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ENDOGENOUS DEMOGRAPHIC CHANGE, RETIREMENT, AND SOCIAL SECURITY

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In this paper, we analyze the effects of demographic change on a pay-as-you-go (PAYG) pension system, financed with a defined contribution scheme. In particular, we examine the relationship between retirement, fertility, and pensions in a three-period overlapping generations model. We focus on both the case of mandatory retirement and the case where the retirement age is freely chosen. In the case of mandatory retirement, increasing longevity has an unambiguously negative impact on fertility and pension payouts and a positive effect on the level of physical capital in the steady state. On the other hand, when agents choose the time of retirement, an increase in life expectancy positively affects physical capital only when the tax rate is sufficiently low and can have a positive impact on pension benefits, because agents may find it optimal to retire later and to decrease fertility less. Finally, the effects of the social security tax on capital per worker are negative with mandatory retirement; however, they could be positive in the optimal retirement case.

Keywords: PAYG Pensions, Endogenous Fertility, Aging, Retirement

1. INTRODUCTION

Pensions systems everywhere are faced with the problem of population ageing. According to the World Population Ageing Report of the United Nations (2015) between now and 2030, the number of people aged 60 years or over is projected to grow by 56% and, at the end of this period, it will outnumber children aged 0–9. This ageing process is especially advanced in Europe and Northern America where people aged 60 or over are already more than a fifth of the population and will be more than a fourth by 2030. This process is driven by both increasing longevity and decreasing fertility. In fact, since 1950 life expectancy at birth has risen by more than 10 years in Northern America and Europe and by about 25

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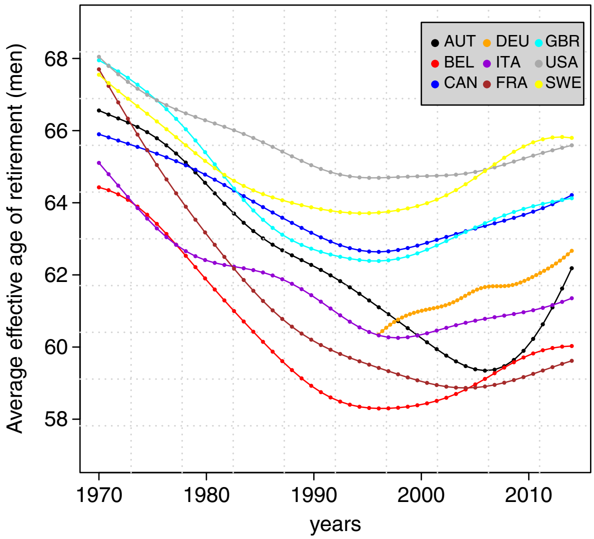

Source: http://www.oecd.org/els/emp/average-effective-age-of-retirement.html

FIGURE 1. Average effective age at which men leave the labor market over time (1970–2014).

years in Africa and Latin America. At the same time, fertility has declined dramatically over the last few decades, reaching unprecedented low levels, so that now nearly half of the world lives in countries with below-replacement level fertility (2.1 children per woman). The result is an old-age dependency ratio in developed countries of about 25%, expected to double by the end of the century. These trends pose a challenge for the sustainability of unfunded pension systems, since they struggle to maintain an adequate level of income support. Faced with this challenge, governments have introduced pension reforms that have strengthened the link between contributions paid during the working life and benefits received during retirement, for example, by switching from a mandatory-defined benefit pay-as-you-go (PAYG) pension plan to a mandatory defined contribution (DC) PAYG pension. Also as a result of these institutional reforms, the labor force participation of older workers, which was decreasing during most of the second half of the 20th century, has now reversed in many countries and is increasing in many developed countries (see Figure 1). In this paper, we study how demographic ageing interacts with the pension system. We use an overlapping generations model where agents live for two periods with certainty and for an extra period with a probability less than one. Agents make the usual choice on intertemporal consumption given their budget constraint, which also includes a labor tax and a DC PAYG pension. Moreover, agents decide about their fertility, since children enter the utility function as durable goods and the budget constraint as a time cost. Finally, another important feature of our model consists in the retirement choice: agents can decide for how long to work in the last period, the alternative

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being retirement. However, we also compare and contrast these results with the mandatory retirement case, which is still prevalent in many countries. Retired agents receive a pension and consume their savings and the government runs a balanced budget. Our main contribution is the setup of a unified framework with pensions, exogenously changing life expectancy, endogenous fertility, and endogenous retirement. The theoretical literature has in fact studied these issues separately.

In the literature, there are two main ways to endogenize fertility: one approach, pioneered by Caldwell (1978), builds on the so-called old-age security motive, that is, parents have children because they provide them with old-age transfers. The second approach includes children in the utility function, because they are regarded as (durable) consumption goods that is desirable in themselves. The pioneer work here is Barro and Becker (1989). There are few attempts to set up models with both channels [Wigger (1999) is one of these] but generally they are considered alternatively. Among the papers that model pensions with endogenous fertility of the first type, the so-called old-age security type, seminal works are Cigno (1993), Zhang and Nishimura (1993), and Bental (1989). On the other hand, some recent contributions following the second approach, that is, the consumption good motive, are Cipriani (2014), Cremer et al. (2011), and van Groezen et al. (2003). The first paper studies the effects of a PAYG pension system in an overlapping generations model under exogenous and endogenous fertility. It shows that increasing longevity may lead to population ageing and adversely affect the pension system. The last paper presents a model where child benefits can be used by a government to replicate the command optimum when rearing children causes a positive externality on the PAYG pension system. Finally, Cremer et al. (2011) have a model with human capital investment that, together with fertility choice, affects the distribution of abilities, thus introducing a further externality in the presence of a PAYG pension system. Then, in a first best environment they study optimal policies on child and education subsidies. In this paper, we model fertility choices like in this second approach, since our purpose is to study a developed economy with retirement and pensions, where children are not expected to contribute to the retirement income of their parents. In particular, we assume the utility function to be logarithmic and additively separable as in van Groezen et al. (2003). Moving now to the other side of the problem, the process of ageing from above, which takes place through an exogenous or endogenous increase in life expectancy, and its effects on a PAYG system have been extensively studied, given their clear policy relevance in a world where individuals live much longer than when the first unfunded pension systems were introduced. Some recent examples, which model this increase in life expectancy in the same way we do here, that is, in an overlapping generations model where life extends probabilistically to a last period, are Tabata (2015), Cipriani and Makris (2012), and Fanti and Gori (2008). The first paper studies how a pension reform from a defined-benefit scheme to a DC scheme affects growth in a model where fertility is assumed to be exogenous. In the second paper, Cipriani and Makris (2012)

