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On the transition to instability for compressible vortex sheets

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We study the linear stability of a vortex sheet in a limit case that corresponds to a transition between a weakly stable regime and a violently unstable regime. We prove an energy estimate that reflects the high degeneracy of the uniform Kreiss–Lopatinskii condition.

1. Introduction

The existence of compressible vortex sheets is a nonlinear hyperbolic free-boundary problem. In three space dimensions, all constant vortex sheets are violently unstable (see, for example, [3]). In two space dimensions, a constant vortex sheet is violently unstable if and only if

$$\|\boldsymbol{u}_{\rm r} - \boldsymbol{u}_{\rm l}\| < 2\sqrt{2}c,$$

where $\boldsymbol{u}_{\rm r}$ and $\boldsymbol{u}_{\rm l}$ are the fluid velocities on either side of the interface and c is the sound speed (which is constant on either side of the interface). In a recent work [2], we have studied the stability of vortex sheets that satisfy

$$\|\boldsymbol{u}_{\rm r} - \boldsymbol{u}_{\rm l}\| > 2\sqrt{2}c,$$

and we have shown that the solutions to the linearized problem obey an *a priori* energy estimate. In this paper, we study the limit case

$$\|\boldsymbol{u}_{\rm r} - \boldsymbol{u}_{\rm l}\| = 2\sqrt{2}c.$$

We shall show that the linearized problem about such a constant vortex sheet still obeys an *a priori* estimate. However, the energy estimate is very weak, due to the unusual fact that the so-called Lopatinskii determinant has a triple root.

To avoid overloading the paper, we shall often refer to [2], where the reader will find detailed calculations and a wider list of references on the subject. As was done

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in [2], we consider the compressible Euler equations in the whole space $\mathbb{R}^2$,

$$\left.\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = 0, \end{aligned}\right\} \tag{1.1}$$

where $p = p(\rho)$ is the pressure law. It is assumed to be $\mathcal{C}^\infty$ and increasing. As in [2], the sound speed is denoted by c. We also decompose the velocity $\boldsymbol{u}$ as follows: $\boldsymbol{u} = (v, u) \in \mathbb{R}^2$.

In this paper, we consider a piecewise constant solution of (1.1) that takes the following form:

$$(\rho, \boldsymbol{u}) := \begin{cases} (\rho, v_{\mathrm{r}}, 0) & \text{if } x_2 > 0, \\ (\rho, v_{\mathrm{l}}, 0) & \text{if } x_2 < 0. \end{cases} \tag{1.2}$$

We are interested in the linear stability of this piecewise constant solution. We assume that the vortex sheet defined by (1.2) satisfies

$$v_{\mathrm{r}} + v_{\mathrm{l}} = 0 \quad \text{and} \quad v_{\mathrm{r}} = \sqrt{2}c > 0. \tag{1.3}$$

2. The linearized equations

Because we deal with a free-boundary problem, it is convenient to fix the (unknown) interface by a change of variables. Then we linearize the nonlinear equations about the particular solution given by (1.2). The linearized equations read (see [2] for details)

$$\left.\begin{aligned} \mathcal{L}W := \mathcal{A}_0 \partial_t W + \mathcal{A}_1 \partial_{x_1} W + \mathcal{A}_2 \partial_{x_2} W = f \quad & \text{if } x_2 > 0, \\ \mathcal{B}(W^{\mathrm{nc}}, \psi) := \underline{M} W^{\mathrm{nc}}|_{x_2=0} + \underline{b} \begin{pmatrix} \partial_t \psi \\ \partial_{x_1} \psi \end{pmatrix} = g \quad & \text{if } x_2 = 0, \end{aligned}\right\} \tag{2.1}$$

where ψ is the unknown perturbed front and W is the following vector,

$$W := \left(\dot{v}_+, \dot{v}_-, -\frac{\dot{\rho}_+}{2\rho} + \frac{\dot{u}_+}{2c}, \frac{\dot{\rho}_-}{2\rho} + \frac{\dot{u}_-}{2c}, \frac{\dot{\rho}_+}{2\rho} + \frac{\dot{u}_+}{2c}, -\frac{\dot{\rho}_-}{2\rho} + \frac{\dot{u}_-}{2c} \right)^{\mathrm{T}},$$

where $\dot{\rho}_+$ (respectively, $\dot{\rho}_-$) denotes the perturbed density on the right (respectively, on the left) of the interface, and so on. In (2.1), the vector W^{nc} is obtained by retaining only the four last components of W. Furthermore, we recall that the matrices $\mathcal{A}_j$ are given by the following formulae,

$$\mathcal{A}_0 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c^2 \end{pmatrix},$$

