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Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies

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We prove uniqueness of solutions to scalar conservation laws with space discontinuous fluxes. To do so, we introduce a partial adaptation of Kružkov's entropies which naturally takes into account the space dependency of the flux. The advantage of this approach is that the proof turns out to be a simple variant of the original method of Kružkov. In particular, we do not need traces, interface conditions, bounded variation assumptions (neither on the solution nor on the flux), or convex fluxes. However, we use a special 'local uniform invertibility' structure of the flux, which applies to cases where different interface conditions are known to yield different solutions.

1. Introduction

We consider the Cauchy problem associated with a scalar conservation law where the flux depends discontinuously on space

$$\left.\begin{aligned}\partial_t u + \partial_x[A(x,u)] = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}_+,\\ u(0,x) = u_0(x) \in L^\infty(\mathbb{R}).\end{aligned}\right\} \tag{1.1}$$

We propose a new method to prove an L^1 contraction principle for a class of solutions to this equation when the space dependence of the flux is discontinuous. As in most of the recent papers that deal with this subject [1, 15], our proof is based on Kružkov's framework. But our idea is to adapt the definition of Kružkov entropies to the discontinuous case and thus to avoid a special treatment of the interface.

When the space dependence of the flux is sufficiently smooth, this scalar equation is quite well known. In particular, Kružkov's theory applies and provides existence and uniqueness of a weak solution to (1.1) that satisfies Kružkov entropy inequalities (see [7, 20, 23]).

We consider in this article the case where the flux is a discontinuous function of x, not necessarily of bounded variation. The first existence results for such a problem were obtained through an analogy with 2×2 hyperbolic systems and the

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study of the related Riemann problem. Indeed, if we assume that the flux is of the form $A(x,u) = f(\gamma(x),u)$, the equation (1.1) can be written as a system in (u,γ) by adding the trivial equation $\partial_t \gamma = 0$. In the 1980s and considering particular forms of the flux, [10, 11, 24] established the global existence of a solution for the corresponding Cauchy problems by proving the convergence of different numerical methods. Later on, existence results were extended to more general fluxes by using convergence of numerical schemes [1, 6, 12, 13, 15, 16, 18, 19, 25, 26] or regularization of the coefficients [14, 17, 21, 22].

Here we are interested in the problem of the uniqueness of solutions. The first results about this topic were obtained in the mid 1990s, and different methods have been investigated. In [8], Diehl considered a flux on the form $A(x,u) = H(x)f(u) + (1 - H(x))g(u)$, where $H(x)$ is the Heaviside function, and proved existence and uniqueness locally in time by introducing a coupling condition Γ at the interface. In the same year, in [18], Klingenberg and Risebro considered a multiplicative flux $A(x,u) = k(x)f(u)$ such that f is a convex function that satisfies $f(0) = f(1) = 0$ and $k(x) \geqslant k > 0$ is a bounded variation (BV) piecewise smooth function with a finite number of discontinuity points; they proved uniqueness for a solution that satisfies a wave entropy condition (see also [17], where the authors proved continuous dependence on the coefficient k and on the initial data for the same problem). In [12], Greenberg *et al*. considered a convex additive flux $A(x,u) = f(u) + a(x)$, where f is even and convex and a is piecewise constant, and proved a contraction principle for the solution that they constructed by solving Riemann problems with an appropriate interaction of waves when seen as a 2×2 system. In [21], Ostrov proposed another approach: he proved uniqueness of a solution of the Hamilton–Jacobi equation obtained as the limit of viscosity solutions for regularized coefficient cases. Then he concluded for (1.1) by using the equivalency between the Hamilton–Jacobi equations and scalar conservation laws. He extended the uniqueness result to fluxes of the form $A(t,x,u) = f(k(t,x),u)$, where f is convex in u and satisfies a superlinear growth condition, and k is bounded and discontinuous along a finite number of curves and is Lipschitz continuous away from these curves. Towers [25] came back to the multiplicative case and established an L^1 contraction principle for a class of solutions that satisfies Kružkov-type entropy inequalities; it means that the solution satisfies classical Kružkov entropy inequalities away from the discontinuities of the flux and also satisfies a geometric condition at the discontinuity points that can be interpreted as an interface entropy condition. To give a sense to this new entropy condition, he needed to assume some additional regularity conditions on the solution, namely that u is piecewise C^1 and possesses traces on the discontinuities of k. In this earlier work, the flux was assumed to be convex in u, but this approach was further investigated by Karlsen *et al*. in [15], and the uniqueness result has been extended to non convex fluxes on the form $A(x,u) = f(k(x),u)$, where k is a piecewise C^1 BV function with a finite number of discontinuities and f is Lipschitz continuous in u and k and satisfies a given crossing condition. In that paper and in [22], the existence of traces for u is proved for particular additive/multiplicative fluxes, but should be assumed in the general case. Very recently, Adimurthi *et al*. [1] introduced another interface entropy condition—still coupled with classical Kružkov entropy inequalities away from the discontinuity—and also proved an L^1 contraction principle for this new class of solutions. They considered a Heaviside flux type where f and g have only one global minimum and no local minimum, and assumed the existence of traces on the discontinuity. For particular fluxes and initial data it can be proved that the interface conditions in [15] and [1] do not select the same solution. We prove in § 5 that our method selects the solution derived from the interface condition of [1].

