MADNESS IN VECTOR SPACES

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Abstract. We consider maximal almost disjoint families of block subspaces of countable vector spaces, focusing on questions of their size and definability. We prove that the minimum infinite cardinality of such a family cannot be decided in ZFC and that the "spectrum" of cardinalities of mad families of subspaces can be made arbitrarily large, in analogy to results for mad families on ω. We apply the author's local Ramsey theory for vector spaces [32] to give partial results concerning their definability.

§1. Introduction. Recall that two infinite subsets x and y of the natural numbers ω are *almost disjoint* if $x \cap y$ is finite. A collection $\mathcal{A} \subseteq [\omega]^\omega$, where $[\omega]^\omega$ is the set of infinite subsets of ω, is an *almost disjoint family* if its elements are pairwise almost disjoint, and is a *maximal almost disjoint family*, or *mad family*, if it is not properly contained in another such family. While any finite (almost) partition of ω forms a mad family, our focus here is confined to infinite mad families.

It is well known that every almost disjoint family is contained in a mad family and every infinite mad family is uncountable. The former is an application of Zorn's Lemma, while the later a straightforward diagonalization.

A large almost disjoint family can be obtained as follows: identifying ω with $2^{<\omega}$, consider

$$\mathcal{A} = \{\{x \restriction n : n \in \omega\} : x \in 2^\omega\}. \tag{1}$$

It is easy to see that $\mathcal{A}$ is almost disjoint and of size $\mathfrak{c}$, thus can be extended to a mad family of size $\mathfrak{c}$. Note that $\mathcal{A}$ is (topologically) closed as it is a homeomorphic image of 2^ω. Here, we identify $[\omega]^\omega$ as a subspace of 2^ω via characteristic functions, from which it inherits a Polish topology.

Two fundamental questions about infinite mad families one might ask are:

1. How big (or small) can they be?
2. How definable can they be?

One way of addressing Question 1 is to determine the value of the cardinal invariant

$$\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is an infinite mad family}\}.$$

This could mean which $\aleph_\alpha$ is such that $\mathfrak{a} = \aleph_\alpha$, or how $\mathfrak{a}$ relates to other well-studied cardinal invariants (see [5] or [36]) between $\aleph_1$ and $\mathfrak{c}$. By our comments

above, $\aleph_1 \leq \mathfrak{a} \leq \mathfrak{c}$, and a modification of that diagonalization argument shows that $\mathfrak{b} \leq \mathfrak{a}$, where $\mathfrak{b}$ is the minimum size of an unbounded family of functions $\omega \to \omega$ (see [ibid.]). However, the value of $\mathfrak{a}$ cannot be decided in ZFC: both the Continuum Hypothesis CH and Martin's Axiom MA (see [21] or [22]) imply that $\mathfrak{a} = \mathfrak{c}$, and thus, consistently $\aleph_1 < \mathfrak{a} = \mathfrak{c}$, while Kunen [21] showed that in the model obtained by adding $\aleph_2$-many Cohen reals to a model of CH, $\aleph_1 = \mathfrak{a} < \mathfrak{c} = \aleph_2$. In [19], Hrušák showed[1] that the latter also holds in the models obtained by adding $\aleph_2$-many Sacks reals iteratively or "side-by-side" to a model of CH.

A more sophisticated version of Question 1 might ask for the "spectrum" of cardinalities between $\aleph_1$ and $\mathfrak{c}$ that mad families can posses. This was first addressed by Hechler [14], who produced a method for obtaining arbitrarily large continuum and, simultaneously, mad families of all cardinalities κ for $\aleph_1 \leq \kappa \leq \mathfrak{c}$. While beyond the scope of our investigations here, these questions have been the focus of much deep work in recent decades, notably Brendle's [6], which establishes the consistency of $\mathfrak{a} = \aleph_\omega$, Shelah's [29], which establishes the consistency of $\mathfrak{d} < \mathfrak{a}$, and Shelah and Spinas' [30], which gives a nearly sharp characterization of possible mad spectra.

Question 2 above seeks to understand to what extent the nonconstructive methods used to obtain mad families are necessary. A result of Mathias [23] says that an infinite mad family can never be analytic (i.e., a continuous image of a Borel set). Under large cardinal hypotheses, this can be pushed further to show that there are no definable mad families at all, in the sense that there are none in $\mathbf{L}(\mathbb{R})$ (see [9, 23, 35], and for a consistency result without large cardinals, [16]). Mathias' result is also sharp; Miller [24] proved that there is a coanalytic (i.e., the complement of an analytic set) mad family assuming $\mathbf{V} = \mathbf{L}$, work later refined by Törnquist [34].

This article is concerned with an analogue of mad families arising in vector spaces. Throughout, E will be a countably infinite-dimensional vector space over a countable (possibly finite) field F.

DEFINITION 1.1. We say that two infinite-dimensional subspaces X and Y of E are *almost disjoint* if $X \cap Y$ is finite-dimensional.

Due to their more tractable nature, we will focus on *block subspaces*, that is, those having a basis in "block position" with respect to a fixed basis for E (see Section 2 for the definition). Every infinite-dimensional subspace contains a block subspace, so this is a relatively mild restriction.

DEFINITION 1.2. A collection $\mathcal{A}$ of infinite-dimensional (block) subspaces of E is an *almost disjoint family of (block) subspaces* if its elements are pairwise almost disjoint and is a *maximal almost disjoint family of (block) subspaces*, or *mad family of (block) subspaces*, if it is not properly contained in another such family.

Note that, by our prior comment, being maximal with respect to arbitrary subspaces is equivalent to being maximal with respect to block subspaces.

While the topic of almost disjoint families of subspaces seems very natural, it appears to have been little studied except for an article by Kolman [20], wherein

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[1]Given the comments in [19], this result was likely known earlier.

they are called "almost disjoint packings,"[2] and indirectly in the recent work of Brendle and García Ávila [7] discussed below.

In light of the above questions for mad families on ω, we ask the analogous questions for infinite mad families of subspaces:

1. How big (or small) can they be? In particular, what is

$$\mathfrak{a}_{\mathrm{vec},F} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is an infinite mad family of block subspaces}\}?$$

2. How definable can they be?

Two related notions have been studied for separable Hilbert spaces, that of "almost orthogonal" and "almost disjoint" families of closed infinite-dimensional subspaces, where "almost" is measured by considering the corresponding projection operators modulo the compact operators. Results concerning Question 1 in these settings were obtained in articles of Wofsey [37] and Bice [4], respectively. While not directly related[3] to our setting, these articles provide both motivation for, and ideas used in, the results in Section 3 below.

When F is the finite field of order 2, vectors may be identified with elements of FIN, the set of nonempty finite subsets of ω, via their supports. Sums of vectors in block position correspond to unions of the corresponding supports. This is the setting of Hindman's Theorem [15] on disjoint unions of finite subsets of ω. During the preparation of this article, an independent work of Brendle and García Ávila [7] appeared on maximal almost disjoint families of combinatorial subspaces of FIN. Among other results, they show that $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{FIN}}$, where $\mathrm{non}(\mathcal{M})$ is the minimum size of a nonmeager subset of $\mathbb{R}$ and $\mathfrak{a}_{\mathrm{FIN}}$ is the minimum size of an infinite mad family in FIN, or in our language, a mad family of block subspaces when $|F| = 2$. Together with known results, this shows the consistency of $\mathfrak{a} < \mathfrak{a}_{\mathrm{FIN}}$.

This article is organized as follows: In Section 2, we consider issues of cardinality and address Question 1 using only ZFC techniques, showing that mad families of block subspaces of cardinality ≥ 2 are always uncountable (Proposition 2.5) and that $\mathfrak{b} \leq \mathfrak{a}_{\mathrm{vec},F}$ (Proposition 2.6). We then adapt the aforementioned work of Brendle and García Ávila to show, moreover, that $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},F}$, for general F (Corollary 2.11). In Section 3, we use forcing to establish consistency results regarding $\mathfrak{a}_{\mathrm{vec},F}$ in analogy to those mentioned above for $\mathfrak{a}$, showing that is equal to $\aleph_1$ in the Cohen (Theorem 3.3) and Sacks (Theorem 3.6) models, and that the spectrum of cardinalities of mad families of block subspaces can be made arbitrarily large (Theorem 3.7). In Section 4, we consider issues of definability. We use the Ramsey-theoretic results from the author's [32] to give a partial solution for

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[2]Several proofs in [20] appear to use a stronger property than almost disjointness, namely, that whenever $X_0, \ldots, X_n \in \mathcal{A}$ are distinct, then $X_i \cap (\sum_{j \neq i} X_j)$ is finite-dimensional. It easy to construct almost disjoint families of subspaces for which this fails, e.g., $X_0 = \langle (e_{2n})_{n \in \omega} \rangle$, $X_1 = \langle (e_{2n+1})_{n \in \omega} \rangle$, and $X_2 = \langle (e_{2n} + e_{2n+1})_{n \in \omega} \rangle$, where (e_n) is a basis for E. This can be extended to an infinite almost disjoint family of subspaces by our Proposition 2.5. As such, we reprove several of the results appearing in [20].

[3]Almost orthogonal families of closed subspaces of Hilbert space appear more closely related to almost disjoint families on ω than does our setting. For instance, countable almost orthogonal families arise as images of countable almost disjoint families on ω via the "diagonal map" (cf. Lemma 5.34 in [10]), and, consistently, some mad families on ω remain maximal when passed through this map [37]. Less is understood about the notion of almost disjointness for closed subspaces, e.g., it appears to be open whether the corresponding cardinal invariant is $\aleph_1$ in ZFC.

"full" mad families of subspaces (Theorem 4.10). The existence of such families is established under certain set-theoretic hypotheses (Theorem 4.12). Section 4 can be read independently from the other sections. We conclude in Section 5 with further remarks, conjectures, and open questions.

§**2. Cardinality:** ZFC **results.** Throughout, we fix (e_n) an F-basis for E (e.g., $E = \bigoplus_{n \in \omega} F$ and e_n is the nth unit coordinate vector). If X is a subset of E, or a sequence of vectors in E, we write $\langle X \rangle$ for its linear span.

For $x \in E$, the *support* of x is given by

$$\mathrm{supp}(x) = \{n \in \omega : x = \sum a_i e_i \Rightarrow a_n \neq 0\}.$$

For nonzero vectors $x, y \in E$, and $M \in \omega$, we write $x > M$ if $\min(\mathrm{supp}(x)) > M$, and $x < y$ if $\max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y))$. We call a sequence of nonzero vectors (x_n) a *block sequence* if $x_n < x_{n+1}$ for all n. A space spanned by a block sequence is a *block subspace*.

As mentioned in Section 1, every infinite-dimensional subspace of E contains a block subspace (Lemma 2.1 in [32]), and the block sequence forming the basis of a block subspace is unique, up to scaling. Unless otherwise specified, a block subspace is always assumed to be infinite-dimensional.

We begin with the following easy facts:

PROPOSITION 2.1. *Every almost disjoint family of* (block) *subspaces is contained in a mad family of* (block) *subspaces.*

PROOF. This is a standard Zorn's Lemma argument.   ⊣

PROPOSITION 2.2. *There is an almost disjoint family of block subspaces, and thus a mad family of block subspaces, of size* $\mathfrak{c}$.

PROOF. Let $\mathcal{A}$ be an almost disjoint family on ω of size $\mathfrak{c}$, as in (1) above. The image of $\mathcal{A}$ under the injective map $x \mapsto \langle (e_n)_{n \in x} \rangle$ is easily seen to be an almost disjoint family of subspaces.   ⊣

Given an infinite-dimensional subspace Y and an $M \in \omega$, we write Y/M for the set of $y \in Y$ with $y > M$; Y/M is always an infinite-dimensional subspace of Y. Given a vector x, we write Y/x for $Y/\max(\mathrm{supp}(x))$. The following lemma will be key to much of what follows.

