda)})^{-1}(\alpha)$ is a negative root. Thus $t^{w(\lambda)} \in {}^K\tilde{W}$. This finishes the proof. \square

Corollary 9.6. *The dimension of $X(\mu, b)_K$ is bounded as*

$$\dim X(\mu, b)_K \leq \langle \mu, 2\rho \rangle.$$

If equality holds, then b is basic.

Proof. By Theorem 9.2, we have

$$\begin{aligned} \dim X(\mu, b)_K &= \dim_{\check{K}}(\check{K} \text{Adm}(\mu)\check{K} \cap [b]) - \langle v_b, 2\rho \rangle \\ &\leq \dim_{\check{K}}(\check{K} \text{Adm}(\mu)\check{K}) - \langle v_b, 2\rho \rangle \\ &= \langle \mu, 2\rho \rangle - \langle v_b, 2\rho \rangle, \end{aligned}$$

where we used Proposition 9.5 in the last line. If $\dim X(\mu, b)_K = \langle \mu, 2\rho \rangle$, we have $\langle v_b, 2\rho \rangle = 0$ and thus $[b]$ is the unique basic σ -conjugacy class in $B(\mathbf{G}, \mu)$. \square

Remark 9.7. Whereas $\check{K} \text{Adm}(\mu)\check{K}$ is equidimensional, the corresponding statement is not true for $X(\mu, b)_K$.

10. Statement of results

10.1. Criterion for maximal dimension

We introduce

$$\begin{aligned} W(\underline{\mu})_{K, \text{fin}} &= \{\lambda \in W_0(\underline{\mu}) \mid t^\lambda \in {}^K\tilde{W}, W_{\text{supp}_\sigma(t^\lambda)} \text{ is finite}\} \\ &= \{\lambda \in W_0(\underline{\mu}) \mid t^\lambda \in {}^K\text{Adm}(\mu)_0\}, \end{aligned} \tag{10.1}$$

where we use the notation of equation (3.1) in the first line and of equation (3.3) in the last line. We simply write $W(\underline{\mu})_{\text{fin}}$ for $W(\underline{\mu})_{\emptyset, \text{fin}}$. Note that since t^λ is an element of $\text{Adm}(\mu)$ of maximal length, it is a maximal element of ${}^K\text{Adm}(\mu)_0$ with respect to the partial order $\leq_{K, \sigma}$. The following theorem gives a classification of those cases when equality holds in Corollary 9.6.

Theorem 10.1. *Let \check{K} be a σ -stable parahoric subgroup of \check{G} of type K , and $[b] \in B(\mathbf{G}, \mu)$. If $\dim X(\mu, b)_K = \langle \mu, 2\rho \rangle$, then $[b] = [\tau]$ is basic, \mathbf{J}_τ is quasi-split and $\underline{\mu}$ is minuscule. When \check{K} is an Iwahori subgroup, then the converse holds.*

For general \check{K} , $\dim X(\mu, b)_K = \langle \mu, 2\rho \rangle$ if and only if $[b]$ is basic and $W(\mu)_{K, \text{fin}} \neq \emptyset$. In this case, the irreducible components of $X(\mu, b)_K$ of dimension $\langle \mu, 2\rho \rangle$ are the irreducible components of $\overline{X_{K, t^\lambda}(b)}$, where $\lambda \in W(\mu)_{K, \text{fin}}$.

The proof is given in Section 12.

10.2. Classification of maximal equidimensional cases

The following theorem gives a classification of all cases when $X(\mu, \tau)_K$ is equidimensional of maximal dimension:

Theorem 10.2. *Assume that \mathbf{G} is quasi-simple over F and that μ is not central. Write the Tits datum of (\mathbf{G}, μ) as $(\text{Res}_{F_d/F}(\tilde{\Delta}, \sigma_d), (\mu_1, \dots, \mu_d))$.*

Then $X(\mu, \tau)_K$ is equidimensional of dimension equal to $\langle \mu, 2\rho \rangle$ if and only if we are in one of the following cases:

- (1) *The tuple $(\tilde{\Delta}, \sigma_d)$ is $(\tilde{A}_{n-1}, \varrho_{n-1})$, where ϱ_{n-1} denotes rotation by $n-1$ steps, precisely one μ_i is noncentral (say μ_1) and $\mu_1 = \omega_1^\vee$. Furthermore, $K = \emptyset$.*
- (2) *The tuple $(\tilde{\Delta}, \sigma_d)$ is $(\tilde{A}_3, \varrho_2, \emptyset)$, where ϱ_2 denotes rotation by two steps, precisely one μ_i is noncentral (say μ_1) and $\mu_1 = \omega_2^\vee$. Furthermore, $K = \emptyset$.*
- (3) *The tuple $(\tilde{\Delta}, \sigma_d)$ is $(\tilde{A}_{n-1}, \text{id})$, there exist $i \neq i'$ such that μ_j is central for all $j \neq i, i'$ and $(\mu_i, \mu_{i'}) = (\omega_1^\vee, \omega_{n-1}^\vee)$. Furthermore, $K = \emptyset$.*

The proof is given in Section 13.

Example 10.3. Here we consider the example of Stamm from [35, Thm. 3]. The corresponding Tits datum is $(\tilde{\Delta}, \{\lambda\})$, where $\tilde{\Delta}$ is of type $\tilde{A}_1 \times \tilde{A}_1$, $\tilde{\mathfrak{S}} = \{s_0, s_1, s_{0'}, s_{1'}\}$, $\lambda = ((1, 0), (1, 0))$ and we consider the Iwahori level structure $K = \emptyset$. The Frobenius morphism σ induces a bijective map on $\tilde{\mathfrak{S}}$, which permutes s_0 with $s_{0'}$ and s_1 with $s_{1'}$. Let τ be the length 0 element in \tilde{W} with $\kappa(\tau) = \kappa(\lambda)$. Then the action of $\text{Ad}(\tau)$ on $\tilde{\mathfrak{S}}$ permutes s_0 with s_1 and $s_{0'}$ with $s_{1'}$. Therefore the action of $\text{Ad}(\tau) \circ \sigma$ permutes s_0 with $s_{1'}$ and s_1 with $s_{0'}$. We have

$$\text{Adm}(\mu) = \{\tau, s_0\tau, s_1\tau, s_{0'}\tau, s_{1'}\tau, s_0s_{0'}\tau, s_0s_{1'}\tau, s_1s_{0'}\tau, s_1s_{1'}\tau\}.$$

In this case, $\check{\mathcal{I}} \text{Adm}(\mu) \check{\mathcal{I}} \cap [\tau] = \overline{\check{\mathcal{I}}s_0s_{1'}\tau\check{\mathcal{I}} \cup \check{\mathcal{I}}s_1s_{0'}\tau\check{\mathcal{I}}}$ and $\overline{\check{\mathcal{I}}s_0s_{1'}\tau\check{\mathcal{I}} \cap \check{\mathcal{I}}s_1s_{0'}\tau\check{\mathcal{I}}} = \check{\mathcal{I}}\tau\check{\mathcal{I}}$. Hence $X(\mu, \tau)$ has two irreducible components, both of dimension 2, and their intersection is of dimension 0.

On the other hand, if $K = \{s_0, s_0'\}$, then

$${}^K\text{Adm}(\mu) = \{\tau, s_1\tau, s_1'\tau, s_1s_1'\tau\}.$$

In this case, $\check{K} \text{Adm}(\mu)\check{K} \cap [\tau] = \overline{\check{K} \cdot_{\sigma} \check{I} s_1 \tau \check{I}} \cup \overline{\check{K} \cdot_{\sigma} \check{I} s_1' \tau \check{I}}$ and $\overline{\check{K} \cdot_{\sigma} \check{I} s_1 \tau \check{I}} \cap \overline{\check{K} \cdot_{\sigma} \check{I} s_1' \tau \check{I}} = \check{K} \cdot_{\sigma} \check{I} \tau \check{I}$. Hence $X(\mu, \tau)_K$ has two irreducible components, both of dimension 1, and their intersection is of dimension 0.