In [1, 15, 22, 25] the uniqueness proof is based on the use of classical Kružkov entropies which leads to the following entropy inequalities

$$\partial_t|u-k| + \partial_x[(A(x,u)-A(x,k))\operatorname{sgn}(u-k)] + \operatorname{sgn}(u-k)\partial_x A(x,k) \leqslant 0. \qquad (1.2)$$

Thus an interface entropy condition has to be introduced by the authors to deal with the discontinuities of the flux and to give sense to the last term of the left-hand side. Here we propose, pushing further an argument in [4], to adapt the definition of Kružkov entropies to the discontinuous case by introducing *partially adapted Kružkov entropies*

$$E_\alpha(x,u) = |u-k_\alpha(x)|,$$

where $k_\alpha(x)$ satisfies

$$A(x,k_\alpha(x)) = \alpha.$$

This new definition allows us to remove the problematic term in the entropy inequalities (1.2), since we arrive at

$$\partial_t|u-k_\alpha(x)| + \partial_x[(A(x,u)-A(x,k_\alpha(x)))\operatorname{sgn}(u-k_\alpha(x))] \leqslant 0. \qquad (1.3)$$

Thus the interface does not need a special treatment and no interface entropy condition is needed. Uniqueness then follows from arguments very close to Kružkov's original proof and the main difficulty is now to deal with the family(ies) of functions $k_\alpha(x)$.

This new method allows us to remove the hypothesis about the traces of the solution on the discontinuities of the flux and the BV bounds on the space dependence of the flux and on the initial data. Also we can deal with an infinite number of discontinuity points and we do not need convexity assumptions or crossing conditions. However, we need some other hypothesis on the flux—and more particularly on the u dependence of the flux—to be able to define our partially adapted Kružkov entropies.

The outline of the paper is as follows. In § 2 we list the hypotheses on the flux and we comment on them along with some examples. In § 3 we define the partially adapted Kružkov entropies, and in § 4 we prove the L^1 contraction principle.

2. Hypotheses on the flux

In this work we assume the following hypotheses on the flux A.

(H1) $A(x,u)$ is continuous at all points of $\mathbb{R}\setminus\mathcal{N}\times\mathbb{R}$, where $\mathcal{N}$ is a closed zero-measure set,

(H2) $\exists(f,g)\in(C^0(\mathbb{R}))^2$ such that, $\forall x\in\mathbb{R}$, $f(u)\leqslant|A(x,u)|\leqslant g(u)$. We assume that $f(u\neq 0)>0$ and $|f(\pm\infty)|=+\infty$.

(H3) For $x\in\mathbb{R}\setminus\mathcal{N}$, $A(x,\cdot)$ is a locally Lipschitz one-to-one function from $\mathbb{R}$ to $\mathbb{R}$.

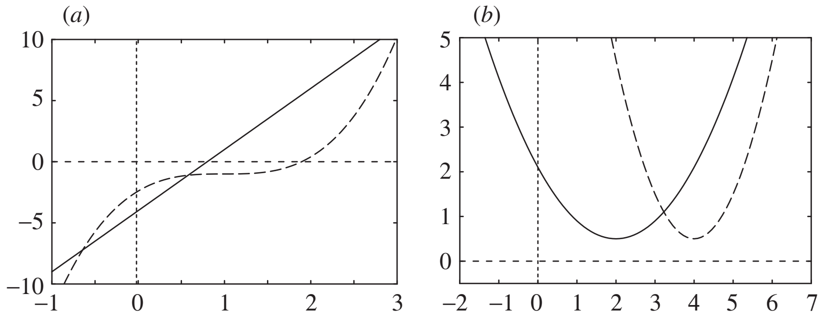

Figure 1. Admissible fluxes (Heaviside type): (a) hypothesis (H3); (b) hypothesis (H3′).

