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The geometric dimension of an equivalence relation and finite extensions of countable groups

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Abstract. We say that the geometric dimension of a countable group G is equal to n if any free Borel action of G on a standard Borel probability space (X, μ), induces an equivalence relation of geometric dimension n on (X, μ) in the sense of Gaboriau. Let $\mathcal{B}$ be the set of all finitely generated amenable groups all of whose subgroups are also finitely generated, and let $\mathcal{A}$ be the subset of $\mathcal{B}$ consisting of finite groups, torsion-free groups and their finite extensions. In this paper we study finite free products K of groups in $\mathcal{A}$. The geometric dimension of any such group K is one: we prove that also geom-dim$(G_f(K)) = 1$ for any finite extension $G_f(K)$ of K, applying the results of Stallings on finite extensions of free product groups, together with the results of Gaboriau and others in orbit equivalence theory. Using results of Karrass, Pietrowski and Solitar we extend these results to finite extensions of free groups. We also give generalizations and applications of these results to groups with geometric dimension greater than one. We construct a family of finitely generated groups $\{K_n\}_{n\in\mathbb{N},n>1}$, such that geom-dim$(K_n) = n$ and geom-dim$(G_f(K_n)) = n$ for any finite extension $G_f(K_n)$ of K_n. In particular, this construction allows us to produce, for each integer $n > 1$, a family of groups $\{K(s, n)\}_{s\in\mathbb{N}}$ of geometric dimension n, such that any finite extension of $K(s, n)$ also has geometric dimension n, but the finite extensions $G_f(K(s, n))$ are non-isomorphic, if $s \neq s'$.

1. *Introduction*

The notion of a countable measure-preserving equivalence relation was introduced by Feldman and Moore [**12**] in their investigations of the orbit properties of actions of countable groups on a standard Borel space. The case of amenable equivalence relations was studied in the fundamental work of Connes *et al* [**7**], and Ornstein and Weiss [**32**]. For some recent applications, see [**9, 10**].

Recently, new results have been obtained in the study of non-amenable equivalence relations, see the review of Shalom [**40**]. If each almost-everywhere connected component of a Borel equivalence relation does not have loops, then this equivalence relation is called *treeable*: we present a more precise definition in §2.5 below. The study of treeable equivalence relations is important in the theory of measure-preserving equivalence relations. The basic results in this area were obtained by Adams [**1, 2**], Gaboriau [**13, 15, 16**], Hjorth [**18, 19**], Jackson *et al* [**21**], Kechris and Miller [**25**], and Pemantle and Peres [**33**].

A countable group is called *strongly treeable* if each free Borel action of G on a standard Borel space with G-invariant probability measure induces a treeable equivalence relation. This definition was introduced by Gaboriau [**13**], who also gave examples of treeable non-amenable groups. The simplest amongst them are free products of countable amenable groups, in particular, free groups and free products of finite groups.

In [**8**] we considered finite extensions of strongly treeable groups, and showed that there are many examples where these finite extensions are again strongly treeable groups. In particular, we showed that any strongly treeable group has such finite extensions. It is natural to conjecture that any finite extension of a strongly treeable group is again strongly treeable: we showed in [**8**] that this conjecture holds for free groups F_n with $n < \infty$.

In this paper, we consider analogous problems for finite extensions of free countable groups with any number of generators and free products of a finite number of finitely generated (f.g.) amenable groups. We develop another approach to this problem in §§3 and 4, independent of results and methods of [**8**].

Finite extensions of free groups were considered in combinatorial group theory in relation to Serre's conjecture [**38**]: a torsion-free group with a free subgroup of finite index is again free. This problem was solved by Stallings [**41**] for f.g. groups, and by Swan for the general case [**43**]. Their methods are a combination of algebraic and topological ideas. Subsequently, Karrass *et al* [**22**], using some constructions from [**42**], gave the general construction of any finite extension of a free group. They showed that it is given by a special HNN (Higman–Neumann–Neumann) extension of a tree product of finite groups (see §3.3). We refer to these as *KPS groups*. They also proved that any finite extension of a free group F_n with finitely many generators is KPS. Cohen [**4**] proved similar results for any group which is a finite extension of a free group with countably many generators, applying Serre's theory of graphs of groups [**39**] together with results from [**41**] and [**43**] (see also [**6**]). The general case was treated by Scott and Wall [**36, 37**].

In §3 we prove that if a countable group is KPS, then it is strongly treeable (Theorem 3.1). In order to prove this, we need to develop methods due to Gaboriau [**13**] and Kechris and Miller [**25**]. In particular, we apply the theory of free products of Borel equivalence relations and their HNN extensions [**13, 25**]. As any finite extension of a free group is KPS, we can conclude that they are all strongly treeable groups.

Note that if a countable group G is a finite extension of a free group, then any finite extension H of G is again an extension of that free group, and hence H is again strongly treeable. We use this simple observation in Lemma 3.16 to show that the free product of abelian groups with uniformly bounded orders is a finite normal extension of a free group. We prove a similar assertion in §4 for any free product of a finite number of f.g. groups.

Let $\mathcal{B}$ be the subset of all f.g. amenable groups all of whose subgroups are also f.g., and let $\mathcal{A}$ be the subset of $\mathcal{B}$ containing all finite groups, torsion-free groups from $\mathcal{B}$ and their finite extensions. In §4 we consider strongly treeable groups of the form $K = K_1 * \cdots * K_t$ where each $K_i \in \mathcal{A}$ and $t < \infty$. Let $G_f(K)$ be an extension of K by a finite group G_f.

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We prove that $G_f(K)$ is a strongly treeable group in Theorems 4.3 and 4.4. Furthermore, if the group K has torsion then it is a finite normal extension of a group of the form $N_1 * \cdots * N_s$ where each N_i is a torsion-free group from $\mathcal{A}$ and $s < \infty$. To prove this result, we need a different approach than in §3, applying some results due to Stallings [**42**] on the structure of finite extensions of free products of groups and the Neumann subgroup theorems. Free products $K = K_1 * \cdots * K_t$, where $K_i = \mathbb{Z}^{n_i}$ for $n_i \in \mathbb{N}$ are used in §5.

In §5, we discuss Gaboriau's [**14, 15**] notion of *geometric dimension* of a measure-preserving Borel equivalence relation R on a standard Borel probability space (X, μ), denoted by geom-dim$(R) \in \mathbb{N} \cup \{\infty\}$. We give the definition and some properties of this concept in §5.1: it turns out that the equivalence relation R is treeable if and only if geom-dim$(R) = 1$.

It is natural to extend the notion of a strongly treeable group in the following way. We say that a countable group H has *geometric dimension* equal to n if any measure-preserving free Borel action of G on a standard Borel probability space (X, μ) induces an equivalence relation E^X_G of geometric dimension n.

It is easy to see that G is a strongly treeable group if and only if the geometric dimension of G is one. Examples of groups G with geometric dimension greater than one are given in [**14**]; we review and extend these examples in §5.

The geometric dimension of a countable group K is related to Gaboriau's notion [**14**, Definition 6.4] of ergodic dimension of this group K, erg-dim$(K) \in \mathbb{N} \cup \{\infty\}$, an analogue of Serre's cohomological dimension of a group [**3**, Ch. VIII, §2].

It is easy to see that if geom-dim$(K) = n$, then erg-dim$(K) = n$. One can prove further that erg-dim$(G_f(K)) = n$ for any finite extension of $G_f(K)$ of K. However, it is not known whether any Borel action of $G_f(K)$ on a Borel space (X, μ) as above must induce an equivalence relation $E^X_{G_f(K)}$ of geometric dimension n. This problem is not simple even for the case $n = 1$, as one can see from §§3 and 4.

It is thus natural to ask to what extent the results §§3 and 4 hold for finite extensions of groups K with geometric dimension $n > 1$. We provide an answer to these questions in §5. In particular, we construct a family $\{F(n) \mid n \in \mathbb{N}, n > 1\}$ of f.g. groups such that geom-dim$(F(n)) = n$ and if $G_f(F(n))$ is a finite extension of $F(n)$, then geom-dim$(G_f(F(n)) = n$, see Theorem 5.11. This construction is based on the results of §4. We also can produce an infinite family of non-isomorphic groups with this property for any $n > 1$ (Theorem 5.13).

2. *Preliminaries*

2.1. *Relations.* A *relation* R on a set X is a set of ordered pairs from X, $R \subset X^2$. If R is a relation we write

$$xRy \Leftrightarrow (x, y) \in R.$$

If $Y \subseteq X$, the restriction of R to Y, $R|_Y$ is defined by $R|_Y = R \cap Y^2$.

A *graph* $\mathcal{G}$ with vertex set X is a non-reflexive (i.e. $(x, x) \notin \mathcal{G}$ for all $x \in X$), symmetric (i.e. $\mathcal{G} = \mathcal{G}^{-1}$) relation on X. A $\mathcal{G}$*-path* from x to y is a finite sequence of vertices

$x = x_0, x_1, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in \mathcal{G}$, for all $i < n$, and $x_i \neq x_j$ if $i \neq j$. Consider the equivalence relation on X given by

$$x E y \Leftrightarrow \text{there exists } a\ \mathcal{G}\text{-path from } x \text{ to } y.$$

Its equivalence classes are the *connected components* of $\mathcal{G}$. A *cycle* is a $\mathcal{G}$-path $x_0, x_1, \ldots, x_n = x_0$, starting and ending at the same point. A graph $\mathcal{G}$ is *acyclic* if it contains no $\mathcal{G}$-cycles. An acyclic graph containing only one connected component is called a *tree*.

2.2. *Countable Borel equivalence relations.* Let X be a standard Borel space. An equivalence relation E on X is called *Borel* if it is a Borel subset of the product space $X \times X$. A Borel equivalence relation E is *countable* if every equivalence class $[x]_E$, $x \in X$, is countable.

If Γ is a countable group Γ and $(g, x) \mapsto g \cdot x$ is a Borel action of Γ on X, then the orbit equivalence relation

$$x E_\Gamma^X y \Leftrightarrow \text{there exists } g \in \Gamma \text{ such that } g \cdot x = y$$

is countable. The converse assertion is also true. This well-known result is due to Feldman and Moore [**12**], see also [**25**, Theorem 15.1].

A Borel subset S of X is called a *complete section* if it meets every equivalence class.

We denote by $[E]$ the set of all Borel isomorphisms f of X with $f(x)Ex$ for all $x \in E$, and by $[[E]]$ the set of all partial Borel isomorphisms $f : A \to B$, where A, B are Borel subsets of X, with $f(x)Ex$, for all $x \in A$.

Let μ be a measure on a standard Borel space X and E a countable Borel equivalence relation on X. As usual, we say that μ is a *finite* measure if $\mu(X) < \infty$ and *a probability* measure if $\mu(X) = 1$.

We say that μ is *E-invariant* if there is a countable group Γ and a Borel action of Γ on X with $E_\Gamma^X = E$, such that μ is Γ-invariant. Properties of E-invariant measures are given in [**25**, §16].

2.3. *Graphings.* *A graph* on a standard Borel space $(X, \mathcal{B})$ is a graph $\mathcal{G}$ on the set X, such that $\mathcal{G} \subseteq X^2$ is Borel, and every $x \in X$ has at most countably many neighbours. Let E be a countable Borel equivalence relation. A *Borel graphing* of E is a graph $\mathcal{G}$ such that the connected components of $\mathcal{G}$ are exactly the E-equivalence classes. This notion was introduced by Adams [**1**].