Example 10.4. Here we consider the case $(\tilde{A}_{n-1} \times \tilde{A}_{n-1} {}^1\zeta_0, (\omega_1^{\vee}, \omega_{n-1}^{\vee}), \emptyset)$ for $n \geq 3$, where ${}^1\zeta_0$ is the automorphism of $\tilde{A}_{n-1} \times \tilde{A}_{n-1}$ which exchanges the two factors. By Theorem 10.1, if $\check{S} \setminus K$ contains $\{s_i, s_{i+1}, s_{i'}, s_{(i+1)'}\}$ for some i , then $X(\mu, \tau)_K$ has dimension $\langle \mu, 2\rho \rangle$. But only when $K = \emptyset$ is $X(\mu, \tau)_K$ equidimensional of dimension $\langle \mu, 2\rho \rangle$.

11. Critical index set

11.1. Critical index set

Recall that \mathfrak{a} denotes the base alcove. For any $x \in \tilde{W}$, we define the *critical index set* for x by

$$\text{Crit}(x) = \{v \mid v \text{ is a common vertex of } \mathfrak{a} \text{ and } x(\mathfrak{a})\}. \tag{11.1}$$

Note that if $x = w\tau$ for $w \in W_a$ and $\tau \in \Omega$, $\text{Crit}(x) = \text{Crit}(w)$, and this is a nonempty set if and only if $W_{\text{supp}(w)}$ is finite.

11.2. Quasi-rigid set

Let $\tau \in \Omega$, i.e., a length 0 element in \tilde{W} . We introduce the *quasi-rigid set* for τ as follows:

$$\text{Q-Rig}(\tau) = \{w\tau \text{ with } w \in W_a \mid W_{\text{supp}(w)} \text{ is finite}\}. \tag{11.2}$$

In other words, $\text{Q-Rig}(\tau) = \text{Q-Rig}(1)\tau$ consists of all elements x in $W_a\tau$ such that the critical index set for x is nonempty.

For any length-preserving automorphism θ of \tilde{W} , we introduce the *θ -rigid set* for τ :

$$\text{Rig}(\tau, \theta) = \{x \in W_a\tau \mid W_{\text{supp}_{\theta}(x)} \text{ is finite}\} \tag{11.3}$$

(cf. [4]). Note that

$$\begin{aligned} \text{supp}(w) &\subset \text{supp}_{\theta}(w\tau) = \cup_{i \in \mathbb{Z}} (\text{Ad}(\tau) \circ \theta)^i \text{supp}(w), \\ \text{supp}(w) &= \text{supp}_{\text{Ad}(\tau)^{-1}}(w\tau). \end{aligned}$$

Hence

- (1) for any length-preserving automorphism θ of \tilde{W} , $\text{Q-Rig}(\tau) \supset \text{Rig}(\tau, \theta)$; and
- (2) $\text{Q-Rig}(\tau) = \text{Rig}(\tau, \text{Ad}(\tau)^{-1})$.

The following theorem compares ${}^K\text{Adm}(\mu)$ and $\text{Q-Rig}(\tau)$:

Theorem 11.1. *Assume that \tilde{W} is irreducible. Let $K \subset \check{S}$ with W_K finite, i.e., $K \neq \check{S}$. Then ${}^K\text{Adm}(\mu) \subset \text{Q-Rig}(\tau)$ if and only if $(\tilde{\Delta}, \sigma, \mu) = (\tilde{A}_{n-1}, \zeta_1, \omega_1^{\vee})$ (up to isomorphism), in which case ${}^K\text{Adm}(\mu) = \text{Q-Rig}(\tau) \cap {}^K\tilde{W}$.*

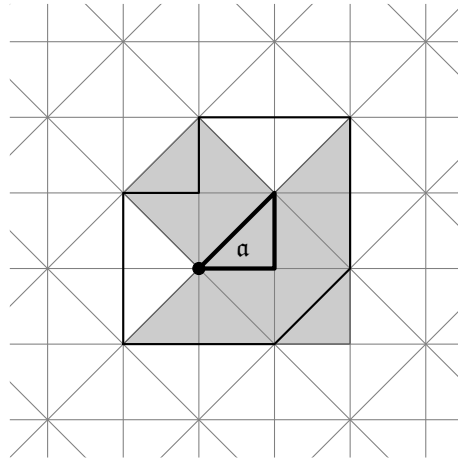


Figure 1. Admissible set (shaded grey) for \tilde{B}_2 , $\mu = \omega_1^\vee$, and quasi-rigid set for $\tau = \tau(t^\mu)$ (inside the thick lines).

Remark 11.2. The case where $K = \emptyset$ is Proposition 5.3. The proof of that proposition does not show the general case, since there are fewer elements in ${}^K\text{Adm}(\mu)$ as K becomes larger. Therefore we have to use more advanced techniques here.

Proof of Theorem 11.1. Let \mathbf{H} be a connected reductive group over F with Iwahori–Weyl group over \tilde{F} isomorphic to \tilde{W} and where the induced action of the Frobenius on \tilde{W} equals $\text{Ad}(\tau)^{-1}$. By item (2) we have $\text{Q-Rig}(\tau) = \text{Rig}(\tau, \sigma)$. Hence, by assumption, for any $x \in {}^K\text{Adm}(\mu)$, $W_{\text{supp}_\sigma(x)}$ is finite. Hence, by Proposition 3.2, $\check{K} \cdot_\sigma \check{I} x \check{I} \subset [\tau]$. By equation (2.12), we see that $X(\mu, b)_K = \emptyset$ if b is not basic. By Theorem 2.4, $B(\mathbf{G}, \mu) = \{[\tau]\}$ is then a singleton. Then by [25, §6], $(\tilde{\Delta}, \mu) = (\tilde{A}_{n-1}, \varsigma_1, \omega_1^\vee)$ (up to isomorphism). \square

Remark 11.3. The concept of a critical index is due to Drinfeld [6]. The fact that in the Drinfeld case $(\tilde{A}_{n-1}, \varsigma_1, \omega_1^\vee)$ any element of ${}^K\text{Adm}(\mu)$ has a critical index is crucial in his proof of p -adic uniformisation of the Drinfeld RZ-space. The proof in [6] is by linear algebra. Note that Theorem 11.1 answers the question raised in [34, §3].

Note that the study of $\text{Q-Rig}(\tau)$ can be reduced to the case where G is adjoint and \tilde{W} is irreducible. The following result describes the translation elements in $\text{Q-Rig}(\tau)$ in the case where \tilde{W} is irreducible:

Proposition 11.4. *Suppose that \tilde{W} is irreducible. Let t^λ be a translation element in \tilde{W} , and let $\tau \in \Omega$ with $t^\lambda \in W_\alpha \tau$. Then $t^\lambda \in \text{Q-Rig}(\tau)$ if and only if there exists a length-preserving automorphism θ of \tilde{W} such that $\theta(\lambda)$ is a dominant minuscule coweight.*

Furthermore, if t^λ is noncentral, then t^λ has exactly one critical index, and the critical index corresponds to a special vertex.

As the proof will show, if \mathbf{G} is adjoint and θ exists, then θ can be chosen as conjugation by a length 0 element of \tilde{W} .

Proof. If $\theta(\lambda)$ is dominant minuscule, then we have $t^{\theta(\lambda)} = \tau' w_K w_0$ for some $\tau' \in \Omega$ and $K \subsetneq \mathbb{S}$. Thus $t^\lambda = \theta^{-1}(\tau' w_K w_0) = \theta^{-1}(\tau')\theta^{-1}(w_K w_0)$. Since $t^\lambda \in W_a \tau$, we have $\theta^{-1}(\tau') = \tau$. In this case, $t^\lambda = \tau\theta^{-1}(w_K w_0) = (\tau\theta^{-1}(w_K w_0)\tau^{-1})\tau$. Moreover, $\text{supp}(\tau\theta^{-1}(w_K w_0)\tau^{-1}) = \text{Ad}(\tau) \circ \theta^{-1}(\mathbb{S})$. Therefore $t^\lambda \in \text{Q-Rig}(\tau)$.