Throughout this article we also consider an alternative case by replacing hypothesis (H3) by

(H3′) There is a function $u_M(x)$ from $\mathbb{R}$ to $\mathbb{R}$ such that, for $x \in \mathbb{R} \setminus \mathcal{N}$, $A(x, \cdot)$ is a locally Lipschitz one-to-one function from $[-\infty, u_M(x)]$ and $[u_M(x), +\infty]$ to $[0, +\infty]$ that satisfies $A(x, u_M(x)) = 0$.

Two examples of Heaviside-type fluxes $A(x,u) = H(x)f(u) + (1 - H(x))g(u)$ that satisfy hypothesis (H3) and (H3′) are presented in figure 1. For the case of (H3), it is enough that f and g are increasing one-to-one functions.

An example of application, with hypothesis (H3), is the classical transport equation but with a discontinuous coefficient $S(x)$ in the flux

$$\partial_t u + \partial_x[S(x)u] = 0. \tag{2.1}$$

However, note that in this case our result is less general than those in [5].

The alternative case involving hypothesis (H3′) is obviously related to the discontinuous Burger–Hopf equation

$$\partial_t u + \partial_x[S(x)u^2] = 0. \tag{2.2}$$

In both examples (2.1) and (2.2), hypothesis (H3) (respectively (H3′)) is satisfied, hypothesis (H1) gives the admissible discontinuous form of S and hypothesis (H2) is equivalent to

$$\exists (m_S, M_S) \quad \text{for a.e. } x \in \mathbb{R}, \quad 0 < m_S \leqslant S(x) \leqslant M_S < +\infty.$$

But hypothesis (H3′) is also able to cover more general crossing convex fluxes of the form $A(x,u) = k_\pm(u - \beta_\pm)^2 - \alpha$ (see figure 1b).

The discontinuous flux case is important in numerous applications: sedimentation, two-phase flow in porous media, road traffic, etc. Let us also mention that the discontinuous flux case has natural links with Saint-Venant models: the modelling of blood flow with the Saint-Venant system exhibits the Young modulus of arteries (which can be discontinuous after surgery) as a coefficient in the pressure flux (see [9]); for a stationary flow, the coupled transport equation is of the form $A(x,u) = a(x)u$, where $a(x)$ is the velocity of the flow, which can be discontinuous (see [2]).

3. Partially adapted Kružkov entropies

Given $\alpha \in \mathbb{R}$, an immediate consequence of hypotheses (H1) and (H3) is the existence and the uniqueness of a function k_α from $\mathbb{R}$ to $\mathbb{R}$ such that

$$A(x, k_\alpha(x)) = \alpha \quad \text{for a.e. } x \in \mathbb{R}. \tag{3.1}$$

The alternative case (H3′) leads to similar conclusions: given $\alpha \in [M, +\infty]$ and $x \in \mathbb{R} \setminus \mathcal{N}$, there are two unique real numbers $k_\alpha^+(x) \in [u_M(x), +\infty]$ and $k_\alpha^-(x) \in [-\infty, u_M(x)]$ such that

$$A(x, k_\alpha^\pm(x)) = \alpha. \tag{3.2}$$

In the following, when relations are valid under hypothesis (H3) or (H3′), and to avoid redundancy, $k_\alpha(x)$ will denote either $k_\alpha(x)$ or $k_\alpha^\pm(x)$. Also, M will denote either the minimum of the flux $A(x, \cdot)$ under hypothesis (H3′) or $-\infty$ under hypothesis (H3).

These definitions allow us to introduce partially adapted Kružkov entropies, which are a natural way to extend classical Kružkov entropies to the discontinuous flux case.

Definition 3.1. Let u (respectively v) $\in L^\infty([0, T] \times \mathbb{R}) \cap C^0([0, T], L^1_{\text{loc}}(\mathbb{R}))$. We say it is an entropy subsolution (respectively supersolution) of (1.1) if and only if, for all $\alpha \in [M, +\infty]$,

$$\partial_t(u - k_\alpha(x))_+ + \partial_x[(A(x, u) - A(x, k_\alpha(x))) \operatorname{sgn}_+(u - k_\alpha(x))] \leqslant 0 \tag{3.3}$$

and

$$\partial_t(v - k_\alpha(x))_- + \partial_x[(A(x, v) - A(x, k_\alpha(x))) \operatorname{sgn}_-(v - k_\alpha(x))] \leqslant 0, \tag{3.4}$$

respectively.