In this paper we use another equivalent concept of a graph which is called an L-graph (L stands for Levitt). This is a countable family $\Phi = \{\varphi_i\}$, $i \in I$, of partial Borel isomorphisms $\varphi_i : A_i \to B_i$ where A_i, B_i are Borel subsets of X, $\varphi_i \in [[E]]$. We say that Φ is an *L-graphing* of E if Φ generates E, i.e. xEy means that $x = y$ or there is a sequence $i_1, \ldots, i_k \in I$ and $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$ such that $x = \varphi_{i_1}^{\varepsilon_1} \cdots \varphi_{i_k}^{\varepsilon_k}(y)$.

The connection between these two notions is explained in [**25**, §17].

2.4. *Cost of an equivalence relation.* Let E be a countable Borel equivalence relation on X and μ a finite E-invariant measure. If $\Phi = \{\varphi_i\}_{i\in I}$, $\varphi_i \subseteq [[E]]$ is an L-graphing, define its cost by

$$\begin{aligned} C_\mu(\Phi) &= \sum_{i\in I} \mu(\mathrm{dom}(\varphi_i)) \\ &= \sum_{i\in I} \mu(\mathrm{rng}(\varphi_i)) \\ &= \frac{1}{2}\int \sum_{i\in I}(\chi_{A_i}(x) + \chi_{B_i}(x))\, d\mu(x) \end{aligned}$$

where $A_i = \mathrm{dom}(\varphi_i)$, $B_i = \mathrm{rng}(\varphi_i)$ and χ_A is the indicator function of A.

Now we can define the *cost* of E as

$$\begin{aligned} C_\mu(E) &= \inf\{C_\mu(\mathcal{G}) \mid \mathcal{G} \text{ is a graphing of } E \text{ a.e.}\} \\ &= \inf\{C_\mu(\Phi) \mid \Phi \text{ is an } L\text{-graphing of } E \text{ a.e.}\}. \end{aligned}$$

This important notion was introduced by Levitt [**28**]. It is clear that $0 \le C_\mu(E) \le \infty$.

2.5. *Treeings of an equivalence relation.* An L-graphing $\Phi = \{\varphi_i\}$ of E is called an *L-treeing* [**1, 2**] if for each non-empty reduced word $w = \varphi_{i_1}^{\varepsilon_1} \cdots \varphi_{i_n}^{\varepsilon_n}$ $(\varepsilon_i = \{+1, -1\})$, in the symbols $\{\varphi_i\}$, the set $\{x \mid x \in \mathrm{dom}\, w \text{ and } w(x) = x\}$ is empty.

Φ is called an *L-treeing almost everywhere* if $\mu(\{x \mid x \in \mathrm{dom}(w) \text{ and } w(x) = x\}) = 0$.

The importance of treeing to costs is demonstrated by the following key result of Gaboriau [**13**, Proposition I.11] or [**25**, Proposition 19.1].

THEOREM 2.1. *Let E be a countable equivalence relation, μ an E-invariant measure with $C_\mu(E) < \infty$. If Φ is an L-graphing of E which attains the cost of E, i.e. $C_\mu(E) = C_\mu(\Phi)$, then Φ is an L-treeing of E almost everywhere.*

An equivalence relation E is called *treeable* if E admits an L-treeing.

Note that for each $\alpha \in [1, \infty]$, there exists a treeable equivalence relation E with E-invariant probability measure μ such that $C_\mu(E) = \alpha$ (see [**13**]).

2.6. *Treeable and strongly treeable groups.* A countable group G is called *treeable* [**25**, Proposition 3.1] if there is a measure-preserving free Borel action of G on a standard Borel probability space such that the induced equivalence relation E_G^X is treeable μ-almost everywhere.

This notion was introduced by Pemantle and Peres [**33**]. Another related notion was introduced by Gaboriau [**13**]. A countable group G is called *strongly treeable* if for every free measure-preserving Borel action of G on a standard Borel probability space (X, μ), the induced relation E_G^X is treeable.

Note that here we use the terminology of Kechris and Miller [**25**], which differs from that of Gaboriau [**13**].

Interesting examples of treeable groups were given in [**13**], specifically the fundamental group $\pi_1(\Sigma_g)$ of an orientable surface of genus $g \ge 2$. Some generalizations of these were given in [**8**].

The simplest examples of strongly treeable groups are countable amenable groups, $SL(2, \mathbb{Z})$, any free group and any free product of these groups. These and other examples were presented in [**13**]. In this paper we give some new examples of strongly treeable groups.

In [**8**], we observed that if G is strongly treeable and H is a subgroup of G of finite index, then H is also strongly treeable. It is interesting to consider the dual situation when H is a strongly treeable subgroup of G and $[G : H] < \infty$, and ask whether G is also strongly treeable. In [**8**] we presented many examples when this is the case. We showed in particular that if G is strongly treeable and K is any finite group then the direct product $G \times K$ is again strongly treeable.

However, the general problem turned out to be rather complicated, and we treated only the case where H is a free subgroup: we were able to prove the conjecture in the case where H is f.g. We develop this theory further in §§3 and 4.

2.7. *Free product with amalgamation and HNN extensions.* We recall from [**29**] and [**6**], some basic constructions of combinatorial group theory which we use in the following.

Let K and H be two countable groups. We say that $G = K * H$ is a *free product* of K and H if each element g of G has the form $g = k_1h_1k_2h_2 \cdots k_nh_n, n < \infty, k_i \in K, h_i \in H$, $1 \leqq i \leqq n$.

Let A be a subgroup of K and B a subgroup of H and suppose that there is an isomorphism $\varphi : A \to B$. Consider the group $G = \langle a \in K * H; a = \varphi(a), a \in A\rangle$, where G is the quotient of the free product of $K * H$ by the normal subgroup of $K * H$ generated by $\{a\varphi(a)^{-1} \mid a \in A\}$. We call G a *free product with amalgamation* of K and H, or *amalgamated free product* of K and H with A amalgamated: $G = K *_A H$. This construction was introduced by Schreier [**35**] in 1926.

Let G be a group, A and B be subgroups of G with $\varphi : A \to B$ an isomorphism. The HNN *extension* of G *relative* to A, B and φ is the group $G^* = \langle G, p : p^{-1}ap = \varphi(a), a \in A\rangle$.

The group G is called the *base* of G^* and p is called the *stable letter*, A and B are called *associated subgroups*. Again, G^* is the quotient of the free product $P * G$, where P is generated by p, by the normal subgroup of $P * G$ generated by $\{p^{-1}ap\varphi(a)^{-1}, a \in A\}$.

This construction was introduced by Higman, Neumann and Neumann [**17**] in 1949. Note that these two constructions are parallel and there is a single axiomatization of both constructions given by Stalling's concept of a bipolar structure [**29**, Ch. IV, §.6].

The structure of a HNN extension is described by the normal form theorem, which we present below.

Definition 2.2. The sequence $g_0, p^{\varepsilon_1}, g_1, \ldots, p^{\varepsilon_n}, g_n, \varepsilon = \pm 1, n > 0$, where $g_i \in G$, is said to be *reduced*, if there is no consecutive subsequence of the form p^{-1}, g_i, p with $g_i \in A$ or p, g_i, p^{-1} with $g_i \in B$.

Definition 2.3. A *normal form* is a sequence $g_0, p^{\varepsilon_1}, g_1, \ldots, p^{\varepsilon_n}, g_n (n \geq 0)$ where:
(i) g_0 is an arbitrary element of G;
(ii) if $\varepsilon_i = -1$, then g_i is a representative of a coset of A in G;
(iii) if $\varepsilon_i = +1$, then g_i is a representative of a coset of B in G;
(iv) there is no consecutive subsequence of the form $p^{\varepsilon}, 1, p^{-\varepsilon}$.

THEOREM 2.4. (See [**29**, IV.2.1]) *Let* $G^* = \langle G, p;\ p^{-1}ap = \varphi(a), a \in A >$ *be a HNN extension. Then we have the following results.*

(i) *(Britton's lemma) The group* G *is embedded in* G^* *by the map* $g \mapsto g$. *If* $g_0 p^{\varepsilon_1} \cdots p^{\varepsilon_n} g_n = 1$, $\varepsilon_i = \pm 1$, $1 \leq i \leq n$ *in* G^*, *where* $n \geq 1$, *then* $g_0, p^{\varepsilon_1}, \ldots, p^{\varepsilon_n}, g_n$ *is not reduced.*

(ii) *Every element* w *of* G^* *has a unique representation as* $w = g_o p^{\varepsilon_1} \cdots p^{\varepsilon_n} g_n$ *where* $g_0, p^{\varepsilon_1}, \ldots, p^{\varepsilon_n}, g_n$ *is a normal form (see Definition 2.3).*

Note that there are also constructions of free product, free product with amalgamation and HNN extension for Borel equivalence relations on a standard Borel space introduced by Gaboriau [**13**], see also [**25**]. In particular, if a countable group $G = K * H$ has a measure-preserving free Borel action on a standard Borel probability space (X, μ) with induced Borel equivalence relation E_G^X, then $E_G^X = E_K^X * E_H^X$ where E_K^X and E_H^X are the Borel equivalence relations on (X, μ) induced by K and H, respectively. Similar results hold for amalgamated free products and HNN extensions (see [**13**, §IV], [**20**] and [**25**, §§27, 37]).

Now let G be a countable group, $\{(A_i, B_i)\}$ a collection of pairs of subgroups of G and φ_i isomorphisms $\varphi_i : A_i \to B_i$. Then one can construct the HNN extension of the base group G with associated subgroups (A_i, B_i) and stable letters p_i, $i = 1, 2, \ldots,$ by

$$G^* = \langle G, p;\ p_i^{-1} a_i p_i = \varphi_i(a_i), a_i \in A_i \rangle.$$

Again G^* is the quotient of the free product $P * G$ where P is a free product of p_i, $i = 1, 2, \ldots,$ by the normal subgroup N of $P * G$ generated by $p_i^{-1} a_i p_i (\varphi_i(a_i))^{-1}$, $i = 1, 2, \ldots.$ The reader may find the details in [**6**].

3. *Finite extensions of free groups*

Karass *et al* [**22**] presented a construction of a group which contains a free subgroup of finite index. We refer to their construction as the *KPS construction*, and to groups constructed by it as *KPS groups*. The exact definition is given in Definition 3.14.

In this section we prove the following assertion.

THEOREM 3.1. *Every countable KPS group is strongly treeable.*

To prove this theorem we apply methods of orbit equivalence theory.

COROLLARY 3.2. *Any countable group G, which is a finite extension of a f.g. free group, is strongly treeable.*

Proof. Indeed, it was proved in [**22**] that any group G as in the statement is KPS. □

Note that we gave a proof of this result in [**8**] which did not use the KPS construction.

COROLLARY 3.3. *Any countable group G, which is a finite extension of a free group, is strongly treeable.*

Proof. Cohen [**4**] proved that any such group G is KPS, using the theory of graphs of groups [**39**]. One can find a more recent proof of this result in [**5**, Theorem 8.55]. □

We use Corollary 3.3 twice in this paper, specifically in Proposition 3.15 and Theorem 5.9, where we deal with the free group of infinite rank. The remainder of the paper is devoted to f.g. groups.

Note also that the KPS construction allows one to find interesting examples of finite extensions of free groups, see also Remark 3.17.

3.1. *HNN extensions of strongly treeable groups.* In this section we prove the following assertion, which is a step towards the proof of Theorem 3.1.

PROPOSITION 3.4. *Let G be a strongly treeable group, $\{(A_i, B_i)\}$ a set of pairs of finite subgroups of G and $\{\varphi_i\}$ a set of isomorphisms $\varphi_i : A_i \to B_i$. Then the HNN extension G^* with the base G, associated subgroups A_i, B_i and stable letters p_i, $i = 1, 2, \ldots,$ is strongly treeable.*

We first prove several auxiliary lemmas.