Now we prove the other direction. Suppose that $t^\lambda \in \text{Q-Rig}(\tau)$. Let $\mathfrak{a}' = t^\lambda(\mathfrak{a})$ be the alcove obtained from the base alcove \mathfrak{a} by translation. Then \mathfrak{a} and \mathfrak{a}' have a common vertex, say v .

Note that the vertices of \mathfrak{a} are $\frac{\omega_i^\vee}{\langle \omega_i^\vee, \beta \rangle}$ for $i \in \mathbb{S}$ and 0. Here β is the highest root and ω_i^\vee is the fundamental coweight associated to i . Thus the vertices of \mathfrak{a}' are $\frac{\omega_j^\vee}{\langle \omega_j^\vee, \beta \rangle} + \lambda$ for $j \in \mathbb{S}$ and λ . Then we have one of the following:

- (1) $v = \frac{\omega_i^\vee}{\langle \omega_i^\vee, \beta \rangle}$ and $\lambda = \frac{\omega_i^\vee}{\langle \omega_i^\vee, \beta \rangle} - \frac{\omega_j^\vee}{\langle \omega_j^\vee, \beta \rangle}$ for some $i \neq j \in \mathbb{S}$,
- (2) $v = \lambda = \frac{\omega_i^\vee}{\langle \omega_i^\vee, \beta \rangle}$,
- (3) $v = 0$ and $\lambda = -\frac{\omega_j^\vee}{\langle \omega_j^\vee, \beta \rangle}$ or
- (4) $v = \lambda = 0$.

In case (1), we have $\frac{1}{\langle \omega_i^\vee, \beta \rangle} = \langle \lambda, \alpha_i \rangle \in \mathbb{Z}$ and $\frac{1}{\langle \omega_j^\vee, \beta \rangle} = -\langle \lambda, \alpha_j \rangle \in \mathbb{Z}$, where α_i is the simple root associated to the simple reflection s_i . Thus both ω_i^\vee and ω_j^\vee are minuscule coweights. Hence both v and $v - \lambda$ are special vertices in the base alcove. In cases (2)–(4), we can show by a similar (but easier) argument that v and $v - \lambda$ are still special vertices in the base alcove.

The group of length 0 elements acts transitively on the set of special vertices of \mathfrak{a} , so after applying the length-preserving automorphism of \tilde{W} induced by such an element, we may assume that $v - \lambda$ is the origin in the base alcove. In other words, $v = \lambda$ is a special vertex in the base alcove, and hence λ is a minuscule coweight (recall that we excluded the possibility that λ is central in our assumptions). □

Corollary 11.5. *Assume that \tilde{W} is irreducible and t^λ is non-central. If $K \subset \tilde{\mathbb{S}}$ with $K \supsetneq \text{supp}(t^\lambda \tau^{-1})$, then $K = \tilde{\mathbb{S}}$.*

Proof. By Proposition 11.4, $\text{supp}(t^\lambda \tau^{-1}) = \tilde{\mathbb{S}}$ or $\tilde{\mathbb{S}} \setminus \{s\}$ for some simple reflection s , corresponding to a special vertex. Thus if $K \supsetneq \text{supp}(t^\lambda \tau^{-1})$, then $\text{supp}(t^\lambda \tau^{-1}) = \tilde{\mathbb{S}} \setminus \{s\}$ and $K = \tilde{\mathbb{S}}$. □

12. Maximal dimension

In this section, we prove Theorem 10.1.

12.1. Preparations

The following result gives an explicit description of the set $W(\underline{\mu})_{\text{fin}}$ introduced in Section 10.1:

Proposition 12.1. *Suppose that G is quasi-simple over F , i.e., σ acts transitively on the set of irreducible components of \tilde{W} . Suppose that $\underline{\mu}$ is noncentral in G , i.e., the restriction of $\underline{\mu}$ to some irreducible component of \tilde{W} is noncentral. Then*

$$W(\underline{\mu})_{\text{fin}} = \{\lambda \in W_0(\underline{\mu}) \mid t^\lambda \text{ has an } \text{Ad}(\tau) \circ \sigma\text{-stable critical index}\}.$$

In particular, for any $\lambda \in W(\underline{\mu})_{\text{fin}}$, λ is minuscule, t^λ has a unique $\text{Ad}(\tau) \circ \sigma$ -stable critical index and the corresponding vertex is special.

Proof. Without loss of generality, we may assume that G is adjoint. In this case, $\tilde{W} = \tilde{W}_1 \times \tilde{W}_2 \times \dots \times \tilde{W}_d$ and $\tilde{S} = \tilde{S}_1 \times \tilde{S}_2 \times \dots \times \tilde{S}_d$, where $\tilde{W}_1 \cong \tilde{W}_2 \cong \dots \cong \tilde{W}_d$ are irreducible. We have $\underline{\mu} = (\mu_1, \mu_2, \dots, \mu_d)$. We may assume that μ_1 is noncentral in \tilde{W}_1 . Let $\tau = (\tau_1, \tau_2, \dots, \tau_d)$.

For any subset $K \subseteq \tilde{S}$, W_K is finite if and only if in each component of the Dynkin diagram there is at least one vertex not contained in K . Hence, as we have remarked before, λ has a critical index if and only if $W_{\text{supp}(t^\lambda \tau^{-1})}$ is finite. In the case where case the critical index is unique, we have that $\text{supp}(t^\lambda) = \text{supp}_\sigma(t^\lambda)$ if and only if the critical index is $\text{Ad}(\tau) \circ \sigma$ -stable.

Since $\underline{\mu}$ is noncentral, elements of $W_0(\underline{\mu})$ have at most one critical index, and we obtain that the right-hand side is a subset of $\tilde{W}(\underline{\mu})_{\text{fin}}$.

Conversely, let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d) \in W_0(\underline{\mu})$ be an element in $W(\underline{\mu})_{\text{fin}}$. By Proposition 11.4, μ_1 is minuscule, λ_1 is of the form $\theta_1(\mu_1)$ and t^{λ_1} has a unique critical index. Note that $\text{supp}(t^{\lambda_1} \tau_1^{-1}) = \tilde{S}_1 \setminus \{s_1\}$ for some simple reflection s_1 that corresponds to the critical index of t^{λ_1} . For $1 \leq i \leq d$, let $s_i = (\text{Ad}(\tau) \circ \sigma)^{i-1}(s_1) \in \tilde{W}_i$. Then $\tilde{S} \setminus \{s_1, s_2, \dots, s_d\} \subset \text{supp}_\sigma(t^\lambda)$. Note that for any $K \supseteq \tilde{S} \setminus \{s_1, s_2, \dots, s_d\}$, W_K is an infinite group. Thus we have $\tilde{S} \setminus \{s_1, s_2, \dots, s_d\} = \text{supp}_\sigma(t^\lambda)$. In particular, $(\text{Ad}(\tau) \circ \sigma)^d(s_1) = s_1$. And for each $1 \leq i \leq d$, either λ_i is central or λ_i is minuscule noncentral and s_i is the simple reflection corresponding to the critical index of t^{λ_i} . Hence t^λ has a critical index which corresponds to $s_1 s_2 \dots s_d$. Moreover, by construction, this is the unique $\text{Ad}(\tau) \circ \sigma$ -stable critical index.

The final part follows from Proposition 11.4, or from the equality of the two sets, since all elements of the right-hand side have these properties. □

Proposition 12.2. *The set $W(\underline{\mu})_{\text{fin}}$ is nonempty if and only if \mathbf{J}_τ is quasi-split and $\underline{\mu}$ minuscule.*

Proof. Since $[\tau]$ is basic, \mathbf{J}_τ is an inner form of \mathbf{G} . It is quasi-split if and only if there exists a collection $\Pi \subset \tilde{S}$ of special vertices, one in each connected component of the affine Dynkin diagram, such that $\text{Ad}(\tau) \circ \sigma(\Pi) = \Pi$, i.e., the subset is fixed by the twisted Frobenius corresponding to \mathbf{J}_τ . If $W(\underline{\mu})_{\text{fin}}$ is nonempty, then $\underline{\mu}$ is minuscule and Proposition 12.1 implies that \mathbf{J}_τ is quasi-split.

