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Group Gradings on Matrix Algebras

Dedicated to the 60th birthday of Robert Moody

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Abstract. Let Φ be an algebraically closed field of characteristic zero, G a finite, not necessarily abelian, group. Given a G-grading on the full matrix algebra $A = M_n(\Phi)$, we decompose A as the tensor product of graded subalgebras $A = B \otimes C$, $B \cong M_p(\Phi)$ being a graded division algebra, while the grading of $C \cong M_q(\Phi)$ is determined by that of the vector space Φ^n. Now the grading of A is recovered from those of A and B using a canonical "induction" procedure.

1 Introduction

In various branches of algebra the researchers are interested in graded algebras and the ways gradings can be given to algebras. Of special interest are gradings of simple algebras. In the case of simple Lie algebras there are gradings by groups and semigroups arising from the root decomposition with respect to a Cartan subalgebra. But there are other ways of grading simple Lie algebras, and in several papers authored by J. Patera (see *e.g.* [5]) he and his co-authors produce, study and apply some gradings, whose nature is rather different. In [8], [12], [4], [13], [6] the authors study gradings in various classes of algebras both associative and non-associative.

In this paper we consider gradings of the full matrix algebra $M_n(\Phi)$ of order n over a field Φ by an arbitrary finite group G. Our main Theorem 5.1 describes such gradings in the case where Φ is algebraically closed of characteristic 0. Gradings on algebras have been studied in a number of publications for many years. Among early books on this topic one can specifically invoke [10] where the authors study various instances of the theory of graded rings. In their study of primitive graded rings, that is, rings possessing a faithful graded simple module, they introduce graded matrix algebras over graded division algebras and endow them with certain gradings (in our terminology "induced gradings"). There are two particular cases of induced gradings: one where the graded division algebra in question has trivial grading. These gradings of a matrix algebra of order n are completely determined by the gradings of the canonical n-dimensional module; they are called "*elementary*". The opposite case is where the primitive algebra is a graded division algebra. Then all nonzero homogeneous elements are invertible. It turns out that in this case the grading is "*fine*" in the sense that every nonzero homogeneous component is one-dimensional. In [3] the authors show that the gradings of a matrix algebra over an algebraically closed field by abelian torsion-free groups are always elementary. In [14] the authors extend

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the previous result to the case of Artinian simple rings. In [1] the authors completely describe G-gradings in the case where G is a finite abelian group. Among other results is the description of graded simple algebras and graded division algebras. In this paper we start with an arbitrary G-graded matrix algebra $M_n(\Phi)$ as above and by direct argument decompose it as the tensor product $M_p(\Phi) \otimes M_q(\Phi)$ where the grading on the first tensor factor is fine and on the second elementary. The way we introduce gradings on tensor products makes the first factor into a graded division algebra D and the whole of $M_n(\Phi)$ into a graded matrix algebra $M_q(D)$ over D. Thus the grading described in [10] becomes a general pattern for the matrix algebras over algebraically closed fields of characteristic zero. With our theorem established, one can now recover a G-graded irreducible simple module making $M_n(\Phi)$ into a G-graded primitive algebra (see concluding remarks in Section 5).

2 Elementary and Fine Gradings

In this section we recall about two of the most important types of gradings on the matrix algebra and give some auxiliary statements that will be used in what follows. In the next section we will establish a connection between the "fine" gradings and projective representations. Let Φ be an arbitrary field,

$$R = M_n(\Phi) = \bigoplus_{g\in G} R_g \tag{1}$$

a G-grading on the algebra of $n \times n$-matrices over Φ, and G a group. We denote by G^n the n-th direct power of the G, and by E_{ij} the matrix units of R, $i, j = 1, \ldots, n$.

Definition 2.1 The grading (1) is called *elementary*, if there exists an n-tuple $\mathbf{g} = (g_1, \ldots, g_n) \in G^n$ such that $E_{ij} \in R_{g_i^{-1} g_j}$. The grading (1) is called *fine* if for any $g \in G$ we have $\dim R_g \leq 1$.

It is obvious that the tuple $(g_1, \ldots, g_n)$ for a given elementary grading is defined in a non-unique way. For example the n-tuple $(gg_1, \ldots, gg_n)$ defines the same grading. In particular one may always assume that $g_1 = e$ is the identity of G.