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$$\mathcal{A}_1 := \begin{pmatrix} v_{\rm r} & 0 & -c^2 & 0 & c^2 & 0 \\ 0 & v_{\rm l} & 0 & c^2 & 0 & -c^2 \\ -c^2 & 0 & 2c^2 v_{\rm r} & 0 & 0 & 0 \\ 0 & c^2 & 0 & 2c^2 v_{\rm l} & 0 & 0 \\ c^2 & 0 & 0 & 0 & 2c^2 v_{\rm r} & 0 \\ 0 & -c^2 & 0 & 0 & 0 & 2c^2 v_{\rm l} \end{pmatrix},$$

$$\mathcal{A}_2 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2c^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2c^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c^3 \end{pmatrix},$$

while $\underline{M}$ and $\underline{b}$ are defined as follows:

$$\underline{b} := \begin{pmatrix} 0 & v_{\rm r} - v_{\rm l} \\ 1 & v_{\rm r} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2v_{\rm r} \\ 1 & v_{\rm r} \\ 0 & 0 \end{pmatrix}, \qquad \underline{M} := \begin{pmatrix} -c & c & -c & c \\ -c & 0 & -c & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}. \tag{2.2}$$

Before stating our energy estimate for (2.1), we introduce some notation. First define the half-space

$$\Omega := \{(t, x_1, x_2) \in \mathbb{R}^3 \text{ such that } x_2 > 0\} = \mathbb{R}^2 \times]0, +\infty[.$$

The boundary $\partial\Omega$ is identified to $\mathbb{R}^2$. For all real number s and all $\gamma \geqslant 1$, we define the following norm on the Sobolev space $H^s(\mathbb{R}^2)$,

$$\|u\|_{s,\gamma}^2 := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} (\gamma^2 + |\xi|^2)^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi,$$

where $\hat{u}$ is the Fourier transform of any function u defined on $\mathbb{R}^2$. The space $L^2(\mathbb{R}^+; H^s(\mathbb{R}^2))$ is equipped with the norm

$$|||u|||_{s,\gamma}^2 := \int_0^{+\infty} \|u(\cdot, x_2)\|_{s,\gamma}^2 \,\mathrm{d}x_2.$$

In the sequel, the variable in $\mathbb{R}^2$ is (t, x_1), while x_2 is the variable in $\mathbb{R}^+$.

Introducing $\tilde{W} := \exp(-\gamma t)W$ and $\tilde{\psi} := \exp(-\gamma t)\psi$, we find that (2.1) is equivalent to

$$\left.\begin{aligned} \mathcal{L}^\gamma \tilde{W} &:= \gamma \mathcal{A}_0 \tilde{W} + \mathcal{L}\tilde{W} = \exp(-\gamma t) f && \text{if } x_2 > 0, \\ \mathcal{B}^\gamma(\tilde{W}^{\rm nc}, \tilde{\psi}) &:= \underline{M}\tilde{W}^{\rm nc}|_{x_2=0} + \underline{b}\begin{pmatrix} \gamma\tilde{\psi} + \partial_t\tilde{\psi} \\ \partial_{x_1}\tilde{\psi} \end{pmatrix} = \exp(-\gamma t) g && \text{if } x_2 = 0. \end{aligned}\right\} \tag{2.3}$$

The main result of this paper is the following theorem.

