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THE HADAMARD PRODUCT OF TWO BROWNIAN MATRICES: ANALYTIC INVERSE AND DETERMINANT

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Abstract

The explicit inverse and determinant of a class of matrices is given. The class is the Hadamard product of two already known classes. Its elements are defined by $3n - 1$ parameters, analytical expressions of which compose the Hessenberg form inverse. These expressions enable a recursive formula to be obtained, which gives the inverse in $O(n^2)$ multiplications/divisions and $O(n)$ additions/subtractions.

1. Introduction

Linear models are frequently used in the theory of digital signal processing. A common characteristic of such models is that they lead to linear systems of general form $Bx = c$, where B denotes a generalized resultant matrix obtained from certain *Brownian matrices*. This is due to the fact that *discrete-time Brownian motion* is a proper model for *discrete-time signals*, of which, in turn, the covariance matrix is Brownian (see, for example, [3], [9], and references therein). Furthermore, if there are two discrete-time-signal devices, represented by their covariance Brownian matrices, B_1 and B_2, respectively, then their coupling results in a generalized resultant matrix $B = B_1$ .op. B_2, where the operator .op. may be (i) matrix multiplication or (ii) Hadamard product. If B_1^{-1} and B_2^{-1} can be computed by "fast" algorithms, then the same could also be true for B^{-1} in the case (i). In the present paper a fast algorithm is developed for computing B^{-1} in the case (ii), when B_1 and B_2 are given Brownian matrices, $B_1 = N$ and $B_2 = A$.

In particular, in [10] the tridiagonal inverse of a symmetric class of matrices $N = [a_{ij}]$ with elements $a_{ij} = k_i$, $i \geq j$, has been obtained. In [11] a more general

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type of the matrix N has been presented; namely, the matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} a_j & i \geq j, \\ b_i & i < j, \end{cases}$$

the inverse of which is of upper Hessenberg form. In this paper we present the explicit inverse and determinant of a class represented by the Hadamard product of these two classes, that is, $B = N \circ A$, where B is the class under consideration and $\circ$ denotes the Hadamard product.

If we consider the $n \times n$ matrix $P = [p_{ij}]$, with elements

$$p_{ij} = \begin{cases} 1 & i + j = n + 1, \\ 0 & \text{otherwise}, \end{cases}$$

then the matrix PNP is a pure Brownian matrix, while A is a Brownian matrix, in agreement with the definitions of [9] and [3], respectively. In [1] the so-called "diagonal innovation matrices" (DIM) are treated, special cases of which are both the matrices N and A.

The Hadamard product of these matrices, besides its interest from both the mathematical and the digital-signal-processing viewpoint, does also constitute a significant general case of already known classes of *test matrices*. In particular, by assigning proper values to the $3n - 1$ parameters, a variety of matrices for testing computational algorithms can be constructed. It is worth noting that, by restricting the a's, b's, and k's, well-known classes of test matrices become subcases of the present category; namely, the classes appeared in [2], [5] – [8], [10], [11], as well as in [4], pages 42 and 48.

2. The matrix and its inverse

Let us consider a matrix $B = [b_{ij}]$ with elements

$$b_{ij} = \begin{cases} k_i a_j & i \geq j, \\ k_j b_i & i < j. \end{cases} \tag{1}$$

Its inverse $B^{-1} = [\sigma_{ij}]$ is found to be an upper Hessenberg matrix with elements given analytically by the formulae

$$\sigma_{ij} = \begin{cases} (k_{i-1}a_{i+1} - k_{i+1}b_{i-1})/(c_{i-1}c_i) & i = j = 2, \ldots, n-1, \\ a_2/(c_0 c_1) & i = j = 1, \\ k_{n-1}/(k_n c_{n-1}) & i = j = n, \\ (-1)^{i+j} d_j g_{i-1} \prod_{\nu=i+1}^{j-1} k_\nu f_\nu \Big/ \prod_{\nu=i-1}^{j} c_\nu & i < j, \\ -1/c_j & i = j + 1, \\ 0 & i > j + 1, \end{cases} \tag{2}$$

where

$$\begin{array}{lll} c_i = k_i a_{i+1} - k_{i+1} b_i & i = 1, 2, \dots, n-1, \; c_0 = a_1, \; c_n = 1, & \\ d_i = k_i a_{i+1} - k_{i+1} a_i & i = 2, 3, \dots, n-1, \; d_n = 1, & \\ f_i = b_i - a_i & i = 2, 3, \dots, n-1, & \\ g_i = k_i b_{i+1} - k_{i+1} b_i & i = 1, 2, \dots, n-2, \; g_0 = b_1, & \text{and} \\ \displaystyle\prod_{\nu=i+1}^{j-1} k_\nu f_\nu = 1 & \text{whenever } j = i+1. & \end{array} \tag{3}$$

Evidently, the above formulae are valid if

$$k_n \neq 0 \qquad \text{and} \qquad c_i \neq 0 \qquad i = 0, 1, \dots, n-1.$$

3. The proof

To verify (2), we adopt a similar method to that in [11]. By applying a number of row operations to the matrix (1), we transform it to the identity matrix. Then the product of the elementary matrices gives the inverse matrix (2). By adopting the conventions (3) and $k_0 = 1$, the row operations carried out are the following:

1. row $i - k_i/k_{i-1}$ row $(i-1)$, $\quad i = n, n-1, \dots, 2$
2. row $i - k_i g_{i-1}/(k_{i-1} g_i)$ row $(i+1)$, $\quad i = 1, 2, \dots, n-2$
3. row $(n-1) - k_{n-1} g_{n-2}/(k_{n-2} c_{n-1})$ row n and
   row $i - k_i k_i g_{i-1} f_{i+1}/(k_{i-1} g_i c_i)$ row $(i+1)$, $\quad i = n-2, n-3, \dots, 1$.
4. $1/(k_1 a_1)$ row 1 and $k_{i-1}/(k_i c_{i-1})$ row i, $\quad i = 2, 3, \dots, n$.