There is a strong relationship between elementary gradings and gradings induced from vector spaces. Namely, any G-grading on a finite-dimensional vector space V determines a G-grading on the algebra $\operatorname{End} V$ of all linear transformations of V. Specifically, let $V = \bigoplus_{g\in G} V_g$. We call an operator $\varphi \in \operatorname{End} V$ homogeneous of degree $wt(\varphi) = h$, if $\varphi(V_g) \subset V_{hg}$ for all $g \in G$. We denote by π_g the canonical projection $V \to V_g$. Then for any $f \in \operatorname{End} G$ and any $g, h \in G$ the operator $\pi_g f \pi_h$ is homogeneous, with $wt(\pi_g f \pi_h) = gh^{-1}$, and the decomposition

$$f = \sum_{g,h\in G} \pi_g f \pi_h$$

defines a G-grading on $\operatorname{End} V$.

Proposition 2.2 *The matrix algebra $M_n(\Phi) = R = \bigoplus_{g\in G} R_g$ with an elementary grading is isomorphic as a G-graded algebra to the endomorphism algebra* $\operatorname{End} V$ *of some G-graded vector space V.*

Proof Let $R = M_n(\Phi) = \bigoplus_{g\in G} R_g$ be given an elementary grading by means of an n-tuple $(g_1, \dots, g_n) \in G^n$, that is, $E_{ij} \in R_{g_i^{-1} g_j}$. We denote by $V = \Phi^n$ a vector space of dimension n, on which R acts canonically, $V = \operatorname{Span}\{v_1, \dots, v_n\}$, $E_{ij}(v_j) = v_i$, $i, j = 1, \dots, n$, $E_{ij}(v_k) = 0$ for $k \neq j$. If $V = \bigoplus_{g\in G} V_g$ is given a G-grading such that $v_j \in V_{g_j^{-1}}$ then $wt(v_i) = g_i^{-1} = wt\big(E_{ij}(v_j)\big) = g_i^{-1} g_j g_j^{-1} = wt(E_{ij}) wt(v_j)$. This means that the original elementary grading on $M_n(\Phi)$ is defined by a grading on $V = \Phi^n$ and the proof is complete. ■

For proving our main results we need some technical remarks.

Lemma 2.3 (**[14, Lemma 1]**) *Let $R = M_n(D)$ be a matrix ring over a skew field D with some G-grading, $R = \bigoplus_{g\in G} R_g$. If all scalar matrices in R are in the identity component R_e and all matrix units E_{ii}, $i = 1, \dots, n$, are homogeneous then the grading is elementary.*

Lemma 2.4 (**[14, Lemma 3]**) *Let $R = \bigoplus_{g\in G} R_g$ be a ring with a finite G-grading, e the identity element of G, and R_e the identity component with respect to this grading. If R has no nonzero nilpotent ideals then also R_e has no nonzero nilpotent ideals. If R is a ring with the identity E then $E \in R_e$. If R Artinian then R_e is Artinian as well.*

The proof of the next statement for rings R and A can be found in [11, Lemma 3.11] or in [7, Ch. 4, Sect. 4]. The same arguments can be applied also to the case of algebras.

Lemma 2.5 *Let R be an algebra over F with the identity element E and A a subalgebra of R isomorphic to $M_n(F)$. If E_{ij}, $i, j = 1, \dots, n$, are the matrix units from A and $E = E_{11} + \cdots + E_{nn}$ then $R = AC \simeq A \otimes C \simeq M_n(C)$ where C is the centralizer of A in R.*

Lemma 2.6 *Let $M_n(\Phi) = A = \bigoplus_{g\in G} A_g$ be a matrix algebra over a field Φ with a "fine" G-grading. Then $H =$ $\operatorname{Supp} A$ *is a subgroup in G and all homogeneous nonzero elements in A are invertible.*

Proof By Lemma 2.4 the identity matrix is in the identity component A_e, therefore, A_e consists of the scalar matrices. Let $0 \neq a \in A_g$ be an arbitrary homogeneous element. If a is a degenerate matrix then any matrix xay is also degenerate. Then $RaR \cap R_e = 0$ and RaR is a proper ideal in R, a contradiction. Hence a is an invertible matrix. It follows that if $R_g, R_h \neq 0$ then also $R_{gh} \neq 0$, that is, $H = \operatorname{Supp} R$ is a multiplicatively closed subset in G. Since $|H| < \infty$, H is a subgroup in G. ■