LEMMA 3.5. *Let $\{R_i : i = 1, 2\}$, be countable treeable equivalence relations on a standard Borel space (X, μ) and $R = R_1 * R_2$. Suppose that μ is a finite R-invariant measure on X. Then R is also treeable, and if $T(R_i)$ is an L-treeing of R_i then $T(R) = T(R_1) \cup T(R_2)$ is an L-treeing of R.*

Proof. The proof is a consequence of the definition of the free products (or free joins [**25**]) of equivalence relations of R_1 and R_2 and the definition of an L-treeing of an equivalence relation (see [**13, 25**]). □

COROLLARY 3.6. [**13**] *Let G_1 and G_2 be strongly treeable groups. Then $G_1 * G_2$ is also strongly treeable.*

LEMMA 3.7. *Let R_i, $i = 1, 2$, be countable treeable equivalence relations on a standard Borel space (X, μ), and suppose that $R_3 = R_1 \cap R_2$ is a finite equivalence relation. Let $R = R_1 *_{R_3} R_2$. Suppose further that μ is an R-invariant finite measure.*

Then R is also treeable. If, moreover, $T(R_1)$ is an L-treeing of R_1, then there is an L-treeing $T(R)$ of R such that $T(R) \supset T(R_1)$.

Proof. Let D be a fundamental Borel set of R_3. This means that D meets every R_3-orbit in exactly one point. Consider $R_2' = R_2 |_D$. Then $R = R_1 * R_2'$ (see [**13**, Example IV.11]). Let $T(R_1)$ be an L-treeing of R_1 and $T(R_2')$ an L-treeing of $R_2 |_D$ (see [**13**, §II.6(1)]). It is clear that $T(R) = T(R_1) \cup T(R_2')$. □

COROLLARY 3.8. [**13**] *Let G_1 and G_2 be strongly treeable groups and K a finite group. Then the amalgamated free product $G_1 *_K G_2$ of G_1 and G_2 over K is also a strongly treeable group.*

LEMMA 3.9. *Let G be a strongly treeable group, A, B finite subgroups of G and φ an isomorphism $\varphi : A \to B$. Then the HNN extension*

$$G^* = \langle G, p;\ p^{-1}ap = \varphi(a), a \in A\rangle$$

with the base G, associated subgroups (A, B) and stable letter p, is also strongly treeable.

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Moreover, suppose that G^* has a measure-preserving free action on a standard Borel probability space (X, μ). If $T(E^X_G)$ is an L-treeing of E^X_G, then there is an L-treeing $T(E^X_{G^*})$ of $E^X_{G^*}$ such that $T(E^X_{G^*}) \supset T(E^X_G)$.

Proof. We use the bipolar construction from [**13**], see also [**25**, §37]. Let $x \mapsto g\cdot x$, $g \in G^*$ be a free Borel action of G^* on X, and μ a G^*-invariant probability measure on X. Let X_1 and X_2 be two copies of X with $\bar{h} : x_1 \mapsto x_2 = \bar{h}x_1$ the identification map from X_1 to X_2. Let μ_1 and μ_2 be two copies of μ for X_1 and X_2, respectively.

There are also two copies of the action of G^* on X: $(g, x_i) \to g\cdot x_i$ where $x_i \in X_i$, $i = 1, 2$. Note that $g\cdot \bar{h}x_1 = \bar{h}g\cdot x_1$, $x_1 \in X_1$, $g \in G^*$.

Define the isomorphism $\bar{f} : X_1 \to X_2$ as follows:

$$\bar{f}(x) = \bar{h}p\cdot x = p\cdot \bar{h}x, \quad x \in X_1, \tag{1}$$

where p is the stable letter from G^*.

As G is strongly treeable, it follows that $E^{X_1}_G$ and $E^{X_2}_G$ are strongly treeable equivalence relations on X_1 and X_2, respectively. Let Φ_1 be an L-treeing of $E^{X_1}_G$. Then $\Phi_2 = \bar{h}\Phi_1\bar{h}^{-1}$ is an L-treeing of $E^{X_2}_G$.

Let $\bar{X} = X_1 \sqcup X_2$ and $\nu = \mu_1 + \mu_2$. Consider the following equivalence relations on $\bar{X}$:

- R_1, the equivalence relation generated by $E^{X_1}_G$ and $\bar{h}$;
- R_2, the equivalence relation generated by $E^{X_2}_A$ and $x \to \bar{f}(x)$, where $x \in X_1$;
- $R_3 = E^{X_2}_A \cup E^{X_1}_{\varphi(A)}$;
- R, the equivalence relation on $\bar{X}$ generated by R_1 and $x \to \bar{f}(x)$, $x \in X_1$.

It follows from the definition of an amalgamated join of two equivalence relations [**25**, §27] and Theorem 2.4 above that $R = R_1 *_{R_3} R_2$.

Note that X_1 and X_2 are complete Borel sections for R, R_1, R_2, respectively. Hence, $\nu|_{X_1} = \mu_1$, $\nu|_{X_2} = \mu_2$ and

$$R|_{X_1} = E^{X_1}_{G^*}, \quad R_1|_{X_1} = E^{X_1}_G, \quad R_2|_{X_2} = E^{X_2}_A.$$

Now it follows from our assumption on G that $E^{X_1}_G$ is treeable, and hence that R_1 is treeable by [**13**, §II.6(1)]. As A is a finite subgroup of G, it follows that $E^{X_2}_A$ and R_2 are finite equivalence relations. Thus, $R = R_1 *_{R_3} R_2$ is treeable by Lemma 3.7.

We now develop some properties of L-treeings of R. As $R_3 = E^{X_2}_A \cup E^{X_1}_B$ is a finite equivalence relation on $\bar{X}$, it follows that R_3 has a fundamental Borel set $D \in \bar{X}$. Let $D = D_1 \cup D_2$ where $D_i = D \cap X_i$ and D_1 (respectively D_2) is the fundamental set of $E^{X_1}_B$ (respectively $E^{X_2}_A$). Then the restriction of $\bar{f}$ to D_1 defines an isomorphism $\bar{f}|_{D_1} : D_1 \to D_2$.

Note that $R'_2 = R_2|_D$ is generated by $\bar{f}|_{D_1}$, and furthermore $R = R_1 * R'_2$ (see the proof of Lemma 3.7). Recall that we denoted an L-treeing of $R_1|_{X_1}$ by Φ_1 . Hence,

$$\bar{\Phi}_1 = \Phi_1 \cup \bar{h}$$

is an L-treeing of R_1. As $R = R_1 * R'_2$, then

$$\bar{\Phi} = \Phi_1 \cup \bar{h} \cup \bar{f}|_{D_1}$$

is an L-treeing of R. However, $\bar{f} = \bar{h}p$ by (1) and, hence, we can take an L-treeing of R of the following form:

$$\Psi = \Phi_1 \cup p|_{D_1} \cup \bar{h}.$$

As $\Psi|_{X_1} = \Phi_1 \cup p|_{D_1}$, we see that $\Psi|_{X_1}$ is an L-treeing of $R_{X_1} = E_{G^*}^{X_1}$. Thus, we obtain

$$\Phi_1 \subset \Psi|_{X_1}.$$

Hence, the last assertion of our lemma is also proved. □

The next lemma is well known. It describes the universal property of HNN extensions (see [**6**, Proposition 1.30]).

LEMMA 3.10. *Let A and B be subgroups of a group G and let φ be an isomorphism $\varphi : A \to B$. Suppose that there is a homomorphism j from G to a group K, and that K has an element k such that $k^{-1} j(a)k = j(\varphi(a))$. Let H be the HNN extension of G with stable letter p and associated subgroups A and B. Then there is a unique homomorphism $\psi : H \to K$ such that $\psi|_G = j$ and $\psi(p) = k$.*

LEMMA 3.11. *Suppose that a countable group G has two pairs of subgroups $\{A_i, B_i \mid i = 1, 2\}$, and that there exist isomorphisms $\varphi_i : A_i \to B_i$, $i = 1, 2$. Consider the following groups:*

$$G^* = \langle G, p_1, p_2;\ p_i^{-1} a_i p_i = \varphi_i(a_i),\ a_i \in A_i \rangle,$$
$$G_1^* = \langle G, \check{p}_1;\ \check{p}_1^{-1} a_1 \check{p}_1 = \varphi_1(a_1),\ a_1 \in A_1 \rangle,$$
$$G_2^* = \langle G_1^*, \check{p}_2;\ \check{p}_2^{-1} a_2 \check{p}_2 = \varphi_2(a_2),\ a_2 \in A_2 \rangle.$$

Then G^* is isomorphic to G_2^*.

Proof. Let H_1 be a subgroup of G^* generated by p_1 and G. It is clear that there is an isomorphism $j : G_1^* \to H_1$. In order to simplify the notation, we suppose that j is the identity map from a subgroup G of G_1^* onto the subgroup G of H_1 and $j\check{p}_1 = p_1$. Then we have the following relations:

$$j\varphi_1(a_1) = \varphi_1 j(a_1), \quad a_1 \in A_1$$

and

$$j\varphi_2(a_2) = \varphi_2 j(a_2), \quad a_2 \in A_2.$$

Hence, we obtain the following equality:

$$j\varphi_2(a) = \varphi_2 j(a) = p_2^{-1} j(a) p_2, \quad a \in A_2.$$

Now one can apply Lemma 3.10 to G_1^*, G^* and j. It follows from this lemma that there is a unique homomorphism $\psi : G_2^* \to G^*$ such that $\psi|_{G_1^*} = j$ and $\psi(\check{p}_2) = p_2$.

Thus, the homomorphism ψ has the following properties:

$$\psi(\check{p}_1) = p_1,$$
$$\psi(g) = g, \quad g \in G, \tag{2}$$
$$\psi(\check{p}_2) = p_2.$$

It follows from (2) that ψ is a homomorphism onto G^* and that there is a normal subgroup $\check{N}$ of G_2^* such that $G_2^*/\check{N} \approx G^*$.

Equation (2) also shows that we may identify $\check{p}_1, g, \check{p}_2$ with p_1, g, p_2 respectively, for any $g \in G$. Recall that the group G^* has the form $P_2 * G/N$, where P_2 is a free group generated by p_1, p_2, and N is the smallest normal subgroup of $P_2 * G$ generated by the relations $p_i^{-1} a_i p_i \varphi_i(a_i)^{-1} = 1$, $a_i \in A_i$, $i = 1, 2$. This is a consequence of the definition of a HNN extension G^* of G. As G^* is also generated by p_1, p_2, and $g \in G$, then $G^* \approx P_2 * G/M$, where M is a normal subgroup of $P_2 * G$. Since ψ is a homomorphism from G_2^* onto G^* we see that $M \subseteq N$.

Now it follows from the definition of G_2^* that $\check{p}_i^{-1} a_i \check{p}_i = \varphi_i(a_i)$, $a_i \in A_i$, $i = 1, 2$. Hence, our identification of $\check{p}_1, \check{p}_2, g$ with p_1, p_2, g shows that the relations $p_i^{-1} a_i p_i \varphi_i(a_i)^{-1}$, $a_i \in A_i$, $i = 1, 2$, also belong to M. Thus, $M \supseteq N$, and we obtain $M = N$. □

We are now ready to complete our proof of Proposition 3.4.

Proof of Proposition 3.4. Let G and G^* be as in the statement of the proposition. Let $G_0 = G$ and G_i be a subgroup of G^* generated by G and $p_1, p_2, \ldots, p_i$. Suppose that we have a measure-preserving free Borel action of G^* on a standard Borel probability space (X, μ). Let $E_{G_i}^X$ be the Borel equivalence relation on X generated by G_i, for $i = 0, 1, \ldots$. Then $E_{G_0}^X$ is treeable by our assumptions on G, and $E_{G_1}^X$ is also treeable by Lemma 3.9 and our assumptions on the subgroups A_1, B_1 of G. Moreover, if T_0 is an L-treeing of $E_{G_0}^X$, then there is an L-treeing T_1 of $E_{G_1}^X$ such that $T_0 \subset T_1$.