Our motivation to introduce these adapted entropies comes from the contraction property, which still holds true under the form (1.3). It is natural to state Kružkov entropy with the steady-state solution

$$\frac{\partial}{\partial_t} k_\alpha(x) + \frac{\partial}{\partial_x} A(x, k_\alpha(x)) = 0, \tag{3.5}$$

but not with constants.

In a future work, we will prove that this condition can be derived from the vanishing-viscosity method and thus is a natural entropy condition.

4. Uniqueness theorem

Theorem 4.1. *Let u and $v \in L^\infty([0, T], \mathbb{R}) \cap C^0([0, T], L^1_{\text{loc}}(\mathbb{R}))$ be respectively an entropy sub- and supersolution to the initial-value problem (1.1) with initial data $u_0, v_0 \in L^\infty(\mathbb{R})$. Assume hypotheses (H1), (H2) and (H3) or (H3′) on the flux are true. Then, for a.e. $t \in [0, T]$,*

$$\int_a^b (u(x, t) - v(x, t))_+ \, \mathrm{d}x \leqslant \int_{a-Mt}^{b+Mt} (u_0(x) - v_0(x))_+ \, \mathrm{d}x, \tag{4.1}$$

where $M = \sup_{0 \leqslant t \leqslant T, x \in \mathbb{R}} |A_u(x, u(t, x))|$.

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Proof. We define $Q = \mathbb{R} \times [0, T]$. Proving the theorem is equivalent to establishing the following inequality for all $\phi \in C_0^\infty(Q)$ (see [7,23]):

$$\int_Q (u(t,x) - v(t,x))_+ \partial_t \phi \, \mathrm{d}x \, \mathrm{d}t \\ + \int_Q (A(x,u(t,x)) - A(x,v(t,x))) \operatorname{sgn}_+(u(t,x) - v(t,x)) \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t \\ + \int_{\mathbb{R}} (u_0(x) - v_0(x))_+ \phi(x,0) \, \mathrm{d}x \geqslant 0. \tag{4.2}$$

Since $A(\cdot, u)$ is continuous for $x \in \mathbb{R} \setminus \mathcal{N}$, we can define, for a.e. $(x,y,s,t) \in Q^2$, two functions $\tilde{u}(t,x,y)$ and $\tilde{v}(s,y,x)$ from $\mathbb{R}^2 \times [0,T]$ to $\mathbb{R}$ such that

$$\left.\begin{aligned} A(y, \tilde{u}(t,x,y)) &= A(x, u(t,x)), \\ A(x, \tilde{v}(s,y,x)) &= A(y, v(s,y)). \end{aligned}\right\} \tag{4.3}$$

According to the notation of § 3, and under hypothesis (H3), it is equivalent to

$$\left.\begin{aligned} \tilde{u}(t,x,y) &= k_{A(x,u(t,x))}(y), \\ \tilde{v}(s,y,x) &= k_{A(y,v(s,y))}(x). \end{aligned}\right\} \tag{4.4}$$

In the case (H3′), we impose also that

$$\operatorname{sgn}(\tilde{u}(t,x,y) - u_M(y)) = \operatorname{sgn}(u(t,x) - u_M(x)), \\ \operatorname{sgn}(\tilde{v}(s,y,x) - u_M(x)) = \operatorname{sgn}(v(s,y) - u_M(y)).$$

We denote the new sign function by $\widetilde{\operatorname{sgn}}(x,u) = \operatorname{sgn}(u - u_M(x))$. According to the previous notation it means that

$$\left.\begin{aligned} \tilde{u}(t,x,y) &= k^+_{A(x,u(t,x))}(y) \widetilde{\operatorname{sgn}}_+(x, u(t,x)) + k^-_{A(x,u(t,x))}(y) \widetilde{\operatorname{sgn}}_-(x, u(t,x)), \\ \tilde{v}(s,y,x) &= k^+_{A(y,v(s,y))}(x) \widetilde{\operatorname{sgn}}_+(y, v(s,y)) + k^-_{A(y,v(s,y))}(x) \widetilde{\operatorname{sgn}}_-(y, v(s,y)). \end{aligned}\right\} \tag{4.5}$$

Now we write the entropy condition (3.3) for $u(t,x)$ with $\alpha = A(y, v(s,y))$

$$\partial_t (u(t,x) - k_{A(y,v(s,y))}(x))_+ \\ + \partial_x [(A(x,u(t,x)) - A(x, k_{A(y,v(s,y))}(x))) \operatorname{sgn}_+(u(t,x) - k_{A(y,v(s,y))}(x))] \leqslant 0,$$

which leads to

$$\partial_t (u(t,x) - \tilde{v}(s,y,x))_+ \\ + \partial_x [(A(x,u(t,x)) - A(y,v(s,y))) \operatorname{sgn}_+(u(t,x) - \tilde{v}(s,y,x))] \leqslant 0. \tag{4.6}$$