Conversely, suppose that \mathbf{J}_τ is quasi-split and that $\underline{\mu}$ is minuscule, so that t^μ has a critical index. Applying Proposition 12.1, it is enough to show that with $\Pi \subset \tilde{\mathbb{S}}$ as before, there exists a length-preserving automorphism θ of \tilde{W} and $\lambda \in W_0(\underline{\mu})$ such that $\theta(\lambda) = \underline{\mu}$ and $\Pi = \tilde{\mathbb{S}} \setminus \text{supp}(t^\lambda)$. We may assume that \mathbf{G} is adjoint. Then the subgroup of length 0 elements of \tilde{W} acts transitively on the set of special vertices of the base alcove. Let θ be induced by a length 0 element and such that $\lambda := \theta^{-1}(\underline{\mu})$ satisfies $\Pi = \tilde{\mathbb{S}} \setminus \text{supp}(t^\lambda)$. Then $\lambda \in W_0(\underline{\mu})$ and hence $\lambda \in W(\underline{\mu})_{\text{fin}}$. \square

12.2. Proof of Theorem 10.1

First assume that b is basic and $W(\underline{\mu})_{K, \text{fin}} \neq \emptyset$. By Proposition 3.2, $\check{K} \cdot_\sigma \check{I} t^\lambda \check{I} \subset [\tau]$ for $\lambda \in W(\underline{\mu})_{K, \text{fin}}$. By Theorems 9.2 and 9.4, we see that $\dim X(\underline{\mu}, b)_K = \langle \underline{\mu}, 2\rho \rangle$. For $K = \emptyset$, if \mathbf{J}_τ is quasi-split and $\underline{\mu}$ is minuscule, then Proposition 12.2 shows $W(\underline{\mu})_{\text{fin}} \neq \emptyset$ and hence $\dim X(\underline{\mu}, b)_K = \langle \underline{\mu}, 2\rho \rangle$.

Now suppose that $\dim X(\underline{\mu}, b)_K = \langle \underline{\mu}, 2\rho \rangle$. By Corollary 9.6, $[b] = [\tau]$ is basic. We next claim that the irreducible components of $X(\underline{\mu}, \tau)_K$ of dimension $\langle \underline{\mu}, 2\rho \rangle$ are the irreducible components of the $\overline{X_{K, t^\lambda}(\tau)}$ of dimension $\langle \underline{\mu}, 2\rho \rangle$, where $\lambda \in W(\underline{\mu})_{K, \text{fin}}$. Indeed, by equation (2.12),

$$X(\underline{\mu}, \tau)_K = \bigsqcup_{x \in {}^K\text{Adm}(\underline{\mu})} X_{K, x}(\tau).$$

Now for $x \in {}^K\text{Adm}(\underline{\mu})$, $\dim X_{K, x}(\tau) \leq \dim X_x(\tau) = \dim_{\check{I}}(\check{I}x\check{I} \cap [\tau]) \leq \dim_{\check{I}}(\check{I}x\check{I}) = \ell(x)$, using Theorem 9.2 for the first and Theorem 9.4 for the final equality, which proves the claim. In particular, $W(\underline{\mu})_{K, \text{fin}} \neq \emptyset$. On the other hand, $\overline{X_{K, t^\lambda}(\tau)}$ is equidimensional. In fact, $X_{K, t^\lambda}(\tau)$ is a disjoint union of copies of a classical Deligne–Lusztig variety by [10, Prop. 5.7] and [9, Thm. 4.1.1, Thm. 4.1.2].

Finally, the map $X(\underline{\mu}, b) \rightarrow X(\underline{\mu}, b)_K$ is surjective (cf. [19, Thm. 1.1]). Hence we deduce from $\dim X(\underline{\mu}, b)_K = \langle \underline{\mu}, 2\rho \rangle$ that $\dim X(\underline{\mu}, b) = \langle \underline{\mu}, 2\rho \rangle$. The previous reasoning applied to $K = \emptyset$ implies $W(\underline{\mu})_{\text{fin}} \neq \emptyset$, and hence we deduce from Proposition 12.2 that \mathbf{J}_τ is quasi-split and $\underline{\mu}$ minuscule. Theorem 10.1 is proved. \square

Remark 12.3. For any $(\mathbf{G}, \underline{\mu})$ such that $\underline{\mu}$ is minuscule, there exists an inner form \mathbf{H} of \mathbf{G} such that $\dim X^{\mathbf{H}}(\underline{\mu}, \tau) = \langle \underline{\mu}, 2\rho \rangle$, namely the one with Frobenius $\text{Ad}(\tau) \circ \sigma$. In particular, this applies when \mathbf{G} splits over \bar{F} , because then $\underline{\mu} = \mu$.

13. Maximal equidimension

In this section, we prove Theorem 10.2.

13.1. Reduction to the fully Hodge–Newton decomposable case

Suppose that $X(\underline{\mu}, b)_K$ is equidimensional of dimension equal to $\langle \underline{\mu}, 2\rho \rangle$. By Theorem 10.1, $[b] = [\tau]$ is basic and

$$X(\underline{\mu}, \tau)_K = \bigcup_{\lambda \in W(\underline{\mu})_{K, \text{fin}}} \overline{X_{K, t^\lambda}(\tau)}.$$

We claim that (\mathbf{G}, μ) is of fully Hodge–Newton decomposable type. In fact, by Theorem 3.1 it is enough to show that whenever $w \in {}^K\text{Adm}(\mu)$ satisfies $X_{K,w}(\tau) \neq \emptyset$, then $W_{\text{supp}_\sigma(w)}$ is finite. But then $X_{K,w}(\tau) \subseteq X(\mu, \tau)_K$ and the equation gives $X_{K,w}(\tau) \subseteq \overline{X_{K,t^\lambda}(\tau)}$ for some $\lambda \in W(\underline{\mu})_{K,\text{fin}}$. Now Theorem 9.3 shows that

$$\overline{\check{\mathcal{K}} \cdot_\sigma \check{\mathcal{I}} t^\lambda \check{\mathcal{I}}} = \bigsqcup_{\{x \in {}^K \check{W} \mid x \leq_{K,\sigma} t^\lambda\}} \check{\mathcal{K}} \cdot_\sigma \check{\mathcal{I}} x \check{\mathcal{I}},$$

and this implies that

$$\overline{X_{K,t^\lambda}(\tau)} \subseteq \bigsqcup_{\{x \in {}^K \check{W} \mid x \leq_{K,\sigma} t^\lambda\}} X_{K,x}(b).$$

We obtain that $w \leq_{K,\sigma} t^\lambda$ for some $\lambda \in W(\underline{\mu})_{K,\text{fin}}$. This implies $\text{supp}_\sigma(w) \subseteq \text{supp}_\sigma(t^\lambda)$, so $W_{\text{supp}_\sigma(w)}$ is finite.

Hence, by Theorem 3.5,

$$X(\mu, \tau)_K = \bigsqcup_{x \in {}^K \text{Adm}(\mu)_0} X_{K,x}(b).$$

In particular, we have that $X(\mu, b)_K$ is equidimensional of dimension equal to $\langle \mu, 2\rho \rangle$ if and only if the following condition is satisfied:

(★) *The set of maximal elements of ${}^K\text{Adm}(\mu)_0$ with respect to the partial order $\leq_{K,\sigma}$ is equal to $\{t^\lambda \mid \lambda \in W(\underline{\mu})_{K,\text{fin}}\}$.*

We first check which cases satisfy (★) under the additional assumption that μ is noncentral in every irreducible component: in Sections 13.2 and 13.3 we go through the irreducible cases, and in Section 13.4 we check the remaining case, the Hilbert–Blumenthal case. Finally, in Section 13.5 we explain how to deduce the general case where μ is allowed to have central components.