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consider exogenous fertility and endogenous longevity, where the average level of human capital affects life expectancy. In that model, the pension scheme affects agents' wealth and thereby the private accumulation of human capital, which in turn affects longevity in a way which could be inconsistent with the projected path of longevity used to design pensions in the first place. Finally, Fanti and Gori (2008) in a simple overlapping generations model with exogenous fertility and PAYG pensions show that increasing longevity may not always reduce pensions. However, none of these papers considers retirement choice. Only very few papers have modeled retirement in this framework, among these Cabo and García-Gonzále**a**z (2014), Chen and Lau (2016), and Nishimura et al. (2017).[1] The first paper models the strategic interaction between the individual's decision about retirement and the government choice for the generosity of the public pension. The second paper studies the response of retirement age and saving to mortality decline in a computational analysis of a continuous-time overlapping generations model. Finally, Nishimura et al. (2017) have a model with education investment, where education enhances the productivity of labor and the duration of old age. In the old age, there is some disutility from labor, which gives endogenous retirement like in our model. However, the focus of their paper is on the Ben-Porath effect under endogenous longevity, that is, on the fact that an increase in longevity, by increasing the retirement age, increases the returns to education, thus reinforcing the Ben-Porath effect. None of these models, however, has modeled fertility choice. A recent paper which considers retirement policies in a model with endogenous fertility is Cipriani and Pascucci (2018). It shows that policy measures usually adopted to face the problem of the sustainability of a PAYG pension system, like increasing the retirement age or the social security contribution rate, might have negative effects on the fertility rate, thus exacerbating population ageing. However, the closest work to ours is Dedry et al. (2017), who set up a unified model to compare the effects of different social security and retirement regimes. Still, their model assumes exogenous fertility, while in our setting fertility is optimally chosen and therefore population ageing, which is driven both by declining fertility and by increasing longevity, is fully endogenous.

We study two cases: mandatory retirement and optimal retirement. In the steady state, we show that, under some assumptions on the parameters, population ageing from above, that is, from increasing life expectancy, is reinforced by ageing from below, that is, by decreasing fertility. With optimal retirement individuals will extend their working life such that the overall effects on pension payments are not necessarily negative. On the other hand, with mandatory retirement pensions will unambiguously fall. Finally, the effects of increasing longevity on equilibrium capital per worker are positive, like in the case without social security, with mandatory retirement. However, in the optimal retirement case, the introduction of endogenous fertility in the model gives a different result: longevity has a positive effect on capital only if the social security tax rate is sufficiently low. In order to illustrate the results of the model, especially in the cases when analytical

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results are ambiguous, we simulate the steady state and the transition dynamics of the model, under the values of parameters calibrated on Italian data.

The structure of the paper is the following. In the next section we present the model and derive analytical results in the mandatory retirement and optimal retirement settings. In Section 3 we illustrate the steady state and the dynamics with a numerical simulation of a calibrated version of the model on Italian data. A concluding section summarizes our results.

2. THE MODEL

We consider an economy populated by overlapping generations of people who potentially live for three periods. In the first period, they are children and make no decisions; in adulthood, individuals work full time and raise their offspring. Though certain to live through childhood and adulthood, agents are, however, subject to a probability p of surviving to old age. If they survive to the third period, agents will choose either to continue working or to retire. In the case of retirement, agents benefit from a state-funded PAYG pension scheme. For the sake of simplicity, the length of each period is normalized to one.

Preferences of an adult agent born in period t are defined over consumption in adult age, that is, c_a^t, the number of children n^t, and if they survive to old age, from consumption c_o^t and leisure time $(1 - l_t)$ after retirement. Thus, their expected utility function is given by

$$U^t = \ln c_a^t + \theta \ln n^t + \beta p[\ln c_o^t + \delta \ln(1 - l^t)], \tag{1}$$

where $\beta \in (0, 1)$ is the overall weight attached to utility in old age, $\delta \in (0, 1)$ is the weight of retirement, and θ reflects fertility preferences.

In the adult age, parents allocate their wage between consumption c_t^a, saving s^t, raising their children n^t, and paying a tax τ in order to contribute to the PAYG pension scheme. Thus the budget constraint of an adult agent in period t is

$$c_a^t = (1 - \tau - qn^t)w_t - s^t, \tag{2}$$

where $0 < q < 1$ is the fraction of the parental wage required to raise each child[2] and $n_t \leq (1 - \tau)/q$.

In the third period, agents consume their savings and receive a pension benefit b_{t+1} over the period in which they do not work, and when they do work their wage is taxed in order to finance the PAYG scheme. Thus the budget constraint in old age is given as follows:

$$c_o^t = \frac{R_{t+1}s^t}{p} + (1 - \tau)w_{t+1}l^t + b_{t+1}(1 - l^t), \tag{3}$$

where R_{t+1}/p is the rate of return on savings, given that the savings of agents that do not survive to the old age are redistributed to the surviving ones.

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Production occurs according to a constant-returns-to-scale technology, using labor L_t and physical capital K_t. Assuming a Cobb–Douglas production function, output produced at time t is

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha}, \tag{4}$$

where $\alpha \in (0, 1)$ and $A > 0$ is a technological parameter.

Firms operate in a perfectly competitive market. In each period t, firms choose the level of labor L_t and of physical capital K_t so as to maximize profits. Thus the wage rate, w_t, and the rate of return to capital, r_t, are given by

$$w_t = A(1-\alpha)k_t^{\alpha} \tag{5}$$

and

$$R_t = \alpha A k_t^{\alpha-1}, \tag{6}$$

where $k_t = K_t/L_t$.

Labor supply in each period t is given by labor supplied by the adult, that is, $N_t = n^{t-1}N_{t-1}$ and the labor supplied by the old, that is, $N_{t-1}pl^{t-1}$. Thus, labor force supply in period t is $N_{t-1}(n^{t-1} + pl^{t-1})$. In equilibrium, this supply must be equal to the total demand, that is:

$$L_t = N_{t-1}(n^{t-1} + pl^{t-1}). \tag{7}$$

The equilibrium condition in the capital market, under the assumption that physical capital depreciates completely after one period, is

$$K_{t+1} = N_t s_t. \tag{8}$$

Thus the dynamic equation for capital per worker is given by

$$k_{t+1} = \frac{s^t}{n^t + pl^t}. \tag{9}$$

The government has to observe a balanced budget and a PAYG social security scheme. Thus, the revenue from taxing both young, that is, $\tau_{t+1}w_{t+1}n^tN_t$, and old, that is, $\tau_{t+1}w_{t+1}l^tpN_t$ is used to finance retirement pensions. Thus pension benefits are

$$b_{t+1} = \frac{\tau w_{t+1}(n^t + pl^t)}{p(1-l^t)}. \tag{10}$$

Each household chooses n^t, s^t, and l^t so as to maximize the utility function (1) subject to (2), (3), and $l^t \geq 0$ taking as given the wage, the interest rate, the pension benefit, and the tax rate. After substituting for b_{t+1} from equation (10) optimal saving, fertility, and retirement at an interior solution are given by

$$n^t = \frac{(1-\tau)\theta(pw_{t+1} + R_{t+1}w_t)}{R_{t+1}qw_t[1+\theta+\beta p(1+\delta)]}, \tag{11}$$

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$$s^t = \frac{p(1-\tau)[R_{t+1}\beta(1+\delta)w_t - w_{t+1}(1+\theta)]}{R_{t+1}[1+\theta+\beta p(1+\delta)]}, \tag{12}$$

$$l^t = \frac{(1-\tau)\left\{w_t w_{t+1} R_{t+1}\left[\beta p^2 q + pq(1+\theta) - \tau\theta\right] - w_t^2 R_{t+1}^2 \beta\delta pq - p\tau\theta w_{t+1}^2\right\}}{w_t w_{t+1} R_{t+1} pq[1+\theta+\beta p(1+\delta)]}. \tag{13}$$

From the first-order conditions we derive on the one hand that, *ceteris paribus*, a longer life expectancy positively affects agent savings. In fact, in accordance with an extensive literature a higher probability of surviving to the third period of life leads individuals to save more in order to finance increased consumption needs in old age. On the other hand, labor supply in the third period negatively affects savings, because the possibility of working in old age reduces the necessity of saving [see Aísa et al. (2012) and Dedry et al. (2017)] and also it positively affects the number of children through a positive income effect [see Mizuno and Yakita (2013)].