We obtain a similar inequality when we write the partially adapted Kružkov entropy relation (3.4) for $v(s,y)$ with $\alpha = A(x, u(t,x))$:

$$\partial_s (v(s,y) - \tilde{u}(t,x,y))_- \\ + \partial_y [(A(y,v(s,y)) - A(x,u(t,x))) \operatorname{sgn}_-(v(s,y) - \tilde{u}(t,x,y))] \leqslant 0. \tag{4.7}$$

Now for $\varepsilon > 0, \eta > 0$, we introduce two positive functions $\rho, \xi \in C_0^\infty(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \rho(z)\,\mathrm{d}z = \int_{\mathbb{R}} \xi(z)\,\mathrm{d}z = 1, \tag{4.8}$$

and, for $\eta, \varepsilon > 0$, we define two families of functions $\rho_\varepsilon, \xi_\eta \in C_0^\infty(\mathbb{R})$ such that

$$\xi_\eta(z) = \frac{1}{\eta}\xi\left(\frac{z}{\eta}\right), \qquad \rho_\varepsilon(z) = \frac{1}{\varepsilon}\rho\left(\frac{z}{\varepsilon}\right),$$

which provide two approximations of the Dirac mass δ_0. Moreover, we impose that the support of ρ is included in $]-2,-1[$. Then we add (4.7) to (4.6) and we integrate in y, s, x, t against a function $\Phi_{\eta\varepsilon}(x,t,y,s) \in C_0^\infty(Q^2)$ with $\Phi_{\eta\varepsilon}(x,t,y,s) = \phi(x,t)\rho_\varepsilon(t-s)\xi_\eta(x-y)$. Finally, we obtain

$$\left.\begin{aligned}
&\text{(I)} && \int_{Q^2}(u(t,x)-\tilde{v}(s,y,x))_+\partial_t\phi(x,t)\rho_\varepsilon(t-s)\xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\text{(II)} && -\int_{Q^2}((u(t,x)-\tilde{v}(s,y,x))_+ - (v(s,y)-\tilde{u}(t,x,y))_-) \\
& && \qquad\qquad \times \phi(x,t)\rho_\varepsilon'(t-s)\xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\text{(III)} && +\int_{Q^2}(A(x,u(t,x))-A(x,\tilde{v}(s,y,x)))\partial_x\phi(x,t)\rho_\varepsilon(t-s)\xi_\eta(x-y) \\
& && \qquad\qquad \times \operatorname{sgn}_+(u(t,x)-\tilde{v}(s,y,x))\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\text{(IV)} && -\int_{Q^2}(A(x,u(t,x))-A(y,v(s,y)))\phi(x,t)\rho_\varepsilon(t-s)\xi_\eta'(x-y) \\
& && \qquad\qquad \times (\operatorname{sgn}_+(u(t,x)-\tilde{v}(s,y,x)) \\
& && \qquad\qquad\qquad + \operatorname{sgn}_-(v(s,y)-\tilde{u}(t,x,y)))\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\text{(V)} && +\int_{Q\times\mathbb{R}}(u_0(x)-\tilde{v}(s,y,x))_+\phi(x,0)\rho_\varepsilon(-s)\xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x \\
&\text{(VI)} && +\int_{Q\times\mathbb{R}}(v_0(y)-\tilde{u}(t,x,y))_-\phi(x,t)\rho_\varepsilon(t)\xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \geqslant 0.
\end{aligned}\right\} \tag{4.9}$$

The main difference from the classical proof of Kružkov (see [7, 20, 23]) is the presence of terms (II) and (IV). Note that the derivatives that appear in these terms are derivatives of functions that tend to Dirac masses. We will prove that term (IV) is equal to zero for all (η, ε). For term (II), the main idea of the proof is to first consider the limit when η tends to zero, with a fixed ε, and to show that this limit is equal to zero for all ε.

Let us first establish that $\tilde{v}$ and $\tilde{u}$ belong to $L^\infty([0,T]\times\mathbb{R}^2)$. We give the proof for $\tilde{v}$. By hypothesis, $v \in L^\infty([0,T]\times\mathbb{R})$. It follows from (H2) that for a.e. s, y

$$|A(y,v(s,y))| \leqslant \max_{-\|v\|_{L^\infty}\leqslant\sigma\leqslant\|v\|_{L^\infty}} g(\sigma) = M.$$

Since $A(x,\tilde{v}(s,y,x)) = A(y,v(s,y))$, and using (H2) again, we obtain

$$f(\tilde{v}(s,y,x)) \leqslant M.$$

Finally, since $|f(\pm\infty)| = +\infty$, we conclude that $\tilde{v} \in L^\infty([0,T]\times\mathbb{R}^2)$.