13.2. Candidates for the irreducible cases

We first consider the case where \check{W} is irreducible. Since $X(\mu, \tau)_K$ has dimension $\langle \mu, 2\rho \rangle$, we have $W(\underline{\mu})_{K,\text{fin}} \neq \emptyset$. By Proposition 12.1, $\text{Ad}(\tau) \circ \sigma$ fixes a special vertex in the affine Dynkin diagram of \check{W} . The fully Hodge–Newton decomposable cases with \check{W} irreducible and where $\text{Ad}(\tau) \circ \sigma$ fixes a special vertex can be extracted from the table in Theorem 3.3 and are as follows (see the explanation after Theorem 3.4 for the notation):

- (i) $(\check{A}_{n-1}, \varrho_{n-1}, \omega_1^\vee)$ for $n \geq 2$,
- (ii) $(\check{A}_{2m}, \varsigma_0, \omega_1^\vee)$ for $m \geq 1$,
- (iii) $(\check{A}_3, \varsigma_0, \omega_2^\vee)$,
- (iv) $(\check{A}_3, \varrho_2, \omega_2^\vee)$,
- (v) $(\check{B}_n, \text{Ad}(\tau_1), \omega_1^\vee)$ for $n \geq 3$,

- (vi) $(\tilde{C}_2, \text{Ad}(\tau_2), \omega_2^\vee)$,
- (vii) $(\tilde{D}_n, \zeta_0, \omega_1^\vee)$ for $n \geq 4$.

Next we check when (\star) is satisfied.

13.3. Case-by-case analysis

13.3.1. $(\tilde{A}_{n-1}, \text{Ad}(\tau_{n-1}), \omega_1^\vee)$ for $n \geq 2$. Here the only possible K is \emptyset and $\check{K} = \check{I}$. This is the Drinfeld case, and $B(\mathbf{G}, \mu)$ consists of a single element, namely $[\tau]$. In this case, $\check{I} \text{Adm}(\mu) \check{I} \subset [\tau]$ and $X(\mu, \tau)$ is equidimensional of dimension equal to $\langle \mu, 2\rho \rangle$.

13.3.2. $(\tilde{A}_{2m}, \zeta_0, \omega_1^\vee)$ for $m \geq 1$. In this case, $\tilde{S}^{\text{Ad}(\tau)\circ\sigma} = \{s_{m+1}\}$. Thus the only translation element in $\text{Adm}(\mu)_0$ is t^λ , where $\lambda = \text{Ad}(\tau_n)(\omega_1^\vee) \in \tilde{S} \setminus \{s_m\} \tilde{W}$ and $\text{supp}(t^\lambda \tau^{-1}) = \tilde{S} \setminus \{s_{m+1}\}$. Therefore if $\lambda \in W(\underline{\mu})_{K, \text{fin}}$, then $K \subset \tilde{S} \setminus \{s_m\}$. Since $K = \sigma(K)$, we have $K \subset \tilde{S} \setminus \{s_m, s_{m+1}\}$. In this case, $s_{m+1}\tau \in {}^K \text{Adm}(\mu)_0$ and $s_{m+1}\tau \not\prec_{K, \sigma} t^\lambda$. This contradicts (\star) .

13.3.3. $(\tilde{A}_3, \zeta_0, \omega_2^\vee)$. In this case, $\tilde{S}^{\text{Ad}(\tau)\circ\sigma} = \{s_1, s_3\}$. Thus the only translation elements in $\text{Adm}(\mu)_0$ are $s_1 s_2 s_0 s_1 \tau$ and $s_3 s_2 s_0 s_3 \tau$. Therefore if $W(\underline{\mu})_{K, \text{fin}} \neq \emptyset$, then $s_1 \notin K$ or $s_3 \notin K$. Since $K = \sigma(K)$, both s_1 and s_3 are not in K . In this case, $s_1 s_3 \tau \in {}^K \text{Adm}(\mu)_0$, $s_1 s_3 \tau \not\prec_{K, \sigma} s_1 s_2 s_0 s_1 \tau$ and $s_1 s_3 \tau \not\prec_{K, \sigma} s_3 s_2 s_0 s_3 \tau$. This contradicts (\star) .

13.3.4. $(\tilde{A}_3, \text{Ad}(\tau_2), \omega_2^\vee)$. We first consider the case where $K = \emptyset$. In this case, the maximal elements in ${}^K \text{Adm}(\mu)_0$ are $s_2 s_1 s_3 s_2 \tau$, $s_3 s_2 s_0 s_3 \tau$, $s_0 s_1 s_3 s_0 \tau$ and $s_1 s_2 s_0 s_1 \tau$, and (\star) is satisfied.

If $K = \{s_0, s_2\}$, then the maximal elements in ${}^K \text{Adm}(\mu)_0$ are $s_3 s_2 s_0 s_3 \tau$, $s_1 s_2 s_0 s_1 \tau$, $s_1 s_3 s_0 \tau$ and $s_1 s_3 s_2 \tau$. This contradicts (\star) .

13.3.5. $(\tilde{B}_n, \text{Ad}(\tau_1), \omega_1^\vee)$ for $n \geq 3$. By Proposition 12.1, $W(\underline{\mu})_{\text{fin}} = \{\omega_1^\vee, \text{Ad}(\tau_1)(\omega_1^\vee)\}$. Note that

$$t^{\omega_1^\vee} \in {}^S \tilde{W} \text{ and } \text{supp}(t^{\omega_1^\vee} \tau^{-1}) = \tilde{S} \setminus \{s_1\};$$

$$t^{\text{Ad}(\tau_1)(\omega_1^\vee)} \in \tilde{S} \setminus \{s_1\} \tilde{W} \text{ and } \text{supp}(t^{\text{Ad}(\tau_1)(\omega_1^\vee)} \tau^{-1}) = S.$$

Thus if ${}^K \text{Adm}(\mu)_0$ contains some of these translation elements and $K = \sigma(K)$, then $K \subset \tilde{S} \setminus \{s_0, s_1\}$. In this case, $s_0 s_1 \tau \in {}^K \text{Adm}(\mu)_0$, $s_0 s_1 \tau \not\prec_{K, \sigma} t^{\omega_1^\vee}$ and $s_0 s_1 \tau \not\prec_{K, \sigma} t^{\text{Ad}(\tau_1)(\omega_1^\vee)}$. This contradicts (\star) .

13.3.6. $(\tilde{C}_2, \text{Ad}(\tau_2), \omega_2^\vee)$. In this case, $\tilde{S}^{\text{Ad}(\tau)\circ\sigma} = \{s_0, s_1, s_2\}$. The only translation elements in $\text{Adm}(\mu)_0$ are $s_0 s_1 s_0 \tau$ and $s_2 s_1 s_2 \tau$. Therefore if $W(\underline{\mu})_{K, \text{fin}} \neq \emptyset$, then $s_0 \notin K$ or $s_2 \notin K$. Since $K = \sigma(K)$, both s_0 and s_2 are not in K . In this case, $s_0 s_2 \tau \in {}^K \text{Adm}(\mu)_0$, $s_0 s_2 \tau \not\prec_{K, \sigma} s_0 s_1 s_0 \tau$ and $s_0 s_2 \tau \not\prec_{K, \sigma} s_2 s_1 s_2 \tau$. This contradicts (\star) .

13.3.7. $(\tilde{D}_n, \zeta_0, \omega_1^\vee)$ for $n \geq 4$. In this case, the special vertices that are fixed by $\text{Ad}(\tau) \circ \sigma$ are $n - 1$ and n . By Propositions 11.4 and 12.1, the elements of $W(\underline{\mu})_{\text{fin}}$ are of the

form $\theta(\underline{\mu})$, where θ runs over a length-preserving automorphism such that $\theta \circ \text{Ad}(\tau)(\mathbb{S})$ is $\text{Ad}(\tau) \circ \sigma$ -stable. In this case, θ sends the vertices $\{0, 1\}$ to the vertices $\{n - 1, n\}$. We have that $K \subset \tilde{\mathbb{S}} \setminus \{s_{n-1}\}$ or $K \subset \tilde{\mathbb{S}} \setminus \{s_n\}$. Since $K = \sigma(K)$, we have $K \subset \tilde{\mathbb{S}} \setminus \{s_{n-1}, s_n\}$. Then we have $s_{n-1}s_n\tau \in {}^K\text{Adm}(\mu)_0$. On the other hand, we have $\text{supp}(t^{\theta(\underline{\mu})}\tau^{-1}) \subset \tilde{\mathbb{S}} \setminus \{s_{n-1}\}$ or $\text{supp}(t^{\theta(\underline{\mu})}\tau^{-1}) \subset \tilde{\mathbb{S}} \setminus \{s_n\}$. Thus $s_{n-1}s_n\tau \notin {}^K_{K,\sigma} t^{\theta(\underline{\mu})}$. This contradicts (\star) .