In many countries, agents can't choose their retirement age because it is set by the government. This implies that agents in old age work for a fixed amount of time $\bar{l}$. Thus, in this case optimal choices become:

$$n^t = \frac{R_{t+1}(1-\tau)\theta w_t + \bar{l}p\theta w_{t+1}}{R_{t+1}q(1+\theta+\beta p)w_t - \tau\theta w_{t+1}}, \tag{14}$$

$$s^t = \frac{R_{t+1}\beta pq(1-\tau)w_t^2 - w_t w_{t+1}[\tau(1-\tau)\theta + \bar{l}pq(1+\theta)]}{R_{t+1}q(1+\theta+\beta p)w_t - \tau\theta w_{t+1}}. \tag{15}$$

To study the case of mandatory retirement, for the sake of simplicity we normalize the mandatory retirement age $\bar{l}$ to zero.[3] The results obtained in this case are, of course, identical to the optimal retirement ones in the case of a corner solution with full retirement.

2.1. Steady-State Comparative Statics

In the following sub-sections we analyze the comparative statics in the steady state for the two PAYG retirement regime schemes as defined in the previous section, that is, mandatory retirement and optimal retirement.

2.1.1. Mandatory Retirement. In the case of mandatory retirement, the dynamic equation for physical capital per worker is

$$k_{t+1} = \frac{Aq(1-\alpha)\alpha\beta p}{\theta[\alpha+\tau(1-\alpha)]}k_t^{\alpha}. \tag{16}$$

Thus there is one stable steady state given:

$$k_{mr}^* = \left\{\frac{Aq(1-\alpha)\alpha\beta p}{\theta[\alpha+\tau(1-\alpha)]}\right\}^{\frac{1}{1-\alpha}}. \tag{17}$$

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TABLE 1. The effects of social security and aging on equilibrium capital per worker

	Standard case ($\tau = 0$)	DC ($\tau > 0$)
Mandatory retirement		
Exogenous Fertility	$\partial k/\partial p > 0$	$\partial k/\partial \tau < 0$, $\partial k/\partial p > 0$
Endogenous Fertility	$\partial k/\partial p > 0$	$\partial k/\partial \tau < 0$, $\partial k/\partial p > 0$
Optimal retirement		
Exogenous Fertility	$\partial k/\partial p > 0$	$\partial k/\partial \tau < 0$
Endogenous Fertility	$\partial k/\partial p > 0$ if $p < \hat{p}$	$\partial k/\partial \tau > 0$, $\partial k/\partial p > 0$ if $p < \hat{p}$

Note: See Appendix A.1.

The steady-state level of fertility and saving are therefore given by, respectively:

$$n^*_{mr} = \frac{(1-\tau)\theta[\alpha + \tau(1-\alpha)]}{q\alpha\beta p + q(1+\theta)[\alpha + \tau(1-\alpha)]}, \tag{18}$$

$$s^*_{mr} = \frac{\alpha A\beta p(1-\tau)(1-\alpha)k^{*\alpha}}{\alpha\beta p + (1+\theta)[\alpha + \tau(1-\alpha)]}. \tag{19}$$

Unsurprisingly, from equation (17), the presence of social security depresses the steady-state level of physical capital, that is, $\partial k/\partial \tau < 0$, due to the usual saving displacement effect.

As in the standard case with no social security, the increase in longevity positively affects the steady-state level of physical capital per worker. This is because increasing longevity leads to higher saving and to lower fertility. Thus, the positive effect of longevity on k^* through a higher saving is reinforced by the positive effect through a lower fertility. In particular, it is easy to demonstrate that the steady-state level of physical capital rises at increasing rate with respect to longevity (i.e. $\partial k/\partial p > 0$ and $\partial k^2/\partial p^2 > 0$). These results are summarized in Table 1.

From equations (18) and (19), an increase in life expectancy has always got a negative effect on the steady-state level of fertility and has a positive effect on saving. Thus, from equation (10) population ageing has two opposite effects on pension payout in the steady state when agents fully retire: a positive effect through wages and a negative one through the old age dependency ratio:

$$b_{mr} = \tau A(1-\alpha)\left\{\frac{\alpha\beta(1-\alpha)Aq}{\theta[\alpha + \tau(1-\alpha)]}\right\}^{\frac{1}{1-\alpha}} \frac{n_{mr}}{p^{(1-2\alpha)/(1-\alpha)}}. \tag{20}$$

Therefore, in accordance with Cipriani (2014, 2018) if $\alpha < 1/2$, an increase in the probability of surviving to old age negatively affects pensions in the steady state.

Overall, we can conclude that the introduction of endogenous fertility reinforces the main results obtained in a framework with exogenous fertility. In other words, population ageing as a result of increasing life expectancy is reinforced from below by decreasing fertility.

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2.1.2. Optimal retirement. From equations (5), (6), (11)–(13), the dynamic equation for physical capital per worker is given by

$$k_{t+1} = Ak_t^{\alpha}\left[\sqrt{\left(\frac{T(p)}{p}\right)^2 + \Psi} - \frac{T(p)}{p}\right], \tag{21}$$

where $T(p) = [\alpha\theta(1-\tau) + pq(1+\theta+\alpha\beta p)]/2\theta(1-\tau)$ and $\Psi = \alpha q\beta(1-\alpha+\delta)/\theta(1-\tau)$. Hence there are two steady states: one, unstable, is at $k = 0$, which can be disregarded, and one stable steady state given by[4]

$$k_{or}^* = \left\{A\left[\sqrt{\left(\frac{T(p)}{p}\right)^2 + \Psi} - \frac{T(p)}{p}\right]\right\}^{\frac{1}{1-\alpha}}. \tag{22}$$

The steady-state level of fertility, saving, and labor supply in old age are given respectively by

$$n_{or}^* = \frac{(1-\tau)\theta(pk^{*1-\alpha} + \alpha A)}{\alpha Aq[1+\theta+\beta p(1+\delta)]}, \tag{23}$$

$$s_{or}^* = \frac{p(1-\tau)(1-\alpha)[\alpha A\beta(1+\delta)k^{*\alpha} - (1+\theta)k^*]}{\alpha[1+\theta+\beta p(1+\delta)]}, \tag{24}$$

$$l_{or}^* = \frac{1-\tau}{1+\theta+\beta p(1+\delta)}\left[1+\theta+\beta p - \frac{\alpha A\delta\beta}{k^{*1-\alpha}} - \frac{\tau\theta(pk^{*1-\alpha}+\alpha A)}{\alpha Aqp}\right]. \tag{25}$$

As opposed to the mandatory retirement scheme, when agents decide on retirement the introduction of endogenous fertility gives different results than when it is assumed exogenous. In fact, in the case of exogenous fertility, the steady-state level of physical capital negatively depends on social security,[5] whereas it does have a positive relationship when fertility is endogenous. (for technical details see Appendix A.2). We summarize these results in Table 1.