LEMMA 2.3. *Let* Y *be a block*[4] *subspace of* E *and* $x_0, \ldots, x_m$ *nonzero vectors in* E.

(a) *If* $x \in Y$, *then*

$$\langle x_0, \ldots, x_m, x \rangle \cap Y = \langle x_0, \ldots, x_m \rangle \cap Y + \langle x \rangle.$$

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[4]In earlier versions of this article, including in [31], this lemma was stated for *arbitrary* infinite-dimensional subspaces Y, rather than block subspaces. This was in error, as the following counterexample shows: Let $Y = \langle e_0 + e_2, e_1 + e_3, \ldots, e_{2n} + e_{2n+2}, e_{2n+1} + e_{2n+3}, \ldots \rangle$. Note that Y contains either $e_0 + e_{2n+2}$ or $e_0 - e_{2n+2}$, for each n. Thus, if we take $x_0 = e_0$, then for any M, we can find an n with $2n + 2 > M$ and so $e_0 \in \langle e_0, e_{2n+2} \rangle \cap Y \neq \{0\} = \langle e_0 \rangle \cap Y$. This is related to the fact that the basis for Y cannot put in a "row reduced echelon form," and appears to be the essential difficulty in removing "block" from many of the arguments herein.

(b) *There is an $M \in \omega$ (that depends only on Y and $\max(\bigcup_{i=0}^{m} \operatorname{supp}(x_i))$) such that whenever $x > M$ and $x \notin Y$,*

$$\langle x_0, \ldots, x_m, x \rangle \cap Y = \langle x_0, \ldots, x_m \rangle \cap Y.$$

PROOF. (a) is just the special case of the modularity law for subspaces (and holds for arbitrary Y):

$$Z \subseteq Y \quad \text{implies} \quad (X + Z) \cap Y = (X \cap Y) + Z,$$

where $X = \langle x_0, \ldots, x_m \rangle$ and $Z = \langle x \rangle$.

(b) Suppose that $Y = \langle (y_n) \rangle$, where (y_n) is a block sequence. Put $K = \max(\bigcup_{i=0}^{m} \operatorname{supp}(x_i))$ and let N be the largest index such that $\operatorname{supp}(y_N) \cap [0, K] \neq \emptyset$. Set

$$M = \max \left\{ \max(\operatorname{supp}(y_N)), K \right\}.$$

Take $x > M$ with $x \notin Y$. Suppose that

$$v = \lambda_0 x_0 + \cdots + \lambda_m x_m + \lambda x \in Y,$$

with λ_i's not all 0. To prove the result, it suffices to show that $\lambda = 0$. Towards a contradiction, suppose that $\lambda \neq 0$ and write

$$\alpha_0 y_0 + \cdots + \alpha_k y_k = \lambda_0 x_0 + \cdots + \lambda_m x_m + \lambda x,$$

for some $k \in \omega$. Since $x > M$, we must have that $k > N$. But, by our choice of N and the fact that the y_n are in block position, we have that

$$\alpha_0 y_0 + \cdots + \alpha_N y_N = \lambda_0 x_0 + \cdots + \lambda_m x_m,$$

which implies $x = \frac{1}{\lambda}(\alpha_{N+1} y_{N+1} + \cdots + \alpha_k y_k) \in Y$, a contradiction. ⊣

LEMMA 2.4. *Suppose that $Y_0, \ldots, Y_n, Y_{n+1}$ are pairwise disjoint block subspaces and $x_0, \ldots, x_m$ nonzero vectors such that $\langle x_0, \ldots, x_m \rangle \cap Y_k = \{0\}$ for $k \leq n + 1$. Then, there is an $x > x_m$ such that $\langle x_0, \ldots, x_m, x \rangle \cap Y_k = \{0\}$ for $k \leq n + 1$.*

PROOF. By repeatedly applying Lemma 2.3(b), we can obtain an $M_0 \geq \max(\bigcup_{i=0}^{m} \operatorname{supp}(x_m))$ such that whenever $x > M_0$ and not in any of the Y_k's, $\langle x_0, \ldots, x_m, x \rangle \cap Y_k = \{0\}$ for $k \leq n + 1$.

To find such an x, start by picking $x_0' \in Y_0 / M_0$, so $\langle x_0' \rangle \cap Y_0 = \langle x_0' \rangle$ and $\langle x_0' \rangle \cap Y_k = \{0\}$ for $0 < k \leq n + 1$. By repeatedly applying Lemma 2.3, we can obtain an $M_1 \geq M_0$ such that whenever $y \in Y_1 / M_1$, we have that

$$\langle x_0', y \rangle \cap Y_0 = \langle x_0' \rangle \cap Y_0 = \langle x_0' \rangle,$$
$$\langle x_0', y \rangle \cap Y_1 = \langle x_0' \rangle \cap Y_1 + \langle y \rangle = \langle y \rangle,$$
$$\langle x_0', y \rangle \cap Y_k = \langle x_0' \rangle \cap Y_k = \{0\} \quad \text{for } 1 < k \leq n + 1.$$

Pick $x_1' \in Y_1 / M_1$. Continue in this fashion, using Lemma 2.3 to choose an $M_\ell \geq M_{\ell-1}$ and $x_\ell' \in Y_\ell / M_\ell$, for $1 \leq \ell \leq n + 1$, so that

$$\langle x_0', \ldots, x_{\ell-1}', x_\ell' \rangle \cap Y_0 = \langle x_0', \ldots, x_{\ell-1}' \rangle \cap Y_0 = \langle x_0' \rangle,$$

$$\vdots$$

$$\langle x_0', \ldots, x_{\ell-1}', x_\ell' \rangle \cap Y_{\ell-1} = \langle x_0', \ldots, x_{\ell-1}' \rangle \cap Y_{\ell-1} = \langle x_{\ell-1}' \rangle,$$
$$\langle x_0', \ldots, x_{\ell-1}', x_\ell' \rangle \cap Y_\ell = \langle x_0', \ldots, x_{\ell-1}' \rangle \cap Y_\ell + \langle x_\ell' \rangle = \langle x_\ell' \rangle,$$
$$\langle x_0', \ldots, x_{\ell-1}', x_\ell' \rangle \cap Y_k = \langle x_0', \ldots, x_{\ell-1}' \rangle \cap Y_k = \{0\} \quad \text{for } \ell < k \leq n + 1.$$

Then, $x = x_0' + \cdots + x_{n+1}'$ is not in any of the Y_k's, and so $\langle x_0, \ldots, x_m, x \rangle \cap Y_k = \{0\}$ for $k \leq n + 1$.   ⊣

If X is a finite-codimensional subspace, then $\{X\}$ is always a mad family of subspaces. These are the only countable mad families of block subspaces.

PROPOSITION 2.5. *Let $\mathcal{A}$ be a maximal almost disjoint family of block subspaces of size ≥ 2. Then, $\mathcal{A}$ is uncountable.*

PROOF. Suppose first that $\mathcal{A} = \{Y_0, \ldots, Y_n, Y_{n+1}\}$ is a finite almost disjoint family of block subspaces. By replacing each Y_k with a relatively finite-codimensional subspace, we may assume that they are pairwise disjoint. Pick an x_0 not in any of the Y_k's, which can be done as in the proof of Lemma 2.4. By repeatedly applying Lemma 2.4, we can build an infinite block sequence (x_m) such that for each m and $k \leq n + 1$, $\langle x_0, \ldots, x_m \rangle \cap Y_k = \{0\}$. Then, $\langle (x_m) \rangle$ is disjoint from each Y_k, witnessing that $\mathcal{A}$ fails to be maximal.

Suppose that $\mathcal{A} = \{Y_n : n \in \omega\}$ is a countably infinite almost disjoint family of block subspaces. Again, by passing to finite-codimensional subspaces, we may assume that the Y_k are pairwise disjoint. Pick a nonzero $x_0 \in Y_0$. By Lemma 2.3, we can pick $x_1 \in Y_1/x_0$ such that

$$\langle x_0, x_1 \rangle \cap Y_0 = \langle x_0 \rangle,$$
$$\langle x_0, x_1 \rangle \cap Y_k \subseteq \langle x_0, x_1 \rangle \quad \text{for } k \geq 1.$$

In general, given $x_0, \ldots, x_m$, we can apply Lemma 2.3 to obtain $x_{m+1} \in Y_{m+1}/x_m$ such that

$$\langle x_0, \ldots, x_m, x_{m+1} \rangle \cap Y_0 = \langle x_0, \ldots, x_m \rangle \cap Y_0 = \langle x_0 \rangle,$$

$$\vdots$$

$$\langle x_0, \ldots, x_m, x_{m+1} \rangle \cap Y_m = \langle x_0, \ldots, x_m \rangle \cap Y_m \subseteq \langle x_0, \ldots, x_m \rangle,$$
$$\langle x_0, \ldots, x_m, x_{m+1} \rangle \cap Y_k \subseteq \langle x_0, \ldots, x_m, x_{m+1} \rangle \quad \text{for } k \geq m + 1.$$

Thus, (x_m) is an infinite block sequence such that $\langle (x_m) \rangle \cap Y_n \subseteq \langle x_0, \ldots, x_n \rangle$ for each $n \in \omega$, and so again, $\mathcal{A}$ fails to be maximal.   ⊣

For $f, g \in \omega^\omega$, we write $f <^* g$ if there is some N such that $f(n) < g(n)$ for all $n \geq N$. A family of functions $\mathcal{B} \subseteq \omega^\omega$ is *bounded* if there is some $h \in \omega^\omega$ such that $f <^* h$ for all $f \in \mathcal{B}$, and *unbounded* otherwise. We write

$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \text{ is an unbounded family in } \omega^\omega\}.$$

It is easy show that $\mathfrak{b}$ is uncountable and it is well known that $\mathfrak{b} \leq \mathfrak{a}$ (see Proposition 8.4 in [5] or Theorem 3.1 in [36]). The corresponding result for infinite-dimensional block subspaces of FIN was proved in [7], however their proof does not appear to easily generalize; our proof here uses Lemma 2.3 to adapt the usual proof of $\mathfrak{b} \leq \mathfrak{a}$.

PROPOSITION 2.6. $\mathfrak{b} \leq \mathfrak{a}_{\mathrm{vec},F}$.

PROOF. Let $\mathcal{A}$ be an infinite almost disjoint family of block subspaces with $|\mathcal{A}| = \kappa < \mathfrak{b}$. We may enumerate $\mathcal{A}$ as $\{Y_\alpha : \alpha < \kappa\}$. By passing to finite-codimensional subspaces, we may assume that the Y_n, for $n < \omega$, are pairwise disjoint. For $\omega \leq \alpha < \kappa$, define f_α by

$$f_\alpha(n) = \min\{k : Y_\alpha \cap Y_n \subseteq \langle e_0, \ldots, e_k \rangle\}.$$

Define f_m for $m < \omega$ arbitrarily. For each $\alpha < \kappa$, let $g_\alpha \in \omega^\omega$ be such that whenever $y_0 < \cdots < y_k$ are such that $\mathrm{supp}(y_k) \subseteq [0, n]$ and $x > g_\alpha(n)$,

$$\langle y_0, \ldots, y_k, x \rangle \cap Y_\alpha = \begin{cases} \langle y_0, \ldots, y_k \rangle \cap Y_\alpha & \text{if } x \notin Y_\alpha, \\ \langle y_0, \ldots, y_k \rangle \cap Y_\alpha + \langle x \rangle & \text{if } x \in Y_\alpha. \end{cases}$$

Such functions exist by Lemma 2.3 (we are using the fact that the M in Lemma 2.3(b) depends only on the given subspace and the maximum of the *supports* of the given finite sequence). As $\kappa < \mathfrak{b}$, there is an $h \in \omega^\omega$, which we may take strictly increasing, with $\max\{f_\alpha, g_\alpha\} <^* h$ for all $\alpha < \kappa$. Define a block sequence $X = (x_n)$ by choosing $x_0 \in Y_0$ and $x_{n+1} \in Y_{n+1}/h(\max(\mathrm{supp}(x_n)))$ for all $n \in \omega$. We claim that $\langle X \rangle$ is almost disjoint from each Y_α.