Term (IV)

We now treat the part involving the sign functions. We prove that

$$\operatorname{sgn}(u(t,x)-\tilde{v}(s,y,x)) = \operatorname{sgn}(\tilde{u}(t,x,y)-v(s,y)) \quad \text{for a.e. } t,x,s,y \in Q^2. \tag{4.10}$$

By definition of $\tilde{u}$ and $\tilde{v}$ in (4.3), we have, for a.e. $t,x,s,y \in Q^2$,

$$A(x,u(t,x)) - A(x,\tilde{v}(s,y,x)) = A(y,\tilde{u}(t,x,y)) - A(y,v(s,y)). \tag{4.11}$$

Under the hypothesis (H3), $A(x,\cdot)$ is monotone and therefore (4.11) implies the result (4.10). Under hypothesis (H3′), it follows from (4.5) that

$$\widetilde{\operatorname{sgn}}(x,u) - \widetilde{\operatorname{sgn}}(y,\tilde{u}) = \widetilde{\operatorname{sgn}}(y,v) - \widetilde{\operatorname{sgn}}(x,\tilde{v}) = 0. \tag{4.12}$$

The case $\widetilde{\operatorname{sgn}}(x,u) = \widetilde{\operatorname{sgn}}(x,\tilde{v})$ reduces to hypothesis (H3) since $A(x,\cdot)$ is monotone on each semi-space $[-\infty,u_M(x)]$ and $[u_M(x),+\infty]$. If $\widetilde{\operatorname{sgn}}(x,u) \neq \widetilde{\operatorname{sgn}}(x,\tilde{v})$, the result (4.10) is an immediate consequence of (4.12).

Then from (4.10) we deduce that for a.e. $t,x,s,y \in Q^2$

$$(A(x,u) - A(y,v))\phi(x,t)\rho_\varepsilon(t-s)\xi'_\eta(x-y)(\operatorname{sgn}_+(u-\tilde{v}) + \operatorname{sgn}_-(v-\tilde{u})) = 0.$$

Since $u,v,\tilde{u},\tilde{v} \in L^\infty$ and, for $\eta,\varepsilon > 0$, $\phi,\xi_\eta,\rho_\varepsilon \in C_0^\infty$, we can apply Lebesgue's theorem and conclude that, for every $\eta,\varepsilon > 0$, term (IV) is equal to zero.

Term (II)

We first observe that

$$\begin{aligned} &|(u(t,x)-\tilde{v}(s,y,x))_+ - (v(s,y)-\tilde{u}(t,x,y))_-| \\ &\qquad\qquad \leqslant |u(t,x)-\tilde{u}(t,x,y)| + |v(s,y)-\tilde{v}(s,y,x)|, \end{aligned}$$

and then it is sufficient to prove that

$$\int_{\mathbb{R}} |v(s,y)-\tilde{v}(s,y,x)|\xi_\eta(x-y)\,\mathrm{d}x \xrightarrow[\eta\to 0]{} 0 \quad \text{for a.e. } s,y \in Q, \tag{4.13}$$

to establish that, for every $\varepsilon > 0$, the limit in η of term (II) is equal to zero. Indeed, once we have (4.13), and since, for $\varepsilon > 0$, all functions are bounded, we can apply dominated convergence to conclude that the integral in s, y, t, x tends to zero. The results for $|u(t,x)-\tilde{u}(t,x,y)|$ are obviously similar. Thus the absolute value of term (II) is bounded by an expression that vanishes with η.

In order to prove (4.13) we now establish that

$$\tilde{v}(s,y,x) \xrightarrow[x\to y]{} \tilde{v}(s,y,y) = v(s,y) \quad \text{for a.e. } s,y \in Q. \tag{4.14}$$

Here we use the assumption (H1), i.e. that A is continuous outside a negligible set. Then, for $y \in \mathbb{R}\setminus\mathcal{N}$,

$$A(x,\tilde{v}(s,y,y)) \xrightarrow[x\to y]{} A(y,\tilde{v}(s,y,y)).$$

Alternatively, we have, by construction of $\tilde{v}$,

$$A(y,\tilde{v}(s,y,y)) = A(y,v(s,y)) = A(x,\tilde{v}(s,y,x)) \quad \text{for a.e. } s,y,x \in Q\times\mathbb{R}.$$

Thus

$$A(x,\tilde v(s,y,x)) - A(x,\tilde v(s,y,y)) \xrightarrow[x\to y]{} 0,$$

and (4.14) is a consequence of the fact that $A(x,\cdot)$ is a one-to-one function thanks to assumption (H3), and in the case of (H3′) we also use the fact that $\operatorname{sgn}(\tilde v(s,y,x)) = \operatorname{sgn}(\tilde v(s,y,y))$ from its very construction in (4.5).