As G_2 is isomorphic to a HNN extension with base G_1, stable letter p_2 and associated subgroups A_2, B_2, it follows, again from Lemma 3.9, that $E_{G_2}^X$ is treeable and there is an L-treeing T_2 of $E_{G_2}^X$ such that $T_1 \subset T_2$.

Thus, by induction we see that $E_{G_i}^X$ is treeable and there is an L-treeing T_i of $E_{G_i}^X$ such that $T_{i-1} \subset T_i$.

We claim that $T = \bigcup_i T_i$ is an L-treeing of $E_{G^*}^X$. First, observe that $G^* = \bigcup_i G_i$. Hence, if $x, y \in X$ and $(x, y) \in E_{G^*}^X$, then there is $g \in G_i$ for some i such that $y = g \cdot x$. As T_i is an L-treeing of $E_{G_i}^X$, then there is a *unique* reduced word w of the form

$$w = \varphi_{j_1}^{\epsilon_1} \cdots \varphi_{j_n}^{\epsilon_n}, \quad n < \infty, \ \epsilon_i = \pm 1, \tag{3}$$

where $\varphi_i \in T_i$, such that $y = w \cdot x$. Hence, T is an L-graphing of $E_{G^*}^X$.

In order to prove that T is an L-treeing of $E_{G^*}^X$, observe that T_i is an L-treeing of $E_{G_i}^X$ for each i, $E_{G^*}^X = \bigcup_i E_{G_i}^X$ and $T_1 \subset T_2 \subset T_3 \subset \ldots$. Hence if x, y and w are as above, w will be a unique reduced word of the form (3) for each L-treeing T_j of $E_{G_j}^X$ for $j \geq i$. Thus, w is the *unique* reduced word of the form (3) for T such that $y = w \cdot x$. This means that T is an L-treeing of $E_{G^*}^X$. □

3.2. *Tree products of strongly treeable groups.* In this section, we consider some properties of a tree products of groups, introduced in [**23**]. In the next section, we use these in the construction of KPS groups.

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We recall some properties of trees. A graph $\mathcal{G}$ consists of two disjoint sets: the set $V(\mathcal{G})$ of *vertices* of $\mathcal{G}$ and the set $E(\mathcal{G})$ of *edges* of $\mathcal{G}$. We have a *start* function $\sigma : E(\mathcal{G}) \to V(\mathcal{G})$, an *inverse* mapping $e \mapsto \bar{e} : E(\mathcal{G}) \to E(\mathcal{G})$ and the *end* function $\tau : E(\mathcal{G}) \to V(\mathcal{G})$.

These mappings have the properties that $e \neq \bar{e}$, $e = \bar{\bar{e}}$ and $\sigma(\bar{e}) = \tau(e)$.

A path of length $n > 0$ in $\mathcal{G}$ is a finite sequence $e_1, \ldots, e_n$ of edges with $\tau(e_i) = \sigma(e_{i+1})$ for $i < n$. A *loop* is a path which ends at the point at which it starts i.e. $\sigma(e_1) = \tau(e_n)$.

A path is called *reducible* if there is some i such that $e_{i+1} = \bar{e}_i$, otherwise it is called *irreducible*. A *tree* is a connected graph all of whose loops of positive length are reducible.

We now present a simple lemma.

LEMMA 3.12. *Let $\mathcal{T}$ be a tree with countable vertex set $V(\mathcal{T})$. Then there is a sequence of finite subtrees $\mathcal{T}_i$ of $\mathcal{T}$ such that $\bigcup_i \mathcal{T}_i = \mathcal{T}$ and the sequence $\mathcal{T}_i$ has the following properties:*

(i) $\mathcal{T}_0 = (v_0)$ *where* $v_0 \in V(\mathcal{T})$*;*

(ii) $V(\mathcal{T}_i) = (v_0, v_1, \ldots, v_i)$, $v_s \in V(\mathcal{T})$, $0 \leq s \leq i$*;*

(iii) $E(\mathcal{T}_i) = (e_1, e_2, \ldots, e_i)$, *where* $\sigma(e_s), \tau(e_s) \in V(\mathcal{T}_i)$, $1 \leq s \leq i$.

In particular, $V(\mathcal{T}_{i+1}) = V(\mathcal{T}_i) \cup v_{i+1}$ *and* $E(\mathcal{T}_{i+1}) = E(\mathcal{T}_i) \cup (e_{i+1})$ *and* $\sigma(e_{i+1}) \in V(\mathcal{T}_i)$, $\tau(e_{i+1}) = v_{i+1}$.

The proof follows easily from the definition of a tree.

We now present the notion of a *tree product of groups*, introduced in [**23**]. First, a *tree of groups* $\mathcal{G}$, consists of:

(i) a tree $\mathcal{T}$;

(ii) a group G_v for each vertex $v \in E(\mathcal{T})$ and a group G_e for each edge $e \in E(\mathcal{T})$ such that $G_e = G_{\bar{e}}$ for all $e \in E(\mathcal{T})$;

(iii) monomorphisms $\tau(e) : G_e \to G_{\tau(e)}$ and $\sigma(e) : G_e \to G_{\sigma(e)}$.

The *tree product of groups* is the group

$$\pi(\mathcal{G}) = (\Pi^*_{v \in V(\mathcal{T})} G_v)/N$$

where N is the normal subgroup of $\Pi^*_{v \in V(\mathcal{T})} G_V$ generated by $(\sigma(e)a)(\tau(e)a)^{-1}$, $a \in G_e$, $e \in E(\mathcal{T})$.

PROPOSITION 3.13. *Let $\mathcal{T}$ be a tree of groups such that the vertex group G_v is a strongly treeable for each $v \in V(\mathcal{T})$ and G_e is a finite group for each $e \in E(\mathcal{T})$. Then the tree product of groups $G = \pi(\mathcal{T})$ is also a strongly treeable group.*

Proof. We use the notation of Lemma 3.12. Let Γ_i be a tree product of groups corresponding to a finite subtree $\mathcal{T}_i$. Then $\Gamma_i \subseteq \Gamma_{i+1}$ and $G = \bigcup_i \Gamma_i$. Moreover, it follows from Lemma 3.12 that

$$\Gamma_{i+1} = \Gamma_i *_{G_{e_{i+1}}} G_{v_{i+1}}. \tag{4}$$

This means that Γ_{i+1} is the free product of the Γ_i and $G_{v_{i+1}}$, by amalgamating the subgroup $\tau(e_{i+1})G_{e_{i+1}}$ of $G_{v_{i+1}}$ and the subgroup $\sigma(e_{i+1})G_{e_{i+1}}$ of Γ_i.

Let H_{i+1} be the subgroup of Γ_{i+1}, elements h of which have the form $h = \sigma(e_{i+1})(g) = \tau(e_{i+1})(g)$ where $g \in G_{e_{i+1}}$.

A free measure-preserving action of G on a standard Borel probability space (X, μ). Then G induces a Borel equivalence relation E_G^X on (X, μ). We consider the following equivalence relations on (X, μ):

$$E_{\Gamma_i}^X, E(v_i) = E_{G_{v_i}}^X, \quad E(e_i) = E_{H_i}^X. \tag{5}$$

It follows from (4) that

$$E_{\Gamma_{i+1}}^X = E_{\Gamma_i}^X *_{E(e_{i+1})} E(v_{i+1}). \tag{6}$$

Recall that each $E(v_i)$, $i = 0, 1, 2, \ldots$ is treeable and $E(e_i)$, $i = 1, 2, \ldots$ is finite by assumption. As $E_{\Gamma_0}^X = E(v_0)$, by our construction (see Lemma 3.12), it follows from (6) and Lemma 3.7 that $E_{\Gamma_1}^X$ is also treeable and there is an L-treeing T_1 of $E_{\Gamma_1}^X$ such that $T_0 \subset T_1$ where T_0 is an L-treeing of $E(v_0)$. Suppose that T_i is an L-treeing of $E_{\Gamma_i}^X$, then (6) and Lemma 3.7 show that there is an L-treeing T_{i+1} of $E_{\Gamma_{i+1}}^X$ such that $T_i \subset T_{i+1}$. Let $T = \bigcup_i T_i$. The argument of the end of the proof of Proposition 3.4 now allows one to conclude that T is an L-treeing of $E_G^X = \bigcup_i E_{\Gamma_i}^X$. □

3.3. *Construction of Karrass, Pietrowski and Solitar*

Definition 3.14. We say that a group G is a *KPS group* if it is a HNN extension of a tree product of a countable set of finite groups with uniformly bounded orders, whose associated subgroups are contained in vertices of the tree product base.

This class of groups was introduced in [**22**], where it was also proved that each KPS group is a finite extension of a free group.

Now we are ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Since G is countable, it follows from Propositions 3.13 and 3.4 that G is a strongly treeable group. □

We now give some applications of Theorem 3.1.

PROPOSITION 3.15. *Let K be a free product of a countable collection of a finite abelian groups $\{A_i \mid i \in \mathbb{N}\}$ with uniformly bounded orders. Then any finite extension of K is a strongly treeable group.*

The proof of this proposition is based on the following lemma.

LEMMA 3.16. *Let K be as in the statement Proposition 3.15. Then K is a finite normal extension of a free group.*

Proof. Let $H \leq K$ be the centralizer subgroup, generated by $xyx^{-1}y^{-1}$, $x, y \in K$. It is well known that H is a free normal subgroup of K and K/H is abelian.

Suppose first that $A_i = \{1\}$ for $i > m \in \mathbb{N}$. Let $A^m = \times_{i=1}^m A_i$, and write $A_i = \{a_1^i, \ldots, a_{p_i}^i\}$ for $i = 1, \ldots, m$, $p_i < \infty$. Then $\{a_j^i \mid 1 \leq i \leq m, 1 \leq j \leq p_i\}$ are generators for K and also for A^m. This allows us to conclude that K/H and A^m are isomorphic. (In fact, further analysis shows that H is a f.g. free group, see Corollary 4.6 below.)

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Suppose now that $\{A_i \mid i \in \mathbb{N}\}$, is such that $A_i \neq \{1\}$ for sufficiently large i. Consider firstly the case when each A_i is a copy of a fixed finite abelian group A, and let $L = \times_{i\in\mathbb{N}} A_i$. Then we have two canonical maps $\varphi_1 : K \to L$ and $\varphi_2 : L \to A$, defined by $\varphi_1(a_{i_1}, \ldots, a_{i_r}) = (a_{i_1}, \ldots, a_{i_r})$, where $a_i \in A_i$, $a_{i_1} \ldots, a_{i_r} \in K$, $(a_{i_1}, \ldots, a_{i_r}) \in L$, and $\varphi_2((a_{i_1}, \ldots, a_{i_r})) = a_{i_1}, \ldots, a_{i_2} \cdot \cdots \cdot a_{i_r}$, where $a_{i_1} \cdot \cdots \cdot a_{i_r} \in A$.

Since $A_i \to \varphi_1(A_i)$ is an isomorphism, for any i, we put $\varphi_1(A_i) = A_i$. Similarly, we put $\varphi_2(A_i) = A$, and let $H_1 = \ker \varphi_1$ and $H_2 = \ker \varphi_2$.