Recall that a unitary graded algebra $R = \bigoplus_{g\in G} R_g$ is a *graded division algebra* if all non-zero homogeneous elements of R are invertible. As a consequence of Lemma 2.6 we immediately get:

Corollary 2.7 *Let $A = M_n(\Phi) = \bigoplus_{g\in G} A_g$ be a matrix algebra over an algebraically closed field Φ with some G-grading. Then A is a graded division algebra if and only if the grading is "fine".*

Proof If the grading is "fine" then A is a graded division algebra by Lemma 2.6. Assume now that any non-zero homogeneous element of A is invertible. By Lemma 2.4 the identity component A_e is a semisimple subalgebra containing all scalar matrices. By our hypothesis A_e is a division algebra over Φ. Since Φ is algebraically closed we have $\dim A_e = 1$. By Lemma 4 from [1] all non-zero subspaces A_g are of dimension 1 and the proof is complete. ■

3 "Fine" Gradings and Projective Representations

In the final section we will show that the problem of describing all gradings on a matrix algebra can be reduced to describing the gradings of two special kinds: elementary and "fine". In case of an abelian grading group a lucid construction of any "fine" grading was presented earlier [1]. Here we will show that the classification of all "fine" gradings in the non-abelian case is equivalent to a well-known group-theoretical problem.

Definition 3.1 Let G be a finite group and V a vector space over Φ. Recall (see, for example, [2]) that a mapping $f\colon G \to \mathrm{GL}(V)$ is called *a projective representation* of G on V if $f(e) = E$, where e is the identity of G and E is the identity transformation on V and $f(g)f(h) = \alpha(g,h)f(gh)$ for any $g, h \in G$ where $\alpha(g,h)$ is some non-zero scalar. A projective representation is called *irreducible* if V has no non-trivial subspaces invariant under all $f(g)$, $g \in G$.

It is not difficult to show that for a group of order n the dimensions of irreducible projective representations are bounded by $\sqrt{n}$. Any Abelian group of the type $\mathbf{Z}_n \times \mathbf{Z}_n$ has an irreducible n-dimensional representation. It is a hard problem to classify all groups of order n^2, which have irreducible n-dimensional projective representations and to list these representations.

By Lemma 2.6 from Section 2 the support of any "fine" grading of a matrix algebra is a finite subgroup H in G. The next theorem shows that the classification of all "fine" gradings on matrix algebras is in some sense equivalent to description of all finite groups with irreducible projective representations of maximal degree.

Theorem 3.2 *Any "fine" grading on a matrix algebra $R = M_n(\Phi)$ over an arbitrary field Φ determines an irreducible projective n-dimensional representation of the group $H = \operatorname{Supp} R$ of order n^2. If Φ is algebraically closed and G is a group of order n^2 then any irreducible n-dimensional projective representation of G determines a "fine" grading on the matrix algebra $M_n(\Phi)$.*

Proof Let $M_n(\Phi) = R = \bigoplus_{g\in G} R_g$ be a "fine" grading on a matrix algebra. Then by Lemma 2.6 we have that $H = \operatorname{Supp} R$ is a subgroup in G and $\dim R_h = 1$ for any $h \in H$. Let us fix in each subspace R_h, $h \in H$, $h \neq e$, any nonzero matrix and denote

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it by $f(h)$. For $h = e$ we set $f(e) = E$. Then $f(g)f(h) = \alpha(g,h)f(gh)$ for some scalar $\alpha(g,h)$, since all components are 1-dimensional. Since $f(g)$ and $f(h)$ are invertible by Lemma 2.6, their product is nonzero, that is, $\alpha(g,h) \neq 0$, and we have obtained a projective representation of H. The irreducibility of f is obvious, and the proof of this claim is complete. Now let G be a group of order n^2 and $f\colon G \to \mathrm{GL}_n(\Phi)$ an irreducible projective representation of G of dimension n over an algebraically closed field Φ. Then the linear span

$$A = \mathrm{Span}\{f(g) \mid g \in G\}$$

is a subalgebra in the matrix algebra $M_n(\Phi)$. Since the space Φ^n is a faithful simple A-module over an algebraically closed field Φ it follows that $k = n$, $A = M_n(\Phi)$ and all the elements $f(g)$, $g \in G$, are linearly independent, that is

$$M_n(\Phi) = A = \bigoplus_{g\in G} A_g, \tag{2}$$

where A_g is a 1-dimensional subspace in $M_n(\Phi)$, generated by $f(g)$. Now from Definition 3.1 of the projective representation we have that $A_g A_h \subseteq A_{gh}$ for any $g, h \in G$, that is, the decomposition (2) is a G-grading on $M_n(\Phi)$, and the proof of Theorem 3.2 is complete. ■