The basic intuition behind these results is that when fertility is exogenous social security negatively affects both saving and labor supply in old age. The negative effect of a lower saving on k^* is higher than the positive effect of reduced labor supply in old age. When fertility is endogenous social security negatively affects saving, labor supply in old age, and fertility. The positive effect of lower labor supply in old age on k^* is reinforced by the positive effect of lower fertility, and thus the overall positive impact on k^* is higher than the negative one.

When considering the impact of longevity, we see that it positively affects the steady-state level of capital per worker, that is, $\partial k^*/\partial p > 0$, if longevity is below a certain threshold $\hat{p}$ (for technical details see Appendix A.2).

ASSUMPTION 1.

$$p < \hat{p} = \left[\frac{\theta(1-\tau)}{q\beta}\right]^{1/2}. \tag{26}$$

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We now move on to see the impact of an increase in longevity on the equilibrium level of fertility and labor supply in old age. Under Assumption 1, from equation (23), we get that the increase in adult survival has two opposite effects on fertility: a direct negative effect and an indirect positive effect through k^* given that fertility is a normal good. Some calculations show that when adult survival is below a certain threshold, the negative effect prevails, whereas above this threshold, the impact is ambiguous. However, when parameters satisfy a sufficient condition, then the relationship between fertility and adult survival is always negative (see Appendix A.4). As far as the impact of longevity on labor supply in old age is concerned, from equation (25), under Assumption 1, we see that an increase in p induces agents to retire later and when p is sufficiently low, a corner solution for optimal retirement arises (for technical details see Appendix A.3). The basic intuition behind the positive relationship between elderly labor supply and adult survival is that an increase in p, *ceteris paribus*, negatively affects the pension benefit and therefore induces agents to increase their labor supply in old age [see Cipriani (2018)].

Finally, from equation (10), the pension payout in the steady state is given by

$$b^* = \frac{\tau A(1-\alpha)k^{*\alpha}(n^* + pl^*)}{p(1-l^*)} = \frac{\tau A(1-\alpha)s^*}{p(1-l^*)k^{1-\alpha}}, \tag{27}$$

thus an increase in p has two opposing effects. On the one hand, it has a positive effect because it increases wages, but on the other hand, it has a negative effect because it increases the pension system dependency ratio $p(1-l^*)/(n^* + pl^*)$ (see Appendix A.6). The overall effect is therefore ambiguous. For this reason, in the next section, we employ a numerical simulation to study the relationship between pension benefits and life expectancy.

3. SIMULATIONS

In this section we simulate the steady state and the dynamics of the model under different retirement schemes. The purpose of this simulation is not to replicate any specific case but to illustrate the results of the model, especially when analytical results are ambiguous. In order to give reasonable values of the parameters, we calibrate the model on Italian data. We use Italian data because they represent a practical example which comes close to the DC mandatory retirement model, given that Italy has introduced a mandatory notional DC pension system since 1995. However, given that we use a simple theoretical framework in order to find our results, we do not expect a perfect replication of the complex demographic and institutional aspects of the Italian pension system, since the model does not lend itself to give a perfect match of the real data. The model is calibrated on the assumption that one period lasts 30 years. The probability of surviving to old age (longevity) is computed using life expectancy at age 30 and considering that the length of adulthood is fixed at 30 years.

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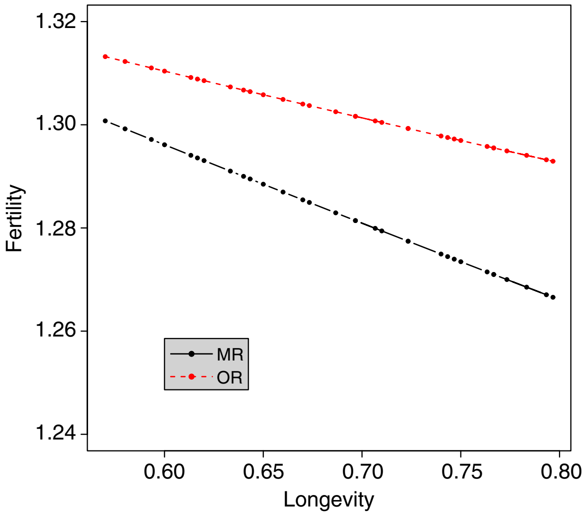

FIGURE 2. Fertility and longevity.

We have set the parameters α, β, and q by either using the data for Italy in 2015, when available, or the parametrization of other quantitative studies. In particular, the parameter α, that is, the capital share in added value, is set to 0.313 as provided by the OECD (2018) table on labor share in Italy in 2012 (the last available at the time of writing). The quarterly utility discount factor, β, in the literature is usually set equal to 0.99; for the entire adult lifespan, it is evaluated as 0.99^{120}, approximately 0.3 [see De La Croix and Doepke (2003)]. The child rearing cost, q, is set equal to 0.3, in line with the empirical literature on children's resource share, which estimates that children account for between 20% and 30% of the households' budget [see, e.g., Letablier et al. (2009) and Apps and Rees (2001)].

The social security tax rate, that is, τ is calibrated using equation (10), under the mandatory retirement scheme, in order to match the gross replacement rate for Italy [OECD (2018)]. In particular given that the gross replacement rate is equal to 83% and that the old age dependency ratio is 35% (World Development Indicators, 2018) we get $\tau = 0.3$. The parameter θ is calibrated from equation (18) in order to replicate the Italian fertility rate. In particular, we use only the data on Italian citizens in order to isolate it from the effect of immigration. In 2015, the Italian fertility rate was 1.27. However, since we have single-sex individuals in the model, we calibrate θ such that fertility is equal to 0.6 per individual. This delivers θ equal to 0.42. Finally, the scale parameter A is calculated by solving the dynamic equation for income (see equation (B1) in Appendix B) in order to match the per capita income in Italy in 2015.

Figure 2 plots the steady-state level of fertility rate. As shown in theory, fertility is declining with longevity, thus exacerbating the ageing process. However, fertility is higher in the optimal retirement scheme and the decrease is steeper with a mandatory retirement setting.

