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SMOOTH BIMODULES AND COHOMOLOGY OF II_1 FACTORS

ALIN GALATAN AND SORIN POPA

Department of Mathematics, University of California, UCLA, Los Angeles, CA 90095-1555, USA (agalatan@math.ucla.edu; popa@math.ucla.edu)

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Abstract We prove that, under rather general conditions, the 1-cohomology of a von Neumann algebra M with values in a Banach M-bimodule satisfying a combination of smoothness and operatorial conditions vanishes. For instance, we show that, if M acts normally on a Hilbert space $\mathcal{H}$ and $\mathcal{B}_0 \subset \mathcal{B}(\mathcal{H})$ is a norm closed M-bimodule such that any $T \in \mathcal{B}_0$ is *smooth* (i.e., the left and right multiplications of T by $x \in M$ are continuous from the unit ball of M with the s^*-topology to $\mathcal{B}_0$ with its norm), then any derivation of M into $\mathcal{B}_0$ is inner. The compact operators are smooth over any $M \subset \mathcal{B}(\mathcal{H})$, but there is a large variety of non-compact smooth elements as well.

Keywords: von Neumann II_1 factors; cohomology; derivations; smooth bimodules

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0. Introduction

Given a von Neumann algebra M and a Banach M-bimodule $\mathcal{B}$, an element $T \in \mathcal{B}$ is *smooth* (over M) if the left and right multiplications of T by elements in M are continuous from the unit ball of M with the s^*-topology, into $\mathcal{B}$ with its norm topology. The space $s^*_M(\mathcal{B})$ (or $s^*(\mathcal{B})$, for simplicity) of all smooth elements in $\mathcal{B}$ is itself a Banach M-bimodule, and we investigate in this paper the question of whether the first cohomology with values in a smooth closed M-bimodule $\mathcal{B}_0 \subset s^*(\mathcal{B})$ vanishes. In other words, whether any derivation $\delta : M \to \mathcal{B}_0$ (i.e., a linear map satisfying $\delta(xy) = x\delta(y) + \delta(x)y$, $\forall x, y \in M$) is 'inner', in the sense that there exists $T \in \mathcal{B}_0$ such that $\delta(x) = Tx - xT$, $\forall x \in M$.

We in fact only consider Banach M-bimodules $\mathcal{B}$ that are *operatorial* (over M), i.e., for which the norm on $\mathcal{B}$ satisfies the axiom $\|pTp + (1-p)T(1-p)\| = \max\{\|pTp\|, \|(1-p)T(1-p)\|\}$, for all $T \in \mathcal{B}$ and all projections p in M, and they will usually be assumed dual and normal.

The prototype example of a dual normal operatorial M-bimodule is the space of all linear bounded operators $\mathcal{B}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$ on which M is represented. Its smooth part $s^*(\mathcal{B}(\mathcal{H}))$ is a hereditary C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ that contains the space of compact operators $\mathcal{K}(\mathcal{H})$. But, unless M is a direct sum of matrix algebras, $s^*(\mathcal{B}(\mathcal{H}))$ is

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much bigger, containing a large variety of smooth elements that are non-compact (see 1.6, 1.7 hereafter).

We first prove that, if $\mathcal{B}$ is a dual normal operatorial M-bimodule, then any derivation δ from M into a closed sub-bimodule $\mathcal{B}_0 \subset s^*(\mathcal{B})$ can be 'integrated' to an element in $\mathcal{B}_0$ on any abelian von Neumann subalgebra of M, and show that, if M satisfies some very weak regularity condition (e.g., if M has a Cartan subalgebra [11], or if it has property (Γ) of [21]), then any element in $\mathcal{B}_0$ implementing δ on a diffuse abelian subalgebra automatically implements δ on all M. We also prove a similar statement for *smooth derivations* of M into an arbitrary (not necessarily dual normal) operatorial Banach bimodule $\mathcal{B}_0$, i.e., for derivations that are continuous from the unit ball of M with the s^*-topology into $\mathcal{B}_0$ with its norm topology. The precise statement is as follows.

Theorem 0.1. *Let M be a von Neumann algebra with the property that any* II_1 *factor summand of M contains a wq-regular diffuse abelian von Neumann subalgebra. Let $\mathcal{B}_0$ be an operatorial Banach M-bimodule and $\delta : M \to \mathcal{B}_0$ a derivation. Assume that either δ is smooth, or that $\mathcal{B}_0$ is a closed sub-bimodule of the smooth part of a dual normal operatorial M-bimodule $\mathcal{B}$, i.e., $\mathcal{B}_0 \subset s^*(\mathcal{B})$. Then there exists $T \in \mathcal{B}_0$ such that $\delta = \mathrm{ad}T$, $\|T\| \leqslant \|\delta\|$.*

The *wq-regularity* condition for a diffuse von Neumann subalgebra $B \subset M$ requires that one can 'reach out' from B to M inductively, by a chain of algebras, $B = N_0 \subset N_1 \subset \cdots \subset N_\imath = M$, such that, for all $\jmath < \imath$, $N_{\jmath+1}$ is generated by unitaries $u \in M$ with the property that $uN_\jmath u^* \cap N_\jmath$ is diffuse. We in fact prove a stronger statement than 0.1, where one merely requires the existence of a wq-regular subalgebra that satisfies an asymptotic commutativity property in M, which we call property (C'), generalizing property (Γ) in [21] and (C) in [29]. We mention that, in the case when a derivation δ takes values into a smooth M-bimodule $\mathcal{B}_0$ that is contained in a dual normal M-bimodule, we actually prove that δ follows automatically smooth, without the wq-regularity condition.

Our second main result shows that, in the specific case when $\mathcal{B} = \mathcal{B}(\mathcal{H})$, 0.1 holds true even without the wq-regularity assumption: any smooth-valued derivation of any von Neumann subalgebra M into $\mathcal{B}(\mathcal{H})$ is inner. More precisely, we prove the following.

Theorem 0.2. *Let $M \subset \mathcal{B}(\mathcal{H})$ be an arbitrary von Neumann algebra normally represented on a Hilbert space $\mathcal{H}$, $\mathcal{B}_0 \subset s^*(\mathcal{B}(\mathcal{H}))$ a Banach M sub-bimodule, and $\delta : M \to \mathcal{B}_0$ a derivation. Then there exists $T \in \mathcal{B}_0$ such that $\delta = \mathrm{ad}T$, $\|T\| \leqslant \|\delta\|$.*

In particular, since the compact operators are smooth over any von Neumann algebra, the above theorem recovers a result in [30], showing that any derivation of a von Neumann algebra with values into the space of compact operators is implemented by a compact operator. It also provides an answer to a question posed by Gilles Pisier [27], motivated by the similarity problem, concerning derivations from a C*-algebra with a trace (M_0, τ) (such as the reduced C*-algebra of a free group, $C_r^*(\mathbb{F}_n)$, $2 \leqslant n \leqslant \infty$), represented on a Hilbert space $\mathcal{H}$, with values in $\mathcal{B}(\mathcal{H})$, satisfying $\| \, \|_2$ - $\| \, \|_{\mathcal{B}(\mathcal{H})}$ type continuity conditions. Indeed, the existence of such a derivation δ implies that the representation of M_0 is automatically 'normal with respect to τ' on the non-degenerate part of δ, and that δ

extends to a smooth derivation of $M = M_0''$ into $s_M^*(\mathcal{B}(\mathcal{H}))$, so 0.2 applies, to give the following.

Corollary 0.3. *Let M_0 be a C^*-algebra with a faithful trace τ, and let $M_0 \subset \mathcal{B}(\mathcal{H})$ be a faithful representation of M_0. Let $\delta : M_0 \to \mathcal{B}(\mathcal{H})$ be a derivation. Assume that δ is continuous from the unit ball of M_0 with the topology given by the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M_0$, to $\mathcal{B}(\mathcal{H})$ with the operator norm topology. Then there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\delta = \mathrm{ad}T$ and $\|T\| \leqslant \|\delta\|$.*

As we mentioned before, the prototype example of a smooth operatorial M-bimodule is the space of compact operators $\mathcal{K}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$ on which M is represented, a case studied in [16, 29, 30]. The initial motivation for our work has in fact been to provide an abstract setting for this case, and to find the largest degree of generality for which arguments in the spirit of [16, 29, 30] can be carried over. At the same time, we were hoping to find M-bimodules for which the vanishing cohomology on abelian subalgebras (as in [16]) and automatic extension properties (as in [29]) do hold, while the arguments in [30], showing the vanishing 1-cohomology for arbitrary II_1 factors, do not.

Results that show vanishing of the 1-cohomology with coefficients in $\mathcal{B}(\mathcal{H})$ for arbitrary algebras, like in 0.2 or 0.3 above, can be relevant for the similarity problem (see [6, 20, 26]). In turn, the existence for some II_1 factor M of a smooth M-bimodule $\mathcal{B}_0$ for which M has non-inner derivations would imply, via Theorem 0.1, that M has no diffuse abelian wq-regular subalgebra (so, in particular, M would have no Cartan subalgebras, would be prime, etc). This falls within the larger scope of finding a cohomology theory for II_1 factors that is non-vanishing (and if possible calculable) and that could detect important properties of II_1 factors, like absence of regularity, or infinite generation. We do not provide smooth such $\mathcal{B}_0$ here, and it may be that in fact 0.1, 0.2 hold true for all II_1 factors and all smooth bimodules. We leave the clarification of this aspect as an open problem.