CASE 1. $\alpha = m < \omega$. Let $N > m$ be such that $g_m(n) < h(n)$ for all $n \geq N$. Note that $\max(\mathrm{supp}(x_N)) \geq N$. If $k \geq N$, then $x_{k+1} \in Y_{k+1}/g_m(\max(\mathrm{supp}(x_k)))$ and, since Y_{k+1} and Y_m are disjoint,

$$\langle x_N, \ldots, x_k, x_{k+1} \rangle \cap Y_m = \langle x_N, \ldots, x_k \rangle \cap Y_m = \cdots = \langle x_N \rangle \cap Y_m = \{0\}.$$

This shows that $\langle X/x_{N-1} \rangle$ is disjoint from Y_m.

CASE 2. $\omega \leq \alpha < \kappa$. Let N be such that $\max\{f_\alpha(n), g_\alpha(n)\} < h(n)$ for all $n \geq N$. Again, note that $\max(\mathrm{supp}(x_N)) \geq N$. If $k \geq N$, then $x_{k+1} \in Y_{k+1}/g_\alpha(\max(\mathrm{supp}(x_k)))$, so

$$\langle x_N, \ldots, x_k, x_{k+1} \rangle \cap Y_\alpha = \begin{cases} \langle x_N, \ldots, x_k \rangle \cap Y_\alpha & \text{if } x_{k+1} \notin Y_\alpha, \\ \langle x_N, \ldots, x_k \rangle \cap Y_\alpha + \langle x_{k+1} \rangle & \text{if } x_{k+1} \in Y_\alpha. \end{cases}$$

However, as $x_{k+1} > f_\alpha(k+1)$ and $x_{k+1} \in Y_{k+1}$, it must be that $x_{k+1} \notin Y_\alpha$. Then, as in Case 1,

$$\langle x_N, \ldots, x_k, x_{k+1} \rangle \cap Y_\alpha = \langle x_N, \ldots, x_k \rangle \cap Y_\alpha = \cdots = \langle x_N \rangle \cap Y_\alpha = \{0\},$$

showing, again, that $\langle X/x_{N-1} \rangle$ is disjoint from Y_α.

Thus, $\mathcal{A}$ fails to be maximal, and so $\mathfrak{b} \leq \mathfrak{a}_{\mathrm{vec},F}$. $\dashv$

Recall that FIN is the collection of all nonempty finite subsets of ω. For $a, b \in$ FIN, we write $a < b$ if $\max(a) < \min(b)$, and call a sequence (a_n) of elements of FIN a *block sequence* if $a_n < a_{n+1}$ for all $n \in \omega$. Let $\mathrm{FIN}^{[\infty]}$ denote the set of infinite block sequences in FIN. For $A = (a_n) \in \mathrm{FIN}^{[\infty]}$,

$$\mathrm{FU}(A) = \{a_{n_0} \cup \cdots \cup a_{n_k} : n_0 < \cdots < n_k\},$$

is the *combinatorial subspace* generated by A. We say that $A, B \in \mathrm{FIN}^{[\infty]}$ are *almost disjoint* if $\mathrm{FU}(A) \cap \mathrm{FU}(B)$ is finite. Following Brendle and García Ávila [7], let $\mathfrak{a}_{\mathrm{FIN}}$ be the minimum cardinality of an infinite maximal almost disjoint family (defined in the obvious way) of block sequences in FIN. As commented in Section 1, this is the same as $\mathfrak{a}_{\mathrm{vec},F}$ when $|F| = 2$.

We denote by $\mathrm{non}(\mathcal{M})$ the minimum size of a nonmeager subset of $\mathbb{R}$. Brendle and García Ávila show that $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{FIN}}$ (Theorem 3 in [7]) by showing $\mathfrak{b} \leq \mathfrak{a}_{\mathrm{FIN}}$ (Proposition 12 in [7]), $\mathrm{non}(\mathcal{M}) = \max\{\mathfrak{b}, \mathfrak{b}(pbd \neq^*)\}$ (Lemma 15 in [7], attributed to folklore), and finally, $\mathfrak{b}(pbd \neq^*) \leq \mathfrak{a}_{\mathrm{FIN}}$ (Theorem 16 in [7]). Here, $\mathfrak{b}(pbd \neq^*)$ is the common (Lemma 14 in [7]) value of the cardinals $\mathfrak{b}_h(p \neq^*)$, where, for $h : \omega \to \omega$ a function with $h(n) \to \infty$ as $n \to \infty$, $\mathfrak{b}_h(p \neq^*)$ is the minimum size

of a family $\mathcal{F} \subseteq \omega^\omega$ such that for all partial $g : \omega \rightharpoonup \omega$ with infinite domain and bounded by h on that domain, there is an $f \in \mathcal{F}$ which is equal to g infinitely often.

For $A = (a_n) \in \mathrm{FIN}^{[\infty]}$, denote by

$$E_A = \bigcup \{a_n : |a_n| = 1\}.$$

A careful reading of their proof reveals that Brendle and García Ávila have shown the following:

THEOREM 2.7 (cf. Theorem 16 in [7]). *Suppose that* $\mathcal{A} \subseteq \mathrm{FIN}^{[\infty]}$ *satisfies the following for all* $A, A' \in \mathcal{A}$:

  (i) E_A *is coinfinite, and*
  (ii) *if* $A \neq A'$, *then* $E_A \cap E_{A'}$ *is finite.*

Then, if $\omega \leq |\mathcal{A}| < \mathfrak{b}(pbd \neq^*)$, *there is a* $B \in \mathrm{FIN}^{[\infty]}$ *which is almost disjoint from each element of* $\mathcal{A}$.

For $X = (x_n)$ a block sequence in E, let $\mathrm{supp}(X) = (\mathrm{supp}(x_n)) \in \mathrm{FIN}^{[\infty]}$. If $\mathcal{A}$ is a collection of infinite-dimensional block subspaces of E, then let

$$\mathrm{supp}(\mathcal{A}) = \{\mathrm{supp}(X) : X \text{ is a block sequence and } \langle X \rangle \in \mathcal{A}\}.$$

Note if X and Y are block sequences spanning the same subspace, then $\mathrm{supp}(X) = \mathrm{supp}(Y)$. The proof of the following is easy and omitted.

LEMMA 2.8. *For any block sequence X in E, if $A \in \mathrm{FIN}^{[\infty]}$ is such that $\mathrm{FU}(A) \subseteq \mathrm{FU}(\mathrm{supp}(X))$, then there is a block sequence Y in E with $\langle Y \rangle \subseteq \langle X \rangle$ and $\mathrm{supp}(Y) = A$.*[5]

LEMMA 2.9. *If $\mathcal{A}$ is a family of infinite-dimensional block subspaces of E and $A \in \mathrm{FIN}^{[\infty]}$ is almost disjoint (in the sense of FIN) from every element of $\mathrm{supp}(\mathcal{A})$, then for any block sequence X in E with $\mathrm{supp}(X) = A$, $\langle X \rangle$ will be almost disjoint (in the sense of E) from every $Y \in \mathcal{A}$.*

PROOF. Let $\mathcal{A}$ and A be as described, and suppose that there is some block sequence X with $\mathrm{supp}(X) = A$, and a subspace in $\mathcal{A}$, with block basis Y, such that $\langle Y \rangle \cap \langle X \rangle$ is infinite-dimensional. Let Z be an infinite block sequence in $\langle Y \rangle \cap \langle X \rangle$. Then, $\mathrm{supp}(Z)$ will witness that A fails to be almost disjoint from $\mathrm{supp}(Y)$. ⊣

LEMMA 2.10. *If $\mathcal{B}$ is an infinite almost disjoint family of block subspaces of E, then $\mathcal{A} = \mathrm{supp}(\mathcal{B})$ satisfies conditions (i) and (ii) in Theorem 2.7.*

PROOF. This follows immediately from the observation that if $A = \mathrm{supp}(X)$ for X a block sequence in E, and $n \in E_A$, then $e_n \in \langle X \rangle$. ⊣

Putting Lemmas 2.8, 2.9, and 2.10 together with Proposition 2.6 and Theorem 2.7, we have:

COROLLARY 2.11. $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},F}$.

§3. Cardinality: Consistency results. It follows from Proposition 2.5 that under CH, every mad family of block subspaces is of size $\mathfrak{c}$. Likewise, since $\mathrm{MA}_\kappa(\sigma\text{-centered})$ implies $\kappa < \mathfrak{p}$ (cf. [3]), and $\mathfrak{p} \leq \mathfrak{b}$ (see [36]), these together with Proposition 2.6 yield $\kappa < \mathfrak{a}_{\mathrm{vec},F}$. We give here a direct proof of this fact:

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[5]This lemma implies that the supp map is a *projection*, in the sense of forcing, between block sequences in E and those in FIN. See the related discussion in Section 6 of [32].

THEOREM 3.1 ($\mathrm{MA}_\kappa(\sigma\text{-centered})$). $\kappa < \mathfrak{a}_{\mathrm{vec},F}$.

PROOF. Let $\mathcal{A}$ be an infinite almost disjoint family of block subspaces. Define a poset $\mathbb{P}$ to be all pairs (s, F) where s is a finite normalized (i.e., leading coefficients are equal to 1) block sequence in E and F a finite subset of $\mathcal{A}$. We order elements of $\mathbb{P}$ by $(s', F') \leq (s, F)$ if $s' \sqsupseteq s$, $F' \supseteq F$, and $\forall X \in F (\langle s' \rangle \cap X \subseteq \langle s \rangle)$. Note that if $(s, F'), (s, F) \in \mathbb{P}$, for a fixed s, then $(s, F' \cup F) \in \mathbb{P}$ and extends both conditions. As there are only countably many such s, this shows that $\mathbb{P}$ is σ-centered. If G is a filter in $\mathbb{P}$, then we let $X_G = \langle \bigcup \{s : \exists F ((s, F) \in G)\} \rangle$.

Observe that if $X \in \mathcal{A}$, then the set $D_X = \{(s, F) \in \mathbb{P} : X \in F\}$ is dense, and if G is a filter in $\mathbb{P}$ with $G \cap D_X \neq \emptyset$, then $X_G \cap X$ is finite dimensional. For $n \in \omega$, let $E_n = \{(s, F) \in \mathbb{P} : |s| \geq n\}$. In order to see that the sets E_n are dense, it suffices to show that a given (s, F) in $\mathbb{P}$ can be extended to an $(s {}^\frown x, F)$ in $\mathbb{P}$. This can be accomplished by using Lemma 2.3 to obtain an M for which whenever $x > M$ and not in $\bigcup F$, $\langle s {}^\frown x \rangle \cap X = \langle s \rangle \cap X$ for each of the finitely many $X \in F$. Then, for any such x, $(s {}^\frown x, F) \leq (s, F)$.