13.4. The reducible case

We consider the case where \tilde{W} is reducible (cf. Theorem 3.4). Let us first assume that μ is noncentral in each factor, so it is of type $(\tilde{A}_{n-1} \times \tilde{A}_{n-1}, {}^1\zeta_0, (\omega_1^\vee, \omega_{n-1}^\vee))$. There are two copies of the affine Dynkin diagram of type \tilde{A}_{n-1} , and we label the vertices by i and i' , respectively, where $i, i' \in \mathbb{Z}/n\mathbb{Z}$. The Frobenius σ acts by ${}^1\zeta_0$, which exchanges the vertex i with i' for any i . The $\text{Ad}(\tau) \circ \sigma$ -orbits on $\tilde{\mathbb{S}}$ are $\{s_i, s_{(i-1)'}\}$ for $i \in \mathbb{Z}/n\mathbb{Z}$. If $K = \emptyset$, then the maximal elements in ${}^K\text{Adm}(\mu)_0$ are $(s_i s_{i-1} \cdots s_{i-n+2})(s_{(i-n+1)'} \cdots s_{(i-2)'} s_{(i-1)'})\tau$ for $i \in \mathbb{Z}/n\mathbb{Z}$. They are all translation elements. Hence (\star) is satisfied.

Now suppose that $K \neq \emptyset$. Without loss of generality, we may assume that $\{s_0, s_{0'}\} \subset K$. Then $(s_{n-1}s_{n-2} \cdots s_2)(s_{1'}s_{2'} \cdots s_{(n-1)'})\tau$ is a maximal element in ${}^K\text{Adm}(\mu)_0$. This contradicts (\star) .

13.5. The general case

Finally, let us reduce the general case to the case where μ is noncentral in each component. Given (\mathbf{G}, μ) , we may assume that \mathbf{G} is adjoint, and we construct (\mathbf{G}', μ') as in Section 2.7. Since we have already shown that (\mathbf{G}, μ) is fully Hodge–Newton decomposable, μ is minute. This implies that μ' is minute, and hence we see that the Dynkin type of (\mathbf{G}', μ') is one of the types in Theorem 3.3. The only possibilities for (\mathbf{G}, μ) then are the following:

- All μ_φ , except for one, are central, and the component where μ is noncentral is as in Theorem 3.3.
- All μ_φ , except for two, are central, and the two components where μ is noncentral give rise to the Hilbert–Blumenthal case $(\tilde{A}_{n-1} \times \tilde{A}_{n-1}, (\omega_1^\vee, \omega_{n-1}^\vee))$.

The components where μ is central do not contribute to the set ${}^K\text{Adm}(\mu)_0$, so the analysis of whether (\star) is satisfied is exactly the same as in the previous sections.

14. Lattice interpretation of the maximal equidimensional cases

In this section, we go through the list in Theorem 10.2 under the assumption that $\underline{\mu}$ is noncentral in each factor of \tilde{W} and give lattice interpretations of $X(\mu, \tau)_K$ in each case.

14.1. The Drinfeld case

Let (N, ϕ) be an \check{F} -vector space of dimension n , equipped with a σ -linear automorphism isoclinic of slope 0. Then we have

$$X(\mu, \tau)_K = \bigsqcup_{v \in \mathbb{Z}} \{M_\bullet \mid M_{i+1} \supset \phi(M_i), \forall i, \text{vol}(M_0) = v\}. \tag{14.1}$$

Here M_\bullet is a periodic $O_{\bar{F}}$ -lattice chain with period n . The decomposition indexed by v corresponds to the decomposition of the affine flag variety into connected components.

In this case, we can identify the set in equation (14.1) as the set of points of a π -adic formal scheme, as follows. We fix an embedding of F into an algebraic closure $\bar{\mathbb{Q}}_p$ of \mathbb{Q}_p . Let B be a central division algebra over F with invariant $1/n$. Up to isomorphism, there is a unique *special formal O_B -module* of F -height n^2 over $\bar{\mathbb{F}}_p$ (cf. [33, Lem. 3.60]). Taking this as a *framing object* over $\bar{\mathbb{F}}_p$, we obtain a formal scheme \mathcal{N} over $\mathrm{Spf} O_{\bar{F}}$ which represents the functor of special formal O_B -modules together with a quasi-isogeny framing (cf. [33, § 3.59]). It is a π -adic formal scheme [33, Prop. 3.62] which is flat over $O_{\bar{F}}$ [33, § 3.69, Thm. 3.72]. Then the set in equation (14.1) can be identified with $\mathcal{N}(\bar{\mathbb{F}}_p)$, and the elements of the set in equation (14.1) for $v = 0$ with the connected component $\mathcal{N}^o(\bar{\mathbb{F}}_p)$ of height zero elements. Indeed, let \tilde{F}/F be an unramified subfield of B of degree n , with a fixed embedding $\tilde{F} \hookrightarrow \bar{F}$, and let Π denote a uniformiser in O_B which satisfies $\Pi^n = \pi$ such that Π normalises \tilde{F} and induces on \tilde{F} the Frobenius generator of the Galois group $\mathrm{Gal}(\tilde{F}/F)$. Let (\check{V}, Φ) be the F -isocrystal of the framing object. Let

$$\check{V} = \bigoplus_{k \in \mathbb{Z}/n} \check{V}_k$$

be the eigenspace decomposition under \tilde{F} . Then Φ is an endomorphism of degree 1, and so is Π . Then set $N = \check{V}_0$, $\phi = \Pi^{-1}\Phi$. The decomposition $O_{\bar{F}} \otimes_{O_F} O_{\bar{F}} = \bigoplus_{k \in \mathbb{Z}/n} O_{\bar{F}}$ induces for the O_F -Dieudonné module \tilde{M} of a special formal O_B -module in $\mathcal{N}(\bar{\mathbb{F}}_p)$ a decomposition

$$\tilde{M} = \bigoplus_{k \in \mathbb{Z}/n} \tilde{M}_k.$$

Then $\Phi(\tilde{M}_k) \subset \tilde{M}_{k+1}$ and $\Pi(\tilde{M}_k) \subset \tilde{M}_{k+1}$, with both inclusions of colength 1. Then the lattice chain M_\bullet in the set in equation (14.1) is given as $M_i = \Pi^{-i} \tilde{M}_{[i]}$, where $[i] \in \mathbb{Z}/n$ denotes the residue class of i .

The formal scheme \mathcal{N} is an RZ-space $\mathcal{N}_{\mathcal{D}_{\mathbb{Z}_p}}$, corresponding to the following RZ-data \mathcal{D} (cf. [33, Def. 3.18]). Let V be a free B -module of rank 1. Let $\check{V} = V \otimes_F \bar{F}$. Let $b \in \mathrm{GL}_B(\check{V})$ such that the *relative isocrystal* $(\check{V}, b(\mathrm{id} \otimes \sigma))$ is isoclinic of slope $1/n$. The conjugacy class μ is given by $(1, 0, \dots, 0)$ for the fixed embedding of F into $\bar{\mathbb{Q}}_p$ in an identification of $\mathrm{GL}_B(V)$ with GL_n after extension of scalars to \bar{F} , and trivial for all other embeddings of F into $\bar{\mathbb{Q}}_p$. The integral RZ-data $\mathcal{D}_{\mathbb{Z}_p}$ are given by the unique maximal order O_B of B and the periodic lattice chain $\mathcal{L} = \{\Pi^i O_B \mid i \in \mathbb{Z}\}$.