The paper is organized as follows. In § 1, we define the smooth part of a Banach M-bimodule, prove the main properties, and provide examples. In particular, we show that, if $M \subset \mathcal{B}(\mathcal{H})$ is a non-atomic finite von Neumann algebra, then the smooth part of the M-bimodule $\mathcal{B}(\mathcal{H})$ contains infinite-dimensional projections along 'Gaussian', 'canonical anticommutation relation (CAR)', and 'free' directions (see 1.7, 1.8). In § 2, we consider smooth-valued derivations and show that they can be 'integrated' on abelian subalgebras (see 2.5). The proof of this result follows ideas and techniques from [16]. In § 3, we introduce the notion of wq-regularity for subalgebras, and prove Theorem 0.1 (see 3.7, 3.8), by showing that, if a smooth-valued derivation δ has been integrated to a smooth element T on a diffuse abelian subalgebra that is wq-regular in M, then T automatically implements δ on all M. As we mentioned before, we in fact prove a stronger result, involving a property that generalizes properties (Γ) in [21] and (C) in [29], called property (C') (see 3.6). The proofs in this section consist of a refinement of arguments in [29].

In § 4, we prove Theorem 0.2 and deduce Corollary 0.3 (see 4.1 and 4.5). The proof of 4.1 follows the same strategy as in the case of compact-valued derivations in [30], but the additional technical difficulties are significant. We overcome them by using

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the incremental patching technique in [31, 33, 34]. Section 5 contains general remarks, including a generalization of Theorem 4.1 that recovers results in [36] (see 5.1), the definition of smooth n-cohomology for $n \geqslant 2$ (5.2), and some final comments on smooth cohomology (5.3). For convenience, we have included an Appendix with the proof of a general 'continuity principle', extracted from [29, 30], used several times in this paper. For the basics of von Neumann algebras, we refer the reader to the classic monographs [9, 42].

1. Smooth bimodules

Recall that if M is a unital Banach algebra (which we will in fact always assume in this paper to be a von Neumann algebra), then a *Banach M-bimodule* $\mathcal{B}$ is an M-bimodule with the property that $1_M T = T 1_M = T$, $\forall T \in \mathcal{B}$, and the left and right multiplication operations $M \times \mathcal{B} \ni (x, T) \mapsto xT \in \mathcal{B}$, $\mathcal{B} \times M \ni (T, x) \mapsto Tx \in \mathcal{B}$ are bounded bilinear maps. For simplicity, we will in fact always assume that $\|xT\| \leqslant \|x\|\|T\|$, $\|Tx\| \leqslant \|T\|\|x\|$, $\forall x \in M, T \in \mathcal{B}$.

If in addition $\mathcal{B}$ is the dual of a Banach space $\mathcal{B}_*$ and for each $x \in M$ the maps $\mathcal{B} \ni T \mapsto xT \in \mathcal{B}$, $\mathcal{B} \ni T \mapsto Tx \in \mathcal{B}$ are continuous with respect to the $\sigma(\mathcal{B}, \mathcal{B}_*)$ topology (also called weak*-topology), then $\mathcal{B}$ is called a *dual M-bimodule*. Finally, if M is a von Neumann algebra, $\mathcal{B}$ is a dual M-bimodule, and for each $T \in \mathcal{B}$ the maps $M \ni x \mapsto xT \in \mathcal{B}$, $M \ni T \mapsto Tx \in \mathcal{B}$ are continuous from $(M)_1$ with the $\sigma(M, M_*)$-topology (also called ultraweak topology), then we say that the dual M-bimodule $\mathcal{B}$ is *normal*.

Definitions 1.1. (1°) Let M be a von Neumann algebra and $\mathcal{B}$ a Banach M-bimodule. An element $T \in \mathcal{B}$ is *smooth* (with respect to M) if the maps $x \mapsto xT$, $x \mapsto Tx$ are continuous from $(M)_1$ with the s^*-topology to $\mathcal{B}$ with the norm topology. We denote by $s^*_M(\mathcal{B})$ (or simply $s^*(\mathcal{B})$ if no confusion is possible) the set of smooth elements $T \in \mathcal{B}$, and call it the *smooth part* of the M-bimodule $\mathcal{B}$. The Banach M-bimodule $\mathcal{B}$ is *smooth* if $s^*(\mathcal{B}) = \mathcal{B}$. A subset $\mathcal{S}$ of a Banach M-bimodule $\mathcal{B}$ is *uniformly smooth* if, $\forall \varepsilon > 0$, there exists a s^*-neighborhood $\mathcal{V}$ of 0 in M such that, if $x \in (M)_1 \cap \mathcal{V}$, then $\|Tx\|, \|xT\| \leqslant \varepsilon$, $\forall T \in \mathcal{S}$. Note that, if (M, τ) is a finite von Neumann algebra with a faithful normal trace τ (the case of most interest for us), then this amounts to the existence of some $\alpha > 0$ such that, if $x \in (M)_1$, $\|x\|_2 \leqslant \alpha$, then $\|xT\|, \|Tx\| \leqslant \varepsilon$, where as usual $\|x\|_2 = \tau(x^*x)^{1/2}$ denotes the Hilbert norm implemented by τ.

(2°) An M-bimodule is *operatorial* if, for any $p \in \mathcal{P}(M) = \{p \in M \mid p = p^* = p^2\}$ and any $T \in \mathcal{B}$, we have $\|pTp + (1-p)T(1-p)\| = \max\{\|pTp\|, \|(1-p)T(1-p)\|\}$.

Proposition 1.2. *Let $\mathcal{B}$ be a Banach M-bimodule and $\mathcal{B}_0 \subset \mathcal{B}$ a Banach sub-bimodule.*

(1°) *$s^*(\mathcal{B})$ is a Banach M-bimodule.*

(2°) *If $M_0 \subset M$ is a von Neumann subalgebra, then $s^*_M(\mathcal{B}) \subset s^*_{M_0}(\mathcal{B})$.*

(3°) *$s^*(\mathcal{B})/\mathcal{B}_0 \subset s^*(\mathcal{B}/\mathcal{B}_0)$.*

(4°) *If $\mathcal{B}$ is operatorial, then $\mathcal{B}/\mathcal{B}_0$, with its quotient norm and M-bimodule structure, is an operatorial Banach M-bimodule.*

(5°) *The bidual $\mathcal{B}^{**}$ of $\mathcal{B}$ has a natural dual M-bimodule structure, and, if $\mathcal{B}$ is operatorial, then so is $\mathcal{B}^{**}$.*

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(6°) *Let* $\mathcal{B}$ *be a dual normal* M*-bimodule. Then its predual* $\mathcal{B}_*$ *has a natural Banach* M*-bimodule structure which is smooth (i.e.,* $s_M^*(\mathcal{B}_*) = \mathcal{B}_*$*). But if* $\mathcal{B}$ *is operatorial, then the norm on* $\mathcal{B}_*$ *satisfies* $\|\varphi(p \cdot p) + \varphi((1-p)\cdot(1-p))\| = \|\varphi(p \cdot p)\| + \|\varphi((1-p)\cdot(1-p))\|$*, for* $\varphi \in \mathcal{B}_*$*,* $p \in \mathcal{P}(M)$*, and thus it is not operatorial in general.*

Proof. (1°) $s^*(\mathcal{B})$ is clearly closed to addition and scalar multiplication. Also, if $T_n \in s^*(\mathcal{B})$ and $\lim_n \|T_n - T\| = 0$ for some $T \in \mathcal{B}$, then for any $\varepsilon > 0$ there exists n_0 such that $\|T - T_{n_0}\| \leqslant \varepsilon/2$. Also, since $T_{n_0} \in s^*(\mathcal{B})$, there exists an s^*-neighborhood $\mathcal{V}$ of 0 in M such that if $x \in M \cap \mathcal{V}$ then $\|xT_{n_0}\|, \|T_{n_0}x\| \leqslant \varepsilon/2$. Thus, for such x we also have $\|xT\| \leqslant \|xT_{n_0}\| + \|x(T - T_{n_0})\| \leqslant \varepsilon$. This shows that $T \in s^*(\mathcal{B})$.

If $\|xT\|, \|Tx\| \leqslant \varepsilon$ for $x \in \mathcal{V} \cap (M)_1$, for some s^*-neighborhood $\mathcal{V}$ of 0 in $(M)_1$, then in particular it holds true for $x \in \mathcal{V} \cap (M_0)_1$, for any subalgebra $M_0 \subset M$. Moreover, $\|xT\|_{\mathcal{B}/\mathcal{B}_0}, \|Tx\|_{\mathcal{B}/\mathcal{B}_0} \leqslant \varepsilon$ as well, proving 2° and 3°.