4 Induced Gradings on Tensor Products

Let $A = \bigoplus_{g\in G} A_g$ be any G-graded algebra, $B = M_n(\Phi) = \bigoplus_{g\in G} B_g$ be a matrix algebra over Φ with an elementary grading given by an n-tuple $(g_1, \dots, g_n) \in G^n$, that is, $E_{ij} \in B_{g_i^{-1}g_j}$. Then the direct computations show that $R = A \otimes B$ will be given a G-grading if one sets

$$R_g = \mathrm{Span}\{a \otimes E_{ij} \mid a \in A_h, g_i^{-1} h g_j = g\}.$$

Definition 4.1 The grading just defined will be called *induced*.

It is obvious that $A = A \otimes E$ is embedded in R as a G-graded algebra. Similarly, if A is an algebra with identity then also $B = 1 \otimes B$ is a graded subalgebra in R. We notice that if $S = \mathrm{Supp}\, A$ and $T = \mathrm{Supp}\, B$ are two commuting subsets in G then the induced grading assumes a more habitual form of

$$(A \otimes B)_g = \bigoplus_{st=g} A_s \otimes B_t.$$

In particular, if S and T are arbitrary groups and $G = S \times T$, then $A \otimes B$ will be naturally endowed with a G-grading for any S-graded algebra A and T-graded algebra B. Thus the induced grading is a natural generalization of the tensor product of graded algebras to the case of noncommuting supports, but under the restriction that B is a matrix algebra with an elementary grading.

Note that gradings on $A \otimes B \cong M_n(A)$ of similar type were considered also in [10] in case where A is a graded division algebra.

5 Noncommutative Gradings on the Matrix Algebra

The main result of this paper is as follows.

Theorem 5.1 *Let $M_n(\Phi) = R = \bigoplus_{g\in G} R_g$ be a matrix algebra over an algebraically closed field Φ graded by a group G. Then there exists a decomposition $n = pq$, a subgroup $H \subseteq G$ of order p^2 and a q-tuple $\mathbf{g} = (g_1, \dots, g_q) \in G^q$ such that $M_n(\Phi)$ is isomorphic as a G-graded algebra to the tensor product $M_p(\Phi) \otimes M_q(\Phi)$ with an induced G-grading where $M_p(\Phi)$ is an H-graded algebra with "fine" H-grading and $M_q(\Phi)$ is endowed with an elementary G-grading, determined by $\mathbf{g}$.*

Proof Let us set $A = R_e$ where e is the identity element of G. By Lemma 2.4 A is semisimple and contains the identity of R. We decompose A into the sum of simple components, $A = A^{(1)} \oplus \cdots \oplus A^{(k)}$. Since Φ is algebraically closed, all $A^{(i)} \cong M_{q_i}(\Phi)$ are matrix algebras, $i = 1, \dots, k$. Denote by $e_1, \dots, e_k$ the identity elements of the algebras $A^{(1)}, \dots, A^{(k)}$, respectively. Then $R^{(i)} = e_i R e_i$ is a simple subalgebra in R, which is homogeneous in the G-grading. By Lemma 2.5, $R^{(i)} = A^{(i)}C^{(i)} \cong A^{(i)} \otimes C^{(i)}$ where $C^{(i)}$ is the centralizer of $A^{(i)}$ in $R^{(i)}$. Here $C^{(i)}$ is simple and homogeneous in the G-grading and $C^{(i)} \cap A^{(i)}$ is of dimension 1. Then by Lemma 2.6 from [1], $C^{(i)}$ is an algebra with a "fine" grading and $\operatorname{Supp} C^{(i)} = H^{(i)}$ is a subgroup in G by Lemma 2.6. We now decompose the identity elements of the algebras $A^{(1)}, \dots, A^{(k)}$ into the sum of minimal idempotents, that is, the diagonal matrix units. For this we denote by $e^i_{\alpha\beta}$, $1 \le \alpha, \beta \le q_i$ the matrix units of $A^{(i)}$, $i = 1, \dots, k$. Then $e_i = e^i_{11} + \cdots + e^i_{q_i q_i}$. Since