If $|\mathcal{A}| \leq \kappa$, by $\mathrm{MA}_\kappa(\sigma\text{-centered})$, there is a filter $G \subseteq \mathbb{P}$ which meets the sets D_X and E_n, for $X \in \mathcal{A}$ and $n \in \omega$. Then, X_G witnesses that $\mathcal{A}$ fails to be maximal. $\dashv$

Let $\mathbb{B}_\kappa$ be κ-*random forcing*, the set of all positive measure Borel subsets of 2^κ ordered by containment modulo null sets, where $\kappa \geq \omega$ and 2^κ is given the product measure. By the *random model*, we mean the generic extension of a model of CH obtained by forcing with $\mathbb{B}_{\aleph_2}$. It is well known that in the random model, $\mathfrak{b} = \mathfrak{d} = \mathfrak{a} = \aleph_1$ and $\mathrm{non}(\mathcal{M}) = \mathfrak{c} = \aleph_2$ (see, e.g., Section 11.4 of [5]). Thus, by Corollary 2.11, we have:

COROLLARY 3.2.   *In the random model,* $\aleph_1 = \mathfrak{a} < \mathfrak{a}_{\mathrm{vec},F} = \aleph_2$.

Let $\mathbb{C}_\kappa$ be κ-*Cohen forcing*, the set of all finite partial functions p with $\mathrm{dom}(p) \subseteq \kappa \times \omega$ and $\mathrm{ran}(p) \subseteq 2$, ordered by extension. We identify $\mathbb{C}_{\aleph_0}$ with the set $\mathbb{C}$ of all finite partial functions p with $\mathrm{dom}(p) \subseteq \omega$ and $\mathrm{ran}(p) \subseteq 2$. By the *Cohen model*, we mean the generic extension of a model of CH obtained by forcing with $\mathbb{C}_{\aleph_2}$. Theorem 3.3 is stated as Theorem 3.7 in [20], however the proof given is just a reference to [21]. We give a complete proof here. See also Theorem 4 in [7] for the analogous result for FIN.

THEOREM 3.3.   *In the Cohen model,* $\aleph_1 = \mathfrak{a}_{\mathrm{vec},F} < \mathfrak{c}$.

PROOF. We follow the proof of the corresponding result for mad families of subsets of ω, Theorem 2.3 in Chapter VIII of [21]. We define a maximal almost disjoint family $\mathcal{A} = \{X_\xi : \xi < \omega_1\}$ of block subspaces having the property that it remain maximal after forcing with $\mathbb{C}$. By standard properties of Cohen forcing (Lemma 2.2 in Chapter VIII of [21]), this suffices.

Using CH in the ground model, let (p_ξ, τ_ξ) for $\omega \leq \xi < \omega_1$ enumerate all pairs (p, τ) such that $p \in \mathbb{C}$ and τ is a nice $\mathbb{C}$-name for a subset of E (in the sense of Definition 5.11 in Chapter VII of [21]). We recursively pick block subspaces X_ξ as follows: Let X_n, $n < \omega$, be any sequence of infinite-dimensional almost disjoint block subspaces. If $\omega \leq \xi < \omega_1$, and we have chosen X_η for all $\eta < \xi$, choose X_ξ almost disjoint from each of the (countably many) X_η for $\eta < \xi$ and so that if

$$p_\xi \Vdash_{\mathbb{C}} \tau_\xi \text{ is an infinite-dimensional subspace and } \forall \eta < \xi \ \dim(\tau_\xi \cap \check{X}_\eta) < \infty \quad (2)$$

then

$$\forall n\forall q \le p_\xi \exists r \le q \exists v > n(v \in X_\xi \text{ and } r \Vdash_{\mathbb{C}} \check{v} \in \tau_\xi).$$

To see that X_ξ can be chosen, assume that (2) holds. Let Y_i enumerate $\{X_\eta : \eta < \xi\}$ and let q_i enumerate $\{q : q \le p_\xi\}$. By (2), for each i, $q_i \Vdash_{\mathbb{C}} \dim(\tau_\xi \cap \check{Y}_i) < \infty$. We construct $r_i \in \mathbb{C}$ and $x_i \in E$ inductively in i. Pick $r_0 \le q_0$ and x_0 a nonzero vector so that $r_0 \Vdash_{\mathbb{C}} \check{x}_0 \in \tau_\xi$. Having chosen $r_0, \ldots, r_n$ and $x_0 < \cdots < x_n$ so that $r_i \le q_i$ and

$$r_i \Vdash_{\mathbb{C}} \check{x}_i \in \tau_\xi \wedge \forall k \le i(\langle \check{x}_0, \ldots, \check{x}_i \rangle \cap \check{Y}_k \subseteq \langle \check{x}_0, \ldots, \check{x}_k \rangle),$$

for each $i \le n$, apply Lemma 2.3 to find $r_{n+1} \le q_{n+1}$ and $x_{n+1} > x_n$ so that

$$r_{n+1} \Vdash_{\mathbb{C}} \check{x}_{n+1} \in \tau_\xi \wedge \forall k \le n+1(\langle \check{x}_0, \ldots, \check{x}_n, \check{x}_{n+1} \rangle \cap \check{Y}_k \subseteq \langle \check{x}_0, \ldots, \check{x}_k \rangle).$$

This is done as in the infinite case of Proposition 2.5. Let $X_\xi = \langle (x_n) \rangle$.

Clearly $\mathcal{A}$ is an almost disjoint family. It suffices to show that it is maximal in $\mathbf{V}[G]$, where G is $\mathbf{V}$-generic for $\mathbb{C}$. Towards a contradiction, suppose that for some (p_ξ, τ_ξ) with $p_\xi \in G$,

$$p_\xi \Vdash_{\mathbb{C}} \tau_\xi \text{ is an infinite-dimensional subspace and } \forall X \in \check{\mathcal{A}}(\dim(\tau_\xi \cap X) < \infty).$$

In particular, (2) holds at ξ. But $p_\xi \Vdash_{\mathbb{C}} \dim(\tau_\xi \cap \check{X}_\xi) < \infty$, so there is a $q \le p_\xi$ and an N so that $q \Vdash_{\mathbb{C}} \tau_\xi \cap \check{X}_\xi \subseteq \langle \check{e}_0, \ldots, \check{e}_N \rangle$, contradicting that

$$\exists r \le q \exists x > N(x \in X_\xi \wedge r \Vdash_{\mathbb{C}} \check{x} \in \tau_\xi). \qquad \dashv$$

Given a notion of forcing $\mathbb{P}$, we say that a mad family of subspaces $\mathcal{A}$ is $\mathbb{P}$-*indestructible* if $\mathcal{A}$ remains maximal after forcing with $\mathbb{P}$. The proof of Theorem 3.3 above shows that, assuming CH, there is a $\mathbb{C}$-indestructible mad family of subspaces.

Let $\mathbb{S}$ be *Sacks forcing*, the collection of all perfect subtrees of $2^{<\omega}$, ordered by inclusion. $\mathbb{S}$ enjoys the *Sacks property* (cf. Lemma 2.1 in [2]): whenever $p \in \mathbb{S}$ and $\dot{g}$ is an $\mathbb{S}$-name for an element of ω^ω, there is a $q \le p$ and a function $F : \omega \to \mathcal{P}(\omega)$ such that for all n, $|F(n)| \le 2^n$ and $q \Vdash \forall n(\dot{g}(n) \in F(n))$. It follows that $\mathbb{S}$ is ω^ω-*bounding*: every element of ω^ω in the generic extension is bounded by some element of the ground model. We note that $\mathbb{S}$ is proper.[6]

THEOREM 3.4 (CH). *If $\mathbb{P}$ is a proper poset of size $\aleph_1$ having the Sacks property, then there is a $\mathbb{P}$-indestructible mad family of block subspaces.*

PROOF. Using CH and properness, we can construct a sequence of pairs (p_ξ, τ_ξ), $\xi < \omega_1$, so that:

(i) τ_ξ is a nice $\mathbb{P}$-name for an infinite block sequence in E, with all antichains occurring in τ_ξ countable, and

(ii) $p_\xi \in \mathbb{P}$ is such that if there are τ and $p \in \mathbb{P}$ forcing that τ is an infinite block sequence, then there is a ξ such that $p_\xi \le p$ and $p_\xi \Vdash \tau = \tau_\xi$.

We construct a family of block sequences $\mathcal{A} = \{X_\alpha : \alpha < \omega_1\}$ recursively as follows: Begin by letting $\{X_i : i \in \omega\}$ be any almost disjoint family of block sequences (i.e., the corresponding subspaces are almost disjoint).

At stage $\alpha \ge \omega$: If

$$p_\alpha \nVdash \forall \xi < \alpha(\dim(\langle \tau_\alpha \rangle \cap \langle \check{X}_\xi \rangle) < \infty),$$

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[6]See, e.g., [25] for more details on properness.

then choose X_α to be any block sequence almost disjoint from all of the X_ξ for $\xi < \alpha$. Otherwise, enumerate by $(\dot{v}_n)$ and $(\dot{I}_n)$ $\mathbb{P}$-names for vectors (in block position) and intervals containing their supports, respectively, which are forced by p_α to make up τ_α. Enumerate α as $(\xi_n)_{n<\omega}$.

As the X_{ξ_n} are almost disjoint, there is an $f \in \omega^\omega$ so that for all n, $X_{\xi_0}/f(0), \ldots, X_{\xi_n}/f(n)$ are disjoint. By our assumption on p_α, there is a $\mathbb{P}$-name $\dot{g}$ for an element of ω^ω so that

$$p_\alpha \Vdash \forall n(\langle \tau_\alpha/\dot{g}(n)\rangle \cap \langle \check{X}_{\xi_n}\rangle = \{0\}).$$

CLAIM 3.5. *If $Y_0, \ldots, Y_n, Y_{n+1}$ are disjoint block sequences and $x_0 < \cdots < x_n$ so that for all $k \le n$, $\langle x_0, \ldots, x_n\rangle \cap \langle Y_k\rangle = \{0\}$, then there is an M so that whenever $x > M$ and not in any of $\langle Y_0\rangle, \ldots, \langle Y_n\rangle, \langle Y_{n+1}\rangle$, then for all $k \le n+1$, $\langle x_0, \ldots, x_n, x\rangle \cap \langle Y_k\rangle = \{0\}$.*

PROOF OF CLAIM. See the proof of Lemma 2.4.   ⊣

By the claim, there is a $\mathbb{P}$-name $\dot{h}$ for an element of ω^ω so that

$$p_\alpha \Vdash \forall n[(i_0 < \cdots < i_n \text{ and } \dot{h}(0) < \dot{v}_{i_0}, \ldots, \dot{h}(n) < \dot{v}_{i_n})$$
$$\Rightarrow \forall k \le n\langle \dot{v}_{i_0}, \ldots, \dot{v}_{i_n}\rangle \cap \langle \check{X}_{\xi_k}/\check{f}(k)\rangle = \{0\}].$$

As $\mathbb{P}$ is ω^ω-bounding, there is a $p \le p_\alpha$, and a function $m \in \omega^\omega$ so that

$$p \Vdash \forall n(m(n) \ge \max\{\check{f}(n), \dot{g}(n), \dot{h}(n)\}),$$

and so p forces that m shares the relevant properties of f, $\dot{g}$, and $\dot{h}$ above. Further, by ω^ω-bounding, there is an increasing sequence of intervals $(J_n)_{n<\omega}$, and a $p' \le p$, so that

$$p' \Vdash \forall n\exists m(\dot{I}_m \subseteq J_n).$$

Choose a further increasing sequence of intervals $(K_n)_{n<\omega}$ so that K_n contains at least 2^n many intervals of the form J_m, all of which are above $m(n)$.