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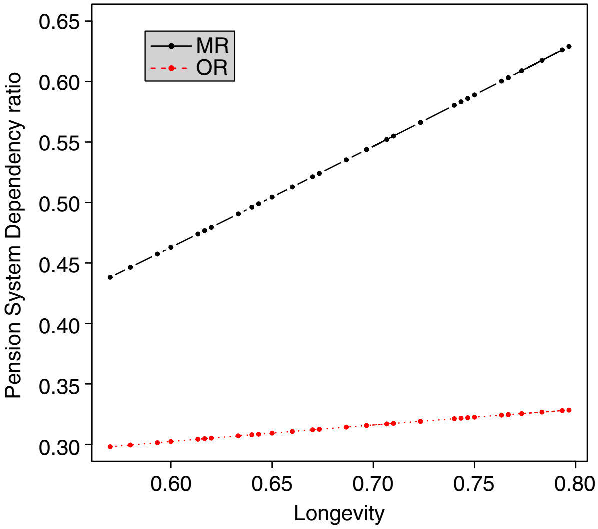

FIGURE 3. Dependency ratio and longevity.

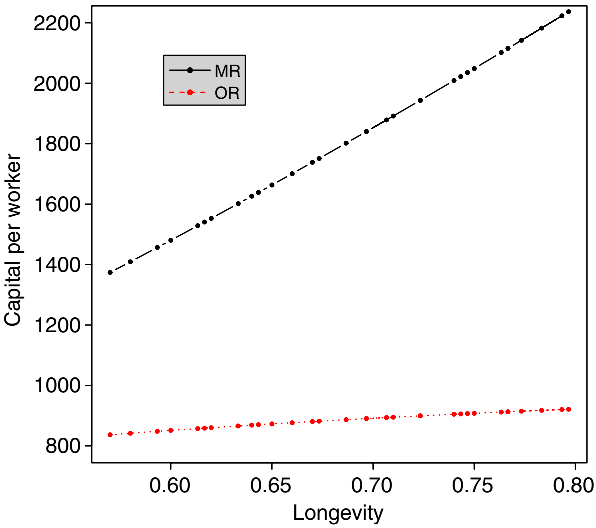

FIGURE 4. Capital per worker and longevity.

Figure 3 shows, in accordance with the theory, that the pension system dependency ratio is increasing in longevity, but this increase is obviously much stronger when retirement is mandatory.

In Figure 4 we plot the steady-state values of capital per worker in the case of mandatory retirement versus optimal retirement. In both cases, steady-state level of physical capital per worker is increasing in longevity. Since forcing an early

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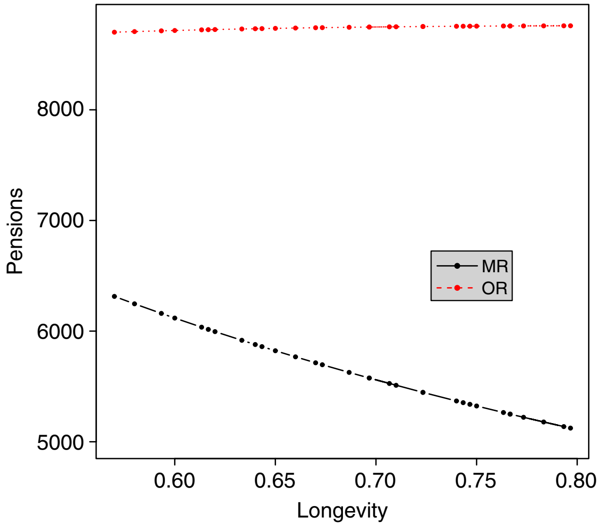

FIGURE 5. Pensions and longevity.

retirement increases savings, under the mandatory retirement case, the increase in capital per worker is steeper than with optimal retirement. Moreover, in the optimal retirement scheme, the increase in saving is partially compensated by the increase in the old-age labor supply.

Pension benefits decrease with longevity if retirement is mandatory. On the other hand, with endogenous retirement, our results show that this effect is ambiguous. In Figure 5 we show that benefits increase with longevity given our set of parameters. Therefore, population ageing might not be problem for the level of pensions if retirement is allowed to be optimally chosen.

Next, in Figure 6, we plot lifetime utility in the two cases. Utility always increases with longevity but the mandatory retirement case dominates optimal retirement. This is the result of the steeper increase in k^* with mandatory retirement, which compensates for the utility loss on the labor supply in the last period of life. This trade-off between the dynamic efficiency gain and the static efficiency loss is the same as that studied by Dedry et al. (2017). In fact, the steeper increase in capital per worker when longevity increases is due to the steeper increase in savings, because working time is inelastically supplied under mandatory retirement. On the other hand, this additional constraint on labor supply must have a negative effect on lifetime utility. The conclusion from Figure 6 is that the former effect must be greater than the latter so that, overall, there is a larger increase in the steady-state level of utility with respect to the optimal retirement case.

Finally, Figure 7 plots the steady-state level of income per worker as a function of the social security tax rate when $p = 0.8$. As shown in the theory, the relationship is negative under the mandatory retirement case and positive in the optimal retirement case.

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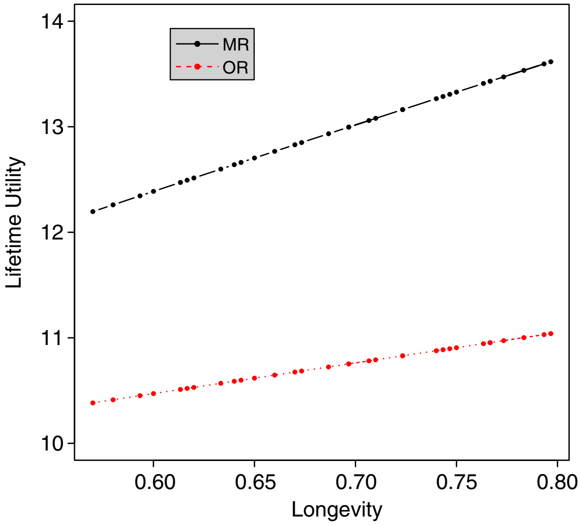

FIGURE 6. Lifetime utility and longevity.

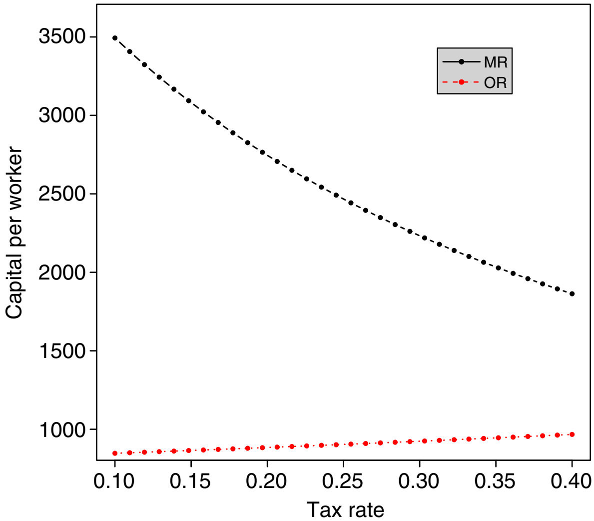

FIGURE 7. Capital per worker and the social security tax rate.