$$e_k R \cdots R e_2 R e_1 R e_2 R \cdots R e_k = e_k R e_k \neq 0,$$

in R one can find homogeneous elements $x_{12}, x_{23}, \dots, x_{k-1,k}, x_{k,k-1}, \dots, x_{32}, x_{21}$ such that $x_{ij} \in e_i R e_j$ and

$$e_k x_{k,k-1} e_{k-1} \cdots e_2 x_{21} e_1 x_{12} e_2 x_{23} \cdots x_{k-1,k} e_k \neq 0.$$

Now let us notice that $\operatorname{Supp} R^{(i)} = \operatorname{Supp} C^{(i)} = H^{(i)}$, $i = 1, \dots, k$. We set $wt(x_{12}) = a_2, \dots, wt(x_{k-1,k}) = a_k$, $wt(x_{21}) = b_2, \dots, wt(x_{k,k-1}) = b_k$. Since $x_{21}x_{12} \in e_2 R e_2 = R^{(2)}$ then $t_2 = b_2 a_2 \in H^{(2)}$. Let $0 \neq z_2 \in C^{(2)}$, with $wt(z_2) = t_2^{-1}$. Then z_2 is invertible by Lemma 2.6 and commutes with e_2. We replace now x_{21} by $x'_{21} = z_2 x_{21}$ and x_{32} by $x'_{32} = x_{32} z_2^{-1}$. Then again

$$e_k x_{k,k-1} e_{k-1} \cdots e_3 x'_{32} e_2 x'_{21} e_1 x_{12} e_2 x_{23} \cdots x_{k-1,k} e_k \neq 0$$

and $x'_{21} x_{12} \in R^{(2)}$ where $wt(x'_{21}) = wt(x_{12})^{-1}$ in G. Similarly we may replace x'_{32} by $x''_{32} = z_3 x'_{32}$ and x_{43} by $x'_{43} = x_{43} z_3^{-1}$ (where $z_3 \in C^{(3)}$) in such a way that $wt(x''_{32}) = wt(x_{23})^{-1}$. As a final result of the procedure described we obtain $\bar{x}_{k,k-1}, \dots, \bar{x}_{21}$, for which

$$e_k \bar{x}_{k,k-1} e_{k-1} \cdots e_2 \bar{x}_{21} e_1 x_{12} e_2 \cdots e_k \neq 0 \tag{3}$$

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and $\bar{x}_{i+1,i}x_{i,i+1} \in R_e \cap R^{(i+1)}$. Then, obviously, $x_{i,i+1}\bar{x}_{i+1,i} \in R_e \cap R^{(i)}$, $i = 1, \dots, k-1$. By (3) there exist $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k$ such that

$$e^k_{\beta_k\beta_k}\bar{x}_{k,k-1}e^{k-1}_{\beta_{k-1}\beta_{k-1}} \cdots e^2_{\beta_2\beta_2}\bar{x}_{21}e^1_{\beta_1\beta_1}e^1_{\alpha_1\alpha_1}x_{12}e^2_{\alpha_2\alpha_2}x_{23}\cdots x_{k-1,k}e^k_{\alpha_k\alpha_k} \neq 0. \tag{4}$$

We set

$$y_{i,i+1} = e^i_{1\alpha_i}e^i_{\alpha_i\alpha_i}x_{i,i+1}e^{i+1}_{\alpha_{i+1}\alpha_{i+1}}e^{i+1}_{\alpha_{i+1}1}, \quad y_{i+1,i} = e^{i+1}_{1\beta_{i+1}}e^{i+1}_{\beta_{i+1}\beta_{i+1}}\bar{x}_{i+1,i}e^i_{\beta_i\beta_i}e^i_{\beta_i1},$$

in which case the elements $y_{12}, \dots, y_{k-1,k}, y_{21}, \dots, y_{k,k-1}$ are homogeneous,