Parts 4°, 5° are trivial by the definitions. The fact that $s_M^*(\mathcal{B}_*) = \mathcal{B}_*$ in 6° is Lemma 5 in [40], and the last part of 6° is trivial. □

Proposition 1.3. (1°) *If* M *is embedded as a* **-subalgebra in a unital* C^**-algebra* $\mathcal{B}$ *with* $1_M = 1_{\mathcal{B}}$*, then* $\mathcal{B}$ *with its left and right multiplications by elements in* M *is an operatorial Banach* M*-bimodule and* $s_M^*(\mathcal{B})$ *is a hereditary* C^**subalgebra of* $\mathcal{B}$ *which is both an* M*-bimodule and an* $M' \cap \mathcal{B}$*-bimodule.*

(2°) *Assume that* $M \subset \mathcal{B}(\mathcal{H})$ *is a normal representation of* M*. Then the hereditary* C^**-algebra* $s_M^*(\mathcal{B}(\mathcal{H})) \subset \mathcal{B}(\mathcal{H})$ *contains the space of compact operators* $\mathcal{K}(\mathcal{H})$ *and it is both an* M*-bimodule and an* M'*-bimodule.*

(3°) *With* $M \subset \mathcal{B}(\mathcal{H})$ *as in 2°, let* $P \subset M$ *be a von Neumann subalgebra and* $\mathcal{H}_0 \subset \mathcal{H}$ *a Hilbert subspace such that the projection* p *of* $\mathcal{H}$ *onto* $\mathcal{H}_0$ *commutes with* P*. Then* $ps_M^*(\mathcal{B}(\mathcal{H}))p \subset s_{Pp}^*(\mathcal{B}(\mathcal{H}_0))$*, with equality if* $P = M$*.*

(4°) *If* (M, τ) *is a finite von Neumann algebra,* $P \subset M$ *is a von Neumann subalgebra, and* e_P *denotes the orthogonal projection of* L^2M *onto* L^2P*, then for any* $T \in s_M^*(\mathcal{B}(L^2M))$ *we have* $e_PT, Te_P \in s_M^*(\mathcal{B}(L^2M))$*. Also, with the usual identifications, we have* $e_Ps_M^*(\mathcal{B}(L^2M))e_P = s_P^*(\mathcal{B}(L^2P))$*.*

Proof. (1°) If $T_1, T_2 \in s^*(\mathcal{B})$ and $x_i \in (M)_1$ converges s^* to 0, then $\|x_i(T_1T_2)\| \leqslant \|x_iT_1\|\|T_2\| \to 0$ and $\|(T_1T_2)x_i\| \leqslant \|T_1\|\|T_2x_i\| \to 0$, showing that $T_1T_2 \in s^*(\mathcal{B})$. Also, $\|x_iT_1^*\| = \|T_1x_i^*\| \to 0$; thus $s^*(\mathcal{B})$ is actually a C^*-subalgebra of $\mathcal{B}$. Moreover, if $T_0 \in \mathcal{B}$ with $0 \leqslant T_0 \leqslant T \in s^*(\mathcal{B})$, then $\|x_iT_0\| \leqslant \|x_iT^{1/2}\|\|T^{1/2}\| \to 0$, and so $T_0 \in s^*(\mathcal{B})$. Thus, $s^*(\mathcal{B})$ is hereditary.

The smoothness of an element is clearly invariant to left–right multiplication by elements commuting with M.

(2°) For any bounded net of operators $x_i \in \mathcal{B}(\mathcal{H})$ converging s^* to 0, and any $T \in \mathcal{K}(\mathcal{H})$, $\lim_i \|x_iT\| = \lim_i \|Tx_i\| = 0$, so in particular $\mathcal{K}(\mathcal{H}) \subset s^*(\mathcal{B}(\mathcal{H}))$.

(3°) follows trivially from the fact that smoothness over P is invariant to left–right multiplication by elements in P'.

(4°) If $x_n \in (M)_1$ satisfies $\tau(x_n^*x_n) = \|x_n\|_2^2 \to 0$ and we denote by E_P the τ-preserving conditional expectation of M onto P, then, by using that $e_Pye_P = E_P(y)e_P$, $\forall y \in M$,

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and the fact that $E_P(x_n^* x_n)$ converges to 0 in the s^*-topology, we get

$$\|x_n e_P T\|^2 = \|T^* e_P x_n^* x_n e_P T\| = \|T^* e_P E_P(x_n^* x_n) T\| \leqslant \|T\| \|E_P(x_n^* x_n) T\| \to 0.$$

Similarly, $\|T e_P x_n\| \to 0$. The same calculation shows that, if $T_0 \in s_P^*(\mathcal{B}(L^2 P))$, then $e_P T_0 e_P$ viewed as an element in $\mathcal{B}(L^2 M)$ is smooth/M. Combined with 3°, this shows that $e_P s_M^*(\mathcal{B}(L^2 M)) e_P = s_P^*(\mathcal{B}(L^2 P))$. □

Remark 1.4. In the same spirit as 1.3(1°) above, a more general class of operatorial M-bimodules can be obtained by taking two (unital *-) representations π_1, π_2 of M into a C^*-algebra $\mathcal{B}$ and letting $x \cdot T \cdot y = \pi_1(x) T \pi_2(y)$, $\forall T \in \mathcal{B}$, $x, y \in M$. But, as we will later see, we are mainly interested in dual normal M-bimodules, with M a II_1 factor. This means that the C^*-algebra $\mathcal{B}$ should be a von Neumann algebra as well, with π_1, π_2 normal embeddings of M into $\mathcal{B}$. When $\mathcal{B} = \mathcal{B}(\mathcal{H})$, this amounts to normal representations π_1, π_2 of M on the Hilbert space $\mathcal{H}$, where without loss of generality we may assume that $\dim_{\pi_1(M)} \mathcal{H} \geqslant \dim_{\pi_2(M)} \mathcal{H}$. If $\pi_1(M)$ is identified with M, then this implies that $\pi_2(x) = U x U^* p'$, where $U \in \mathcal{U}(\mathcal{H})$ is a unitary element and p' is a projection in the commutant M' of M in $\mathcal{B}(\mathcal{H})$. Thus, the above M-bimodule structure on $\mathcal{B} = \mathcal{B}(\mathcal{H})$ is actually equivalent to the one given by $x \cdot T \cdot y = x T y p'$. Since this class of bimodules does not offer a significant generalization with respect to the case $\pi_1 = \pi_2$, we have opted to only consider this latter case, for the sake of simplicity. On the other hand, it may be interesting to characterize all operatorial Banach M-bimodules $\mathcal{B}$ from an 'operator space perspective': for instance, to examine whether any such $\mathcal{B}$ is 'essentially' equivalent to a quotient of $\mathcal{B}(\mathcal{H})$.

Lemma 1.5. *Let (M, τ) be a finite von Neumann algebra, $M \subset \mathcal{B}(\mathcal{H})$ a normal representation of M, $\mathcal{H}_0 \subset \mathcal{H}$ a Hilbert subspace, and $p = \mathrm{proj}_{\mathcal{H}_0}$.*

(1°) *$p \in s^*(\mathcal{B}(\mathcal{H}))$ iff $\lim_{\alpha \to 0} \sup\{\|x(\xi)\| \mid x \in (M)_1, \|x\|_2 \leqslant \alpha, \xi \in (\mathcal{H}_0)_1\} = 0$ and iff $\lim_n \|q_n p q_n\| = 0$ (equivalently, $\|(1 - q_n) p (1 - q_n) - p\| \to 0$), for any sequence of projections $\{q_n\}_n \subset \mathcal{P}(M)$ that decreases to 0.*

(2°) *If $\mathcal{H} = L^2 M$, then a sufficient condition for p to be in $s^*(\mathcal{B}(L^2 M))$ is that $\mathcal{H}_0 \subset \hat{M}$ and $\exists C > 0$ such that $\|x\| \leqslant C \|x\|_2$, $\forall x \in \mathcal{H}_0$.*

Proof. The first equivalence in 1° is just a reformulation of the condition of smoothness for the orthogonal projection $\mathrm{proj}_{\mathcal{H}_0}$. If $q_n \in \mathcal{P}(M)$ decrease to 0, then they converge to 0 in the strong operator topology, so $\|q_n p\| \to 0$, as required. Conversely, if this latter condition is satisfied for all sequence of orthogonal projections, then by Lemma A.1 it follows that $\|x_n p\| \to 0$ for any sequence $\{x_n\}_n \subset (M)_1$ with $\tau(x_n^* x_n) \to 0$.

(2°) If $\hat{x} \in \mathcal{H}_0 \subset \hat{M} \subset L^2 M$, satisfies $\|x\|_2 \leqslant 1$ and we take $y \in M$, then $\|y(\hat{x})\|_2 = \|yx\|_2 \leqslant \|y\|_2 \|x\| \leqslant C \|y\|_2$. This shows in particular that, if $y \in (M)_1$ satisfies $\|y\|_2 \leqslant \alpha$, then $\|yp\| \leqslant C\alpha$, so p is smooth. □

Definition 1.6. Let (M, τ) be a finite von Neumann algebra. An orthogonal projection p of $L^2(M)$ onto a Hilbert subspace $\mathcal{H}_0 \subset L^2 M$ is *strongly smooth* (or *s-smooth*) if condition 1.5(2°) is satisfied, i.e., there exists some $C > 0$ such that $\|x\| \leqslant C \|x\|_2$, $\forall x \in \mathcal{H}_0$.

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The next two results provide a large variety of concrete examples of smooth and strongly smooth non-compact elements in the case when M is a finite von Neumann algebra and the target bimodule is $\mathcal{B}(L^2M)$. We distinguish three remarkable classes of infinite-dimensional Hilbert subspaces $\mathcal{H} \subset L^2M$ with $p = \text{proj}_{\mathcal{H}}$ smooth.

1.6.1. *Gaussian–Hilbert subspace* $\mathcal{H} \subset L^2M$, generated inside an arbitrary diffuse abelian von Neumann subalgebra A of M by an orthonormal system $\{\xi_n\}_n \subset L^2A$ consisting of a sequence of independent and identically distributed real-valued Gaussian random variables ξ_n, $n \geqslant 1$ (see 1.7(2°)). This can be obtained as follows: endow the real line $\mathbb{R}$ with the probability measure given by $d\mu_0(t) = (2\pi)^{-1/2}e^{-t^2}\, d\lambda_0(t)$, where $d\lambda_0$ is the Lebesgue measure on $\mathbb{R}$, and denote by (X, μ) the infinite product probability space $\Pi_{i=1}^{\infty}(\mathbb{R}, \mu_0)_i$; let ξ_n denote the projection of X onto the nth term $\mathbb{R}$ of the infinite product; viewing A as $L^\infty(X)$ and $L^2(X)$ as L^2A, the sequence $\{\xi_n\}_n \subset L^2(X)$ satisfies the conditions.

1.6.2. *CAR Hilbert subspace* $\mathcal{H} \subset L^2M$, having as orthonormal basis a sequence of self-adjoint unitary elements $\{u_n\}_n \subset \hat{M}$ generating an arbitrary hyperfinite II_1 subfactor $R \subset M$, satisfying the canonical anticommutation relations (CARs) $u_i u_j = -u_j u_i$, $\forall i \neq j$, with $\tau(u_i) = 0$, $\forall i$, as in [38] (see 1.7(4°)).