We now investigate potential differences between the short-term and long-term impacts of increasing longevity on the physical capital per worker, pension benefit, and lifetime utility (see Appendix B for the dynamic equations). In particular, we focus on the transition between two steady states resulting from an increase

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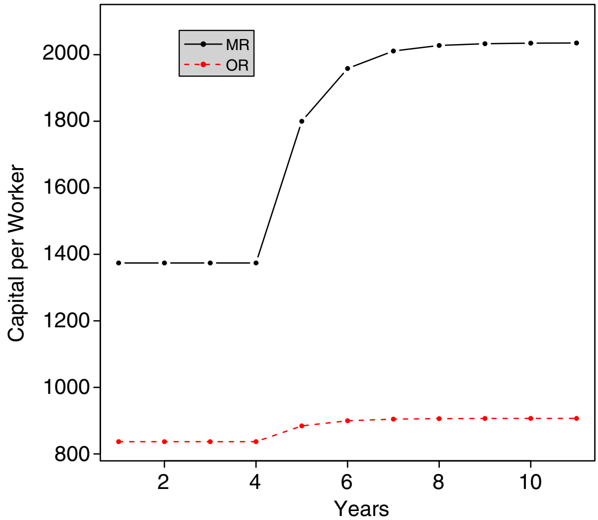

FIGURE 8. Capital per worker dynamics.

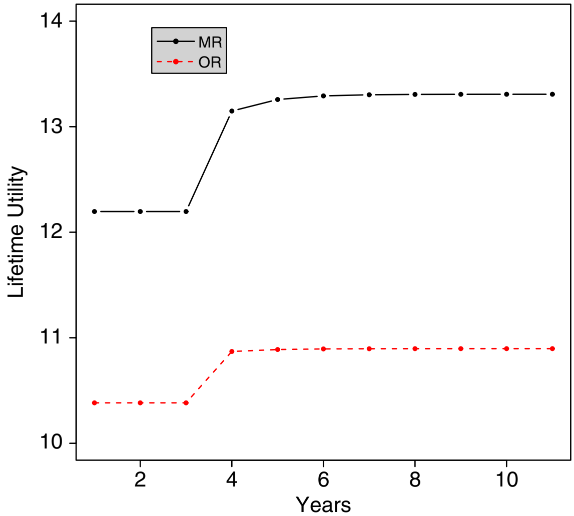

FIGURE 9. Lifetime utility dynamics

in longevity from 0.57 to 0.75. We consider 10 time periods, each with a duration of 30 years and assume that longevity equals 0.57 until period 3 and rises to 0.75 in period 4 and is constant throughout all the remaining periods. This change leads to a decrease in fertility by 2% in the mandatory regime and by 1.2% in the optimal retirement regime. Additionally, in the optimal retirement regime, there is an increase in the labor supply in old age of 38%. Figures 8–10 show that the

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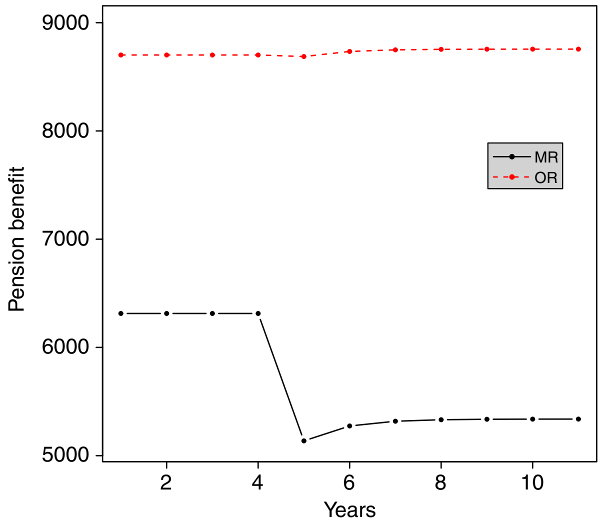

FIGURE 10. Pensions dynamics.

transition in capital accumulation and lifetime utility is monotonic, whereas the path of pension benefit is not.

Capital per worker in period 5 depends on the savings made by the adult generation, that is, those born in the previous period, and the total size of the workforce in the same period. In the mandatory pension scheme, capital per worker increases as higher income per capital leads generation four to increase its savings in order to spread consumption over a longer lifespan as well as opting for lower fertility. In the optimal retirement scheme, the transition is smoother, because the increase in labor supply from the older generation is added to these two factors observed in the mandatory scheme. This increase in labor supply, in fact, leads to a lower growth rate of capital per worker in the optimal retirement scheme with respect to the mandatory scheme. The growth rate of capital per worker, in period 5, is in fact 30% in the mandatory scheme versus 5% in the optimal retirement scheme.

As far as the pension benefit is concerned, in Figure 10, we can observe in the case of mandatory retirement a downward spike in period 5 followed by an increase during period 5–6, which then stabilizes from period 6 onwards. In the optimal retirement case, even though there is a similar path, nevertheless the order of magnitude is considerably lower. Overall, we can observe that the decrease in pension benefit in period 5 is due to the fact that the growth rate in per capita income (8.8% in mandatory retirement and 1.7% in the optimal retirement) is lower than the increase in the old-age dependency ratio (33% in mandatory retirement and 1.9% in the optimal retirement). From period 6 onwards, pension benefits increase because of the increase in per capita income.

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4. CONCLUSIONS

Population ageing due to higher longevity and lower fertility poses a challenge to the sustainability of the pension systems. Unsurprisingly, a large theoretical and empirical literature has developed in the recent years to study these issues. We build on this literature and present a theoretical overlapping generations model where fertility is chosen by optimizing agents and the government operates a fully balanced DC PAYG pension system. We show that, in the steady-state equilibrium, if the retirement age is allowed to be decided by the workers, the pension system dependency ratio may not dramatically worsen, like in the case of mandatory retirement, since agents might find it optimal to work longer and to decrease fertility less. As a consequence, pension benefits may not be decreasing with ageing in the case of optimal retirement.

We think that our findings have some important policy implications. When setting pension policies, governments should also consider how they trigger changes in individual behavior, including fertility and labor supply. The link between fertility and pensions is, of course, very well known in the literature [for an extensive review see Cigno and Werding (2007)]. The novelty of our approach is to consider a setting with endogenous retirement and show that in a pure DC pension setting, if retirement is freely chosen by agents, the pension consequences of increasing life expectancy can be ameliorated, even though there is still a fertility decline and a consequent population ageing.

We are aware that our model presents some caveats and that can be extended to include some important features. First of all, a straightforward extension is to consider endogenous longevity. In this case, agents' lifetime could change as a result of their choices and this could have different implications under alternative pension regimes. However, in the present framework, we have not included endogenous longevity, because this would complicate the analysis and take the focus away from the main issue given the extensive literature which discusses the main factors affecting longevity [see, e.g., Preston (1975) and Livi Bacci (2007) among others]. Secondly, another interesting extension could be to include leisure in adulthood. This would certainly enrich the model and lead to some form of trade-off or complementary between labor supply in adulthood and old age [see, e.g., Matsuyama (2008)]. Thirdly, one could study a defined benefit setting and compare the results with those obtained in the DC case. Defined benefits pension plans have lost their dominance in the occupational pension systems of many countries; however, they are still present in many countries, which often present a combination of the different types of pension arrangements [for an overview see OECD (2016)].