THEOREM 2.1. *Assume that (1.3) holds. Then there exists a positive constant C such that, for all $\gamma \geqslant 1$ and for all $(\tilde{W}, \tilde{\psi}) \in H^4(\Omega) \times H^4(\mathbb{R}^2)$, the following estimate holds:*

$$\gamma |||\tilde{W}|||_0^2 + \|\tilde{W}^{\rm nc}|_{x_2=0}\|_0^2 + \|\tilde{\psi}\|_{1,\gamma}^2 \leqslant C\left(\frac{1}{\gamma^7}|||\mathcal{L}^\gamma \tilde{W}|||_{3,\gamma}^2 + \frac{1}{\gamma^6}\|\mathcal{B}^\gamma(\tilde{W}^{\rm nc}, \tilde{\psi})\|_{3,\gamma}^2\right). \tag{2.4}$$

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Recall that, under the assumption $v_{\mathrm{r}} > \sqrt{2}c$, the main energy estimate we have proved in [2] involves the loss of only one derivative on the boundary and one derivative in the interior domain, that is,

$$\gamma|||\tilde{W}|||_0^2 + \|\tilde{W}^{\mathrm{nc}}|_{x_2=0}\|_0^2 + \|\tilde{\psi}\|_{1,\gamma}^2 \leqslant C\biggl(\frac{1}{\gamma^3}|||\mathcal{L}^\gamma \tilde{W}|||_{1,\gamma}^2 + \frac{1}{\gamma^2}\|\mathcal{B}^\gamma(\tilde{W}^{\mathrm{nc}}, \tilde{\psi})\|_{1,\gamma}^2\biggr).$$

At the opposite, when $v_{\mathrm{r}} < \sqrt{2}c$, the linearized equations (2.1) are violently ill-posed in any Sobolev or Hölder space (see [3]). In the limit case $v_{\mathrm{r}} = \sqrt{2}c$ that we are considering here, the Lopatinskii determinant associated with (2.1) has a triple root, which yields a very poor energy estimate. The transition case is thus really different from the one considered in [4]. As a matter of fact, it was shown in [1] that a situation where the Lopatinskii determinant has a multiple root corresponds to a transition between weak stability (here, the region $v_{\mathrm{r}} > \sqrt{2}c$) and violent instability (here, the region $v_{\mathrm{r}} < \sqrt{2}c$). However, we note that the result of [1] is derived when the root is double, and not triple as in our case.

3. Proof of the main result

We drop the tilde for convenience. Using the same argument as in [2, paragraph 4.1], we claim that it is sufficient to prove theorem 2.1 in the special case $\mathcal{L}^\gamma W \equiv 0$. Performing a Fourier transform in (t, x_1) and eliminating the unknown front in the boundary conditions, we are led to consider the following boundary-value problem,

$$\begin{gathered}(\tau\mathcal{A}_0 + \mathrm{i}\eta\mathcal{A}_1)\hat{W} + \mathcal{A}_2\frac{\mathrm{d}\hat{W}}{\mathrm{d}x_2} = 0 \quad \text{if } x_2 > 0,\\ \beta(\tau,\eta)\hat{W}^{\mathrm{nc}}(0) = \hat{h},\end{gathered} \tag{3.1}$$

where $\hat{h}$ is a source term related to $\mathcal{B}^\gamma(W^{\mathrm{nc}}, \psi)$ and β is defined by

$$\beta(\tau,\eta) := \begin{pmatrix} -1 & -1 & 1 & 1 \\ -c(\tau + \mathrm{i}v_{\mathrm{l}}\eta) & c(\tau + \mathrm{i}v_{\mathrm{r}}\eta) & -c(\tau + \mathrm{i}v_{\mathrm{l}}\eta) & c(\tau + \mathrm{i}v_{\mathrm{r}}\eta)\end{pmatrix} \quad \forall(\tau,\eta) \in \varSigma.$$

The definition of the hemisphere $\varSigma$ is the one we adopted in [2],

$$\varSigma := \{(\tau,\eta) \in \mathbb{C}\times\mathbb{R} \text{ such that } \operatorname{Re}\tau \geqslant 0 \text{ and } |\tau|^2 + v_{\mathrm{r}}^2\eta^2 = 1\}.$$

Moreover, β is homogeneous of degree 0 with respect to (τ, η).

We emphasize that it is still possible to eliminate the front in the limit case $v_{\mathrm{r}} = \sqrt{2}c$. Lemma 1 of [2] also applies in this case.