1.6.3. *Free Hilbert subspace* $\mathcal{H} \subset L^2M$, having as orthonormal basis a sequence of unitaries in $\hat{M} \subset L^2M$ that are free independent with respect to the trace on M (see 1.8(1°)).

As we will see below, s-smoothness is a purely non-commutative phenomenon, which only occurs when M is II_1 (e.g., in CAR and free directions, as in 1.6.2 and 1.6.3). We are grateful to Assaf Naor and Gideon Schechtman for their help with the Gaussian example 1.7(2°), and to Gilles Pisier for his help with the CAR example 1.7(4°), below.

Proposition 1.7. *Let (M, τ) be a finite von Neumann algebra and $M \subset \mathcal{B}(L^2M)$ its standard representation.*

(1°) *If M is atomic, then $s_M^*(\mathcal{B}(L^2M)) = \mathcal{K}(L^2M)$.*

(2°) *Assume that M is diffuse, and let $A \subset M$ be an arbitrary separable diffuse abelian von Neumann subalgebra of M. Given any sequence $\{\xi_n\}_n \subset L^2(A, \tau)$ of independent and identically distributed real-valued Gaussian random variables in A, as in 1.6.1 above, the orthogonal projection p of L^2M onto the Hilbert subspace $\mathcal{H} \subset L^2A$ generated by $\{\xi_n\}_n$ satisfies $p \in s_M^*(\mathcal{B}(L^2M))$. Thus, $\mathcal{K}(L^2M) \subsetneq s_M^*(\mathcal{B}(L^2M))$.*

(3°) *If M is diffuse abelian, then any s-smooth projection $p \in \mathcal{B}(L^2M)$ must have finite rank.*

(4°) *Assume that M is of type II_1, and let $R \subset M$ be an arbitrary hyperfinite II_1 subfactor. If $\{u_n\}_n \subset \mathcal{U}(R)$ is a sequence of self-adjoint unitaries of trace 0 satisfying the CARs $u_i u_j = -u_j u_i$, $\forall i \neq j$, as in 1.6.2 above, and we denote by $\mathcal{H} \subset L^2M$ the Hilbert space with orthonormal basis $\{u_n\}_n$, then the projection of L^2M onto $\mathcal{H}$ is s-smooth.*

Proof. (1°) If M has finite-dimensional center, then M is finite dimensional, and the statement becomes trivial. If in turn $\mathcal{Z}(M)$ is infinite dimensional, then it follows isomorphic to $\ell^\infty(\mathbb{N})$ (because M has a faithful trace). Let $\{e_k\}_{k\geqslant 1}$ be the atoms in $\mathcal{Z}(M)$, and let $p_n = \Sigma_{i=1}^n e_k$. Then p_n are finite-rank projections, i.e., $p_n \in \mathcal{K} = \mathcal{K}(L^2M)$, and they increase to 1. Thus, if $T \in \mathcal{B} = \mathcal{B}(L^2M)$ is a smooth element, then $\|(1-p_n)T\| \to 0$, so in the Calkin algebra $\mathcal{B}/\mathcal{K}$ we have $\|T\|_{\mathcal{B}/\mathcal{K}} = \|(1-p_n)T\|_{\mathcal{B}/\mathcal{K}} \leqslant \|(1-p_n)T\| \to 0$, showing that $T \in \mathcal{K}$.

(2°) By 1.3(4°), it is sufficient to prove that, if $q_n \in A$ are projections with $\tau(q_n) \to 0$, then $\sup\{\|q_n\xi\|_2 \mid \xi = \xi^* \in (\mathcal{H})_1\} \to 0$. If $\xi = \Sigma_j c_j \xi_j$ with $c_j \in \mathbb{R}$, $\Sigma_j |c_j|^2 = 1$, then ξ is still a Gaussian with the same distribution as the ξ_n. Since the L^2-norm of the restriction of a Gaussian ξ to the set $Y_n = \{t \mid |\xi(t)| \geqslant n\}$ decays exponentially in n^2, it follows that the L^2-norm of the restriction of ξ to an arbitrary set $Y \subset X$ of measure $\mu(Y) \leqslant \mu(Y_n)$ is majorized by $\|\xi\chi_{Y_n}\|_2$, and thus tends to 0 as n tends to ∞, independently of ξ.

(3°) It is well known that, if $\mathcal{H}_0 \subset L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$ has dimension $1 \leqslant n \leqslant \infty$ and satisfies the property that $\|x\| \leqslant C\|x\|_2$, $\forall x \in \mathcal{H}_0$, then $C \geqslant \sqrt{n}$ (see, e.g., [24], or [25]).

(4°) The CAR unitaries $\{u_n\}_n \subset R \subset M$ satisfy the following properties: $u_n = u_n^*$; $\tau(u_n) = 0$; $u_n u_m = -u_m u_n$, $\forall n \neq m$. But then, for any finitely many real scalars $c_k \in \mathbb{R}$, the element $x = \Sigma_k c_k u_k \in R$ satisfies the identities

$$\begin{aligned} x^*x &= (\Sigma_i c_i u_i)(\Sigma_j c_j u_j) = \Sigma_k c_k^2 + \Sigma_{i\neq j} c_i c_j u_i u_j \\ &= \|x\|_2^2 1_M + \Sigma_{i<j} c_i c_j (u_i u_j + u_j u_i) = \|x\|_2^2 1_M. \end{aligned}$$

Thus, $x^*x = \|x\|_2^2 1_M$. This shows that, if we denote by $\mathcal{H}_0$ the span of $\{u_k \mid k \geqslant 1\}$, then for any $x = x^* \in \mathcal{H}_0$ we have $x^*x = \|x\|_2^2 1_M$, so in particular $\|x\| = \|x\|_2$. For arbitrary $x \in \mathcal{H}_0$, we thus get

$$\|x\| \leqslant \|\Re x\| + \|\Im x\| = \|\Re x\|_2 + \|\Im x\|_2 \leqslant \sqrt{2}\|x\|_2.$$

Hence, if we let $\mathcal{H}$ be the closure of $\mathcal{H}_0 \subset L^2R \subset L^2M$, then $p = \mathrm{proj}_{\mathcal{H}}$ verifies condition 1.6, and is thus s-smooth. □

Recall that a subset S of a group Γ is *free* if the subgroup generated by S is the free group with generators in S, $\mathbb{F}_S$.

Proposition 1.8. *With (M, τ) as in 1.5, assume that $L(\Gamma) \subset M$ is a group von Neumann subalgebra. Let $S \subset \Gamma$ be an infinite subset, and denote by p_S the orthogonal projection of L^2M onto the Hilbert space $\mathcal{H}_S$ having orthonormal basis $\{u_g \mid g \in S\}$.*

(1°) *If S is a free subset of Γ, then p_S is s-smooth.*

(2°) *If p_S is s-smooth, then Γ is non-amenable.*

Proof. (1°) By [1, 5], if $S \subset \Gamma$ is a free set, then for any $x \in \ell^2(S)$ we have $\|x\| \leqslant 2\|x\|_2$.

(2°) If Γ is amenable, then by Kesten's characterization of amenability in [19], for any $x = \Sigma_g c_g u_g \in L(\Gamma)$ with $c_g \geqslant 0$ and $\Sigma_g c_g = 1$, one has $\|x\| = 1$. This shows in particular that an element of the form $x_n = n^{-1}\Sigma_{i=1}^n u_{s_i}$, with $s_i \in S$, has norm $\|x_n\| = 1$ while $\|x_n\|_2 = n^{-1/2}$. Letting n tend to infinity shows that p_S cannot be s-smooth. □

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Lemma 1.9. *Let M be a von Neumann algebra, $\mathcal{B}$ an operatorial Banach M-bimodule, and $T \in s^*(\mathcal{B})$ a smooth element.*

(1°) *Given any $\varepsilon > 0$, there exist a normal state φ on M and $\alpha > 0$ such that, if $p_1, \ldots, p_n \in \mathcal{P}(M)$ is a partition of 1 with projections that commute with T and satisfy $\varphi(p_i) \leqslant \alpha$, $\forall i$, then $\|T\| \leqslant \varepsilon$.*

(2°) *If T commutes with a diffuse von Neumann subalgebra $B \subset M$, then $T = 0$.*

Proof. (1°) Since T is smooth, given any $\varepsilon > 0$, there exist a normal state φ on M and $\alpha > 0$ such that, if $p \in \mathcal{P}(M)$ satisfies $\varphi(p) \leqslant \alpha$, then $\|pT\| \leqslant \varepsilon$. Thus, if $p_1, \ldots, p_n \in M$ is a partition of 1 with projections satisfying $\varphi(p_i) \leqslant \alpha$, $\forall i$, by using the fact that $T = (\Sigma_i p_i)T = \Sigma_i p_i T p_i$, it follows that

$$\|T\| = \|\Sigma_i p_i T p_i\| = \max_i \|p_i T p_i\| \leqslant \max_i \|p_i T\| \leqslant \varepsilon.$$

(2°) This is now immediate from part 1°, because B diffuse implies that for any normal state φ on M and any $\alpha > 0$ there exist partitions of 1 in B of φ-mesh less than α. □

2. Derivations into smooth bimodules

Definition 2.1. Let $\mathcal{B}$ be a Banach M-bimodule. A *derivation* of M into $\mathcal{B}$ is a linear map $\delta : M \to \mathcal{B}$ satisfying the property that $\delta(xy) = x\delta(y) + \delta(x)y$, $\forall x, y \in M$. Recall from [40] that any derivation is automatically continuous from M with the operator norm topology to $\mathcal{B}$ with its norm topology. We say that δ is *smooth* if it is continuous from $(M)_1$ with the s^*-topology to $\mathcal{B}$ with its norm topology. Thus, if M is a finite von Neumann algebra with a faithful normal tracial state τ, this amounts to the continuity of δ from $(M)_1$ with the Hilbert norm $\| \ \|_2$ given by τ to $\mathcal{B}$ with its norm.