By the Sacks property, there is a $p'' \le p$ and a function F with domain ω so that for each n, $|F(n)| \le 2^n$ and each element of $F(n)$ is a collection of vectors in E, in block position, so that

$$p'' \Vdash \forall n(\{\dot{v}_k : \dot{I}_k \subseteq \check{K}_n\} \in \check{F}(n)),$$

and for all n and $A \in F(n)$, there is a $q \le p''$ with

$$q \Vdash \{\dot{v}_k : \dot{I}_k \subseteq \check{K}_n\} = \check{A}.$$

For each n, let $A_0, \ldots, A_{|F(n)|-1}$ enumerate $F(n)$. We pick vectors u_n^0 recursively as follows: Let u_n^0 be the first element of A_0. Having defined $u_n^0 < \cdots < u_n^j$, with $u_n^i \in A_i$, choose u_n^{j+1} to the first element of A_{j+1} with support above u_n^j. Note that this can be done as each A_k must contain elements with supports in each of 2^n distinct intervals J_m. Let $X_\alpha = (u_0^0, \ldots, u_0^{|F(0)|-1}, u_1^0, \ldots, u_1^{|F(1)|-1}, \ldots)$. Observe that our choice of m ensures that X_α is a block sequence and is almost disjoint from each X_ξ for $\xi < \alpha$. That

$$p'' \Vdash \dim(\langle \tau_\alpha\rangle \cap \langle X_\alpha\rangle) = \infty$$

is ensured by the construction. It is then easy to show that $\mathcal{A} = \{\langle X_\alpha\rangle : \alpha < \omega_1\}$ is forced to be a mad family of subspaces by any condition in $\mathbb{P}$.   ⊣

By the *Sacks model*, we mean the generic extension of a model of CH obtained by forcing with a countable support iteration of Sacks forcing of length ω_2, see, e.g., [2] or [19]. Theorem 3.6 below is also a corollary of Theorem 3.3 and a general theorem of Zapletal (Theorem 0.2 in [38]), though the latter makes use of large cardinals which are not necessary here.

THEOREM 3.6. *In the Sacks model, $\aleph_1 = \mathfrak{a}_{\mathrm{vec},F} < \mathfrak{c}$.*

PROOF. This is proved using Theorem 3.4, exactly as Theorem III.2 in [19], which the reader may consult for details.   ⊣

We note that it follows directly from Theorem 3.4 that in the model obtained by forcing over a model of CH with the "side-by-side" (i.e., countable support product of) Sacks forcing [1] of length ω_2, $\mathfrak{a}_{\mathrm{vec},F} = \aleph_1$ as well. This is because any reals added in the side-by-side model are added by a product of ω_1 many copies of Sacks forcing, which is proper, has the Sacks property, and preserves CH in the intermediate model.

Lastly, following [14], we turn to the problem of producing a "large spectrum" of cardinalities of mad families of subspaces. Given an uncountable regular cardinal κ, let

$$\mathcal{D}_\kappa = \{(\alpha, \beta) \in \kappa \times \kappa : \alpha \text{ is an uncountable limit ordinal and } \beta < \alpha\}.$$

Let $\mathbb{Q}_\kappa$ be the set of all functions $p : F_p \times n_p \to E$ where $F_p \in [\mathcal{D}_\kappa]^{<\omega}$, $n_p \in \omega$, and for each $(\alpha, \beta) \in F_p$, $(p(\alpha, \beta, 0), \ldots, p(\alpha, \beta, n_p - 1))$ is a block sequence in E. We say $q \leq p$ if $q \supseteq p$ and whenever $(\alpha, \beta), (\alpha, \gamma) \in F_p$ with $\beta \neq \gamma$, we have that

$$\langle (q(\alpha, \beta, i))_{i < n_q} \rangle \cap \langle (q(\alpha, \gamma, i))_{i < n_q} \rangle = \langle (p(\alpha, \beta, j))_{j < n_p} \rangle \cap \langle (p(\alpha, \gamma, j))_{j < n_p} \rangle.$$

THEOREM 3.7. *Let κ be an uncountable regular cardinal. If G is $\mathbf{V}$-generic for $\mathbb{Q}_\kappa$, then in $\mathbf{V}[G]$, for every uncountable cardinal $\lambda < \kappa$ there is a mad family of block subspaces of E of cardinality λ. In this model, $\kappa \leq \mathfrak{c} \leq (\kappa^{\aleph_0})^{\mathbf{V}}$.*

Typically, $\kappa = \kappa^{\aleph_0}$ and so $\mathfrak{c} = \kappa$ in the extension. Thus, it is consistent that $\mathfrak{c} > \aleph_2$ (or even $\mathfrak{c} > \aleph_{\omega_1}$, etc) and for every uncountable cardinal $\lambda \leq \mathfrak{c}$, there is a mad family of size λ. We will proceed with a series of lemmas.

LEMMA 3.8. *$\mathbb{Q}_\kappa$ is ccc and $|\mathbb{Q}_\kappa| = \kappa$.*

PROOF. Suppose that $\{p_\xi : \xi < \aleph_1\} \subseteq \mathbb{Q}_\kappa$. By thinning down, we may assume that there is some fixed n for which $n_{p_\xi} = n$ for all $\xi < \aleph_1$. By the Δ-system lemma, we may further thin down so that the F_{p_ξ} form a Δ-system, that is, there is some finite set $R \subseteq \mathcal{D}_\kappa$ for which $F_{p_\xi} \cap F_{p_\eta} = R$ for all $\xi \neq \eta < \aleph_1$. But as there are only countably many functions $R \times n \to E$, uncountably many of the p_ξ agree on $R \times n$. Given such p_ξ and p_η, it is then immediate that $q = p_\xi \cup p_\eta$ is a common extension. That $|\mathbb{Q}_\kappa| = \kappa$ is easy to check.   ⊣

LEMMA 3.9. *Let $p \in \mathbb{Q}_\kappa$. For any $(\alpha, \beta) \in \mathcal{D}_\kappa$, there is a $q \leq p$ with $(\alpha, \beta) \in F_q$.*

PROOF. If $(\alpha, \beta) \notin F_p$, we can define $q \leq p$ so that $F_q = F_p \cup \{(\alpha, \beta)\}$, $n_q = n_p$, and $(q(\alpha, \beta, 0), \ldots, q(\alpha, \beta, n_q - 1))$ any block sequence in E whatsoever.   ⊣

LEMMA 3.10. *Let $p \in \mathbb{Q}_\kappa$. For any $M > 0$, there is a $q \leq p$ so that $n_q = n_p + 1$ and $q(\alpha, \beta, n_p) > M$ for all $(\alpha, \beta) \in F_q$.*

PROOF. Let $q(\alpha, \beta, i) = p(\alpha, \beta, i)$ for $i < n_p$ and $(\alpha, \beta) \in F_p$, as required. Fix α occurring as a first coordinate in F_p. Enumerate by $\beta_0, \ldots, \beta_k$ those β with $(\alpha, \beta) \in F_p$. Let $Y_j = \langle p(\alpha, \beta_j, 0), \ldots, p(\alpha, \beta_j, n_p - 1) \rangle$ for $j \leq k$. By repeated

applications of Lemma 2.3 (we are applying it to a finite-dimensional space Y, however the lemma remains true by essentially the same proof), there is an $N_0 \geq M$ so that whenever $x > N_0$ and not in Y_j,

$$\langle q(\alpha, \beta_0, 0), \ldots, q(\alpha, \beta_0, n_p - 1), x \rangle \cap Y_j = Y_0 \cap Y_j,$$

for $0 < j \leq k$. Let $q(\alpha, \beta_0, n_p)$ be any vector $x > N_0$ and not in $\bigcup_{j \leq k} Y_j$. Let $Y_0' = \langle q(\alpha, \beta_0, 0), \ldots, q(\alpha, \beta_0, n_p - 1), q(\alpha, \beta_0, n_p) \rangle$.

Continue in this fashion, choosing $N_\ell \geq M$ so that whenever $x > N_\ell$ and not in Y_i' or Y_j,

$$\langle q(\alpha, \beta_\ell, 0), \ldots, q(\alpha, \beta_\ell, n_p - 1), x \rangle \cap Y_i' = Y_\ell \cap Y_i' = Y_\ell \cap Y_i,$$

and

$$\langle q(\alpha, \beta_\ell, 0), \ldots, q(\alpha, \beta_\ell, n_p - 1), x \rangle \cap Y_j = Y_\ell \cap Y_j,$$

for $i < \ell$ and $\ell < j \leq k$. Let $q(\alpha, \beta_\ell, n_p)$ be any vector $x > N_\ell$ and not in $\bigcup_{i < \ell} Y_i' \cup \bigcup_{\ell < j \leq k} Y_j$. At the end of the construction, q will be a condition with domain $F_p \times (n_p + 1)$ extending p and having the desired property. $\dashv$

PROOF OF THEOREM 3.7. Let G be **V**-generic for $\mathbb{Q}_\kappa$. By Lemmas 3.9 and 3.10, $\bigcup G : \mathcal{D}_\kappa \times \omega \to E$ so that for each $(\alpha, \beta) \in \mathcal{D}_\kappa$, $G_{\alpha,\beta}(\cdot) = \bigcup G(\alpha, \beta, \cdot)$ is an infinite block sequence in E.

Given an uncountable limit $\alpha < \kappa$, we claim that $\langle G_{\alpha,\beta} \rangle \cap \langle G_{\alpha,\gamma} \rangle$ is finite-dimensional, for $\beta \neq \gamma < \alpha$. Let $p \in \mathbb{Q}_\kappa$ be given with $(\alpha, \beta), (\alpha, \gamma) \in F_p$. By the definition of $\leq$ in $\mathbb{Q}_\kappa$, we have that

$$p \Vdash \langle \dot{G}_{\alpha,\beta} \rangle \cap \langle \dot{G}_{\alpha,\gamma} \rangle = \langle (\check{p}(\alpha, \beta, i))_{j < n_p} \rangle \cap \langle (\check{p}(\alpha, \gamma, i))_{j < n_p} \rangle.$$

Thus, $\langle G_{\alpha,\beta} \rangle \cap \langle G_{\alpha,\gamma} \rangle$ is forced to be finite-dimensional and $\mathcal{A}_\alpha = \{ \langle G_{\alpha,\beta} \rangle : \beta < \alpha \}$ is an almost disjoint family of subspaces. As $\mathbb{Q}_\kappa$ preserves cardinals, $|\mathcal{A}_\alpha| = |\alpha|$. It remains to show that each $\mathcal{A}_\alpha$ is maximal.

Fix α as above and let τ be a nice $\mathbb{Q}_\kappa$-name for a subset of E. As $\mathbb{Q}_\kappa$ is ccc, there is a countable set of conditions $A \subseteq \mathbb{Q}_\kappa$, which decides which vectors are in τ and whether τ is an infinite-dimensional subspace. That is, if $p \Vdash \check{v} \in \tau$, for some $v \in E$ and $p \in \mathbb{Q}_\kappa$, then there is a $q \in A$ with $q \Vdash \check{v} \in \tau$, and likewise if $p \Vdash \tau$ is a subspace. A is contained in

$$\mathbb{Q}_{\kappa,S} = \{ p \in \mathbb{Q}_\kappa : (\alpha, \gamma) \in F_p \Rightarrow \gamma \in S \}$$

for some countable $S \subseteq \alpha$. Suppose that

$p \Vdash \tau$ is an infinite-dimensional subspace of E and $\forall \gamma \in \check{S}(\dim(\tau \cap \langle \dot{G}_{\alpha,\gamma} \rangle) < \infty)$

for $p \in \mathbb{Q}_{\kappa,S}$. Fix $\xi \in \alpha \setminus S$. We claim that for all $M > 0$, the set of conditions $q \in \mathbb{Q}_\kappa$ such that

$$q \Vdash \exists v > M(v \in \tau \cap \langle \dot{G}_{\alpha,\xi} \rangle)$$

is dense below p. Let $p' \leq p$. We may assume that $(\alpha, \xi) \in F_{p'}$. Let $p'' = p' \restriction (\{ (\alpha, \gamma) : \gamma \in S \} \times n_{p'}) \in \mathbb{Q}_{\kappa,S}$. Then, $p'' \leq p$, and so

$p'' \Vdash \tau$ is an infinite-dimensional subspace of E and $\forall \gamma \in \check{S}(\dim(\tau \cap \langle \dot{G}_{\alpha,\gamma} \rangle) < \infty)$

By Lemmas 2.3 and 3.10, there is a $p''' \leq p''$ in $\mathbb{Q}_{\kappa,S}$ and a $v > M$ so that

$$p''' \Vdash \check{v} \in \tau \wedge \forall (\alpha, \gamma) \in \check{F}_{p''}(\check{v} \notin \langle \dot{G}_{\alpha,\gamma} \rangle),$$

and moreover, there is a condition $q \in \mathbb{Q}_\kappa$ so that $F_q = F_{p'} \cup F_{p'''}$, $n_q = n_{p'''} + 1$, $q(\alpha, \xi, n_{p'''}) = v$, and $q \leq p'$. But then,

$$q \Vdash \exists v > M(v \in \tau \cap \langle \dot{G}_{\alpha,\xi} \rangle),$$

as claimed. Thus, $\mathcal{A}_\alpha$ is forced to be a mad family of block subspaces.