Using the two first scalar equations in (3.1), we obtain the following system of ordinary differential equations,

$$\left.\begin{aligned} \frac{\mathrm{d}\hat{W}^{\mathrm{nc}}}{\mathrm{d}x_2} &= \mathcal{A}(\tau,\eta)\hat{W}^{\mathrm{nc}} \quad && \text{if } x_2 > 0,\\ \beta(\tau,\eta)\hat{W}^{\mathrm{nc}}(0) &= \hat{h} && \text{if } x_2 = 0,\end{aligned}\right\} \tag{3.2}$$

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where $\mathcal{A}(\tau,\eta)$ is defined by

$$\mathcal{A}(\tau,\eta) := \begin{pmatrix} \mu_{\mathrm{r}} & 0 & -m_{\mathrm{r}} & 0 \\ 0 & \mu_{\mathrm{l}} & 0 & -m_{\mathrm{l}} \\ m_{\mathrm{r}} & 0 & -\mu_{\mathrm{r}} & 0 \\ 0 & m_{\mathrm{l}} & 0 & -\mu_{\mathrm{l}} \end{pmatrix},$$

with

$$\mu_{\mathrm{r,l}} := \frac{(1/c)(\tau + \mathrm{i}v_{\mathrm{r,l}}\eta)^2 + \frac{1}{2}c\eta^2}{\tau + \mathrm{i}v_{\mathrm{r,l}}\eta}, \qquad m_{\mathrm{r,l}} := \frac{\frac{1}{2}c\eta^2}{\tau + \mathrm{i}v_{\mathrm{r,l}}\eta}.$$

If we denote by ω_{r} (respectively, ω_{l}) the roots of the negative real part of the equation

$$\omega^2 = \frac{1}{c^2}(\tau + \mathrm{i}v_{\mathrm{r}}\eta)^2 + \eta^2 \quad \left(\text{respectively, } \omega^2 = \frac{1}{c^2}(\tau + \mathrm{i}v_{\mathrm{l}}\eta)^2 + \eta^2\right),$$

then the stable subspace of $\mathcal{A}(\tau,\eta)$ has dimension 2 and is spanned by the two vectors

$$E_{\mathrm{r}}(\tau,\eta) := \left(\tfrac{1}{2}c\eta^2, 0, \frac{1}{c}(\tau + \mathrm{i}v_{\mathrm{r}}\eta)^2 + \tfrac{1}{2}c\eta^2 - (\tau + \mathrm{i}v_{\mathrm{r}}\eta)\omega_{\mathrm{r}}, 0\right)^{\mathrm{T}},$$

$$E_{\mathrm{l}}(\tau,\eta) := \left(0, \tfrac{1}{2}c\eta^2, 0, \frac{1}{c}(\tau + \mathrm{i}v_{\mathrm{l}}\eta)^2 + \tfrac{1}{2}c\eta^2 - (\tau + \mathrm{i}v_{\mathrm{l}}\eta)\omega_{\mathrm{l}}\right)^{\mathrm{T}}.$$

The Lopatinskii determinant is defined in the classical way,

$$\Delta(\tau,\eta) := \det[\beta(\tau,\eta)(E_{\mathrm{r}}(\tau,\eta), E_{\mathrm{l}}(\tau,\eta))]. \tag{3.3}$$

It is continuous on the whole closed hemisphere Σ (while the symbol $\mathcal{A}$ has some poles on the boundary of Σ). The following result describes the failure of the uniform Lopatinskii condition.

PROPOSITION 3.1. *Assume that (1.3) holds. Then one has* $\Delta(\tau,\eta) = 0$ *if and only if* $\tau = 0$*. Furthermore, there exists a neighbourhood* $\mathcal{V}$ *of* $(0, 1/v_{\mathrm{r}})$ *in* Σ *and a* $\mathcal{C}^\infty$ *function* h *defined on* $\mathcal{V}$ *such that*

$$\Delta(\tau,\eta) = \tau^3 h(\tau,\eta) \quad \textit{and} \quad h(0, 1/v_{\mathrm{r}}) \neq 0 \quad \forall(\tau,\eta) \in \mathcal{V}.$$

A similar result holds near $(0, -1/v_{\mathrm{r}})$.