If $\mathcal{B}$ is a dual M-bimodule and $\delta : M \to \mathcal{B}$ is a derivation, then we say that δ is *weakly continuous* if it is continuous from the unit ball of M with the ultraweak topology (i.e., the $\sigma(M, M_*)$-topology) to $\mathcal{B}$ with its w*-topology (i.e., the $\sigma(\mathcal{B}, \mathcal{B}_*)$-topology). Recall from [40] that, if the dual M-bimodule $\mathcal{B}$ is normal, then any derivation of M into $\mathcal{B}$ is automatically weakly continuous in this sense.

If δ is a derivation of M in a Banach M-bimodule $\mathcal{B}$, then we denote by K_δ^0 the norm closure of the convex hull of $\{\delta(u)u^* \mid u \in \mathcal{U}(M)\}$. If in addition $\mathcal{B}$ is a dual M-bimodule, then we denote by K_δ the $\sigma(\mathcal{B}, \mathcal{B}_*)$-closure in $\mathcal{B}$ of K_δ^0. More generally, if B is a von Neumann subalgebra of M, we denote by $K_{\delta,B}$ the $\sigma(\mathcal{B}, \mathcal{B}_*)$ closure in $\mathcal{B}$ of the convex hull of $\mathrm{co}\{\delta(u)u^* \mid u \in \mathcal{U}(B)\}$. Since $\|\delta(v)v^*\| \leqslant \|\delta\|$, $\forall v \in \mathcal{U}(M)$, it follows that $K_\delta, K_{\delta,B}$ are $\sigma(\mathcal{B}, \mathcal{B}_*)$-compact subsets of the ball of radius $\|\delta\|$ of $\mathcal{B}$.

We will prove in this section that smooth-valued derivations of von Neumann algebras can be 'integrated in abelian directions'.

Proposition 2.2. *Assume that $\delta : M \to \mathcal{B}$ is a smooth derivation of a von Neumann algebra M into a Banach M-bimodule $\mathcal{B}$. Then we have the following.*

(1°) *δ is continuous from the unit ball of M with the ultraweak topology to $\mathcal{B}$ with the $\sigma(\mathcal{B}, \mathcal{B}^*)$ topology (i.e., δ is weakly continuous as a derivation of M into the dual M-bimodule $\mathcal{B}^{**}$).*

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(2°) *If $\mathcal{B}$ is a dual M-bimodule, then δ is weakly continuous.*

(3°) *We have $\delta(M) \subset s^*(\mathcal{B})$. If in addition M is finite and $\mathcal{B}$ is a dual M-bimodule, then K_δ is a uniformly smooth subset of $s^*(\mathcal{B})$.*

Proof. (1°) If $\varphi : \mathcal{B} \to \mathbb{C}$ is a continuous functional, then $(M)_1 \ni x \mapsto \varphi(\delta(x)) \in \mathbb{C}$ is continuous with respect to the s^*-topology, and thus also continuous with respect to the ultraweak topology on $(M)_1$. This shows that δ is weakly continuous as a derivation into $\mathcal{B}^{**}$.

(2°) follows from 1°.

(3°) Recall that, for any $y \in M$, the multiplication maps $x \mapsto xy$, yx are s^*-continuous on the unit ball of M.

Let $\varepsilon > 0$. By the smoothness of δ, there exists an s^*-neighborhood $\mathcal{V}$ of 0 in $(M)_1$ such that if $y \in \mathcal{V}$ then $\|\delta(y)\| \leqslant \varepsilon/2$. Given $x \in (M)_1$, let $\mathcal{V}_0 \subset \mathcal{V}$ be an s^*-neighborhood of 0 in $(M)_1$ such that if $y \in \mathcal{V}_0$ then $xy, yx \in \mathcal{V}$. Thus, since $y\delta(x) = \delta(yx) - \delta(y)x$, if we take $y \in \mathcal{V}_0$, then we have the following estimates for $\delta(x)$:

$$\|y\delta(x)\| \leqslant \|\delta(yx)\| + \|\delta(y)x\| \leqslant \|\delta(yx)\| + \|\delta(y)\| \leqslant \varepsilon,$$

and similarly $\|\delta(x)y\| \leqslant \varepsilon$. Thus, $\delta(x) \in s^*(\mathcal{B})$.

If M is finite, then a basis of neighborhoods of 0 in the s^*-topology is given by $\| \|_2$-neighborhoods with respect to traces on M. Thus, for any $\varepsilon > 0$, there exists a normal trace τ on M and $\alpha > 0$ such that if $y \in (M)_1$, $\|y\|_2 \leqslant \alpha$ then $\|\delta(y)\| \leqslant \varepsilon/2$. Thus, if we take an arbitrary $x \in (M)_1$ and we choose $\alpha_0 > 0$ such that $x \in (M)_1$ $\|x\|_2 \leqslant \alpha_0$ implies $\|\delta(x)\| \leqslant \varepsilon_0/2$, then for any $u \in \mathcal{U}(M)$ we have $\|x\delta(u)u^*\| \leqslant \|\delta(x)\| + \|\delta(xu)\| \leqslant \varepsilon_0$. Thus, if in addition $\mathcal{B}$ is a dual M-bimodule, by the inferior semicontinuity of the norm on $\mathcal{B}$ with respect to the $\sigma(\mathcal{B}, \mathcal{B}_*)$-topology, it follows that $\|xT\| \leqslant \varepsilon_0$, $\forall T \in K_\delta$. Similarly, $\|Tx\| \leqslant \varepsilon_0$, $\forall T \in K_\delta$. Hence, $K_\delta \subset s^*(\mathcal{B})$ and K_δ is uniformly smooth. □

Lemma 2.3. *Let M be a von Neumann algebra, $\mathcal{B}$ a dual M-bimodule, and $\delta : M \to \mathcal{B}$ a derivation. Let $\emptyset \neq K \subset K_\delta$ be a w^*-closed convex subset. For each $T \in \mathcal{B}$ and $u \in \mathcal{U}(M)$ denote $\mathcal{T}_u(T) = uTu^* + \delta(u)u^*$.*

(1°) *$\mathcal{T}_u$ are w^*-continuous affine transformations satisfying $\mathcal{T}_u \circ \mathcal{T}_{u'} = \mathcal{T}_{uu'}$, $\forall u, u' \in \mathcal{U}(M)$, and $\mathcal{T}_u(K_\delta) \subset K_\delta$.*

(2°) *If $u \in \mathcal{U}(M)$, $T \in \mathcal{B}$, then $\delta(u) = Tu - uT$ iff $\mathcal{T}_u(T) = T$.*

(3°) *Let $\mathcal{U}_0 \subset \mathcal{U}(M)$ be an amenable group, and assume that $\mathcal{T}_u(K) = K$, $\forall u \in \mathcal{U}_0$. Then there exists $T \in K$ such that $\delta = \mathrm{ad}T$ on $\mathrm{sp}\mathcal{U}_0$.*

(4°) *If $A \subset M$ is an abelian von Neumann subalgebra such that $\mathcal{T}_u(K) = K$, $\forall u \in \mathcal{U}(A)$, then there exists $T \in K$ such that $\delta = \mathrm{ad}T$ on A.*

(5°) *If $B \subset M$ is a von Neumann algebra with an amenable subgroup $\mathcal{U}_0 \subset \mathcal{U}(B)$ satisfying $\mathcal{U}_0'' = B$, $\mathcal{T}_u(K) = K$, $\forall u \in \mathcal{U}(A)$, and $\mathcal{B}$ is normal relative to B, then there exists $T \in K$ such that $\delta = \mathrm{ad}T$ on B.*

(6°) *If $u \in \mathcal{U}(M)$ normalizes a subalgebra $B \subset M$ and $T \in \mathcal{B}$ satisfies $\mathrm{ad}T = \delta$ on B, then $\mathcal{T}_u(T)$ implements δ on B as well.*

Proof. (1°) By noticing that $\mathcal{T}_{u'}(\delta(u)u^*) = \delta(u'u)(u'u)^*$, then, taking convex combinations of elements of the form $\delta(v)v^*$ and $\sigma(\mathcal{B}, \mathcal{B}_*)$-closure, it follows that $\mathcal{T}_u(K_\delta) \subset K_\delta$ and $T_{u'} \circ T_u = T_{u'u}$, $\forall u, u' \in \mathcal{U}(B)$.

(2°) $\mathcal{T}_u(T) = T$ means $uTu^* + \delta(u)u^* = T$, or equivalently $\delta(u) = Tu - uT$.

(3°) By applying the Markov–Kakutani fixed point theorem to the amenable group of affine transformations $\{\mathcal{T}_u \mid u \in \mathcal{U}_0\}$ of the bounded $\sigma(\mathcal{B}, \mathcal{B}_*)$-compact convex set K_δ, we get some $T \in K_\delta$ with the property that $\mathcal{T}_u(T) = T$, $\forall u \in \mathcal{U}_0$, and 2° applies. If $A \subset M$ is abelian, then by taking $\mathcal{U}_0 = \mathcal{U}(A)$ in 3° one gets 4°. Part 5° follows trivially from 3°, while 6° is straightforward to check. □

Corollary 2.4. *Let $\mathcal{B}$ be a dual M-bimodule and $\delta : M \to \mathcal{B}$ a smooth derivation. If $A \subset M$ is an abelian von Neumann subalgebra then there exists $T \in K_{\delta,A} \subset K_\delta \subset s^*(\mathcal{B})$ such that $\delta(a) = Ta - aT, \forall a \in A$.*

Proof. This is now immediate from 2.2(3°) and 2.3(4°). □

We will now show that smooth-valued derivations are inner on finite type I von Neumann algebras. The proof follows ideas from the proof of the case when the target bimodule is the ideal of compact operators, in [16].