14.2. The case $D_{2/4}$

Let (N, ϕ) be an isocrystal of dimension 4, where ϕ is a σ -linear automorphism isoclinic of slope 0. Then we have

$$X(\mu, \tau)_K = \bigsqcup_{v \in \mathbb{Z}} \{M_\bullet \mid M_{i+2} \supset \phi(M_i), \forall i, \mathrm{vol}(M_0) = v\}. \tag{14.2}$$

Here M_\bullet is a periodic lattice chain with period 4. The decomposition indexed by v corresponds to the decomposition of the affine flag variety into connected components.

14.3. The Hilbert–Blumenthal case

Let (N, ϕ) be a σ^2 -isocrystal of dimension n , where ϕ is a σ^2 -linear automorphism isoclinic of slope 0. Then we have

$$X(\mu, \tau)_K = \bigsqcup_{v \in \mathbb{Z}} \{(M_\bullet, M'_\bullet) \mid \pi\phi(M_i) \subset M'_i \subset {}^1 M_i, \forall i, \text{vol}(M_0) = v\}. \tag{14.3}$$

Here M_\bullet and M'_\bullet are maximal periodic lattice chains in N . The decomposition indexed by v corresponds to the decomposition of the affine flag variety into connected components.

15. Application to p -adic uniformisation

In this section we assume $F = \mathbb{Q}_p$. As explained in Section 14.1, the RZ-space corresponding to case (1) of Theorem 10.2 is π -adic. In this section we explain various criteria which show that the corresponding sets in cases (2) and (3) of Theorem 10.2 do not come from RZ-spaces which are π -adic formal schemes. Here we implicitly appeal to the uniqueness result [21, Prop. 4.4] that the RZ-space (which a priori depends on *integral RZ-data* $\mathcal{D}_{\mathbb{Z}_p}$; cf. [21]) depends only on the tuple (\mathbf{G}, μ, b, K) . To apply this result, we assume that \mathbf{G} splits over a tamely ramified extension of F .

15.1. Via change of parahoric

We note the following consequence of Theorem 10.2:

Corollary 15.1. *Assume that \mathbf{G} is quasi-simple over F and that μ is noncentral. Then $X(\mu, \tau)_K$ is equidimensional of dimension equal to $\langle \mu, 2\rho \rangle$ for every parahoric subgroup K if and only if the pair $(\tilde{\Delta}, \sigma)$ is isomorphic to $\text{Res}_{F_d/F}(\tilde{A}_{n-1}, \varrho_{n-1})$, where as before ϱ_{n-1} denotes rotation by $n - 1$ steps and F_d/F is unramified of degree d . Writing $\mu = (\mu_1, \dots, \mu_d)$, there is a unique i such that μ_i is noncentral and $\mu_i = \omega_1^\vee$. In this case, $K = \emptyset$ corresponds to the unique parahoric subgroup.*

The significance of this corollary is given by the following fact. Let E be the reflex field of (\mathbf{G}, μ) , i.e., the field of definition of μ . Let \mathfrak{X} be a formal scheme over $\text{Spf } O_{\tilde{E}}$ with underlying reduced scheme $X(\mu, \tau)_K$. We assume that \mathfrak{X} is flat over $\text{Spf } O_{\tilde{E}}$ and that its generic fibre, i.e., the associated rigid space $\mathfrak{X}^{\text{rig}}$, is smooth of dimension $\langle \mu, 2\rho \rangle$. Let π be a uniformiser of $O_{\tilde{E}}$. Assume that the formal scheme \mathfrak{X} is π -adic, i.e., π generates an ideal of definition of \mathfrak{X} . Equivalently, the ideal \mathcal{J} of $X(\mu, \tau)_K$ satisfies $\mathcal{J} = \text{rad}(\pi O_{\mathfrak{X}})$ (radical ideal). Then $X(\mu, \tau)_K$ is equidimensional of dimension $\langle \mu, 2\rho \rangle$. Indeed, then $X(\mu, \tau)_K$ coincides with the special fibre of \mathfrak{X} , which is equidimensional of the same dimension as its generic fibre.

Let $K \subset K'$. Let \mathfrak{X} (resp., \mathfrak{X}') be a normal flat formal scheme over $\text{Spf } O_{\tilde{E}}$ with underlying reduced scheme $X(\mu, \tau)_K$ (resp., $X(\mu, \tau)_{K'}$), and let $f: \mathfrak{X} \rightarrow \mathfrak{X}'$ be a proper morphism inducing the natural map $X(\mu, \tau)_K \rightarrow X(\mu, \tau)_{K'}$ and such that f is a finite morphism in the generic fibres. Let \mathcal{J} (resp., \mathcal{J}') be the ideal of definition of \mathfrak{X} (resp., \mathfrak{X}').

Lemma 15.2. *The equality $\mathcal{J} = \text{rad}(\pi\mathcal{O}_{\mathfrak{X}})$ holds if and only if $\mathcal{J}' = \text{rad}(\pi\mathcal{O}_{\mathfrak{X}'})$.*

In other words, \mathfrak{X} is a π -adic formal scheme if and only if \mathfrak{X}' is.

Proof. Assume $\mathcal{J}' = \text{rad}(\pi\mathcal{O}_{\mathfrak{X}'})$. The morphism f is adic, hence $f^*(\mathcal{J}')$ is an ideal of definition of \mathfrak{X} which is contained in \mathcal{J} , as the latter is a maximal ideal of definition. Hence $\mathcal{J} = \text{rad}(\pi\mathcal{O}_{\mathfrak{X}})$ is clear. For the other direction, let $\tilde{f}: \mathfrak{X} \rightarrow \tilde{\mathfrak{X}}'$ be the Stein factorisation of f . Then the normality of $\tilde{\mathfrak{X}}'$ implies $\tilde{f}_*(\mathcal{O}_{\mathfrak{X}}) = \mathcal{O}_{\tilde{\mathfrak{X}}'}$. On the other hand, for the maximal ideal of definition $\tilde{\mathcal{J}}'$ of $\tilde{\mathfrak{X}}'$, we have $\tilde{\mathcal{J}}' \subset \tilde{f}_*(\tilde{f}^*(\tilde{\mathcal{J}}')) \subset \tilde{f}_*(\mathcal{J})$. Hence $\tilde{\mathcal{J}}' \subset \tilde{f}_*(\mathcal{J}) = \tilde{f}_*(\text{rad}(\pi\mathcal{O}_{\mathfrak{X}})) = \text{rad}(\pi\tilde{f}_*(\mathcal{O}_{\mathfrak{X}})) = \text{rad}(\pi\mathcal{O}_{\tilde{\mathfrak{X}}'})$, and hence $\tilde{\mathfrak{X}}'$ is π -adic. But the normality of \mathfrak{X}' implies that $\mathcal{O}_{\mathfrak{X}'} \cap \pi\mathcal{O}_{\tilde{\mathfrak{X}}'} = \pi\mathcal{O}_{\tilde{\mathfrak{X}}'}$. Hence, since $\tilde{\mathfrak{X}}'$ is a π -adic formal scheme, so is \mathfrak{X}' . □

15.2. Via formal branches

In this subsection, we argue via the local structure of RZ-spaces. Let (\mathbf{G}, μ, K) be the corresponding local model triple over F , and $\mathbb{M}^{\text{loc}}(\mathbf{G}, \mu)_K$ be the local model over O_E , in the sense of [21]. Then the special fibre $\overline{\mathbb{M}}^{\text{loc}}(\mathbf{G}, \mu)_K$ is a closed subset of the loop group partial affine flag variety $LG'/L^+\check{K}'$,

$$\mathcal{A}(\mu, \tau)_K = \{g\check{K} \in \check{G}'/\check{K}' \mid g \in \check{K}' \text{Adm}(\mu)\check{K}'\}. \tag{15.1}$$