That $\mathfrak{c} \leq \kappa^{\aleph_0}$ in $\mathbf{V}[G]$ follows from standard facts about ccc forcing (cf. Lemma 5.13 of Chapter VII in [21]). ⊣

§**4. Definability and Ramsey theory.** In [23], Mathias showed that there are no infinite analytic mad families on ω. His proof proceeds by showing that, given an infinite almost disjoint family $\mathcal{A}$ on ω, the set $\mathcal{H}$ of subsets of ω not covered by a finite union of elements of $\mathcal{A}$ is a selective coideal.[7] Here, a *coideal* is the complement of an ideal of subsets of ω, and *selectivity* refers to closure under a certain kind of diagonalization. Were $\mathcal{A}$ analytic, an application of the main Ramsey-theoretic dichotomy of [23] shows that there must be an infinite set $x \in \mathcal{H}$ none of whose infinite subsets are in the $\subseteq$-downwards closure of $\mathcal{A}$. Such an x witnesses that $\mathcal{A}$ fails to be maximal.

We would like to replicate this argument to prove that there are no infinite analytic mad families of subspaces of E, considered as subsets of the product space 2^E. As is the case for mad families on ω, such a result would be sharp: assuming $\mathbf{V} = \mathbf{L}$, the methods in [24] can be adapted to show that there is a coanalytic mad family of subspaces. This naïve approach runs into several problems, which we discuss below.

Let us first consider the setting where F is a finite field, in which case almost disjoint sub*spaces* of E are also almost disjoint as sub*sets* of E. This suggests the following strategy: Suppose that $\mathcal{A}$ is an infinite analytic almost disjoint family of subspaces of E and let $\mathcal{H}$ be the collection of all sub*sets* of E, which are not covered by a union of finitely many elements of $\mathcal{A}$. Then, $\mathcal{H}$ is a selective coideal of subsets of E. Applying Mathias' dichotomy as above, we obtain an infinite subset $X \in \mathcal{H}$ all of whose further subsets are disjoint from the downwards closure of $\mathcal{A}$. If $\mathcal{A}$ were maximal, then we would obtain the desired contradiction provided X contains an infinite-dimensional subspace. However, there is no a priori reason why X ought to contain such a subspace.

In the event that $|F| = 2$, hope is provided by Hindman's theorem [15], one formulation of which says that the collection $\mathcal{B}$ of all subsets of E which contain an infinite-dimensional block subspace is a coideal. It would suffice, then, to show that $\mathcal{H} \cap \mathcal{B}$ is a selective coideal. As the union of two ideals is an ideal if and only if one contains the other, we would need to have that $\mathcal{H} \subseteq \mathcal{B}$ (clearly, $\mathcal{B} \not\subseteq \mathcal{H}$). Unfortunately, this is *never* true: take $X \in \mathcal{H}$ which has infinite intersection with infinitely many elements of $\mathcal{A}$ and build a block sequence Y in X with the same property. Taken as a set, $Y \in \mathcal{H}$ but Y contains no subspaces. This argument can be adapted to show that the family of block sequences in E whose spans are in $\mathcal{H}$ fails to be a coideal in the associated Ramsey space of all block sequences, in the sense of [8].

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[7]This is shown for infinite mad families in Proposition 0.7 of [23], but the assumption of maximality is not necessary, see Example 2 on p. 35 of [33].

We now turn to a strategy based on the Ramsey-theoretic results in [32] for block sequences in vector spaces over an arbitrary countable field F.

Following [32], we let $bb^\infty(E)$ denote the space of all infinite block sequences in E, which inherits a Polish topology from E^ω that is compatible with the Borel structure of 2^E. For $X, Y \in bb^\infty(E)$, we write $X \preceq Y$ if $\langle X \rangle \subseteq \langle Y \rangle$, and $X \preceq^* Y$ if $X/n \preceq Y$ for some n, where X/n denotes the tail subsequence of X consisting of those vectors in X with supports above n. Note that $\langle X/n \rangle = \langle X \rangle/n$, where the latter was defined for subspaces in Section 2. A nonempty subset of $bb^\infty(E)$ is a *family* if it is closed upwards with respect to $\preceq^*$. If $X \in \mathcal{H}$, we write $\mathcal{H} \restriction X = \{ Y \in \mathcal{H} : Y \preceq X \}$. The key notions from [32] are as follows:

DEFINITION 4.1. A family $\mathcal{H} \subseteq bb^\infty(E)$ is:

(a) a (p)-*family*, or has the (p)-*property*, if whenever $X_0 \succeq X_1 \succeq \cdots$ is a decreasing sequence with each $X_n \in \mathcal{H}$, there is a $Y \in \mathcal{H}$ with $Y \preceq^* X_n$ for all $n \in \omega$.

(b) *full* if whenever $D \subseteq E$ (not necessarily a subspace) and $X \in \mathcal{H}$ are such that for every $Y \in \mathcal{H} \restriction X$, there is a $Z \preceq Y$ with $\langle Z \rangle \subseteq D$, then there is a $Z \in \mathcal{H} \restriction X$ with $\langle Z \rangle \subseteq D$.

(c) a (p^+)-*family* if it is full and has the (p)-property.

DEFINITION 4.2. The *Gowers game* [11] played below $X \in bb^\infty(E)$, denoted $G[X]$, is as follows: Players I and II alternate, with player I going first and playing block sequences $X_k \preceq X$, and player II responding with vectors $y_k \in \langle X_k \rangle$ subject to the constraint $y_k < y_{k+1}$, for $k \in \omega$. The block sequence (y_k) is the *outcome* of a play of the game.

A *strategy* for II in $G[X]$ is a function α taking sequences $(X_0, \ldots, X_k)$ of possible prior moves by I to vectors $y \in \langle X_k \rangle$, with $\alpha(X_0, \ldots, X_{k-1}) < y$ for all k. Given a set $\mathbb{A} \subseteq bb^\infty(E)$, we say that α is a strategy for *playing into* $\mathbb{A}$ if whenever II follows α (that is, at each turn, given as input I's prior moves, II plays the output of α), the resulting outcome lies in $\mathbb{A}$. These notions are defined likewise for I.

DEFINITION 4.3. The *infinite asymptotic game* [27, 28] played below X, denoted $F[X]$, is as follows: Players I and II alternate, with player I going first and playing $n_k \in \omega$, and player II responding with vectors $y_k \in \langle X/n_k \rangle$ subject to the constraint $y_k < y_{k+1}$, for $k \in \omega$. Again, (y_k) is the *outcome* of a play of the game.

Strategies for I and II in $F[X]$ are defined as above for $G[X]$, as is the notion of having a strategy for *playing into* a set. In both games, we will be agnostic about which player "wins".

DEFINITION 4.4. A family $\mathcal{H} \subseteq bb^\infty(E)$ is *strategic* if whenever α is strategy for II in $G[X]$, when $X \in \mathcal{H}$, there is an outcome of α which is in $\mathcal{H}$.

It is proved in [32] that any sufficiently generic filter for $(bb^\infty(E), \preceq^*)$, viewed as a σ-closed notion of forcing, is a strategic (p^+)-family.

The following theorem from [32] was originally proved by Rosendal [28] in the case $\mathcal{H} = bb^\infty(E)$, which in turn was a discretized version of the dichotomy for block sequences in Banach spaces proved by Gowers in [11].

THEOREM 4.5 (Theorem 1.1 in [32]). *Let $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ be a (p^+)-family. If $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is analytic, then there is a $X \in \mathcal{H}$ such that either*

(i) *I has a strategy in $F[X]$ for playing into $\mathbb{A}^c$, or*
(ii) *II has a strategy in $G[X]$ for playing into $\mathbb{A}$.*

Assuming large cardinal hypotheses, and that $\mathcal{H}$ is strategic, Theorem 4.5 can be extended to all sets $\mathbb{A}$ in $\mathbf{L}(\mathbb{R})$ (Theorem 1.3 in [32]).

In what follows, if $\mathcal{A}$ is an infinite almost disjoint family of subspaces of E (notably, the elements of $\mathcal{A}$ need not be block subspaces), we let

$$\mathcal{H}(\mathcal{A}) = \{X \in \mathrm{bb}^\infty(E) : \exists^\infty Y \in \mathcal{A}(\dim(\langle X \rangle \cap Y) = \infty)\}.$$

Note that $\mathcal{H}(\mathcal{A})$ is always nonempty, as it contains (e_n), is closed upwards with respect to $\preceq^*$, and is thus a family. We let

$$\overline{\mathcal{A}} = \{X \in \mathrm{bb}^\infty(E) : \exists Y \in \mathcal{A}(\langle X \rangle \subseteq Y)\}.$$

Note that $\overline{\mathcal{A}} \cap \mathcal{H}(\mathcal{A}) = \emptyset$, and that if $\mathcal{A}$ is analytic, so is $\overline{\mathcal{A}}$.

LEMMA 4.6. *If $\mathcal{A}$ is an infinite almost disjoint family of subspaces of E, then for any $X \in \mathcal{H}(\mathcal{A})$,*

(a) *I and II have strategies in $G[X]$ and $F[X]$, respectively, for playing into $\mathcal{H}(\mathcal{A})$.*
(b) *If $\mathcal{A}$ is maximal, then I and II have strategies in $G[X]$ and $F[X]$, respectively, for playing into $\overline{\mathcal{A}}$.*

PROOF. (a) Fix an enumeration (Y_n) of a countably infinite subset of $\mathcal{A}$, each Y_n having infinite-dimensional intersection with $\langle X \rangle$, in such a way that each Y_n is repeated infinitely often. To see that I has a strategy in $G[X]$ for playing into $\mathcal{H}(\mathcal{A})$, let I play an infinite block sequence in $\langle X \rangle \cap Y_n$ on their nth move. The resulting outcome will have infinitely many entries in each Y_n and is thus in $\mathcal{H}(\mathcal{A})$. To see that II has a strategy in $F[X]$ for playing into $\mathcal{H}(\mathcal{A})$, let II play the first element of Y_n they can on their nth move.