Theorem 2.5. *Let $M = A_0 \oplus B$ with A_0 abelian diffuse and B finite atomic. Let $\mathcal{B}$ be a dual normal operatorial M-bimodule and $\delta : M \to \mathcal{B}$ a derivation. Assume that δ takes values into a Banach sub-bimodule $\mathcal{B}_0 \subset s^*(\mathcal{B})$. Then there exists a unique $T \in K_\delta$ such that $\delta = \mathrm{ad}T$, and this T lies in $\mathcal{B}_0$.*

Proof. Let first $A \subset M$ be an abelian von Neumann subalgebra. By Lemma 2.3, there does exist $T \in K_{\delta,A} \subset \mathcal{B}$ such that $\delta(a) = Ta - aT$, $\forall a \in A$. We show that any $T \in K_{\delta,A}$ that implements δ on A must in fact belong to $\mathcal{B}_0$. Assume that $T \notin \mathcal{B}_0$, in other words that $\|T\|_{\mathcal{B}/\mathcal{B}_0} > 0$.

Note that for any projection $p \in A$ we have $pT - Tp \in \mathcal{B}_0$, and thus $\|T\|_{\mathcal{B}/\mathcal{B}_0} = \|pTp + (1-p)T(1-p)\|_{\mathcal{B}/\mathcal{B}_0}$. Note also that $\mathcal{B}$ operatorial implies that $\mathcal{B}/\mathcal{B}_0$ is operatorial, i.e., $\|pTp + (1-p)T(1-p)\|_{\mathcal{B}/\mathcal{B}_0} = \max\{\|pTp\|_{\mathcal{B}/\mathcal{B}_0}, \|(1-p)T(1-p)\|_{\mathcal{B}/\mathcal{B}_0}\}$, $\forall T \in \mathcal{B}$, $p \in \mathcal{P}(A)$.

Denote by $\mathcal{P}$ the set of all projections $p \in A$ with the property that $\|pTp\|_{\mathcal{B}/\mathcal{B}_0} = \|T\|_{\mathcal{B}/\mathcal{B}_0}$. Assume that $\mathcal{P}$ has a minimal projection e. If e is even minimal in A, i.e., $Ae = \mathbb{C}e$, then for any $u \in \mathcal{U}(A)$ we have

$$e\delta(u)u^*e = \delta(eu)u^*e - \delta(e)e = z\delta(e)\bar{z}e - \delta(e)e = 0,$$

where $z \in \mathbb{T}$ is so that $eu = ze$. This implies that $eT'e = 0$ for any $T' \in K_{\delta,A}$, so in particular $eTe = 0$, contradicting the fact that $e \in \mathcal{P}$.

Thus, e is not minimal in A. But then, any non-zero projection $f \in Ae$ with $f \neq e$ satisfies $f, e - f \notin \mathcal{P}$, so

$$\begin{aligned}\|T\|_{\mathcal{B}/\mathcal{B}_0} &= \|fTf + (e-f)T(e-f)\|_{\mathcal{B}/\mathcal{B}_0} \\ &= \max\{\|fTf\|_{\mathcal{B}/\mathcal{B}_0}, \|(e-f)T(e-f)\|_{\mathcal{B}/\mathcal{B}_0}\} < \|T\|_{\mathcal{B}/\mathcal{B}_0},\end{aligned}$$

a contradiction.

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This shows that $\mathcal{P}$ cannot have minimal projections. Let $\mathcal{F}$ be a maximal chain in $\mathcal{P}$, and denote f_0 the infimum over all $f \in \mathcal{F}$. Since $\mathcal{P}$ has no minimal projections, $f_0 \notin \mathcal{P}$, i.e., $\|f_0 T f_0\|_{\mathcal{B}/\mathcal{B}_0} < \|T\|_{\mathcal{B}/\mathcal{B}_0}$. Since

$$\begin{aligned}\max\{&\|(f-f_0)T(f-f_0)\|_{\mathcal{B}/\mathcal{B}_0}, \|f_0 T f_0\|_{\mathcal{B}/\mathcal{B}_0}\} \\ &= \|(f-f_0)T(f-f_0) + f_0 T f_0\|_{\mathcal{B}/\mathcal{B}_0} = \|fTf\|_{\mathcal{B}/\mathcal{B}_0} = \|T\|_{\mathcal{B}/\mathcal{B}_0},\end{aligned}$$

it follows that $\|(f-f_0)T(f-f_0)\|_{\mathcal{B}/\mathcal{B}_0} = \|T\|_{\mathcal{B}/\mathcal{B}_0}, \forall f \in \mathcal{F}$, or equivalently $\|f'Tf'\|_{\mathcal{B}/\mathcal{B}_0} = \|T\|_{\mathcal{B}/\mathcal{B}_0}$, $\forall f' \in \mathcal{F}'$, where $\mathcal{F}' = \{f - f_0 \mid f \in \mathcal{F}\}$. Thus, $\mathcal{F}'$ is a chain in $\mathcal{P}$ with $\inf\mathcal{F}' = 0$.

Using this fact, we construct recursively a decreasing sequence of projections $f'_k \in \mathcal{F}'$, such that $\|(f'_k - f'_{k+1})T(f'_k - f'_{k+1})\| > c$, $\forall k \geqslant 1$, where $c = (\|T\|_{\mathcal{B}/\mathcal{B}_0} - \|f_0 T f_0\|_{\mathcal{B}/\mathcal{B}_0})/2 > 0$.

Assume that $f'_1, \ldots, f'_n \in \mathcal{F}'$ have been constructed to satisfy this property for all $1 \leqslant k \leqslant n-1$. Since $\mathcal{F}'$ is a chain decreasing to 0 and $\|f'_n T f'_n\| \geqslant 2c$ (because $f'_n \in \mathcal{F}'$), by the normality of the bimodule structure on $\mathcal{B}$ and the inferior semicontinuity with respect to the $\sigma(\mathcal{B}, \mathcal{B}_*)$-topology of the norm on $\mathcal{B}$, it follows that there exists $f' \in \mathcal{F}'$, $f' \leqslant f'_n$, such that $\|(f'_n - f')T f'_n\| > \|f'_n T f'_n\|/2 \geqslant c$. Applying the same reasoning on the right-hand side, it follows that there exists a projection $f'' \in \mathcal{F}'$, such that $f'' \leqslant f'$ and $\|(f'_n - f')T(f'_n - f'')\| > c$. Since $\|(f'_n - f'')T(f'_n - f'')\| = \|f'_n - f'\| \|(f'_n - f'')T(f'_n - f'')\| \geqslant \|(f'_n - f')T(f'_n - f'')\| > c$, the projection $f'_{n+1} = f''$ satisfies the required condition for $k = n$.

We have thus constructed a sequence of mutually orthogonal projections $e_n = f'_n - f'_{n+1} \in A$, $n \geqslant 1$, such that $\|e_n T e_n\| > c$, $\forall n \geqslant 1$. Since $T \in K_{\delta,A}$, it follows that, for each n, there exists $u_n \in \mathcal{U}(A)$, such that $\|e_n \delta(u_n) u_n^* e_n\| > c$. Let now $u \in \mathcal{A}$ be defined by $u = (1 - \Sigma_n e_n) + \Sigma_n u_n e_n$. Then $u \in \mathcal{U}(A)$, $u e_n = u_n$, $\forall n$, and we have

$$e_n \delta(u) u^* e_n = e_n \delta(e_n u) u_n^* e_n - e_n \delta(e_n) e_n = e_n \delta(u_n) u_n^* e_n,$$

where we have used the fact that for any projection $p \in M$ we have $p\delta(p)p = 0$ (because $p\delta(p)p = p\delta(p^2)p = 2p\delta(p)p$). Thus, $\|e_n \delta(u) u^* e_n\| > c$, $\forall n$, contradicting the fact that $\delta(u)u^* \in s^*(\mathcal{B})$.

Thus, the assumption $T \notin \mathcal{B}_0$ led us to a contradiction. This shows that $T \in \mathcal{B}_0$.

Let now $M = A_0 \oplus B$ with A_0 abelian diffuse and B finite atomic. Let $B = \oplus_{k \in J} B_k$, with each B_k a finite-dimensional factor. Denote $\mathcal{V}_k$ a finite subgroup of $\mathcal{U}(B_k)$ that generates the algebra B_k. Also, for each finite set of indices $F \subset J$, denote $\mathcal{V}_F = \oplus_{k \in F} \mathcal{V}_k \oplus \mathbb{C}(1 - s_F)$, where $s_F = \Sigma_{k \in F} s_k$, with $s_k = 1_{B_k}$ the support projection of B_k.

We let $\mathcal{V} = \cup_F \mathcal{V}_F$, the union being over the finite subsets $F \subset J$. Note that $\mathcal{V}$ is a locally finite group (thus amenable) that commutes with $A = A_0 \oplus \mathbb{C}s$ and $\mathcal{V}'' = \mathbb{C} \oplus B = \mathbb{C}(1-s) + Bs$, where $s = \Sigma_k s_k = 1_B$.