It is clear that H_2 is a normal subgroup of L. We claim that $L = A_i H_2$ for any i. To see this, we first describe the structure of H_2. Let $A = \{a_1, \ldots, a_t\}$ and write $A_i = \{a_1^i, \ldots, a_t^i\}$. Suppose that $(a_1^{i_1}, a_2^{i_2}, \ldots, a_s^{i_s})$, $s < \infty$, belongs to L. Then $\varphi_2(a_1^{i_1}, \ldots, a_s^{i_s}) = a_1, \ldots, a_s \in A$. It is now obvious that φ_2 is a homomorphism from L onto A, and that

$$H_2 = \{(a_1^{i_1}, \ldots, a_s^{i_s}) \in L \mid \varphi_2((a_1^{i_1}, \ldots, a_s^{i_s})) = a_1 \cdot \cdots \cdot a_s = e\}.$$

This shows that, for any i, $L = A_i H_2$, as claimed. Now define $N = \varphi_1^{-1}(H_2)$. Then N is a normal subgroup of K, and since φ_1 is a surjective homomorphism from K onto L, K/N is isomorphic to $L/H_2 \approx A_i \approx A$ by [**27**, Ch. I, §4(v)]. Thus, N is a normal subgroup of finite index in K and $K/N \approx A$.

We now show that N is a free group. As N is a normal subgroup of K we see that $A_i/(A_i \cap N) \approx A_i N/N$ by [**27**, Ch. I, §4(iv)]. Recall that $H_1 \subset N$ is also a normal subgroup of K: hence we obtain $A_i N/N \approx \varphi_1(A_i N)/\varphi_1(N)$ by [**27**, Ch. I, §4(iii)]. But $\varphi_1(A_i N) = \varphi_1(A_i)\varphi_1(N) = A_i H_2 = L$ and $\varphi_1(N) = H_2$, hence $A_i N/N \approx L/H_2 \approx A$. Thus, we see that $A_i/A_i \cap N \approx A$. This is possible if and only if

$$A_i \cap N = \{1\}.$$

Further, for each $k \in K$,

$$k A_i k^{-1} \cap N = k(A_i \cap N)k^{-1} = \{1\}.$$

On the other hand, the Kurosh subgroup theorem [**29**] implies that there is a free group F so that

$$N = F * (*_{i,k}(k A_i k^{-1} \cap N))$$

for some $i \in \mathbb{N}$ and some $k \in K$. One can conclude the relations above that $N = F$, i.e. the group K is a finite normal extension of a free group.

Now we consider the general case. As the groups A_i have uniformly bounded orders, there is only a finite number of them which are pairwise non-isomorphic. Denote these groups as $\{B_i \mid 1 \le i \le t\}$.

To study this case we apply the same approach as above. Thus, let $L = \times_{i=1}^{\infty} A_i$ and $B = \times_{i=1}^{t} B_i$. Consider the canonical maps $\varphi_1 : K \to L$ and $\varphi_2 : L \to B$ defined as above. Again we see that $N = \varphi^{-1}(H_2)$ is a free subgroup of finite index in K with $K/N \approx B$. □

Now we are ready to complete the proof of Proposition 3.15.

Proof of Proposition 3.15. Let K be as in the statement of Proposition 3.15, and N be a free subgroup of K with $[K : N] < \infty$, the existence of which was proved in Lemma 3.16. If

$G_f(K)$ is a finite extension of K, then $[G_f(K) : K] < \infty$. Thus, $[G_f(K) : N] = [G_f(K) : K] \cdot [K : N] < \infty$, and $G_f(K)$ is a finite extension of N. It follows by Corollary 3.3 that $G_f(K)$ is a strongly treeable group. □

Remark 3.17. The KPS construction allows us to give many examples of finite extensions of free groups. There is a similar method of constructing extensions using finite groups of outer automorphisms of a free group. Let F be a free group, Aut F the group of all automorphisms of F, and G a finite subgroup of Aut F. Then one can construct a finite extension of F, for example, as a semi-direct product of F and G. Finite subgroups of Aut F have been considered by different authors, see the review of Roman'kov [**34**]. In particular, interesting examples of periodic automorphisms of free groups can be found in articles of Dyer and Scott [**11**], McCool [**30**] and Meskin [**31**].

4. *Finite extensions of free products of f.g. amenable groups.*

In this section, we consider free products of groups $K = K_1 * K_2 * \cdots * K_k$, $k < \infty$, where each K_i is a f.g. amenable group all of whose subgroups are also f.g. These groups K are strongly treeable (see §2.6).

In §4.1 we consider the case when each K_i is a torsion-free group and prove that any finite extension $G_f(K)$ of K is also a strongly treeable group, see Theorem 4.4. This result is important in §5.

Let $\mathcal{B}$ be the set of all f.g. amenable groups all of whose subgroups are also f.g., and let $\mathcal{A}$ be the subset of $\mathcal{B}$ consisting of finite groups, all torsion-free groups and their finite extensions.

The simplest example of an amenable torsion free f.g. group all of whose subgroups are f.g. is $\mathbb{Z}^n$, $n < \infty$, so we have $\mathbb{Z}^n \in \mathcal{A}$. Another interesting examples is the solvable group of all $n \times n$ quasi-triangular matrices over $\mathbb{Z}$ with determinant equal to one. It is easy to construct semi-direct products of $\mathbb{Z}$ and $(\bigoplus \mathbb{Z})^{\mathbb{Z}}$ which is a f.g. amenable group not belonging to $\mathcal{A}$.

In §4.2, we extend the results of §4.1 to show that if $K_i \in \mathcal{A}$ for any i, then $G_f(K)$ is again strongly treeable, see Theorem 4.5. Furthermore, if the group K has a torsion, then K is a normal finite extension of a free product of torsion-free groups K_i from $\mathcal{A}$, see Corollary 4.6.

4.1. *Free products of f.g. torsion-free amenable groups.*

We first consider the subclass consisting of free products of $\mathbb{Z}^n$ in detail, i.e. the groups $F(n_1, n_2, \ldots, n_k) = \mathbb{Z}^{n_1} * \cdots * \mathbb{Z}^{n_k}$, where $k < \infty$, and each n_i is an integer. We use these groups also in §5.

Recall the *rank* of a group H (rank(H)) is the minimum number of generators of H.

LEMMA 4.1. *Let K be a subgroup of the group $F(n_1, \ldots, n_k)$ as above, of finite rank. Then K is isomorphic to a group of the form $F(n'_1, \ldots, n'_l)$ this subclass, where $k \le l < \infty$. Furthermore, for any n'_j, $1 \le j \le l$, there exists n_i, $1 \le i \le k$ such that $n'_j \le n_i$.*

Proof. Recall that the subgroup K has the form

$$K = F * (*_{j=1}^{s} K_j)$$

where F is a free group, and each K_j is the intersection of K with a conjugate of some factor $\mathbb{Z}^{n_i}$ of $F(n_1, \ldots, n_k)$ by the Kurosh subgroup theorem [**29**]. Furthermore,

$$\mathrm{rank}(K) = \mathrm{rank}(F) + \sum_{j=1}^{s} \mathrm{rank}(K_j),$$

by the Grushko–Neumann theorem [**29**, Ch. IV, Corollary 1.9]. Hence, $s \leq \mathrm{rank}(K) < \infty$, and since $K_j = K \cap g\mathbb{Z}^{n_j} g^{-1}$ for some $g \in F(n_1, \ldots, n_k)$ and $j \leq s$ it follows that $K_j \approx \mathbb{Z}^{m_j}$ where $1 \leq m_j \leq n_j$. Thus, K is isomorphic to a group of the form $F(m_1, \ldots, m_l)$, where $l < \infty$, $1 \leq m_j \leq n_i$, and $1 \leq j \leq l$, $1 \leq i \leq k$. □

LEMMA 4.2. *Let $G = F(n_1, \ldots, n_k)$ be as in the statement of Lemma 4.1, and K be a normal subgroup of G of finite index. Then* $\mathrm{rank}(K) < \infty$*, and K is isomorphic to $F(m_1, \ldots, m_l)$ where $l < \infty$. Furthermore, any $j = 1f \ldots, l$, m_j is equal either to one or to n_i, $1 \leq i \leq k$. For any n_i there is at least one j such that $m_j = n_i$. Thus, K is isomorphic to $F * (*_{i=1}^{k}(*(\mathbb{Z}^{n_i}))^{p_i})$ where $p_i > 0$ is an integer, and F is a free group of a finite rank.*

Proof. Let $d = \sum_{i=1}^{k} n_i < \infty$, and let F_d denote the free group of rank d. We have the canonical surjective homomorphism $\varphi : F_d \to G$. Let $H = \varphi^{-1}(K)$. Then [**27**, I, §4(v)] H is normal in F_d, and the quotient group F_d/H is isomorphic to G/K. Hence, $[F_d : H] = [G : K] < \infty$, and as H is a subgroup of finite index in the free group F_d, we have $\mathrm{rank}(H) < \infty$ (see [**29**, Ch. I, Proposition 3.9]). As $K = \varphi(H)$ we must also have $\mathrm{rank}(K) < \infty$.

It follows from Lemma 4.1 that K has the form $F(m_1, \ldots, m_l)$, where $l \leq \infty$. We apply the Kurosh subgroup theorem [**29**] (version of [**23**, Introduction] or [**6**, Ch. 7, Theorem 8]). It follows that $K = F * (*_{i,g} K_{i,g})$, where F is a free group of finite rank, $K_{j,g} = K \cap g(\mathbb{Z}^{n_j})g^{-1}$ and there is a free subfactor $K_{i,g}$ in K for each i, $1 \leq i \leq k$, and each g in a given set of representatives of the $(K, \mathbb{Z}^{n_i})$ double cosets of G.

As K is normal in G:

$$g(\mathbb{Z}^{n_j})g^{-1}/K_j \approx g(\mathbb{Z}^{n_j})g^{-1}K/K \subseteq G/K$$

by [**27**, Ch. I, §4(iv)]. However, $[G : K] < \infty$, and it follows that K_j is isomorphic to $g(\mathbb{Z}^{n_j})g^{-1}$.

Indeed take a subgroup of $g(\mathbb{Z}^{n_j})g^{-1}$ of the form $W = g(\mathbb{Z})g^{-1}$. Then $K \cap W$ is a normal subgroup of W, and $[W : K \cap W)] \leq [G : K] < \infty$. Hence, $K \cap W$ is isomorphic to $m\mathbb{Z}$ for some $m \in \mathbb{N}$. Thus, K_j is a subgroup of $g(\mathbb{Z}^{n_j})g^{-1}$ with the same number of generators as $g(\mathbb{Z}^{n_j})g^{-1}$ and, hence, $K_{j,g} \approx \mathbb{Z}^{n_j}$. □

Now we are ready to prove the following theorem.

THEOREM 4.3. *A finite extension of the group $H = F(n_1, \ldots, n_k)$ is also a strongly treeable group.*

Proof. We use induction on the rank of H. Note that the statement holds for H of rank one since in this case $H = \mathbb{Z}$. If $\mathrm{rank}(H) = 2$ then either $H = \mathbb{Z}^2$ or $H = F_2$. However, $\mathbb{Z}^2$ is abelian and our statement holds for $\mathbb{Z}^2$. The case $H = F_2$ was proved in §3 above.

Suppose that the statement holds for all $H = F(n_1, \ldots, n_k)$ with rank$(H) = \sum_{i=1}^{k} n_i = m$, and consider a group H of the form $H = F(n_1, \ldots, n_k)$ with rank$(H) = m + 1$. As rank(H) is finite, H has an infinite number of ends [**5**, §2, Example 2]. Since $[G : H] < \infty$, it follows by [**42**], see [**5**, Proposition 2.1] that G is also a f.g. group with an infinite number of ends. Now by [**42**] either:

(i) $G = A *_C B$, where A, B are subgroups of G, C is a finite group, and $A \neq C \neq B$; or

(ii) $G = \langle t, K; tCt^{-1} = \varphi(C)\rangle$, where C is a finite subgroup of K, φ is an isomorphism from C to $\varphi(C) \subset K$.