Proof. We first compute

$$\Delta(\tau,\eta) = -c^2(\tau + \mathrm{i}v_{\mathrm{r}}\eta - c\omega_{\mathrm{r}})(\tau + \mathrm{i}v_{\mathrm{l}}\eta - c\omega_{\mathrm{l}})(\omega_{\mathrm{r}} + \omega_{\mathrm{l}})(\omega_{\mathrm{r}}\omega_{\mathrm{l}} - \eta^2).$$

The first two factors $(\tau + \mathrm{i}v_{\mathrm{r}}\eta - c\omega_{\mathrm{r}})$ and $(\tau + \mathrm{i}v_{\mathrm{l}}\eta - c\omega_{\mathrm{l}})$ do not vanish on Σ. The sum $\omega_{\mathrm{r}} + \omega_{\mathrm{l}}$ does not vanish when τ has positive real part, since both numbers $\omega_{\mathrm{r,l}}$ have negative real part. When τ is purely imaginary, one extends $\omega_{\mathrm{r,l}}$ by continuity. Using the formulae given in [2, paragraph 5.1], one shows that $\omega_{\mathrm{r}} + \omega_{\mathrm{l}} = 0$ if and only if $\tau = 0$. Moreover, there exists a neighbourhood $\mathcal{V}$ of $(0, 1/v_{\mathrm{r}})$ and a $\mathcal{C}^\infty$ function h_1 defined on $\mathcal{V}$ such that

$$\omega_{\mathrm{r}} + \omega_{\mathrm{l}} = \tau h_1(\tau,\eta) \quad \text{and} \quad h_1(0, 1/v_{\mathrm{r}}) \neq 0 \quad \forall(\tau,\eta) \in \mathcal{V}.$$

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If the last term $(\omega_{\mathrm{r}}\omega_{\mathrm{l}}-\eta^2)$ vanishes, then $\eta \neq 0$, and we compute that $V := \tau/(\mathrm{i}\eta)$ satisfies

$$V^2(V^2-6c^2)=0.$$

Here we have used the relation $v_{\mathrm{r}}=\sqrt{2}c$. When $V=\sqrt{6}c$, we use the formulae given in [2, paragraph 5.1], and compute

$$\omega_{\mathrm{r}}=-\mathrm{i}\eta\sqrt{7+4\sqrt{3}}, \qquad \omega_{\mathrm{l}}=-\mathrm{i}\eta\sqrt{7-4\sqrt{3}}.$$

Therefore, $\omega_{\mathrm{r}}\omega_{\mathrm{l}}=-\eta^2\neq\eta^2$ and $\tau=\mathrm{i}\sqrt{6}c\eta$ is not a root of the Lopatinskii determinant. Similar calculations show that $\tau=-\mathrm{i}\sqrt{6}c\eta$ is not a root either. When $\tau=0$, we have

$$\omega_{\mathrm{r}}=-\mathrm{i}\eta \quad\text{and}\quad \omega_{\mathrm{l}}=\mathrm{i}\eta,$$

so we have $\omega_{\mathrm{r}}\omega_{\mathrm{l}}=\eta^2$. If we define $\Omega_{\mathrm{r,l}}:=\omega_{\mathrm{r,l}}/(\mathrm{i}\eta)$, then the function

$$f(V):=\Omega_{\mathrm{r}}\Omega_{\mathrm{l}}+1$$

is holomorphic near $V=0$. Using the relations

$$\Omega_{\mathrm{r,l}}^2=\frac{1}{c^2}(V+v_{\mathrm{r,l}})^2-1,$$

one shows that 0 is a double root of f, so we have

$$f(V)=V^2g(V),$$

for a suitable holomorphic function g that does not vanish at 0. This shows that there exists a neighbourhood $\mathcal{V}$ of $(0,1/v_{\mathrm{r}})$ and a $\mathcal{C}^\infty$ function h_2 defined on $\mathcal{V}$ such that

$$\omega_{\mathrm{r}}\omega_{\mathrm{l}}-\eta^2=\tau^2h_2(\tau,\eta) \quad\text{and}\quad h_2(0,1/v_{\mathrm{r}})\neq 0 \quad \forall(\tau,\eta)\in\mathcal{V}.$$

This completes the proof. □

In order to construct a *degenerate* symmetrizer near the roots of the Lopatinskii determinant, we need to precise the behaviour of the matrix $\beta(\tau,\eta)$ restricted to the stable subspace of $\mathcal{A}(\tau,\eta)$. This is summarized in the following lemma.