(b) Suppose that $\mathcal{A}$ is maximal. Take $Y \in \mathcal{A}$ having infinite-dimensional intersection with $\langle X \rangle$. To see that I has a strategy in $G[X]$ for playing into $\overline{\mathcal{A}}$, let I play, repeatedly, any infinite block sequence Z contained in $\langle X \rangle \cap Y$. The resulting outcome will be below Y. To see that II has a strategy in $F[X]$ for playing into $\overline{\mathcal{A}}$, observe that so long as II plays in Y, which they may always do, the outcome will be contained in Y. ⊣

LEMMA 4.7. *For X an infinite-dimensional subspace, Y a block subspace, and $z_0 < \cdots < z_\ell$ in E, if $X \subseteq Y + \langle z_0, \ldots, z_\ell \rangle$, then there is an M such that $X/M \subseteq Y$.*

PROOF. Let (y_n) be a block basis for Y. Let $N = \max\{\mathrm{supp}(z_i) : i \leq \ell\}$ and suppose that $y_0, \ldots, y_k$ are those basis vectors in Y whose supports are not above N. Let $M = \max\{N, \max(\mathrm{supp}(y_k))\}$. We claim that $X/M \subseteq Y$. Take $x \in X/M$. By assumption, $x = y + w$ where $y \in Y$ and $w \in \langle z_0, \ldots, z_\ell \rangle$. Write $y = y' + y''$, where $y' \in \langle y_0, \ldots, y_k \rangle$ and $y'' \in \langle y_{k+1}, y_{k+2}, \ldots \rangle$, so that $x - y'' = y' + w$. If either side of this equation is nonzero, then $\mathrm{supp}(x - y'') > M$, but $\mathrm{supp}(y' + w) \leq M$, a contradiction. Thus, $x = y'' \in Y$. ⊣

LEMMA 4.8. *If $\mathcal{A}$ is an infinite mad family of subspaces, then $\mathcal{H}(A)$ is strategic and has the (p)-property.*

Proof. That $\mathcal{H}(\mathcal{A})$ is strategic is immediate from Lemma 4.6(a), as whenever α is a strategy for II in $G[X]$, for $X \in \mathcal{H}(\mathcal{A})$, we may let I use their strategy for playing into $\mathcal{H}(\mathcal{A})$ in response.

In what follows, if (Z_n) is a sequence in $\mathrm{bb}^\infty(E)$ and $Z \in \mathrm{bb}^\infty(E)$ is such that $Z/n \preceq Z_n$ for all $n \in \omega$, we will call Z a *diagonalization* of (Z_n).

To see that $\mathcal{H}(\mathcal{A})$ has the (p)-property, let $X_0 \succeq X_1 \succeq X_2 \succeq \cdots$ be a decreasing sequence contained within $\mathcal{H}(\mathcal{A})$. Let $X^0 \in \mathrm{bb}^\infty(E)$ be a diagonalization of (X_n) and take $Y^0 \in \mathcal{A}$ having infinite-dimensional intersection with $\langle X^0 \rangle$. Following the proof of Proposition 0.7 in [23], we will construct sequences (X^m) and (Y^m) in $\mathrm{bb}^\infty(E)$ where each Y^m is a distinct element of $\mathcal{A}$, $\langle X^m \rangle$ has infinite-dimensional intersection with Y^m, and X^m a further diagonalization of (X_n).

For each n, construct a countably infinite pairwise disjoint family of block sequences $\mathcal{A}_n$ below X_n such that

(i) for all $Y \in \mathcal{A}_n$, there is a $Y' \in \mathcal{A}$ with $\langle Y \rangle \subseteq Y'$, and
(ii) for all $Y \in \mathcal{A}_n$, $\langle Y \rangle$ is disjoint from Y^0.

This can be accomplished as $X_n \in \mathcal{H}(\mathcal{A})$; simply take a countably infinite $\mathcal{A}'_n \subseteq \mathcal{A}$ not containing Y^0, all of whose elements have infinite-dimensional intersection with $\langle X_n \rangle$, and let $\mathcal{A}_n$ be a set of block bases of subspaces witnessing this. Pairwise disjointness and disjointness from Y^0, for elements in $\mathcal{A}_n$, can be ensured by passing to tail block sequences. Enumerate $\mathcal{A}_n = \{ Y^n_i : i \in \omega \}$ in such a way that each element is repeated infinitely often.

Next, we build a decreasing sequence $X^0_0 \succeq X^0_1 \succeq X^0_2 \succeq \cdots$ in $\mathcal{H}(\mathcal{A})$ such that for each n, $X^0_n \preceq X_n$, and $\langle X^0_n \rangle$ is almost disjoint from Y^0. We will denote by $X^0_n = (x^0_{n,i})_{i \in \omega}$.

Let $x^0_{0,0}$ be the first entry of Y^0_0. There must be a nonzero $x \in \langle Y^1_0 \rangle$ above $x^0_{0,0}$ such that no linear combination of x and $x^0_{0,0}$ is in Y^0, otherwise $Y^1_0 \preceq^* Y^0$ by Lemma
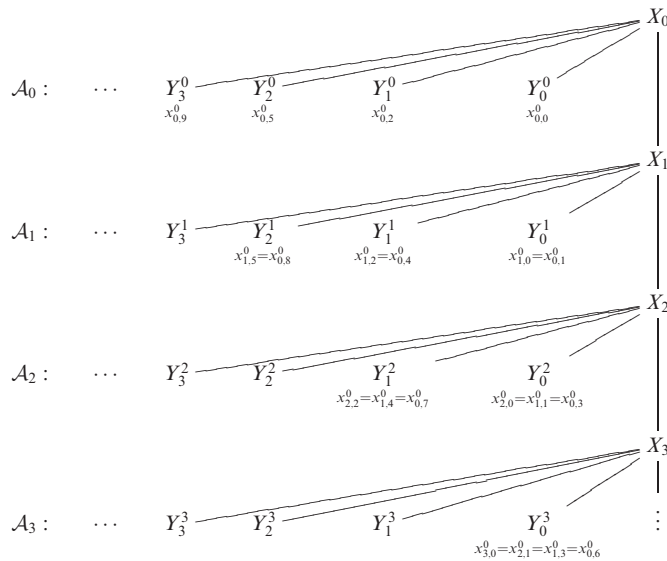

Figure 1. The construction of the X^0_n's

4.7. Let $x_{1,0}^0 = x_{0,1}^0 \in Y_0^1$ be such a vector. We continue in this fashion, following the diagram in Figure 1, with $X_0^0 = (x_{0,n}^0)$, $X_1^0 = (x_{1,n}^0)$, $X_2^0 = (x_{2,n}^0)$, $X_3^0 = (x_{3,n}^0)$, etc. on.

That is, let $x_{0,2}^0 \in Y_1^0$ be a vector above $x_{0,1}^0$ such that no linear combination of it with $x_{0,0}^0$ and $x_{0,1}^0$ lies in Y^0. Next, let $x_{0,3}^0 = x_{1,1}^0 = x_{2,0}^0 \in \langle Y_0^2 \rangle$ be a vector above $x_{0,2}^0$ such that no linear combination of it with $x_{0,0}^0$, $x_{0,1}^0$ and $x_{0,2}^0$ lies in Y^0. And so on.

By construction, $X_0^0 \succeq X_1^0 \succeq X_2^0 \succeq \cdots$ as each X_n^0 is a subsequence of the previous ones, and each $\langle X_n^0 \rangle$ is disjoint from Y^0. Moreover, each $\langle X_n^0 \rangle$ has infinite-dimensional intersection with $\langle Y \rangle$, for each $Y \in \mathcal{A}_n$, and $X_n^0 \in \mathcal{H}(\mathcal{A})$. Let X^1 be a diagonalization of (X_n^0), and thus also a diagonalization of the original (X_n) as well. Let $Y^1 \in \mathcal{A}$ have infinite-dimensional intersection with $\langle X^1 \rangle$. Note that we must have $Y^1 \neq Y^0$.

We continue this process to obtain (X^m) and (Y^m) as desired. Let $i : \omega \to \omega$ be an everywhere infinity-to-one surjection and consider the sequence of pairs $(X^{i(m)}, Y^{i(m)})$. Construct $X = (x_m)$ so that each $x_m \in \langle X^{i(m)}/m \rangle \cap Y^{i(m)}$. Then, $X \in \mathcal{H}(\mathcal{A})$, and moreover, for all n, if $x \in \langle X/n \rangle$, then x is a linear combination of elements of $X^{i(m_0)}/n, \ldots, X^{i(m_k)}/n$, each of which is $\preceq X_n$. So, $X/n \preceq X_n$ for all n. ⊣

DEFINITION 4.9. An infinite mad family $\mathcal{A}$ of subspaces is *full* if $\mathcal{H}(\mathcal{A})$ is full.

The preceding lemmas, and Theorem 4.5, yield the following:

THEOREM 4.10. *There are no analytic full mad families of subspaces.*

PROOF. Suppose that $\mathcal{A}$ was an analytic full mad family of subspaces. By Lemma 4.8, $\mathcal{H}$ is a (p^+)-family. Applying Theorem 4.5 to the analytic set $\overline{\mathcal{A}}$, there is an $X \in \mathcal{H}(\mathcal{A})$ such that either I has a strategy in $F[X]$ for playing into $\overline{\mathcal{A}}^c$, or II has a strategy in $G[Y]$ for playing into $\overline{\mathcal{A}}$. However, the latter contradicts Lemma 4.6(a), while the former contradicts Lemma 4.6(b). ⊣

Under large cardinal hypotheses, an identical proof, using Theorem 1.3 in [32], shows that no full mad family of subspaces can be in $\mathbf{L}(\mathbb{R})$.

Must a mad family of subspaces be full? Unfortunately, we are only able to show that, consistently, there are such mad families. It remains an open question whether mad families must be full (we suspect not), and if not, whether full mad families exist in ZFC. First, we need a variation on Theorem 3.1, localized to a fixed block subspace X.

LEMMA 4.11 ($\mathrm{MA}_\kappa(\sigma\text{-centered})$). *Suppose that X is a block subspace of E and $\mathcal{C}$ an almost disjoint family of block subspaces of E such that each has infinite-dimensional intersection with X. If $|\mathcal{C}| \leq \kappa$, then there is a block subspace Y of X almost disjoint from every element of $\mathcal{C}$.*

PROOF. We mimic the proof of Theorem 3.1. Define a poset $\mathbb{P}$ to be all pairs (s, F) where s is a finite normalized block sequence in X and F a finite subset of $\mathcal{C}$. We order elements of $\mathbb{P}$ by $(s', F') \leq (s, F)$ if $s' \sqsupseteq s$, $F' \supseteq F$, and $\forall W \in F(\langle s' \rangle \cap W \subseteq \langle s \rangle)$. As before, $\mathbb{P}$ is σ-centered, and if G is a filter in $\mathbb{P}$, we let $X_G = \langle \bigcup\{s : \exists F((s, F) \in G)\} \rangle$.

If $W \in \mathcal{C}$, then the set $D_W = \{(s, F) \in \mathbb{P} : W \in F\}$ is dense, and if $G \cap D_W \neq \emptyset$, then $X_G \cap W$ is finite dimensional. For $n \in \omega$, let $E_n = \{(s, F) \in \mathbb{P} : |s| \geq n\}$. In

order to see that the sets E_n, as before we use Lemma 2.3 to obtain an M for which whenever $x > M$ and not in $\bigcup F$, $\langle s^\frown x \rangle \cap W = \langle s \rangle \cap W$ for each of the finitely many $W \in F$. Then, for any such x in X (which can be found in $W \cap X$ for some $W \in \mathcal{C}$), $(s^\frown x, F) \leq (s, F)$.

If $|\mathcal{C}| \leq \kappa$, by $\mathrm{MA}_\kappa(\sigma\text{-centered})$, there is a filter $G \subseteq \mathbb{P}$ which meets the sets D_W and E_n, for $W \in \mathcal{C}$ and $n \in \omega$, so $Y = X_G$ is as desired.   ⊣

It will be useful to note that if $\mathcal{A} \subseteq \mathcal{B}$ are infinite almost disjoint families of subspaces, then $\mathcal{H}(\mathcal{A}) \subseteq \mathcal{H}(\mathcal{B})$.