Finally given that we have a setting with endogenous fertility, the model could be extended to study the effect of other policy measures, such as child benefits or child-related pension benefits. We leave these extensions for further work.

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NOTES

1. See also Miyazaki (2017) and Zhang and Zhang (2009).
2. This way of modeling child cost has been extensively used in the literature. See, for instance, Wigger (1999); Boldrin and Jones (2002) and Fanti and Gori (2014).
3. The presence of $\bar{l} \geq 0$ complicates the analysis but does not affect the qualitative results.
4. Note that $\partial k_{t+1}/\partial k_t > 0$ and $\partial^2 k_{t+1}/\partial k_t^2 < 0$.
5. If $q = 0$ it does not depend on social security [see Cipriani (2018)].

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APPENDIX

APPENDIX A: STEADY STATE ANALYSIS

A.1. EXOGENOUS FERTILITY

When fertility is exogenous, the steady state level of physical capital, in the case of mandatory retirement, is given by [see also Cipriani (2018) for the case in which $q = 0$]:

$$k^*_{MR} = \left\{ \frac{A\alpha\beta p(1-\tau-nq)(1-\alpha)}{n[\alpha(1+\beta p)+\tau(1-\alpha)]} \right\}^{\frac{1}{1-\alpha}}, \qquad \textbf{(A1)}$$

where it easy to note that $\partial k^*_{MR}/\partial p > 0$ and $\partial k^*_{MR}/\partial \tau < 0$.

In the case of optimal retirement the steady state level of physical capital is given by:

$$k^*_{or} = \left[\frac{A\alpha\beta(\delta-\alpha+1)(1-\tau-nq)p}{(1-\tau)[\alpha\beta p^2+\alpha\beta n(1+\delta)p+p+\alpha n]} \right]^{\frac{1}{1-\alpha}}. \qquad \textbf{(A2)}$$

Simple calculations show that $\partial k/\partial p > 0$ given that $n > \beta p^2$ and $\partial k/\partial \tau < 0$ because $n < 2(1-\tau)/q$ always holds given that $n < (1-\tau)/q$ from equation (2).

A.2. PHYSICAL CAPITAL OPTIMAL RETIREMENT

From equation (22), $\partial k^*_{or}/\partial \tau > 0$ if:

$$\frac{1}{2}\left[\left(\frac{T(p)}{p} \right)^2 + \psi \right]^{-1/2} \left[\frac{2T(p)}{p} \frac{\partial [T(p)/p]}{\partial \tau} + \frac{\partial \psi}{\partial \tau} \right] - \frac{\partial [T(p)/p]}{\partial \tau} > 0, \qquad \textbf{(A3)}$$

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where after some simplifications $\frac{\partial k^*_{or}}{\partial \tau} > 0$ if:

$$\frac{\partial \psi/\partial \tau}{2\partial (T(p)/p)/\partial \tau} + \frac{T(p)}{p} > \left[\left(\frac{T(p)}{p} \right)^2 + \psi \right]^{1/2} > 0, \tag{A4}$$

where squaring both sides and after some simplifications we get:

$$\left[\frac{\partial \psi/\partial \tau}{2\partial (T(p)/p)/\partial \tau} \right]^2 + \frac{T(p)}{p} \left[\frac{\partial \psi/\partial \tau}{\partial (T(p)/p)/\partial \tau} - \frac{\psi}{T(p)/p} \right] > 0 \tag{A5}$$

since:

$$\frac{\partial \psi/\partial \tau}{\psi} - \frac{\partial (T(p)/p)/\partial \tau}{T(p)/p} = \frac{\alpha\theta}{\alpha\theta(1-\tau) + pq(1+\theta+\alpha\beta p)} > 0 \tag{A6}$$

From equation (22), $\partial k^*_{or}/\partial p > 0$ if the following necessary and sufficient condition holds:

$$\frac{\partial [T(p)/p]}{\partial p} \left[\frac{\frac{T(p)}{p}}{\sqrt{\left(\frac{T(p)}{p}\right)^2 + \Psi}} - 1 \right] > 0, \tag{A7}$$

where, given that the term inside the brackets is negative and that:

$$\frac{\partial [T(p)/p]}{\partial p} = \frac{\alpha q \beta p^2 - \alpha\theta(1-\tau)}{2\theta(1-\tau)p^2} \tag{A8}$$

then it follows that:

$$if\, p > \left[\frac{\theta(1-\tau)}{q\beta} \right]^{1/2} \Rightarrow \frac{\partial [T(p)/p]}{\partial p} > 0 \Rightarrow \partial k^*_{or}/\partial p < 0, \tag{A9}$$

$$if\, p < \left[\frac{\theta(1-\tau)}{q\beta} \right]^{1/2} \Rightarrow \frac{\partial [T(p)/p]}{\partial p} < 0 \Rightarrow \partial k^*_{or}/\partial p > 0. \tag{A10}$$

Thus under the assumption that:

$$\theta > \frac{q\beta}{1-\tau}. \tag{A11}$$

eq. (A10) is always satisfied.

Finally:

$$\lim_{p\to 0} k^*_{or} = 0 \tag{A12}$$

A.3. LABOR SUPPLY IN OLD AGE

From equation (25), in the steady state, a corner solution with full retirement, that is $l^*_{or} = 0$ arises if:

$$(1+\theta+\beta p){k^*_{or}}^{1-\alpha} < \alpha A\delta\beta + \frac{\tau\theta(p{k^*_{or}}^{1-\alpha} + \alpha A){k^*_{or}}^{1-\alpha}}{\alpha A q p}, \tag{A13}$$

where the left hand side is increasing with respect to p and it is equal to zero when $p = 0$. Thus $l^* > 0$ when p is above a certain threshold. To find this threshold we use equations

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(17) and (22). Simple calculations show that $k^*_{mr} \geq k^*_{or}$ if:

$$\alpha\beta q p^2 + (1+\theta)\mu q p - \frac{\theta\mu[(1-\alpha)\tau + \delta\mu]}{(1-\alpha)} \geq 0, \tag{A14}$$

where $\mu = \alpha + \tau(1-\alpha)$. Thus $k^*_{mr} \geq k^*_{or}$ if $p \geq \bar{p}$:

$$\bar{p} = \frac{\sqrt{[(1+\theta)\mu q]^2 + 4\alpha\beta\theta\mu q \frac{\tau(1-\alpha)+\delta\mu}{1-\alpha}} - (1+\theta)\mu q}{2\alpha\beta q}, \tag{A15}$$

Thus since there exists a unique value of $p = \bar{p}$, such that when $p \geq \bar{p}$ then $k^*_{mr} \geq k^*_{or}$ we can conclude that $l^* \geq 0$ for each $p \geq \bar{p}$. Also, since $\bar{p} < 1$ it must be that $q > \theta\mu[\tau(1-\alpha) + \delta\mu]/(1-\alpha)[\alpha\beta + (1+\theta)\mu]$.