First suppose that (i) holds. If A and B are finite, the conclusion follows (see Lemma 3.7). Assume, therefore, that A is infinite. Since H has trivial intersection with any conjugate of C in G, we may apply the Neumann subgroup theorem [**23, 24, 29**, Ch. IV, Theorem 6.6] to deduce that $H = (H \cap A) * (H \cap B) * \cdots$. As $[G : H] < \infty$ and $[G : A] = \infty$ it follows that $H \cap A \neq H$.

By the Grushko–Neumann theorem [**29**], we have rank$(H \cap A) <$ rank$(H) < \infty$, and as above, $H \cap A$ is isomorphic to the group $F(n'_1, \ldots, n'_s)$, $s < \infty$ by Lemma 4.1.

We claim that $H \cap A \neq \{1\}$. As $[G : H] < \infty$, there is a normal subgroup H' of G such that $H' \subset H$ and $[G : H'] < \infty$. If $H \cap A = \{1\}$, then also $H' \cap A = \{1\}$, and $A = A/(H' \cap A) \approx AH'/H' \subset G/H'$. As $[G : H'] < \infty$, this means that A is a finite group, which contradicts our assumption on A, and hence proves our claim.

We thus have $\{1\} \neq (H \cap A) \neq H$. Furthermore, because $[A : (H' \cap A)] \leq [G : H'] < \infty$ and $[A : (H \cap A)] \leq [A : (H' \cap A)] < \infty$, we see that A is a finite extension of $H \cap A$. As rank$(H \cap A) <$ rank$(H) < \infty$ and $H \cap A$ has the same form as H (Lemma 4.1), one can apply the inductive hypothesis to conclude that A is a strongly treeable group. The same argument shows that B is also strongly treeable. Now $G = A *_C B$ is strongly treeable by Corollary 3.8.

Now suppose that (ii) above holds. If K is a finite group, then the conclusion follows from Lemma 3.9. Assume therefore that K is infinite. As H and K both have finite index in G, it follows $K \cap H \neq H$. As $[G : H] < \infty$ we can use the argument of (i) to prove that $K \cap H \neq \{1\}$. Thus, we have, as above, $\{1\} \neq (H \cap K) \neq H$. Since H intersects the conjugates of C in G trivially, the Neumann subgroup theorem gives $H = (H \cap K) * \cdots$, and hence rank$(H \cap K) <$ rank(H). Moreover, as in (i), H is a finite extension of $H \cap K$, and $H \cap K$ has the same form as H by Lemma 4.1. Hence, one can apply the inductive hypothesis to K and $H \cap K$ to conclude that K is a strongly treeable group. It now follows from Lemma 3.9 that G is strongly treeable. □

It is not too hard to generalize Theorem 4.3 to a free product of elements in $\mathcal{A}$.

THEOREM 4.4. *Let G be a finite extension of a group H of the form $H = *_{i=1}^{k} H_i$, $k < \infty$, where H_i is a torsion free f.g. amenable group from $\mathcal{A}$. Then G is a strongly treeable group.*

Proof. Note first that each amenable group H_i is strongly treeable. Thus, H is a f.g. group, and it has infinitely many ends in the sense of Stallings [**42**]. Hence, G also has this property, and we can apply results of [**42**] to G and use induction on rank(H) as in the proof of Theorem 4.3. It is easy to see that a lemma analogous to Lemma 4.1 holds. Hence, it is easy to modify the approach of Theorem 4.3 to this more general case. □

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4.2. *Free products of f.g. amenable groups with a torsion.* In this section, we extend Theorem 4.4 to a free product of f.g. amenable groups with torsion.

THEOREM 4.5. *For $k \in \mathbb{N}$, choose groups $\{H_i \in \mathcal{A} : 1 \le i \le k\}$. Let G be a finite extension of the group $H = *_{i=1}^{k} H_i$. Then G is strongly treeable.*

Proof. To simplify the proof, we suppose that each H_i is a f.g. abelian group. The general case has some obvious technical complications, which are left to the reader. Then each H_i has the form $H_i = \mathbb{Z}^{n_i} \times A_i$, where n_i is an integer and A_i is a finite abelian group by [**27**, Ch. I, §10, Theorem 8].

Let K be the normalizer subgroup of H generated by elements $xyx^{-1}y^{-1}$, $x, y \in H$. As is well known, $H/K \approxeq \times_{i=1}^{k}(\mathbb{Z}^{n_i} \times A_i)$. If $Z = \times_{i=1}^{k}\mathbb{Z}^{n_i}$ and $A = \times_{i=1}^{k} A_i$, then the homomorphism $\varphi : H \to Z \times A$ is defined as follows: $\varphi(z_1 a_1 \cdots z_k a_k) = (\prod_{i=1}^{k} z_i, \prod_{i=1}^{k} a_i)$ where $z_i \in \mathbb{Z}^{n_i}$, $a_i \in A_i$, $z_1 a_1 \cdots z_k a_k \in H$ and $\prod_i z_i \in Z$, $\prod_i a_i \in A$.

This definition of φ shows that φ is surjective and $\ker \varphi = K$. Since Z is a normal subgroup of finite index in $Z \times A$, it follows from [**27**, Ch. I, §4(v)] that $N = \varphi^{-1}(Z)$ is a normal subgroup of finite index in H.

It is obvious that N is a torsion-free subgroup of H. Indeed, $K \subset N \subset H$, where K is a free subgroup of H. Choose a family $\{\gamma_i\}_{i \in \mathbb{N}}$ of coset representatives of N/K, where $\gamma_i \in \mathbb{Z}$, and let n be an element of a finite order t from N.

Then $n = \gamma_s k$ where $\gamma_s \in \{\gamma_i\}$, $k \in K$. Now we have $\varphi(n^t) = (\varphi(n))^t = \gamma_s^t = e$, and hence $\gamma_s = e$, because Z is torsion free. Therefore, $n = k \in K$ and, hence, $n = e$ as K is a free group.

Since N is torsion free, we obtain

$$N \cap (\mathbb{Z}^{n_i} \times A_i) = N \cap \mathbb{Z}^{n_i},$$

and since N is a normal subgroup of H then for any $h \in H$,

$$N_{i,h} = N \cap h(\mathbb{Z}^{n_i} \times A_i)h^{-1} = h(N \cap \mathbb{Z}^{n_i})h^{-1}.$$

It follows from the Kurosh subgroup theorem that

$$N = F * (*_{i,h} N_{i,h}),$$

for some $h \in H$, $1 \le i \le k$, $k < \infty$.

Now we have assumed that $d = \operatorname{rank}(H) < \infty$. Hence, as above, we have the canonical surjective homomorphism $\psi : F_d \to H$ where as usual F_d is a free group of rank d. Let $N' = \psi^{-1}(N)$: then N' is a normal subgroup of F_d and $F_d/N' \approx H/N$ by [**27**, Ch. I, §4,(v)]. As $[H : N] < \infty$ by construction, N' is a finite index subgroup of F_d, and hence $\operatorname{rank}(N') < \infty$ by [**29**, Ch. I, Proposition 3.9]. It follows that $\operatorname{rank}(N) = \operatorname{rank}(\psi(N')) < \infty$ also. As N is a free product of groups F and $N_{i,h}$ as above, we have by the Grushko–Neumann theorem

$$\operatorname{rank}(N) = \operatorname{rank}(F) + \sum_{i,h} \operatorname{rank}(N_{i,h}) < \infty.$$

It follows from this equality that $\operatorname{rank}(F) < \infty$ and only a finite number of terms $\operatorname{rank}(N_{i,h})$ in this sum differ from the zero. This means that N is a free product of a

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finite number of groups of the form $\mathbb{Z}^{n_i}$, $n_i \in \mathbb{N}$. Finite extensions of these class of groups were studied in Theorem 4.3.

Now, if G is a finite extension of H, then G is also a finite extension of N, because $[G : N] = [G : H] \cdot [H : N] < \infty$. It now follows from Theorem 4.3 that G is strongly treeable. □

The proof of Theorem 4.5 has the following corollaries.

COROLLARY 4.6. *Let H be as in the statement of Theorem 4.5. If H has an element of finite order, then H is a finite normal extension of a torsion free group N of the form $N = *_{i=1}^{k} N_i$ where $k < \infty$, and each N_i is a f.g. torsion-free amenable group (cf. Theorem 4.4). In particular, if H is a free product of finite groups, then N is a f.g. free group.*

COROLLARY 4.7. *Let $H = *_{i=1}^{k} H_i$ be as in the statement of Theorem 4.5. If each H_i is a f.g. abelian group, then H is a strongly treeable group.*

5. *The geometric dimension of an equivalence relation and finite extensions of groups*

In this section we give some applications of results of §§3 and 4. We consider free actions of groups and their finite extensions on the standard Borel space (X, μ) which define equivalence relations with geometric dimension (see [**14, 15**]) greater than one. We present a generalization of results of §§3 and 4 for such groups: Theorems 5.9, 5.11 and 5.13.

5.1. *Geometric dimension.*

We recall Gaboriau's definition of the geometric dimension of a measure-preserving equivalence relation R on a standard Borel probability space (X, μ), see §2.2.

Definition 5.1. [**14, 15**]. A simplicial complex with standard left R-action, or more briefly an R-simplicial complex Σ, consists of following data.

- A discrete left R-space $\Sigma^{(0)} \to X$ (space of vertices);
- For each $n \in \mathbb{N}$, a Borel subset $\Sigma^{(n)} \subset \Sigma^{(0)} * \cdots * \Sigma^{(0)}$ ($n + 1$ times) called the space n-simplices (possible empty for large n), satisfying the following four conditions:
  - (i) (permutation) $\Sigma^{(n)}$ is invariant under permutation of the coordinates;
  - (ii) (non-degeneracy) if $(v_0, v_1, \ldots, v_n) \in \Sigma^{(n)}$ then $v_0 \neq v_1$;
  - (iii) (boundary conditions) $(v_1, v_2, \ldots, v_n) \in \Sigma^{(n-1)}$;
  - (iv) (invariance) $R \cdot \Sigma^{(n)} = \Sigma^{(n)}$.

The data in the fiber of each $x \in X$ is just an ordinary (countable) simplicial complex, denoted by Σ_x.

The R-simplicial complex Σ is *n-connected*, respectively *contractible*, respectively *n-dimensional* if for almost all x in X, the simplicial complex Σ_x has corresponding properties. For some examples, see [**14, 15**].

Definition 5.2. [**14, 15**]. The geometric dimension of an equivalence relation R (geom-dim(R)) is the smallest dimension of a contractible R-simplicial complex.

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Examples 5.3. We give several examples which will be crucial in what follows.

(i) Choose a finite sequence of integers $\{p_1, \ldots, p_l\}$. Let

$$G_l = G_l(p_1, \ldots, p_l) = F_{p_1} \times \cdots \times F_{p_l},$$

where F_{p_i} is a free group with p_i generators. It follows from [**14**] that if $p_i > 1$ for $i = 1, 2, \ldots, l$ then each free Borel action of G_l on a standard Borel space (X, μ) with G_l-invariant probability measure μ induces an equivalence relation $E^X_{G_l}$ such that geom-dim$(E^X_{G_l}) = l$. In particular, if $l = 1$, then $G_1 = F_{p_1}$ is a free group and, as is well known, geom-dim$(E^X_{G_1}) = 1$.

(ii) Let A be a countably infinite amenable group and G_l as in (i). It is shown in [**14**] that geom-dim$(E^X_A) = 1$ and geom-dim$(E^X_{A\times G_l}) = l + 1$.