Let $K_0 = \{T \in K_{\delta,A} \mid Ta - aT = \delta(a), \forall a \in A\}$. This is clearly a w^*-closed convex subset which by the first part is non-empty and contained in $\mathcal{B}_0$. Further, let K denote the w^*-closure of the convex hull of elements of the form $\mathcal{T}_u(T)$, $T \in K_0, u \in \mathcal{V}$. By 2.3(6°), $\mathrm{ad}T = \delta$ on A, $\forall T \in K$. Note also that $(1-s)K(1-s) = (1-s)K_0(1-s)$ and that any element in $sK(1-s)$ lies in $xK_0(1-s)$ for some $x \in Ms = Bs$. Since $K_0 \subset \mathcal{B}_0$, this shows that $sK(1-s) \subset \mathcal{B}_0$, and thus $K(1-s) \subset \mathcal{B}_0$ as well. Similarly, $(1-s)K \subset \mathcal{B}_0$.

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On the other hand, by 2.3(5°), there exists $T \in K$ such that $\delta = \mathrm{ad}T$ on $\mathrm{sp}\mathcal{V}$. Since $\delta = \mathrm{ad}T$ on A as well, we have $\delta = \mathrm{ad}T$ on $\overline{\mathrm{sp}A\mathcal{V}} = B$. Let us prove that $T \in \mathcal{B}_0$. From the above, we see that it is sufficient to show that $sTs \subset \mathcal{B}_0$.

Assume that $\|sTs\|_{\mathcal{B}/\mathcal{B}_0} > 0$, and consider the set $\mathcal{P}$ of all projections $p \in \mathcal{P}(\mathcal{Z}(Ms))$ with the property that $\|pTp\|_{\mathcal{B}/\mathcal{B}_0} = \|T\|_{\mathcal{B}/\mathcal{B}_0}$. If $e \in \mathcal{P}$ were a minimal projection in $\mathcal{P}$, then $s_k \leqslant e$ for some k. But $s_k T s_k \in \mathcal{B}_0$. Indeed, this is because for each $u \in \mathcal{U}(B)$ we have $s_k\delta(u)u^*s_k = s_k\delta(s_k u)u^*s_k - s_k\delta(s_k)s_k = s_k\delta(s_k u)u^*s_k$ and us_k run over the unitary group $\mathcal{U}(B)s_k = \mathcal{U}(B_k)$, which is compact in the (operator) norm of B. Thus, $s_k K_{\delta,B} s_k = s_k K^0_{\delta,B} s_k \subset \mathcal{B}_0$. But then $\|(e - s_k)T(e - s_k)\|_{\mathcal{B}/\mathcal{B}_0} = \|eTe\|_{\mathcal{B}/\mathcal{B}_0} = \|T\|_{\mathcal{B}/\mathcal{B}_0}$, contradicting the minimality of e.

But if $\mathcal{P}$ does not have a minimal projections, then one can proceed exactly like in the argument in 1° above, to get a sequence of mutually orthogonal projections $e_n \in \mathcal{Z}(B)$ such that $\|e_n T e_n\| > c > 0$, $\forall n$. This implies that for each n there exists a unitary element $u_n \in \mathcal{U}(B)$ such that $\|e_n\delta(u_n)u_n^*e_n\| > c$. The unitary element $u = \Sigma_n u_n e_n + (1 - \Sigma e_n) \in M$ will then satisfy $e_n\delta(u)u^*e_n = e_n\delta(u_n)u_n^*e_n$, so that on the one hand we have $\|e_n\delta(u)u^*e_n\| = \|e_n\delta(u_n)u_n^*e_n\| \geqslant c$, $\forall n$, but on the other hand, by the smoothness of $\delta(u)u^* \in \mathcal{B}_0$, we have $\lim_n \|e_n\delta(u)u^*e_n\| = 0$, a contradiction. This ends the proof of the fact that sTs (and thus T) lies in $\mathcal{B}_0$.

To prove the uniqueness of $T \in K_{\delta,B}$ implementing δ on B, let $T' \in K_\delta \cap \mathcal{B}_0$ be another element satisfying $\mathrm{ad}T' = \delta$. Then $[T - T', M] = 0$, so in particular $(T - T')s_k = s_k(T - T')$, $\forall k$. Since $M(1 - s) = A_0(1 - s)$ is diffuse, by Lemma 1.9 we have $T(1 - s) = T'(1 - s)$, $(1 - s)T = (1 - s)T'$, and we only need to prove that $s_k T s_k = s_k T' s_k$, $\forall k$, as well.

To do this, note that $s_k T s_k$ is in the norm closure of the convex hull of the set $\{s_k\delta(v)v^*s_k \mid v \in \mathcal{V}_k\}$. But if for each $v \in \mathcal{V}_k$ we denote by $\mathcal{T}_v$ the transformation of $K_k = s_k K_{\delta,B} s_k$ given by $K_k \ni \xi \mapsto \mathcal{T}_v(\xi) = v\xi v^* + s_k\delta(v)v^*s_k$, then for each $\xi \in K_k$ the map $K_k \ni \xi \mapsto \pi(\xi) = \int_{\mathcal{V}_k} \mathcal{T}_v(\xi)\, d\mu(v)$, where μ is the Haar measure on $\mathcal{V}_k$, satisfies $\pi(T_{v'}(\xi)) = \pi(\xi)$, for any $v' \in \mathcal{V}_k$, because of the invariance of the Haar measure. By taking convex combinations and norm closure, this shows that π has a single point range. Since the operator T implements δ, we also have $\mathcal{T}_v(s_k T s_k) = s_k T s_k$, so $\pi(s_k T s_k) = s_k T s_k$, thus showing that any $T \in K_\delta$ that implements δ coincides with the single point $\pi(K)$ under s_k.

This finishes the proof that there exists a unique $T \in K_\delta$ implementing δ on M, and that this T lies in $\mathcal{B}_0$. □

Theorem 2.6. *Let $M = A_0 \oplus B$ with A_0 abelian diffuse and B finite atomic. Let $\mathcal{B}$ be a dual operatorial M-bimodule and $\delta : M \to \mathcal{B}$ a smooth derivation. Assume that δ takes values into a Banach sub-bimodule $\mathcal{B}_0 \subset s^*(\mathcal{B})$. Then there exists a unique $T \in K_\delta$ such that $\delta = \mathrm{ad}T$, and this T lies in $\mathcal{B}_0$.*

Proof. Note first that there are only two places in the proof of 2.5 where we used the fact that the dual operatorial bimodule $\mathcal{B}$ is normal: (a) to deduce in the first part (abelian case) of the proof that if a net $p_i \in \mathcal{P}(A)$ increases to some projection $p \in A$, then $\sup_i \|p_i T\| = \|pT\|$, $\sup_i \|Tp_i\| = \|Tp\|$, where $T \in \mathcal{B}$ is an element in K_δ that implements δ on the abelian von Neumann subalgebra A; and (b) to deduce in the second part that,

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since δ and ad$T$ are weakly continuous, if they coincide on the *-subalgebra sp$\mathcal{V}A$, which is weakly dense in M, then they must coincide on all M.

But, by 2.2(3°), if $\mathcal{B}$ is merely a dual operatorial M-bimodule (not necessarily normal), but δ is assumed smooth, then K_δ follows a uniformly smooth subset of $s^*(\mathcal{B})$. Thus, if we define T as in the first part of the proof of 2.5, then T lies in K_δ, so it will be smooth. In particular it will satisfy property (a) above. With this in hand, the second part of the proof of 2.5 applies unchanged by using the fact that T smooth implies $\delta' = \delta - \mathrm{ad}T$ smooth, so, if $\delta' = 0$ on a dense *-subalgebra sp$\mathcal{V}A$ of M, then $\delta' = 0$ on all M. □

3. Vanishing 1-cohomology results

We show in this section that, once a smooth-valued derivation δ is implemented by a smooth element on a diffuse abelian von Neumann subalgebra satisfying some very weak regularity properties, then that element automatically implements δ on the entire von Neumann algebra.

We will also prove that smooth-valued derivations are automatically smooth, i.e., if $\delta : M \to s^*(\mathcal{B}) \subset \mathcal{B}$, then δ is continuous from $(M)_1$ with the s^*-topology to $\mathcal{B}$ with its norm topology.

Note that these results can be formulated in terms of properties of the 1-cohomology of M with coefficients in the smooth part of a bimodule $\mathcal{B}$. Thus, if M is a von Neumann algebra and $\mathcal{B}$ is a Banach M-bimodule, then we denote by $Z^1(M, \mathcal{B})$ (respectively, $Z^1_s(M, \mathcal{B})$) the space of derivations (respectively, smooth derivations) of M into $\mathcal{B}$, by $B^1(M, \mathcal{B})$ (respectively, $B^1_s(M, \mathcal{B})$) the subspace of inner derivations (respectively, derivations implemented by smooth elements of $\mathcal{B}$) and by $H^1(M, \mathcal{B}) = Z^1(M, \mathcal{B})/B^1(M, \mathcal{B})$ (respectively, $H^1_s(M, \mathcal{B}) = Z^1_s(M, \mathcal{B})/B^1_s(M, \mathcal{B})$) the corresponding quotient space.

With this terminology, our results show that, if $\mathcal{B}$ is a dual normal operatorial M-bimodule, then $Z^1(M, s^*(\mathcal{B})) = Z^1_s(M, \mathcal{B})$, $H^1(M, s^*(\mathcal{B})) = H^1_s(M, \mathcal{B})$ (M arbitrary), and that if $\mathcal{B}_0 \subset s^*(\mathcal{B})$, then $H^1(M, \mathcal{B}_0) = 0$ whenever M contains a diffuse abelian subalgebra that satisfies the weak regularity property. We will also prove that, under this same regularity condition for M, if $\mathcal{B}$ is an arbitrary smooth operatorial Banach M-bimodule (not necessarily dual normal), then $H^1_s(M, \mathcal{B}) = 0$, i.e., any smooth derivation of M into $\mathcal{B}$ is inner.

The weak regularity property that we consider is in the spirit of a similar concept considered for groups in (Definition 2.3 of [32]; see also 1.2 in [14] for a related concept).