To study the relationship between labour supply in old age and adults survival it is useful to write equation (25) as follows:

$$l^*_{or} = \frac{1}{(1+\delta)(1-\alpha)}\left[(1-\alpha)(1-\tau) - \frac{\alpha\delta s^*_{or}}{pk^*_{or}} - \frac{(1-\alpha)\tau(1+\delta)n^*_{or}}{p}\right] \tag{A16}$$

Thus:

$$\frac{\partial l^*_{or}}{\partial p} = -\alpha\delta\frac{\partial(s^*_{or}/pk^*_{or})}{\partial p} - (1-\alpha)\tau(1+\delta)\frac{\partial n^*_{or}}{\partial p} > 0 \tag{A17}$$

because from equation (24) it is easy to note that $\partial(s^*_{or}/pk^*_{or})/\partial p < 0$ and as shown below $\partial n^*_{or}/\partial p < 0$.

A.4. FERTILITY

Under assumption 1, equations (23) and (A7) , $\partial n^*_{or}/\partial p < 0$ if

$$\frac{\alpha[\theta(1-\tau) - q\beta p^2]}{2\theta(1-\tau)\sqrt{(T(p))^2 + \Psi p^2}} < \frac{\alpha A\beta(1+\delta) - k^{*1-\alpha}(1+\theta)}{k^{*1-\alpha}[1+\theta+\beta p(1+\delta)]} \tag{A18}$$

where both the LHS and RHS of (A18) are functions of p. From equation (A20) $RHS > 0$ for each $0 \leq p \leq 1$. Given $0 \leq p \leq 1$:

$$\begin{aligned} &\frac{\partial LHS(p)}{\partial p} < 0; \\ &\lim_{p\to 0} LHS(p) = 1 \\ &\frac{\partial RHS(p)}{\partial p} < 0; \\ &\lim_{p\to 0} RHS(p) = \infty. \end{aligned} \tag{A19}$$

Note that $RHS(p) > 1$ for each $0 \leq p \leq 1$ if the parameters satisfy the condition $q < 2\alpha(1+\delta)\theta(1-\tau)/(1+\delta-2\alpha)[2(1+\theta)+\beta(1+\delta)]$.

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A.5. SAVING

From equation (24) $s^*_{or} > 0$ because:

$$\frac{\alpha A\beta(1+\delta)}{1+\theta} > k^{*(1-\alpha)}_{or} \tag{A20}$$

holds for each $0 \leq p \leq 1$. In fact, when $p = 0$ $k^* = 0$ and therefore equation (A20) holds. Thus given that $\partial k^*/\partial p > 0$, equation A20 holds because it is satisfied when $p = 1$. In particular when $p = 1$ it is easy to show that:

$$\frac{\alpha\beta(1+\delta)}{1+\theta} + T(1) > [T(1) + \Psi]^{1/2} \tag{A21}$$

where squaring both sides and substituting for T(1) and Ψ we get:

$$\left[\frac{\beta(1+\delta)}{1+\theta} + 1\right]\left[\frac{q}{\theta(1-\tau)} + \frac{1+\delta}{1+\theta}\right] > 0 \tag{A22}$$

A.6. PENSION SYSTEM DEPENDENCY RATIO

The equilibrium old age dependency ratio is given by:

$$DR = \frac{p(1-l)}{n^*_{or} + pl^*_{or}} = \frac{pk^*_{or}(1 - l*_{or})}{s^*_{or}} \tag{A23}$$

Thus from equation (25) it follows that:

$$\frac{\partial DR}{\partial p} = \frac{\partial(pk^*_{or}/s^*_{or})}{\partial p}\left(\frac{\delta+\tau}{1+\delta}\right) + \tau\left[\frac{\partial(pk^*_{or}/s^*_{or})}{\partial p}\frac{n^*_{or}}{p} + \frac{pk^*_{or}}{s^*_{or}}\frac{\partial(n^*_{or}/p)}{\partial p}\right] > 0 \tag{A24}$$

because from equation (24) it is easy to note that $\partial(s^*_{or}/pk^*_{or})/\partial p < 0$ and the following holds:

$$\frac{\partial(pk^*_{or}/s^*_{or})}{\partial p}\frac{n^*_{or}}{p} + \frac{pk^*_{or}}{s^*_{or}}\frac{\partial(n^*_{or}/p)}{\partial p} = 2\mu'(p)(ck^{*\,\alpha-1}_{or} - d)\left(k^{*\,1-\alpha}_{or} + \frac{\alpha A}{p}\right) + \frac{\alpha A}{p^2}\left[\frac{cp^2}{k^{*\,2-\alpha}_{or}} + ck^{*\,\alpha-1}_{or} - d\right] \tag{A25}$$

where $c = \alpha A\beta(1+\delta)$, $d = (1+\theta)$ and $ck^{*\,\alpha-1}_{or} - d > 0$ for each $0 \leq p \leq 1$ (see equation (A20)).

APPENDIX B: DYNAMICS

In the mandatory retirement, from equations (14), (15) and 16, the dynamic equations for the per capita income, fertility and saving are given:

$$y_{t+1} = A\left\{y_t \frac{q(1-\alpha)\alpha\beta p_t}{\theta[\alpha + \tau(1-\alpha)]}\right\}^{\alpha} \tag{B1}$$

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$$n_t = \frac{(1-\tau)\theta[\alpha + \tau(1-\alpha)]}{q\{\,\alpha\beta p_t + (1+\theta)[\alpha + \tau(1-\alpha)]\}}, \tag{B2}$$

$$s_t = \frac{\alpha\beta p_t(1-\alpha)(1-\tau)y_t}{\alpha\beta p_t + (1+\theta)[\alpha + \tau(1-\alpha)]}. \tag{B3}$$

In the optimal retirement, from equations (11), (12) and (13) and 21 the dynamic equations for the per capita income, fertility and saving are given:

$$y_{t+1} = A\left\{y_t\left[\sqrt{\left(\frac{T(p)}{p}\right)^2 + \Psi} - \frac{T(p)}{p}\right]\right\}^{\alpha}, \tag{B4}$$

$$n_t = \frac{(1-\tau)\theta(p_t k_{t+1} + \alpha y_t)}{y_t \alpha q[1+\theta+\beta p_t(1+\delta)]}, \tag{B5}$$

$$l_t = \frac{(1-\tau)[\beta p_t^2 q + p_t q(1+\theta) - \tau\theta - (\alpha y_t \beta\delta q p_t/k_{t+1}) - (p_t \tau\theta k_{t+1}/\alpha y_t)]}{p_t q[1+\theta+\beta p_t(1+\delta)]} \tag{B6}$$

$$s_t = \frac{p_t(1-\tau)(1-\alpha)[\beta(1+\delta)y_t - k_{t+1}(1+\theta)/\alpha]}{1+\theta+\beta p_t(1+\delta)} \tag{B7}$$

where k_{t+1} is given by equation (21).

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