(iii) Let $F(n) = F(1, 1) \times F(2, 2) \times \cdots \times F(n, n)$, $n \in \mathbb{N}$, where the groups $F(i, i) = \mathbb{Z}^i * \mathbb{Z}^i$ were introduced in §4. It follows from [**13**, §VII] and [**14**, §3.5], that any free Borel action of $F(n)$ on (X, μ) as above induces an equivalence relation $E^X_{F(n)}$ such that geom-dim$(E^X_{F(n)}) = n$.

Definition 5.4. Let G be a countably infinite group and let $n \in \mathbb{N}$. We say that the geometric dimension of G is equal to n, geom-dim$(G) = n$, if any free measure-preserving Borel action of G on a standard Borel probability space (X, μ), induces an equivalence relation E^X_G with geom-dim$(E^X_G) = n$.

In Examples 5.3, we have geom-dim$(G_l) = l$, geom-dim$(F_p) = 1$, geom-dim$(A) = 1$, geom-dim$(A \times G_l) = l + 1$, and geom-dim$(F(n)) = n$.

Recall [**13**] that if $l > 1$, then $C(G_l) = 1$, and $C(A \times G_l) = 1$, where $C(H)$ is the cost of a group H. On the other hand, [**13, 14**] if $p > 1$, then $C(G_l * F_p) = p + 1$, and geom-dim$(G_l * F_p) = l$.

Gaboriau [**14**, Definition 6.4] introduced the notion of ergodic dimension for a more general class of groups than in Definition 5.4. This definition was introduced as an analogue of the notion of cohomological dimension (cd) from the theory of infinite groups, and has deep relations with l^2 Betti numbers. A theorem of Serre [**3**, Ch. VIII, §3] states that if K is a torsion-free group and K' is a subgroup of finite index, then cd$(K) =$ cd(K').

Note that if K' is a finite extension of K, then erg-dim$(K) =$ erg-dim(K'). This is easy to see from [**14**, Definition 6.4], and properties of induced actions of groups [**25**, §34.1]. We use similar arguments in the proof of Lemma 5.5. If geom-dim$(K) = n \in \mathbb{N}$, and K' is again a finite extension of K and if K' has finite geometric dimension, then geom-dim$(K') =$ geom-dim(K).

In §§3 and 4, we discussed this question for groups K, where geom-dim$(K) = 1$. We showed the existence of many groups K such that geom-dim$(K') =$ geom-dim$(K) = 1$ for any finite extension K' of K. We show below that for any natural number $n > 1$, there exists a group K_n with geom-dim$(K_n) = n$ such that geom-dim$(G_f(K_n)) = n$ for any finite extension $G_f(K_n)$ of K_n.

5.2. *Finite extensions of groups G_l and $A \times G_l$.*

In this section we consider the geometric dimensions of finite extensions of the groups G_l. The main results are Lemma 5.8 and Theorem 5.9.

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LEMMA 5.5. *Let G be a countable group with* geom-dim$(G) = l \geq 1$, *and let H be a subgroup of G of finite index. Then* geom-dim$(H) = l$.

Proof. Given a free measure-preserving action of H on a standard Borel probability space (X, μ), we construct the induced action of G on the space $Y = X \times (G/H)$, see [**25**, §34.1]. Consider the equivalence relations E^Y_G and E^X_H. It follows from the construction of induced actions that $E^Y_G|_X = E^X_H$, see [**25**, §34.1]. Thus, E^Y_G and E^X_H are stably orbit equivalent (SOE) equivalence relations (see [**14**, §5.1] and [**15**, §1.4]). Hence, geom-dim(E^X_H) = geom-dim$(E^Y_G) = l$, see [**14**, §5.2]. Thus, any free action of H on any standard Borel space (X, μ) defines an equivalence relation E^X_H with geom-dim$(E^X_H) = l$. □

Definition 5.6. If H is a group, we denote by $G_f(H)$ a finite extension of H.

If H has one of the forms from Examples 5.3:

- $H = G_l$;
- $H = A \times G_l$ where A is a countably infinite amenable group;
- $H = F(n)$;

then we say that a finite extension $G_f(H)$ of H has *standard* form if it has one of the three forms (respectively):

- $G_f(H) \subseteq G'_f(G_l) = G_f(F_{p_1}) \times \cdots \times G_f(F_{p_l})$;
- $G_f(H) \subseteq G' f(A \times G_l) = G_f(A) \times G'_f(G_l)$;
- $G_f(H) \subseteq G'_f(F(n)) = G_f(F(1, 1)) \times \cdots \times G_f(F(n, n))$.

Let $G_{fn}(G)$ be a finite *normal* extension of a group G, i.e. G is a normal subgroup of $G_{fn}(G)$ and $[G_{fn}(G) : G] < \infty$.

COROLLARY 5.7.

(i) *We have* geom-dim$(G'_f(G_l)) = l$, *and* geom-dim$(G'_f(A \times (G_l)) = l + 1$.

(ii) *Moreover, if K is a finite extension of G_l (respectively $A \times G_l$) in standard form, then* geom-dim$(K) = l$ *(respectively* geom-dim$(K) = l + 1$*)*.

(iii) *If K is a finite extension of $F(n)$ in standard form, then* geom-dim$(G_f(F(n))$ = geom-dim$(G_f(F(1, 1)) \times \cdots \times G_f(F(n, n)) =$ geom-dim$(F(n)) = n$.

Proof. (i) It follows from Corollary 3.2 and Corollary 3.3 that geom-dim$(G_f(F_p))$ = 1. Thus we have that geom-dim$(G'_f(G_l)) = l$ and geom-dim$(G'_f(A \times G_l)) = l + 1$ by [**14**, §3.5]. For (ii), Lemma 5.5 shows that geom-dim$(K) = l$ (respectively geom-dim(K) $= l + 1$).

(iii) This follows from Theorem 4.3, [**14**, §3.5] and Lemma 5.5. □

A natural question arising from this corollary is to understand whether there is a group of the form G_l or $A \times G_l$ such that its finite extensions all have standard form. We show below the existence of such groups. We need the following lemma.

LEMMA 5.8. *Let $G_l = F_{p_1} \times \cdots \times F_{p_l}$ be as above, $p_1 < p_2 < \cdots < p_l$, and let A be countably infinite amenable group with trivial centre.*

Then $G_{fn}(G_l)$ (respectively $G_{fn}(A \times G_l)$) has standard form. Hence, geom-dim $(G'_{fn}(G_l)) = l$ *(respectively* geom-dim$(A \times G'_{fn}(G_l)) = l + 1$).

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Proof. Consider the case $G_2 = F_{p_1} \times F_{p_2}$, $p_1 < p_2$. The general case is completely similar. For $s \in G_{fn}(G_2)$, $s \notin G_2$, let $\mathrm{Ad}(s) = \alpha$. Then $\alpha(G_2) = G_2$. Note that $\alpha(F_{p_1}) = F_{p_2}$ is impossible because $p_1 \neq p_2$.

We claim that $\alpha(F_{p_i}) = F_{p_i}$, $i = 1, 2$. Indeed, if $x \in F_{p_1}$ then the centralizer C_x of x in G_2 coincides with $(x^n, n \in \mathbb{Z}) \times F_{p_2}$. As the centralizer $C_{\alpha(x)}$ of $\alpha(x)$ must be isomorphic to C_x then a simple analysis shows that $\alpha(x)$ belongs to F_{p_1}. Thus, $\alpha(F_{p_1}) = F_{p_1}$ and $\alpha(F_{p_2}) = F_{p_2}$. Hence, $\alpha = \alpha_1 \times \alpha_2$ where $\alpha_i \in \mathrm{Aut}\, F_{p_i}$, $i = 1, 2$. Since the F_{p_i}, $i = 1, 2$ are free groups, it follows from the basic properties of normal extensions of groups that $G_{fn}(G_2)$ is a subgroup of $G_{fn}(F_{p_1}) \times G_{fn}(F_{p_2})$ where $G_{fn}(F_{p_i})$ is a finite normal extension of F_{p_i}. In fact, it is easy to describe $G_{fn}(F_{p_i})$ explicitly: it is isomorphic to $G_{fn}(G_2)/Z_{F_{p_i}}$ where $Z_{F_{p_i}}$ is the centralizer of F_{p_i} in $G_{fn}(G_2)$.

Thus, the extension $G_{fn}(G_2)$ has standard form, and $\text{geom-dim}(G_{fn}(G_2)) = \text{geom-dim}(G_2) = 2$ by Corollary 5.7. □

THEOREM 5.9. *Let A be a countably infinite amenable group and assume that any normal subgroup $B \trianglelefteq A$ of finite index has trivial centre. Consider the following groups:*

- $K_1 = A \times F_n$, $1 < n \leq \infty$;
- $K_2 = F_n \times F_\infty$, $n < \infty$;
- $K_3 = A \times F_n \times F_\infty$, $n < \infty$.

Then each finite extension $G_f(K_i)$ of K_i has standard form, and in particular, $\text{geom-dim}(G_f(K_i)) = \text{geom-dim}(K_i) = 2$ *if* $i = 1, 2$, *and* $\text{geom-dim}(G_f(K_3)) = \text{geom-dim}(K_3) = 3$.

Proof. If H and K are subgroups of G, we let N_K denote the normalizer of K in G. Recall that if $H \leq N_K$, then there is a surjection $\varphi : H \to HK/K$, with kernel $H \cap K$, and $H/(H \cap K) \approx HK/K$, see [**27**, Ch. I, §4].

We consider in detail only the group K_2, and let $G = G_f(K_2)$. The other cases are similar. As is well known, there exists a normal subgroup N of G such that $N \subset K_2$ and $[G : N] < \infty$. Consider the subgroups $F_n \cap N$ and $F_\infty \cap N$ of G, and note that the normalizer of N is G. It follows from the remark above that

- $[F_n : F_n \cap N] = [F_n N : N] < [G : N] < \infty$,
- $[F_\infty : F_\infty \cap N] < [G : N] < \infty$.

Hence, if $N' = (F_n \cap N) \times (F_\infty \cap N)$, then $[G : N'] = [G : K_2]\cdot [F_n \times F_\infty : N'] < \infty$.

Let us show that N' is also a normal subgroup of G. Note first that $N \cap F_n$ is a free normal subgroup of F_n, where $[F_n : (F_n \cap N)] < \infty$ (see [**29**, Ch. I, §3]). Moreover, as $\mathrm{rank}(F_n \cap N) - 1 = (\mathrm{rank}(F_n) - 1)\cdot [F_n : (F_n \cap N)] < \infty$, see also [**29**, Ch. I, §3], we must have $\mathrm{rank}(F_n \cap N) < \infty$. The same argument shows that $N \cap F_\infty$ is a free normal subgroup of F_∞. It is clear that $\mathrm{rank}(N \cap F_\infty) = \infty$; otherwise, $\mathrm{rank}(F_\infty) < \infty$ which is impossible. Thus, $F_n \cap N$ is not isomorphic to $F_\infty \cap N$ for any $n < \infty$.

It follows from the construction that N' is a normal subgroup of $K_2 = F_n \times F_\infty$. As above, we take $\alpha = \alpha(s) = \mathrm{Ad}s$ for $s \in G$, $s \notin K_2$. It follows from our assumptions on N that $\mathrm{Ad}(s) \in \mathrm{Aut}\, N$. If $a \in N$, $a \notin N'$, then a must have the form $a = a_1 a_2$, where $a_1 \in F_n$, $a_2 \in F_\infty$, and $a_1, a_2 \neq e$. Hence, the centralizer C_a of a in N contains only elements of the form $a_1^n a_2^m$, $n, m \in \mathbb{Z}$. This means that C_a is an abelian subgroup of N.