THEOREM 4.12 ($\mathrm{MA}(\sigma\text{-centered})$[8]). *There is a full mad family of block subspaces.*

PROOF. We will define $\mathcal{A} = \bigcup_{\alpha < \mathfrak{c}} \mathcal{A}_\alpha$ via transfinite recursion on $\mathfrak{c}$. Enumerate by $\{X_\alpha : \alpha < \mathfrak{c}\}$ and $\{D_\alpha : \alpha < \mathfrak{c}\}$ all elements of $\mathrm{bb}^\infty(E)$ and subsets of E, respectively, ensuring that the enumeration X_α repeats each $X \in \mathrm{bb}^\infty(E)$ cofinally often. Fix a bijection $\langle \cdot, \cdot \rangle : \mathfrak{c} \times \mathfrak{c} \to \mathfrak{c}$.

Begin by letting $\mathcal{A}_0$ be any countably infinite almost disjoint family of block subspaces. Given $\alpha < \mathfrak{c}$, suppose that for $\beta < \alpha$, $\mathcal{A}_\beta$ has been defined to be an infinite almost disjoint family of block subspaces with size $\leq |\beta| + \aleph_0$, and that $\mathcal{A}_\beta \subseteq \mathcal{A}_\gamma$ for $\beta \leq \gamma < \alpha$. We define $\mathcal{A}_\alpha$ as follows:

Put $\mathcal{A}'_\alpha = \bigcup_{\beta < \alpha} \mathcal{A}_\beta$. If $\langle X_\alpha \rangle$ is almost disjoint from every element of $\mathcal{A}'_\alpha$, then put $\mathcal{A}''_\alpha = \mathcal{A}'_\alpha \cup \{\langle X_\alpha \rangle\}$. If not, put $\mathcal{A}''_\alpha = \mathcal{A}'_\alpha$. Say $\alpha = \langle \gamma, \delta \rangle$. If $X_\gamma \notin \mathcal{H}(\mathcal{A}''_\alpha)$, then let $\mathcal{A}_\alpha = \mathcal{A}''_\alpha$. Otherwise, let $\mathcal{C}$ be the collection of elements of $\mathcal{A}''_\alpha$ with which X_γ has infinite-dimensional intersection and consider the following cases:

CASE 1. There is a $Z \preceq X_\gamma$ such that $\langle Z \rangle$ is almost disjoint from each $Y \in \mathcal{C}$ and is contained in D_δ. In this case, let $\mathcal{B}$ be a countably infinite almost disjoint family of infinite-dimensional subspaces below Z. Note that if $V \in \mathcal{B}$ is compatible with some $Y \in \mathcal{A}''_\alpha$, then X_γ must be compatible with that Y, so $Y \in \mathcal{C}$, but this yields a contradiction as $\langle Z \rangle$ must be almost disjoint from such a Y. Let $\mathcal{A}_\alpha = \mathcal{A}''_\alpha \cup \mathcal{B}$, an almost disjoint family by the preceding argument. Then, $Z \in \mathcal{H}(\mathcal{A}_\alpha)$.

CASE 2. For every $Y \preceq X_\gamma$ such that $\langle Y \rangle$ is almost disjoint from every element of $\mathcal{C}$, there is no $Z \preceq Y$ with $\langle Z \rangle \subseteq D_\delta$. Note that if this fails, we are in Case 1. As $|\mathcal{C}| \leq |\alpha| + \aleph_0 < \mathfrak{c}$, by $\mathrm{MA}(\sigma\text{-centered})$ and Lemma 4.11, there is a $Y \preceq X_\gamma$ with $\langle Y \rangle$ almost disjoint from each element of $\mathcal{C}$. Let $\mathcal{B}$ be a countably infinite almost disjoint family below Y, and let $\mathcal{A}_\alpha = \mathcal{A}''_\alpha \cup \mathcal{B}$, an almost disjoint family by the same argument as in Case 1. Then, $Y \in \mathcal{H}(\mathcal{A}_\alpha)$.

We claim that $\mathcal{A} = \bigcup_{\alpha < \mathfrak{c}} \mathcal{A}_\alpha$ is as desired. Note that $\mathcal{H}(\mathcal{A}) = \bigcup_{\alpha < \mathfrak{c}} \mathcal{H}(\mathcal{A}_\alpha)$, as whenever $X \in \mathcal{H}(\mathcal{A})$, a countably infinite subset of $\mathcal{A}$ all compatible with X must occur in some initial $\mathcal{A}_\alpha$, as $\mathrm{cf}(\mathfrak{c}) > \aleph_0$. Clearly, $\mathcal{A}$ is a mad family. To verify fullness, let $D \subseteq E$ and $X \in \mathcal{H}(\mathcal{A})$, and suppose that for every $Y \in \mathcal{H}(\mathcal{A}) \upharpoonright X$, there is a $Z \preceq Y$ with $\langle Z \rangle \subseteq D$. We may take $\alpha < \mathfrak{c}$ large enough so that $\alpha = \langle \gamma, \delta \rangle$, $X = X_\gamma$, $D = D_\delta$, and $X_\gamma \in \mathcal{H}(\mathcal{A}''_\alpha)$, for $\mathcal{A}''_\alpha$ as in the construction above. If Case 1 occurred for this α, then there is a $Z \in \mathcal{H}(\mathcal{A}_\alpha) \upharpoonright X \subseteq \mathcal{H}(\mathcal{A}) \upharpoonright X$ with $\langle Z \rangle \subseteq D$. If Case 2 occurred for this α, then there is an $Y \in \mathcal{H}(\mathcal{A}_\alpha) \upharpoonright X \subseteq \mathcal{H}(\mathcal{A}) \upharpoonright X$ having no

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[8]This result seems likely to be true under the weaker assumption that $\mathfrak{a}_{\mathrm{vec},F} = \mathfrak{c}$; the issue is that the relevant almost disjoint family of subspaces, namely those of the form $W \cap \langle X_\gamma \rangle$ for $W \in \mathcal{C}$ in Case 2, need not be *block*.

$Z \preceq Y$ with $\langle Z \rangle \subseteq D$, contrary to assumption. Thus, there is a $Z \in \mathcal{H}(\mathcal{A}) \upharpoonright X$ with $\langle Z \rangle \subseteq D$, as required. $\dashv$

The proof of Theorem 4.12 can be adapted to show how to generically add a full mad family of block subspaces: Let $\mathbb{P}$ be the collection of all countably infinite almost disjoint families of block subspaces, ordered by reverse inclusion. It is easy to see that $\mathbb{P}$ is σ-closed and if G is **V**-generic for $\mathbb{P}$, then $\mathcal{G} = \bigcup G$ is a mad family of block subspaces. The arguments in Cases 1 and 2 above show that, for $\mathcal{A} \in \mathbb{P}$, $X \in \mathcal{H}(\mathcal{A})$, and $D \subseteq E$, the set of all $\mathcal{B} \in \mathbb{P}$ such that $\mathcal{H}(\mathcal{B})$ "witnesses fullness for X and D" is dense below $\mathcal{A}$. In the language of [13], assuming $\mathsf{MA}(\sigma\text{-centered})$, full mad families of block subspaces *exist generically*.

What can we say about analytic mad families of subspaces in the absence of fullness? For a family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ and $X \in \mathcal{H}$, the game $G_\mathcal{H}[X]$ is the variant of $G[X]$ in which I is restricted to playing elements of $\mathcal{H} \upharpoonright X$. A variant of Theorem 4.5, Theorem 3.11.5 in [31], can be used to obtain the following:

THEOREM 4.13. *Let $\mathcal{A}$ be an infinite mad family of subspaces. If $\mathcal{A}$ is analytic, then there is an $Y \in \mathcal{H}(\mathcal{A})$ such that II has a strategy in $G_{\mathcal{H}(\mathcal{A})}[Y]$ for playing into $\overline{\mathcal{A}}$.*

Were $\mathcal{H}(\mathcal{A})$ to be *+-strategic*, that is, whenever α is a strategy for II in $G_{\mathcal{H}(\mathcal{A})}[X]$, for some $X \in \mathcal{H}(\mathcal{A})$, then there is an outcome of α in $\mathcal{H}(\mathcal{A})$, then the conclusion of the above theorem would yield the desired contradiction. However, by Theorem 3.11.9 of [31], this is equivalent to $\mathcal{H}(\mathcal{A})$ being full. These observations suggests that full mad families of subspaces are analogous to *+-Ramsey* mad families on ω, as studied by Hrušák in [18] (see also [13]).[9]

§5. Further remarks, conjectures and open questions.
Many of the arguments above, particularly those dependent on Lemma 2.3 or results from [7], depend on the subspaces involved being block subspaces. For this reason, we incorporated "block" into our definition of the cardinal $\mathfrak{a}_{\mathrm{vec},F}$. It remains unclear whether this is necessary for our results.

QUESTION 5.1. *Given an infinite mad family of (arbitrary) infinite-dimensional subspaces, is there one of the same size consisting only of block subspaces? In particular, can we remove "block" from the definition of $\mathfrak{a}_{\mathrm{vec},F}$?*

We have seen in Corollary 3.2 that it is consistent that $\mathfrak{a} < \mathfrak{a}_{\mathrm{vec},F}$. As in [7], we also ask about the reverse inequality:

QUESTION 5.2. *Is $\mathfrak{a}_{\mathrm{vec},F} < \mathfrak{a}$ consistent with ZFC?*

Given the results in Section 3, it would be interesting to further determine in which "canonical models" $\mathfrak{a}_{\mathrm{vec},F} = \aleph_1$. In particular, as both $\mathfrak{a}$ and $\mathrm{non}(\mathcal{M})$ are $\aleph_1$ in the Miller model (see Section 11.9 in [5]), we suspect that $\mathfrak{a}_{\mathrm{vec},F}$ is as well. The paramterized $\diamondsuit$ principles of Moore, Hrušák, and Džamonja [26] provide a convenient way of isolating such results. For instance, it is shown in [26] that $\diamondsuit(\mathfrak{b})$, which holds in the Cohen, Sacks, and random models, implies that $\mathfrak{a} = \aleph_1$. By Corollary 3.2, this is not the case for mad families of block subspaces. We suspect instead that the "correct" $\diamondsuit$ principle for $\mathfrak{a}_{\mathrm{vec},F}$ is $\diamondsuit(\omega^\omega, =^\infty)$ (cf. Theorem 7.5 in [26]):

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[9]A closer analogue to being +-Ramsey would replace player II with player I in the definition of +-strategic, however this does not seem relevant to the present situation.

CONJECTURE 5.3. $\diamondsuit(\omega^\omega, =^\infty)$ *implies that* $\mathfrak{a}_{\mathrm{vec},F} = \aleph_1$.[10]

As $\diamondsuit(\omega^\omega, =^\infty)$ holds in the Cohen and Sacks models, this would subsume Theorems 3.3 and 3.6. Moreover, $\diamondsuit(\omega^\omega, =^\infty)$ implies that $\mathrm{non}(\mathcal{M}) = \aleph_1$ and thus fails in the random model, consistent with Corollary 3.2.

None of the original results in this article have any dependence on F. What differences, if any, can arise from different choices of F? In particular:

QUESTION 5.4. *Is it consistent with* ZFC *that for some choice of fields F and K (e.g., $|F| = 2$ and $K = \mathbb{Q}$) $\mathfrak{a}_{\mathrm{vec},F} \neq \mathfrak{a}_{\mathrm{vec},K}$?*

The main motivating question for Section 4 remains open:

QUESTION 5.5. *Does there exist an infinite analytic mad family of subspaces of E?*

Since posing this question in an earlier version of this article, it has been answered negatively by Horowitz and Shelah [17] in the special case when $|F| = 2$. The work in Section 4 also raises the following:

QUESTION 5.6. *Must every mad family of subspaces be full? If not, does there exist (in* ZFC*) a full mad family of subspaces?*

This may be analogous to the existence (in ZFC) of a +-Ramsey mad family on ω, recently announced by Osvaldo Guzmán-González [12].

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