$$e^i_{11}y_{i,i+1}e^{i+1}_{11} = y_{i,i+1}, \quad e^{i+1}_{11}y_{i+1,i}e^i_{11} = y_{i+1,i} \tag{5}$$

and by (3)

$$y_{k,k-1}\cdots y_{21}y_{12}y_{23}\cdots y_{k-1,k} \neq 0. \tag{6}$$

It follows from (6) that

$$y_{ij} = y_{i,i+1}y_{i+1,i+2}\cdots y_{j-1,j} \neq 0, \quad y_{ji} = y_{j,j-1}\cdots y_{i+1,i} \neq 0$$

for all $1 \le i < j \le k$ and it easily follows from (5) that y_{ij} are linearly independent. We notice now, that $y_{22} = y_{21}y_{12} \neq 0$, $y_{22} \in A^{(2)}$ and $e^2_{11}y_{22}e^2_{11} = y_{22}$. This means that y_{22} coincides with e^2_{11} up to a scalar factor. If we divide y_{21} by this factor we may assume that $y_{22} = e^2_{11}$, while the relations (5), (6) remain valid. Similarly, $y_{11} = y_{12}y_{21} = \alpha e^1_{11}$. But since y_{22} is an idempotent also $y^2_{11} = y_{11}$, that is, $y_{11} = e^1_{11}$. If we multiply the remaining $y_{i+1,i}$ by respective scalars, if necessary, we obtain

$$y_{i,i+1}y_{i+1,i} = e^i_{11}, \quad y_{i+1,i}y_{i,i+1} = e^{i+1}_{11}.$$

We obtain linearly independent elements y_{ij}, $1 \le i, j \le k$, with multiplication table $y_{ij}y_{rs} = \delta_{jr}y_{is}$, such that $y_{ij} \in e_iRe_j$ and $e^i_{11}y_{ij}e^j_{11} = y_{ij}$. The subspace $\operatorname{Span}\{y_{ij}, 1 \le i, j \le k\}$ is a graded subalgebra in R, isomorphic to $M_k(\Phi)$, such that the matrix units y_{ij} are homogeneous in the G-grading. Therefore by Lemma 2.3 there exist $g_1 = e, g_2, \dots, g_k \in G$ such that $wt(y_{ij}) = g_i^{-1}g_j$. We recall that $R^{(i)} = A^{(i)}C^{(i)}$, the simple subalgebras $A^{(i)}$ and $C^{(i)}$ are pairwise commutative, $A^{(i)} \cong M_{q_i}(\Phi)$ is in the identity component R_e, $C^{(i)} \cong M_{p_i}(\Phi)$ with a "fine" grading and $e_1+\cdots+e_k$ the identity matrix of $R = M_n(\Phi)$. It follows that $p_1q_1+\cdots+p_kq_k = n$ and the matrix rank in R of $e^i_{11} \in A^{(i)}$ is p_i, $i = 1, \dots, k$. Therefore

$$\dim e^i_{11}Re^j_{11} = p_ip_j.$$

In particular, the dimension of the $(C^{(i)}, C^{(j)})$-bimodule $C^{(i)}y_{ij}C^{(j)}$ is at most p_ip_j, since $y_{ij} \in e^i_{11}Re^j_{11}$ and $C^{(i)}e^i_{11}Re^j_{11}C^{(j)} = e^i_{11}C^{(i)}RC^{(j)}e^j_{11}$. On the other hand since all homogeneous elements in $C^{(i)}$ and $C^{(j)}$ are invertible by Lemma 2.6 we have

$$\dim C^{(i)}y_{ij}C^{(j)} \ge p_i^2, p_j^2.$$

Hence $p_i = p_j = p$ for any i, j and all $C^{(1)}, \dots, C^{(k)}$ have the same dimension. Besides,

$$\dim C^{(i)} y_{ij} C^{(j)} = p^2,$$

that is, $C^{(i)} y_{ij} C^{(j)}$ is irreducible as a left graded $C^{(i)}$-module and as right graded $C^{(j)}$-module. Hence