By the *local model diagram*, the singularities of the RZ-space $\mathcal{M}(\mathbf{G}, \mu, b)_K$ corresponding to (\mathbf{G}, μ, b, K) are modeled by $\mathbb{M}^{\text{loc}}(\mathbf{G}, \mu)_K$. More precisely, for any $x \in \mathcal{M}(\mathbf{G}, \mu, b)_K(\bar{k})$, there exists $y \in \mathbb{M}^{\text{loc}}(\mathbf{G}, \mu)_K(\bar{k})$ such that the strict Henselisations at x and at y are isomorphic. Furthermore, for $b = \tau$, under the identification $\mathcal{M}(\mathbf{G}, \mu, \tau)_K(\bar{k}) = X(\mu, \tau)_K$ the point $x_0 = e\check{K}$ is realised by the point $y_0 = \tau \in \mathcal{A}(\mu, \tau)_K$. Hence we have an identification

$$\begin{aligned} \{\text{formal branches of the special fibre of } \mathcal{M}(\mathbf{G}, \mu, \tau)_K \text{ through } x_0\} = \\ \{\text{extreme elements of } {}^K\text{Adm}(\mu)\}. \end{aligned} \tag{15.2}$$

On the other hand, the extreme elements of ${}^K\text{Adm}(\mu)$ can be identified with

$${}^K\text{Adm}(\mu)^\circ := \{\lambda \in W_0(\underline{\mu}) \mid t^\lambda \in {}^K\tilde{W}\}. \tag{15.3}$$

Therefore, we deduce from Theorem 10.1 the following criterion:

Theorem 15.3. *The RZ-space $\mathcal{M}(\mathbf{G}, \mu, \tau)_K$ is π -adic if and only if the inclusion $W(\underline{\mu})_{K, \text{fin}} \subset {}^K\text{Adm}(\mu)^\circ$ is an equality.*

This theorem again excludes cases (2) and (3) of Theorem 10.2. Indeed, in these cases $K = \emptyset$ and the following elements are in $\text{Adm}(\mu)^\circ \setminus W(\underline{\mu})_{\text{fin}}$:

Case (2): $s_1 s_3 s_2 s_0 \tau$.

Case (3): $s_0 s_{n-1} \cdots s_2 s_{0'} s_{(n-1)'} \cdots s_{2'} \tau$.

Here, in the last line, we use the notation from Section 13.4.

15.3. Via non-Archimedean uniformisation

To put the foregoing results into context, let us explain how to view Theorem 10.2 using global methods, i.e., the theory of Shimura varieties. This allows us to ‘see’ all Newton strata at once, which is not possible within one fixed RZ-space.

In each case of Theorem 10.2, we can construct a Shimura pair $(G, \{h_G\})$ of PEL-type which yields after localisation at p the pair (G, μ) . Let $\mathbb{K} = \mathbb{K}^p \mathbb{K}_p \subset G(\mathbb{A}_f) = G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p)$, with $\mathbb{K}_p = K$. Let $\mathbb{E} = \mathbb{E}(G, \{h_G\})$ be the global Shimura field and fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ which determines a p -adic place v of \mathbb{E} with $E = \mathbb{E}_v$.

Let $\mathcal{S}_{\mathbb{K}} = \mathcal{S}(G, \{h_G\})_{\mathbb{K}}$ be the Pappas–Zhu model of the Shimura variety $S(G, \{h_G\})_{\mathbb{K}}$ over O_E . Then the Newton map

$$\delta_{\mathbb{K}}: \mathcal{S}_{\mathbb{K}}(\overline{\mathbb{F}}_p) \rightarrow B(G, \mu) \tag{15.4}$$

is surjective (cf. [23, §9]). In case (1) of Theorem 10.2, the set $B(G, \mu)$ consists only of the unique basic element $[\tau]$ of $B(G, \mu)$ (cf. [25]); in cases (2) and (3) there are additional elements besides $[\tau]$ (in case (2), one additional element). It follows that in cases (2) and (3), the closed subset $\mathcal{S}_{\mathbb{K}, \text{basic}}$ with $\mathcal{S}_{\mathbb{K}, \text{basic}}(\overline{\mathbb{F}}_p) = \delta_{\mathbb{K}}^{-1}([\tau])$ is a proper closed subset of the special fibre $\overline{\mathcal{S}}_{\mathbb{K}}$ of $\mathcal{S}_{\mathbb{K}}$. Hence, in cases (2) and (3), the formal completion $\mathcal{S}_{\mathbb{K}}^{\wedge} / \overline{\mathcal{S}}_{\mathbb{K}, \text{basic}}$ is a formal scheme over $\text{Spf } O_E$ that is not π -adic. However, by non-Archimedean uniformisation [33, Ch. 6], there is an isomorphism of formal schemes over $\text{Spf } O_{\tilde{E}}$,

$$\mathcal{S}_{\mathbb{K}}^{\wedge} / \overline{\mathcal{S}}_{\mathbb{K}, \text{basic}} \times_{\text{Spf } O_E} \text{Spf } O_{\tilde{E}} \simeq G(\mathbb{Q}) \backslash [\mathcal{M}(G, \mu, \tau)_K \times G(\mathbb{A}_f^p) / \mathbb{K}^p].$$

It follows in cases (2) and (3) that the formal scheme $\mathcal{M}(G, \mu, \tau)_K$ is not π -adic.

16. Proof of Theorems 1.4 and 1.5

For Theorem 1.4, all that remains to be shown after Theorem 10.1 is the assertion that $W(\underline{\mu})_{K, \text{fin}}$ parametrises the orbits of $\mathbf{J}_b(F)$ on the set of irreducible components of dimension $\langle \mu, 2\rho \rangle$ of $X(\mu, b)_K$.

By Theorem 10.1, the union of the irreducible components of maximal dimension is equal to $\cup_{\lambda \in W(\underline{\mu})_{K, \text{fin}}} \overline{X_{K, t^\lambda}(b)}$. Note that each $\overline{X_{K, t^\lambda}(b)}$ is stable under the action of $\mathbf{J}_b(F)$. Moreover, the natural map from the set of irreducible components of $X_{K, t^\lambda}(b)$ to the set of irreducible components of $\overline{X_{K, t^\lambda}(b)}$ is bijective and $\mathbf{J}_b(F)$ -equivariant. It remains to show that for any $\lambda \in W(\underline{\mu})_{K, \text{fin}}$, $\mathbf{J}_b(F)$ acts transitively on the set of irreducible components of $X_{K, t^\lambda}(b)$.

The natural projection map $\check{G}/\check{I} \rightarrow \check{G}/\check{K}$ induces the surjection $X_{t^\lambda}(b) \rightarrow X_{K, t^\lambda}(b)$, and this map is $\mathbf{J}_b(F)$ -equivariant. Moreover, since $\lambda \in W(\underline{\mu})_{K, \text{fin}}$, $W_{\text{supp}_\sigma(t^\lambda)}$ is finite. By [9, Prop. 2.2.1], we have $X_{t^\lambda}(b) \cong \mathbf{J}_b(F) \times^{\mathbf{J}_b(F) \cap \check{K}} Y(w)$, where \check{K} is the parahoric subgroup associated to $\text{supp}_\sigma(t^\lambda)$ and $Y(w)$ is the classical Deligne–Lusztig variety associated to w in the finite dimensional flag variety \check{K}/\check{I} . By [29, Ex. 3.10 d)] (cf. also [7, Cor. 1.2]), $Y(w)$ is irreducible. Hence $\mathbf{J}_b(F)$ acts transitively on the set of irreducible components of $X_{t^\lambda}(b)$, and hence transitively on the set of irreducible components of $X_{K, t^\lambda}(b)$.

Theorem 1.5 is deduced from Theorem 10.2 just as Theorems 1.1 and 1.2 are deduced from Theorems 4.1 and 4.2. Corollary 1.6 follows from Theorem 1.5 by the observation

that in cases (2) and (3) there are F -rational parahoric level structures other than the Iwahori level (cf. Corollary 15.1).

Theorem 1.7 follows from the fact that the integral RZ-data $\mathcal{D}_{\mathbb{Z}_p}$ are of extended Drinfeld type if (\mathbf{G}, μ, K) is of type (1) in Theorem 1.4 (here the key is the fact that we assume that the first entry of a rational RZ-datum is a field extension of \mathbb{Q}_p , so that the case of a *fake unitary group* is excluded).

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