On the other hand, if $x \in F_n \cap N$, then the centralizer C_x of x in N has the form $\{x^n \mid n \in \mathbb{Z}\} \times (F_\infty \cap N)$. Hence, the centralizer $C_{\alpha(x)}$ of $\alpha(x)$ must be isomorphic to C_x. This observation shows that $\alpha(x)$ cannot have the form $\alpha(x) = a_1 a_2$, where $a_1 \in F_n$, $a_2 \in F_\infty$, and $a_1, a_2 \neq e$, nor can it have the form $\alpha(x) = a_2 \in F_\infty$, $a_2 \neq e$. There remains only one possibility, that $\alpha(x) = a_1 \in F_n \cap N$, and hence $\alpha(F_n \cap N) = (F_n \cap N)$. The same argument shows that $\alpha(F_\infty \cap N) = F_\infty \cap N$. It follows from these observations that $N' = (F_n \cap N) \times (F_\infty \cap N)$ is a normal subgroup of $G = G_f(F_n \times F_\infty)$ of finite index. Hence, G has the form $G = G_{fn}(N')$, and we are in the situation of Lemma 5.8. It follows from this lemma and Lemma 5.5 that geom-dim(G) = geom-dim(N') = geom-dim $(F_n \times F_\infty) = 2$.

Let us show that $G_f(K_2)$ has standard form. Indeed, we have that $G_f(K_2) = G_{fn}(N')$, but $G_{fn}(N')$ has standard form by Lemma 5.8, and hence $G_{fn}(N') \subset G_{fn}(F_n \cap N) \times G_{fn}(F_\infty \cap N)$. As $F_n \subset G_{fn}(F_n \cap N)$ and $F_\infty \subset G_{fn}(F_\infty \cap N)$ then $G_{fn}(F_n \cap N) \subseteq G_f(F_n)$ and $G_{fn}(F_\infty \cup N) \subseteq G_f(F_\infty$. Hence, $G_f(K_2) \subseteq G_f(F_n) \times G_f(F_\infty)$. □

Remark 5.10. One can prove that geom-dim(K) = geom-dim$(\mathbb{Z} \times F_n) = 2$ for any finite extension K of $\mathbb{Z} \times F_n$. We do not present a proof of this assertion because we do not apply it below. This proof uses other methods than those in Lemma 5.8 and Theorem 5.9. (See [**26**, Part I, §2, 2.4].)

5.3. *Finite extensions of groups $F(n)$.* In Theorem 5.9, we showed that we cannot use only free groups to produce a group K_n, $n \in \mathbb{N}$, with geom-dim$(K_n) = n$ such that geom-dim$(G_f(K_n)) = n$ for all finite extensions $G_f(K_n)$ of K_n for any $n > 1$.

In this section we consider the groups $F(n)$, $n > 1$, introduced in §5.1. We showed that geom-dim$(F(n)) = n$. In this section, we show that any finite extension of $F(n)$ has the same geometric dimension as $F(n)$. This generalizes results of §§3 and 4 where we investigated a class of groups with geometric dimension equal to one. The section is mostly based on results of §4. Recall that $F(i, i) = \mathbb{Z}^i * \mathbb{Z}^i$, $F(n) = F(1, 1) \times \cdots \times F(n, n)$, and an extension of $F(n)$ is of standard for if it has the form $G_f(F(1, 1)) \times \cdots \times G_f(F(n, n))$.

THEOREM 5.11. *Any finite extension $K = G_f(F(n))$ of the group $F(n)$ is of standard form, in particular,* geom-dim(K) = geom-dim$(F(n)) = n$.

We need the following lemma.

LEMMA 5.12. *We have the following results.*

(i) *The groups $F(i, i)$ and $F(j, j)$ are not isomorphic for $i \neq j$.*

(ii) *Let $N(i) \trianglelefteq F(i, i)$ be a normal subgroup of finite index. Then $N(i)$ is not isomorphic to $N(j)$ for $i \neq j$.*

Proof. (i) Let C_x be the centralizer of $x \in F(i, i)$, and $A_i = \{C_x \mid x \in F(i, i)\}$. It is evident that A_i contains centralizers C_x non-isomorphic to centralizers $C_{x'}$ from A_j if $i \neq j$. Hence, $F(i, i)$ is not isomorphic to $F(j, j)$ if $i \neq j$.

(ii) It follows from Lemma 4.2 that $N(i)$ is isomorphic to $F * (*(\mathbb{Z}^i))^m$ for some $m \in \mathbb{Z}$, $2 \leq m < \infty$, where F is a free group of a finite index. Hence, $N(i)$ is not isomorphic to $N(j)$ if $i \neq j$ by (i). □

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Proof of Theorem 5.11. Consider first the case $F(2) = F_2 \times F(2,2)$. Let $K = G_f(F(2))$ be a finite extension of $F(2)$. Then there exists a normal subgroup $N \subset F(2)$ of finite index in K. Now $N \cap F_2$ is a normal subgroup of finite index in F_2, and $N \cap F(2,2)$ is a normal subgroup of finite index in $F(2,2)$, see the proof of Theorem 5.9. Hence, $N' = (N \cap F_2) \times (N \cap F(2,2))$ is a subgroup of finite index in K.

Furthermore, from Lemma 5.12 and the argument on centralizers of elements of $N \cap F_2$, $N \cap F(2,2)$ and N in the proof of Theorem 5.9, it follows that $k(N \cap F(n,n))k^{-1} = N \cap F(2,2)$ for $k \in K$. This means that $N \cap F(2,2)$ is a normal subgroup of K. Hence, both $N \cap F_2$ and $N' = (N \cap F_2) \times (N \cap F(2,2))$ are normal subgroups of K. Now it follows from Lemma 5.8 that $K \subseteq G_{fn}(N \cap F_2) \times G_{fn}(N \cap F(2,2))$. As $F_2 \subseteq G_{fn}(N \cap F_2)$ and $F(2,2) \subseteq G_{fn}(N \cap F(2,2))$, we have $K \subseteq G_f(F_2) \times G_f(F(2,2))$. Hence, $\text{geom-dim}(K) = \text{geom-dim}(G_f(F_2) \times G_f(F(2,2)) = \text{geom-dim}(F(2)) = 2$, by Lemma 5.5 and Corollary 5.7.

The general case is proved similarly. Consider more explicitly the case $F(3)$. Let $K = G_f(F(3))$, and choose $N \subset F(3)$, a normal subgroup of finite index in K. Let $K_i = N \cap F(i,i)$, $i = 1, 2, 3$, and $K_{ij} = K_i \times K_j$, $i < j$, $i, j = 1, 2, 3$.

As before, we can show that K_i is a normal subgroup of K. To see this note that K_{ij} and $K_{i'j'}$ are not isomorphic if $(i, j) \neq (i', j')$. Indeed, one can see this by comparing sets of centralizers of elements in K_{ij} and $K_{i'j'}$, respectively. It is obvious they are different unless $(i, j) = (i', j')$.

Now the centralizer of any element x from K_{i_1} in N is a group of the form $A \times K_{i_2 i_3}$ where A is an abelian subgroup of K_{i_1}, and $i_1 \neq i_2, i_3$. Hence, for $k \in K$, kxk^{-1}, belongs to N: but it cannot belong to K_{i_2} or K_{i_3}. Further, kxk^{-1} cannot have the form $f_p f_q$, where $f_p \in F(p, p)$, $f_q \in F(q, q)$, because the centralizer of $f_p f_q$ in N has the form $A \times F(r, r)$, where A is an abelian subgroup of $F(p, p) \times F(q, q)$ and $r \neq p, q$. By the same reasoning kxk^{-1} cannot have the form $f_i f_2 f_3$ where $f_i \in K_i$. Thus, there is the only one possibility, namely that for all $x \in K_{i_1}$, $kxk^{-1} \in K_{i_1}$. This means that $kK_ik^{-1} \subseteq K_i$ for $i = 1, 2, 3$, and $K_i \trianglelefteq K$. Recall that K_i is a subgroup of finite index in $F(i, i)$, $i = 1, 2, 3$. Thus $N' = K_1 \times K_2 \times K_3$ is a normal subgroup of finite index in $F(3)$, and hence in K. The result is now clear. □

The following theorem summarizes the results of this subsection.

THEOREM 5.13. *We have the following results.*

(i) *For any integer n the group $F(n)$ has* $\text{geom-dim}(F(n)) = n$, *and if $G_f(F(n))$ is any finite extension of $F(n)$ then also* $\text{geom-dim}(G_f(F(n))) = n$.

(ii) *There exists a countable family of pairwise non-isomorphic groups $\{F(k, n) \mid k, n \in \mathbb{N}\}$, such that* $\text{geom-dim}(F(k, n)) = n$, *and if $G_f(F(k, n))$ is a finite extension of $F(k, n)$ then also* $\text{geom-dim}(G_f(F(k, n))) = n$. *Moreover, no group $G_f(F(k, n))$ is isomorphic to any group $G_f(F(k', n))$ if $k \neq k'$.*

Proof. Part (i) follows from Theorem 5.11. Let $F(k, n) = \times_{i=1}^{n} F(k + i, k + i)$ where $F(m, m)$ is a group as at the beginning of this section. It follows, again from Theorem 5.11, that $\text{geom-dim}(F(k, n)) = n$. Let us show that the group $F(k, n)$ is not isomorphic to $F(k', n)$ if $k \neq k'$. Indeed, let $A = \{C_x, x \in F(k, n)\}$ and $A' = \{C_x, x \in F(k', n)\}$,

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where C_x is the centralizer of $x \in F(k, n)$ in $F(k, n)$, or the centralizer of $x \in F(k', n)$ in $F(k', n)$. Now it follows from Lemma 5.12(i) that A and A' contain non-isomorphic groups C_x if $k \neq k'$, hence $F(k, n)$ and $F(k', n)$ are not isomorphic in this case.

Now suppose that $K = G_f(F(k, n))$ is isomorphic to $K' = G_f(F(k', n))$ where $k \neq k'$. It follows from the proof of Theorem 5.11 and the remark on normal subgroups given at the beginning of the proof of Theorem 5.9 that there exist a normal subgroup N of finite index in K, $N \subset F(k, n)$, and an isomorphic normal subgroup N' of finite index in K', $N' \subset F(k', n)$. Furthermore, N and N' have the following structure

$$N = \times_{i=1}^{n} N_i,$$

where N_i is a normal subgroup of finite index in $F(k+i, k+i)$, and

$$N' = \times_{i=1}^{n} N_i',$$

where N_i' is a normal subgroup of finite index in $F(k'+i, k'+i)$. We claim that N is not isomorphic to N'. To see this, we assume the contrary.

Recall that $N_i \approx F * (*(\mathbb{Z}^{k+i}))^{m_i}$ for some integer m_i, where F is a free group, by Lemma 5.12(ii) and Lemma 4.2. Now any element x from N has the form $x = (x_1, \ldots, x_n)$, where $x_i \in N_i$. Suppose that $x_i \neq \{1\}$ for all i. Then the centralizer C_x of x in N has the form $C_x = \times_{i=1}^{n} C'_{x_i}$ where C'_{x_i} is the centralizer of x_i in N_i. Since C'_{x_i} is isomorphic to $\mathbb{Z}^{s_i}$, $1 \leq i \leq k+i$, it follows that $C_x \approx (\times_{i=1}^{n} \mathbb{Z}^{s_i}) \approx \mathbb{Z}^{s(x)}$, where $s(x) = \sum_{i=1}^{n} s_i$ and $s(x) \leq kn + n(n+1)/2$. As $N_i \approx (F * (*\mathbb{Z}^{k+i})^{m_i})$ as above, there exists $y \in N$ such that $s(y) = kn + n(n+1)/2$. As N and N' are isomorphic it follows that $k = k'$. However, this contradicts our assumption. Hence, N is not isomorphic to N'. □

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