$$C^{(i)} y_{ij} = y_{ij} C^{(j)}. \tag{7}$$

Considering the fact that the homogeneous components in $C^{(i)}$ and $C^{(j)}$ are 1-dimensional, one can use (7) to construct a well-defined mapping $C^{(i)} \to C^{(j)}$, which is an isomorphism of graded algebras. Let us denote now for $j = i+1$ each such isomorphism by $\varphi_{i,i+1}$ and we set $\varphi_1 = 1$, $\varphi_2 = \varphi_{12}$, $\varphi_3 = \varphi_{23}\varphi_{12} = \varphi_{23}\varphi_2, \dots, \varphi_k = \varphi_{k-1,k}\varphi_{k-1}$. Then for any $x \in C^{(1)}$ and for $i < j$ one has

$$\varphi_i(x) y_{ij} = y_{ij} \varphi_j(x). \tag{8}$$

We can construct similar isomorphisms $\varphi_{i+1,i} \colon C^{(i+1)} \to C^{(i)}$, $i = 1, \dots, k-1$, and consider an arbitrary $a \in C^{(i)}$. Then

$$a y_{ii} = a y_{i,i+1} y_{i+1,i} = y_{i,i+1} \varphi_{i,i+1}(a) y_{i+1,i} = y_{ii} \big(\varphi_{i+1,i} \varphi_{i,i+1}(a) \big) . \tag{9}$$

Since a and $\varphi_{i+1,i}\big(\varphi_{i,i+1}(a)\big)$ are in $C^{(i)}$, $y_{ii} \in A^{(i)}$ and $A^{(i)}C^{(i)} \cong A^{(i)} \otimes C^{(i)}$, it follows from (9) that $\varphi_{i+1,i}\big(\varphi_{i,i+1}(a)\big) = a$. In its turn this implies that

$$\varphi_i(x) y_{i,i-1} = y_{i,i-1} \varphi_{i,i-1}\big(\varphi_i(x)\big) = y_{i,i-1} \varphi_{i,i-1} \varphi_{i-1,i} \varphi_{i-1}(x) = y_{i,i-1} \varphi_{i-1}(x),$$

so that (8) holds for all $i, j = 1, \dots, k$. Let us consider in R a non-graded subalgebra $C \cong M_p(\Phi)$, whose elements have the form

$$\bar{x} = x + \varphi_2(x) + \cdots + \varphi_k(x),$$

where x runs through the whole of $C^{(1)}$. Since $y_{ij} \in e_i R e_j$, $\varphi_i(x) \in e_i R e_i$ and the idempotents $e_1, \dots, e_k$ are orthogonal, it follows from (8) that

$$\bar{x} y_{ij} = y_{ij} \bar{x} \tag{10}$$

for any $1 \leq i, j \leq k$, $\bar{x} \in C$. It is easy to observe that the elements of the form

$$e^i_{\alpha 1} y_{ij} e^j_{1\beta}, \quad 1 \leq \alpha \leq q_i, \quad 1 \leq \beta \leq q_j,$$

are linearly independent in R and homogeneous in the G-grading. Besides, their linear span D is an algebra isomorphic to $M_q(\Phi)$ where $q = q_1 + \cdots + q_k$. Indeed the isomorphism will be defined if we map $e^i_{\alpha 1} y_{ij} e^j_{1\beta}$ to matrix unit $E_{\mu\nu} \in M_q(\Phi)$ where $\mu = q_1 + \cdots + q_{i-1} + \alpha$, $\nu = q_1 + \cdots + q_{j-1} + \beta$. Since $\bar{x} e^i_{\alpha 1} = \varphi_i(x) e^i_{\alpha 1}$, $e^j_{1\beta} \bar{x} = e^j_{1\beta} \varphi_j(x)$ and $e^i_{\alpha 1}$ is in the centralizer of $C^{(i)}$ it follows from (10) that

$$\bar{x} e^i_{\alpha 1} y_{ij} e^j_{1\beta} = e^i_{\alpha 1} y_{ij} e^j_{1\beta} \bar{x},$$