LEMMA 3.2. *There exists a neighbourhood* $\mathcal{V}$ *of* $(0,1/v_{\mathrm{r}})$ *in* Σ *and a constant* $\kappa_0>0$ *such that the following estimate holds for all* $(\tau,\eta)\in\mathcal{V}$*:*

$$|\beta(\tau,\eta)(E_{\mathrm{r}}(\tau,\eta),E_{\mathrm{l}}(\tau,\eta))Z^-|^2 \geqslant \kappa_0\gamma^6|Z^-|^2 \quad \forall Z^-\in\mathbb{C}^2. \tag{3.4}$$

A similar result holds near $(0,-1/v_{\mathrm{r}})$.

Proof. We have

$$\beta\begin{pmatrix}E_{\mathrm{r}} & E_{\mathrm{l}}\end{pmatrix}=\begin{pmatrix}(\tau+\mathrm{i}v_{\mathrm{r}}\eta)(c^{-1}(\tau+\mathrm{i}v_{\mathrm{r}}\eta)-\omega_{\mathrm{r}}) & (\tau+\mathrm{i}v_{\mathrm{l}}\eta)(c^{-1}(\tau+\mathrm{i}v_{\mathrm{l}}\eta)-\omega_{\mathrm{l}})\\ -c\omega_{\mathrm{r}}(\tau+\mathrm{i}v_{\mathrm{l}}\eta)(c\omega_{\mathrm{r}}-(\tau+\mathrm{i}v_{\mathrm{r}}\eta)) & c\omega_{\mathrm{l}}(\tau+\mathrm{i}v_{\mathrm{r}}\eta)(c\omega_{\mathrm{l}}-(\tau+\mathrm{i}v_{\mathrm{l}}\eta))\end{pmatrix}$$

for all $(\tau,\eta)\in\Sigma$. Consequently, the upper left-hand corner coefficient of $\beta(E_{\mathrm{r}}E_{\mathrm{l}})$ does not vanish near the point $(0,1/v_{\mathrm{r}})$. Writing

$$\beta\begin{pmatrix}E_{\mathrm{r}} & E_{\mathrm{l}}\end{pmatrix}=\begin{pmatrix}\zeta_1 & \zeta_2\\ \zeta_3 & \zeta_4\end{pmatrix},$$

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we easily obtain the equality

$$\begin{pmatrix} 1/\zeta_1 & 0 \\ -\zeta_3/\zeta_1 & 1 \end{pmatrix} \beta \begin{pmatrix} E_{\rm r} & E_{\rm l} \end{pmatrix} \begin{pmatrix} 1 & -\zeta_2 \\ 0 & \zeta_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}.$$

In particular, we obtain the estimate

$$|\beta \begin{pmatrix} E_{\rm r} & E_{\rm l} \end{pmatrix} Z^-|^2 \geqslant \kappa \min(1, |\Delta|^2)|Z^-|^2,$$

for a suitable constant $\kappa > 0$ that is independent on (τ, η) in a neighbourhood of $(0, 1/v_{\rm r})$. One uses the factorization given in proposition 3.1 to conclude the proof. □

We are now able to construct a degenerate Kreiss symmetrizer near the points where the Lopatinskii determinant vanishes. Following [2], we already know that there exists a neighbourhood $\mathcal{V}$ of $(0, 1/v_{\rm r})$ in Σ and a $\mathcal{C}^\infty$ mapping T on $\mathcal{V}$ such that

$$T(\tau, \eta)\mathcal{A}(\tau, \eta)T(\tau, \eta)^{-1} = \begin{pmatrix} \omega_{\rm r} & 0 & 0 & 0 \\ 0 & \omega_{\rm l} & 0 & 0 \\ 0 & 0 & -\omega_{\rm r} & 0 \\ 0 & 0 & 0 & -\omega_{\rm l} \end{pmatrix} \quad \forall(\tau, \eta) \in \mathcal{V}.$$