Now we claim that the integral in (4.13) can be written

$$\int_{\mathbb{R}} |v(s,y) - \tilde v(s,y,y+\eta z)|\xi(z)\,\mathrm{d}z,$$

and then, since all functions are bounded and since the support of ξ is bounded also, we can use the result (4.14) and dominated convergence to conclude (4.13).

Terms (I) and (III) are more classical. The only key point is that we must deal with 'tilde functions', but we will use the result (4.13) to recover the classical proof of Kružkov.

Term (I)

We first observe that

$$\begin{aligned}\bigg|\int_{Q^2}&(u(t,x)-\tilde v(s,y,x))_+\partial_t\phi(x,t)\rho_\varepsilon(t-s)\xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_{Q^2}(u(t,x)-v(s,y))_+\partial_t\phi(x,t)\rho_\varepsilon(t-s)\xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\bigg|\\ &\leqslant\int_{Q^2}|\tilde v(s,y,x)-v(s,y)|\partial_t\phi(x,t)\rho_\varepsilon(t-s)\xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and we use the previous computation (see (4.13)) to claim that the limit in η, ε of term (I) is the same as the limit of

$$\int_{Q^2}(u(t,x)-v(s,y))_+\partial_t\phi(x,t)\rho_\varepsilon(t-s)\xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t.$$

Now we claim it is enough to prove that

$$\int_{Q^2}|v(t,x)-v(s,y)|\partial_t\phi(x,t)\rho_\varepsilon(t-s)\xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \xrightarrow[\eta\to0,\varepsilon\to0]{} 0, \tag{4.15}$$

in order to conclude that the limit of term (I), when η and ε tend to zero, is

$$\int_Q (u(t,x)-v(t,x))_+\partial_t\phi(x,t)\,\mathrm{d}t\,\mathrm{d}x.$$

The proof of (4.15) is also a crucial step of the uniqueness proof when the flux does not depend on the space variable and we refer to [7, 23] for the details.

Term (III)

Finally, we consider the term that contains the fluxes. We define

$$G(x,u,w) = (A(x,u) - A(x,w))\operatorname{sgn}(u-w).$$

Hypotheses (H3) or (H3′) imply that G is a locally Lipschitz function of the third variable. Since $\tilde{v} \in L^\infty([0,T] \times \mathbb{R} \times \mathbb{R})$, it follows that

$$|G(x,u(t,x),\tilde{v}(s,y,x)) - G(x,u(t,x),\tilde{v}(s,y,y))| \leqslant C|\tilde{v}(s,y,x) - \tilde{v}(s,y,y)| = C|\tilde{v}(s,y,x) - v(s,y)|,$$

and it then follows from (4.13) that the limit of term (III) is the same as the limit of

$$\int_{Q^2} (A(x,u(t,x)) - A(x,v(s,y)))\,\mathrm{sgn}(u(t,x) - v(s,y)) \times \partial_x \phi(x,t) \rho_\varepsilon(t-s) \xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t.$$

We use the Lipschitz property on G a second time. Now $v \in L^\infty([0,T] \times \mathbb{R})$ and

$$|G(x,u(t,x),v(s,y)) - G(x,u(t,x),v(t,x))| \leqslant C|v(s,y) - v(t,x)|.$$

Thus, it is sufficient to prove that

$$\int_{Q^2} |v(t,x) - v(s,y)| \partial_x \phi(x,t) \rho_\varepsilon(t-s) \xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \xrightarrow[\eta \to 0, \varepsilon \to 0]{} 0, \qquad (4.16)$$

to conclude that term (III) tends to

$$\int_Q (A(x,u(t,x)) - A(x,v(t,x)))\,\mathrm{sgn}_+(u(t,x) - v(t,x)) \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t.$$

The integral in (4.16) appears in the classical proof. It is very similar to the one in (4.15) and the same arguments lead to the result (4.16).