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that is, D is in the centralizer of C in R. Since $\dim D = q^2$, $\dim C = p^2$, $\dim R = p^2q^2$, $R = CD$ is isomorphic to $C^{(1)} \otimes D$. Let $\varphi\colon C^{(1)} \otimes D \to R$ be the isomorphism, $\varphi(x \otimes E_{\mu\nu}) = \bar{x}E_{\mu\nu}$ where

$$(11) \qquad E_{\mu\nu} = e^i_{\alpha 1} y_{ij} e^j_{1\beta} \quad \mu = q_1 + \cdots + q_{i-1} + \alpha, \quad \nu = q_1 + \cdots + q_{j-1} + \beta,$$

a matrix unit in D. Since all $E_{\mu\nu}$ are homogeneous, the grading on D is elementary, $wt(E_{\mu\nu}) = g_\mu^{-1} g_\nu$ for the element (11). Besides one may assume that $g_\mu = g_i, g_\nu = g_j$, since $e^i_{\alpha 1} \in A^{(i)}$, $e^j_{1\beta} \in A^{(j)}$ and

$$wt(E_{\mu\nu}) = wt(y_{ij}) = g_i^{-1} g_j.$$

Let us set $H = H^{(1)} = \operatorname{Supp} C^{(1)}$ and compute the weight of the element $\bar{x}E_{\mu\nu} = \varphi(x \otimes E_{\mu\nu})$, if $x \in C^{(1)}$, $wt(x) = h \in H$:

$$wt(\bar{x}E_{\mu\nu}) = wt(\bar{x}e^i_{\alpha 1} y_{ij} e^j_{1\beta}) = wt\big(\varphi_i(x) e^i_{\alpha 1} y_{ij} e^j_{1\beta}\big) = wt\big(\varphi_i(x) y_{ij}\big).$$

Let us set $wt\big(\varphi_i(x)\big) = h'$. Then $xy_{1i} = \varphi_1(x) y_{1i} = y_{1i}\varphi_i(x)$ by (8), whence $hg_1^{-1}g_i = g_1^{-1}g_i h'$, that is, $h' = g_i^{-1}hg_j$ since $g_1 = e$. Thus $wt\big(\varphi_i(x)y_{ij}\big) = g_i^{-1}hg_ig_i^{-1}g_j = g_i^{-1}hg_j = g_\mu^{-1}hg_\nu$. We remark now that if we consider on $C^{(1)} \otimes D$ the induced grading (see Section 3) then

$$wt(x \otimes E_{\mu\nu}) = g_\mu^{-1} h g_\nu$$

for $x \in C_h^{(1)}$. This means that φ is an isomorphism of graded algebras, and the proof of Theorem 5.1 is complete. ■

By Theorem 5.1 any graded matrix algebra $A = M_n(\Phi)$ is a tensor product of a matrix algebra $B = M_q(\Phi)$ with an elementary gradings and a matrix algebra $D = M_p(\Phi)$ with a "fine" grading. On the other hand, by Corollary 2.7 D is a graded division algebra and $B \otimes D \cong M_q(D)$ can be considered as the algebra of all Φ-linear transformations of the free left D-module of rank q. We can generalize the notion of elementary grading in the following way.

Definition 5.2 Let $R = \bigoplus_{g \in G} R_g$ be a G-graded algebra over Φ, and $\mathbf{g} = (g_1, \ldots, g_q) \in G^q$. The grading on the matrix algebra $A = M_q(R)$ is called *elementary* defined by the tuple $\mathbf{g}$ if

$$A_t = \operatorname{Span}\{rE_{ij} \mid r \in R_h, g_i^{-1}hg_j = t\}$$

where E_{ij} are the matrix units, $1 \le i, j \le q$.

Using this generalization, Theorem 5.1 and Definition 4.1 of induced grading we can say that any grading on a matrix algebra is elementary.

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Corollary 5.3 *Let $A = M_n(\Phi) = \bigoplus_{g\in G} A_g$ be a G-graded matrix algebra over an algebraically closed field Φ. Then $A \cong M_q(D)$ with an elementary G-grading in the sense of Definition 5.2, where $n = pq$ and D is the $p \times p$-matrix algebra equipped with the structure of a G-graded division algebra.* ■

Corollary 5.4 *Let $M_n(\Phi)$ be graded by a finite group G whose order is not divisible by a square. Then any grading of $M_n(\Phi)$ by G is elementary in the sense of Definition 2.1. That is, any such grading is induced by a G-grading of Φ^n.* ■

Finally, we note that $A = M_n(\Phi)$, as in Corollary 5.3, is a G-graded primitive algebra and it has a faithful graded irreducible module $V = D \otimes F^q \cong D^q$. The action of A on V is natural and the grading is given by $x \otimes e_i \in A_{g_i^{-1}h}$ if $x \in D_h$.

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