The first two columns of $T(\tau, \eta)^{-1}$ are the vectors $E_{\rm r}(\tau, \eta)$ and $E_{\rm l}(\tau, \eta)$. We define our symmetrizer r in the following way,

$$r(\tau, \eta) := \begin{pmatrix} -\gamma^6 & 0 & 0 & 0 \\ 0 & -\gamma^6 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix} \quad \forall(\tau, \eta) \in \mathcal{V},$$

with $K \geqslant 1$ to be fixed large enough. The matrix $r(\tau, \eta)$ is hermitian and we have

$$\operatorname{Re}(r(\tau, \eta)T(\tau, \eta)\mathcal{A}(\tau, \eta)T(\tau, \eta)^{-1}) \geqslant \kappa\gamma \begin{pmatrix} \gamma^6 & 0 & 0 & 0 \\ 0 & \gamma^6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \forall(\tau, \eta) \in \mathcal{V}, \qquad (3.5)$$

for a suitable $\kappa > 0$. We have used the standard notation

$$\operatorname{Re} M := \tfrac{1}{2}M + M^*.$$

Now we let

$$\tilde{\beta}(\tau, \eta) := \beta(\tau, \eta)T(\tau, \eta)^{-1}.$$

Recall that the first two columns of $T(\tau, \eta)^{-1}$ are $E_{\rm r}$ and $E_{\rm l}$. Let

$$Z = (Z^-, Z^+) \in \mathbb{C}^4,$$

with $Z^-, Z^+ \in \mathbb{C}^2$. Writing

$$\tilde{\beta}(\tau, \eta)Z = \tilde{\beta}(\tau, \eta)\begin{pmatrix} Z^- \\ 0 \end{pmatrix} + \tilde{\beta}(\tau, \eta)\begin{pmatrix} 0 \\ Z^+ \end{pmatrix},$$

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and using (3.4), we obtain

$$\kappa_0\gamma^6|Z^-|^2 \leqslant C_0(|\tilde{\beta}(\tau,\eta)Z|^2 + |Z^+|^2),$$

for some appropriate $\kappa_0 > 0$ and $C_0 > 0$. In the definition of the symmetrizer r, we choose $K := 2C_0/\kappa_0 + 1$. This choice yields the inequality

$$\langle r(\tau,\eta)Z, Z\rangle_{\mathbb{C}^4} + \frac{2C_0}{\kappa_0}|\tilde{\beta}(\tau,\eta)Z|^2 \geqslant \gamma^6|Z^-|^2 + |Z^+|^2 \geqslant \gamma^6|Z|^2,$$

that is,

$$r(\tau,\eta) + C(\tilde{\beta}(\tau,\eta))^*\tilde{\beta}(\tau,\eta) \geqslant \gamma^6 I \quad \forall(\tau,\eta) \in \mathcal{V}. \tag{3.6}$$

The construction of the symmetrizer near the other points of Σ is the same as what was done in [2], so we shall not detail it. To derive the energy estimate, we proceed as in [2, paragraph 4.9], using a finite covering of Σ and a partition of unity $(\chi_i)_{1\leqslant i\leqslant I}$. In particular, when the support of χ_i is a neighbourhood of a point where the Lopatinskii condition fails, we use (3.5) and (3.6) to derive an estimate that reads

$$\gamma\chi_i(\tau,\eta)^2\int_0^{+\infty}|\hat{W}^{\mathrm{nc}}(\tau,\eta,x_2)|^2\,\mathrm{d}x_2 + \chi_i(\tau,\eta)^2|\hat{W}^{\mathrm{nc}}(\tau,\eta,0)|^2 \leqslant \frac{C}{\gamma^6}\chi_i(\tau,\eta)^2|\hat{h}|^2(|\tau|^2 + v_{\mathrm{r}}^2\eta^2)^3.$$

When the support of χ_i is a neighbourhood of a point where the Lopatinskii condition is satisfied, we obtain the following energy estimate (see [2] for details):

$$\gamma\chi_i(\tau,\eta)^2\int_0^{+\infty}|\hat{W}^{\mathrm{nc}}(\tau,\eta,x_2)|^2\,\mathrm{d}x_2 + \chi_i(\tau,\eta)^2|\hat{W}^{\mathrm{nc}}(\tau,\eta,0)|^2 \leqslant C\chi_i(\tau,\eta)^2|\hat{h}|^2.$$

Integrating with respect to the frequencies, and using Plancherel's theorem, we obtain (2.4). This completes the proof.

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