The above operations transform the identity matrix to the following forms, respectively:

1. The bidiagonal matrix consisting of the main diagonal $(1, 1, \dots, 1)$ and the lower first diagonal $(-k_2/k_1, -k_3/k_2, \dots, -k_n/k_{n-1})$.
2. The tridiagonal matrix with main diagonal

   $$(k_1 b_2/g_1, k_2(k_1 b_3 - k_3 b_1)/(k_1 g_2), \dots, k_{n-2}(k_{n-3} b_{n-1} - k_{n-1} b_{n-3})/(k_{n-3} g_{n-2}), 1, 1),$$

   upper first diagonal

   $$(-k_1 b_1/g_1, -k_2 g_1/(k_1 g_2), \dots, -k_{n-2} g_{n-3}/(k_{n-3} g_{n-2}), 0),$$

   and lower first diagonal $(-k_2/k_1, -k_3/k_2, \dots, -k_n/k_{n-1})$.
3. The upper Hessenberg matrix of the type (case $n = 5$)

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$$\begin{bmatrix} \frac{k_1 a_2}{c_1} & -\frac{k_1 b_1 d_2}{c_1 c_2} & \frac{k_1 b_1 d_3 k_2 f_2}{c_1 c_2 c_3} & -\frac{k_1 b_1 d_4 k_2 f_2 k_3 f_3}{c_1 c_2 c_3 c_4} & \frac{k_1 b_1 k_2 f_2 k_3 f_3 k_4 f_4}{c_1 c_2 c_3 c_4} \\ -\frac{k_2}{k_1} & \frac{k_2(k_1 a_3 - k_3 b_1)}{k_1 c_2} & -\frac{k_2 d_3 g_1}{k_1 c_2 c_3} & \frac{k_2 d_4 g_1 k_3 f_3}{k_1 c_2 c_3 c_4} & -\frac{k_2 g_1 k_3 f_3 k_4 f_4}{k_1 c_2 c_3 c_4} \\ 0 & -\frac{k_3}{k_2} & \frac{k_3(k_2 a_4 - k_4 b_2)}{k_2 c_3} & -\frac{k_3 d_4 g_2}{k_2 c_3 c_4} & \frac{k_3 g_2 k_4 f_4}{k_2 c_3 c_4} \\ 0 & 0 & -\frac{k_4}{k_3} & \frac{k_4(k_3 a_5 - k_5 b_3)}{k_3 c_4} & -\frac{k_4 g_3}{k_3 c_4} \\ 0 & 0 & 0 & -\frac{k_5}{k_4} & 1 \end{bmatrix}.$$

4. The upper Hessenberg matrix, the elements of which are given by (2).

4. The determinant

The determinant of B is derived easily by carrying out the row operation, row $i - k_i/k_{i-1}$ row $(i-1)$, $i = n, n-1, \dots, 2$, which provides the upper triangular matrix

$$\begin{bmatrix} k_1 a_1 & k_2 b_1 & k_3 b_1 & \dots & k_n b_1 \\ 0 & k_2 c_1/k_1 & k_3 g_1/k_1 & \dots & k_n g_1/k_1 \\ 0 & 0 & k_3 c_2/k_2 & \dots & k_n g_2/k_2 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & k_n c_{n-1}/k_{n-1} \end{bmatrix},$$

using the abbreviations (3). Hence

$$\det(B) = a_1 k_n (k_1 a_2 - k_2 b_1)(k_2 a_3 - k_3 b_2) \dots (k_{n-1} a_n - k_n b_{n-1}),$$

which yields the criteria for the singularity of the matrix B; that is,

$$k_n = 0 \qquad \text{or} \qquad c_i = 0 \qquad \text{for some } i \in \{0, 1, \dots, n-1\}.$$

5. Numerical complexity

The formulae in (2), which give the elements of the inverse matrix above the main diagonal, enable us to get a recursion relation in order to facilitate the evaluation of B^{-1}. This relation provides the recursive algorithm

$$\begin{aligned} \sigma_{i,i+1} &= -\frac{d_{i+1} g_{i-1}}{c_{i-1} c_i c_{i+1}} & & i = 1, 2, \dots, n-1, \\ \sigma_{i,i+s+1} &= -\frac{d_{i+s+1} k_{i+s} f_{i+s}}{d_{i+s} c_{i+s+1}} \sigma_{i,i+s} & & i = 1, 2, \dots, n-2, \ s = 1, 2, \dots, n-i-1, \end{aligned}$$

or, alternatively,

$$\begin{aligned} \sigma_{j-1,j} &= -\frac{d_j g_{j-2}}{c_{j-2}c_{j-1}c_j} & j &= 2,3,\ldots,n, \\ \sigma_{j-s-1,j} &= -\frac{g_{j-s-2}k_{j-s}f_{j-s}}{g_{j-s-1}c_{j-s-2}}\sigma_{j-s,j} & j &= 3,4,\ldots,n,\ s = 1,2,\ldots,j-2. \end{aligned}$$

The above formulae estimate all the elements σ_{ij}, for $i < j$, and reduce the number of multiplications/divisions for the evaluation of the inverse matrix to $n^2/2 + 29n/2 - 25$ in all, since the coefficient of $\sigma_{i,i+s}$ ($\sigma_{j-s,j}$) depends only on the second (first) subscript. Finally, the number of additions/subtractions is $5n - 9$.

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