The computation of terms (V) and (VI) is classical. Thanks to the hypothesis on the support of ρ, term (VI) is equal to zero. For term (V), we claim that the result (4.13) allows us to consider only

$$\int_{Q \times \mathbb{R}} (u_0(x) - v(s,y))_+ \phi(x,0) \rho_\varepsilon(-s) \xi_\eta(x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x \to \int_{\mathbb{R}} (u_0(x) - v_0(x))_+ \phi(x,0)\,\mathrm{d}x,$$

and then the end of the proof is standard (see [7, 23]). □

5. Application: discontinuous convex flux

We pointed out at the end of § 1 that different interface conditions can give rise to different unique solutions. Since our method does not require an interface condition it can be used to discriminate between the existing interface conditions, at least for the cases where our theory can be applied.

Here we propose to study a particular Heaviside-type flux $A(x,u) = H(x)f_+(u) + (1 - H(x))f_-(u)$, where $f_\pm(u) = k_\pm(u - \beta_\pm)^2$, for which the theory in [1, 15] does not give rise to the same solution. This convex flux satisfies hypothesis (H3′) and thus we can exhibit the solution that is selected by our theory.

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We study the Riemann problem associated with the very simple discontinuous convex flux of Heaviside type

$$A(x,u) = \tfrac{1}{2}H(x)u^2 + \tfrac{1}{2}(1-H(x))(u-1)^2, \tag{5.1}$$

and with the constant initial data

$$u_0(x) = \tfrac{1}{2}. \tag{5.2}$$

It is obvious that

$$u(t,x) = \tfrac{1}{2} \tag{5.3}$$

is a weak solution of the Riemann problem (5.1), (5.2). But, more generally, for $u_i \in [0, \frac{1}{2}]$, the function defined by

$$u(t,x) = \begin{cases} \frac{1}{2}, & x \leqslant -\frac{1}{2}t, \\ 1 + \dfrac{x}{t}, & -\frac{1}{2}t < x \leqslant -u_i t, \\ 1 - u_i, & -u_i t < x \leqslant 0, \\ u_i, & 0 < x \leqslant u_i t, \\ \dfrac{x}{t}, & u_i t < x \leqslant \frac{1}{2}t, \\ \frac{1}{2}, & \frac{1}{2}t < x, \end{cases} \tag{5.4}$$

is also a weak solution of the Riemann problem (5.1), (5.2).

Now let us apply our entropy theory to this case. It follows from the definition (3.2) that

$$k_\alpha^\pm(x) = \pm\sqrt{2\alpha} + 1 - H(x). \tag{5.5}$$

Thus the entropy inequality (3.3) becomes

$$\begin{aligned} &\partial_t |u - (\pm\sqrt{2\alpha} + 1 - H(x))| \\ &\quad + \partial_x[(H(x)\tfrac{1}{2}u^2 + (1-H(x))\tfrac{1}{2}(u-1)^2 - \alpha)\operatorname{sgn}(u - (\pm\sqrt{2\alpha} + 1 - H(x)))] \leqslant 0, \end{aligned} \tag{5.6}$$

Let us choose a solution on the form (5.4), with $u_i = 0$:

$$u(t,x) = \begin{cases} \frac{1}{2}, & x \leqslant -\frac{1}{2}t, \\ 1 + \dfrac{x}{t}, & -\frac{1}{2}t < x \leqslant 0, \\ \dfrac{x}{t}, & 0 < x \leqslant \frac{1}{2}t, \\ \frac{1}{2}, & \frac{1}{2}t < x. \end{cases} \tag{5.7}$$

For (x,t) such that $2x \leqslant -t$ or $2x > t$, the entropy inequality (5.6) is obviously satisfied. Now for (x,t) such that $2x \in \,]-t,t]$, the solution can be denoted by

$$u(t,x) = \frac{x}{t} + 1 - H(x).$$

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The entropy inequality (5.6) becomes

$$\partial_t\left|\frac{x}{t} \pm \sqrt{2\alpha}\right| + \tfrac{1}{2}\partial_x\left[\left(\left(\frac{x}{t}\right)^2 - 2\alpha\right)\operatorname{sgn}\left(\frac{x}{t} \pm \sqrt{2\alpha}\right)\right] \leqslant 0.$$

and is also obviously satisfied. Thus the solution given in (5.7) is the entropy solution of the Riemann problem (5.1), (5.2).

Note that, for this particular crossing convex flux, the interface condition in [15] selects the constant solution (5.3), whereas the interface condition in [1] selects the solution (5.7).

Note added in proof

After this paper was accepted, we learned about another method [3] that gives uniqueness without interface conditions.

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