Definition 3.1. Let B be a diffuse von Neumann subalgebra of a II_1 factor M. Consider the well ordered family of intermediate von Neumann subalgebras $B = B_0 \subset B_1 \subset \cdots \subset B_\jmath \subset \cdots$ of M constructed recursively, by transfinite induction, in the following way: (a) for each $\jmath$, $B_{\jmath+1}$ is the von Neumann algebra generated by $v \in \mathcal{U}(M)$ with $vB_\jmath v^* \cap B_\jmath$ diffuse; (b) if $\jmath$ has no 'predecessor', then $B_\jmath = \overline{\cup_{n<\jmath} B_n}^w$. Notice that, if $\imath_0$ is the first ordinal of cardinality $|M|$, then this family is constant from $\imath_0$ on. Let $\imath$ be the first ordinal for which $B_{\imath+1} = B_\imath$. The algebra $B_\imath$ is by definition the *wq-normalizer algebra* of B in M. If the wq-normalizer of B in M is equal to M, then we say that B is *wq-regular* in M. If the wq-normalizer of B in M is equal to B, we also say that B is *wq-malnormal* in M.

With this terminology, we see that the wq-normalizer of B in M is wq-malnormal in M. The converse is in fact also true, so, similarly to the group case in (2.3 of [32]), one can characterize the wq-normalizer of B in M as follows.

Lemma 3.2. *Let $B \subset M$ be a diffuse von Neumann subalgebra and N its wq-normalizer. If an intermediate von Neumann algebra $B \subset Q \subset M$ satisfies the property that any $v \in \mathcal{U}(M)$ with $vQv^* \cap Q$ diffuse must lie in Q, then Q contains N. Thus, the wq-normalizer of B in M is the smallest von Neumann subalgebra $N \subset M$ containing B with the property that there exist no $v \in \mathcal{U}(M) \setminus N$ with $vNv^* \cap N$ diffuse, or equivalently, the smallest wq-malnormal subalgebra of M that contains B.*

Proof. Let $B_0 \subset \cdots \subset B_\imath = N$ be constructed by transfinite induction as in 3.1, with $B_\imath$ the first ordinal having the property that $B_{\imath+1} = B_\imath$. Assume that we have shown that $B_\jmath \subset Q$ for some $0 \leqslant \jmath < \imath$. If $v \in \mathcal{U}(M)$ satisfies $vB_\jmath v^* \cap B_\jmath$ diffuse, then $v \in B_{\jmath+1}$ by the definition of $B_{\jmath+1}$. But, since Q contains $B_\jmath$ we also have $vQv^* \cap Q$ diffuse, so $v \in Q$. This shows that $B_{\jmath+1} \subset Q$. Also, if $\jmath \leqslant \imath$ has no predecessor and $B_n \subset Q$, $\forall n < \jmath$, then the weak closure of $\cup_{n<\jmath} B_n$ is contained in Q. Altogether, these facts imply that $B_\imath \subset Q$. □

Lemma 3.3. *Let M be a von Neumann algebra, $\mathcal{B}$ a dual normal Banach M-bimodule, and $\delta : M \to \mathcal{B}$ a derivation. If δ vanishes on two von Neumann subalgebras $N_1, N_2 \subset M$, then it vanishes on the von Neumann algebra $N_1 \vee N_2$, generated by N_1, N_2. Thus,* $\ker\delta \cap (\ker\delta)^*$ *is the maximal von Neumann subalgebra of M on which δ vanishes, and we will denote it M_δ.*

Proof. Since, if δ is equal to 0 on $x, y \in M$ it is also 0 on the product xy, it follows that δ vanishes on the algebra generated by N_1, N_2, which is a *-algebra. By the weak continuity of δ, it follows that δ also vanishes on $N_1 \vee N_2$. □

Lemma 3.4. *With $M, \mathcal{B}, \delta$ as in 3.3, assume that $\delta(M) \subset s^*(\mathcal{B})$. Let $v \in \mathcal{U}(M)$.*

- (1°) *If $vM_\delta v^* \cap M_\delta$ is diffuse, then $v \in M_\delta$. Thus, M_δ is wq-malnormal in M.*
- (2°) *Assume that M is of type II_1 with a faithful normal trace τ. For any $\varepsilon > 0$ there exists $\alpha > 0$ such that, if $vA_0v^* \subset M_\delta$ for some α-partition $A_0 \subset M_\delta$, then $\|\delta(v)\| \leqslant \varepsilon$. Also, if $x = x^* \in M$ is such that $\{x\}' \cap M_\delta^\omega$ is diffuse, then $x \in M_\delta$.*

Proof. (1°) This is immediate from Lemma 1.9(2°), once we notice that, if $x \in M_\delta$ satisfies $v^*xv \in M_\delta$, then $T = \delta(v)v^*$ commutes with x.

(2°) The first part follows from 1.9(1°), and it implies trivially the last part. □

Theorem 3.5. *Let M be a von Neumann algebra, $\mathcal{B}$ a dual normal operatorial Banach M-bimodule, and $\delta : M \to \mathcal{B}$ a derivation that takes values in a Banach sub-bimodule $\mathcal{B}_0 \subset s^*(\mathcal{B})$, of the smooth part of $\mathcal{B}$.*

- (1°) *δ is automatically smooth. If in addition M is finite, then K_δ is a uniformly smooth subset of $s^*(\mathcal{B})$. More generally, if $T_0 \in s^*(\mathcal{B})$ then $K_\delta(T_0) \stackrel{\text{def}}{=} \overline{\text{co}}^{w^*}\{\mathcal{T}_u(T_0) \mid u \in \mathcal{U}(M)\}$ is a uniformly smooth subset of $s^*(\mathcal{B})$.*

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(2°) *Assume that* $M = Q_0 \oplus Q_1$ *with* Q_1 *of type* II_1 *with atomic center and* Q_0 *having no* II_1 *factor as direct summand. If* $A_1 \subset Q_1$ *is a diffuse abelian von Neumann subalgebra and we denote* $Q = Q_0 \oplus A_1$, *then there exists a unique* $T \in K_\delta \cap \mathcal{B}_0$ *such that* $\mathrm{ad}T = \delta$ *on* Q.

Proof. (1°) Note first that, by Theorem 2.5, δ is implemented by an element in $\mathcal{B}_0 \subset s^*(\mathcal{B})$ on any given abelian von Neumann subalgebra of M. Since $\mathrm{ad}T$ is smooth and takes values in $\mathcal{B}_0$ for any $T \in \mathcal{B}_0$, by subtracting such an 'inner' derivation from δ, it follows that, given any abelian von Neumann subalgebra $A \subset M$, we may assume that $\delta = 0$ on A.

If $M = M_0 \oplus M_1$ with M_0 properly infinite and M_1 finite, then $M_0 \simeq N_0 \overline{\otimes} \mathcal{B}(\ell^2\mathbb{N})$. From the above, it follows that we may assume that $\delta = 0$ on a diffuse MASA (maximal abelian *-subalgebra) $A_0 \subset 1 \otimes \mathcal{B}(\ell^2\mathbb{N})$. But such a MASA is regular in $\mathcal{B}(\ell^2\mathbb{N})$ and in fact, since its commutant contains N_0, it is regular in M_0 as well. So, by 3.4(1°), $\delta = 0$ on all M_0.

We are thus reduced to proving that δ is smooth on M_1, i.e., we may assume that M is a finite von Neumann algebra. By 2.5, by subtracting if necessary an inner derivation implemented by a smooth element, we may also assume that δ vanishes on $\mathcal{Z}(M)$. Let $\{p_i\}_i \subset \mathcal{P}(\mathcal{Z}(M))$ be a net of central projections increasing to 1 such that Mp_i has a faithful normal trace. We already know that for any sequence of mutually orthogonal projections $e_n \in Mp_i$ we have $\lim_n \|\delta(e_n)\| = 0$ (this is because δ is implemented by elements in $s^*(\mathcal{B})$ on any abelian von Neumann subalgebra of M, in particular on the von Neumann algebra generated by $\{e_n\}_n$). By Lemma A.1 in the Appendix, it follows that δ is smooth on each Mp_i. Note now that $\lim_i \|\delta(x(1-p_i))\| = 0$ uniformly in $x \in (M)_1$. Indeed, for, if there exists $c > 0$ such that, for all i, there exist $j > i$ and $x_j \in (M)_1$ with $\|\delta(x_j(1-p_j))\| \geqslant c > 0$, then we can find an increasing sequence of indices $i_1 < i_2 \ldots$ in I such that $\|\delta(x_{i_n}(p_{i_n} - p_{i_{n+1}}))\| \geqslant c/2$. But then $x = \Sigma_n x_{i_n}(p_{i_n} - p_{i_{n+1}}) \in M$, and the mutually orthogonal projections $f_n = p_{i_{n+1}} - p_{i_n}$ satisfy $\|f_n\delta(x)\| \geqslant c/2$, $\forall n$, contradicting the smoothness of $\delta(x)$.

Since δ is smooth, if M is finite, then 2.2(3°) applies to get that K_δ is a uniformly smooth subset of $\mathcal{B}_0$. Since $T_0 \in s^*(\mathcal{B})$ implies that $K_0 = \mathrm{co}\{uT_0u^* \mid u \in \mathcal{U}(M)\}$ is uniformly smooth, the last part of 1° follows as well.

(2°) Let $Q_0 = Q_0^0 \oplus Q_0^1 \oplus Q_0^2$ with Q_0^0 finite atomic, Q_0^1 properly infinite, and Q_0^2 of type II_1 with diffuse center. Let $B = Q_0^0 \oplus A_0^1 \oplus A_0^2 \oplus A_1 \subset Q$, where A_0^1 is a diffuse maximal abelian subalgebra of a copy of $\mathcal{B}(\ell^2\mathbb{N})$ that splits off Q_0^1 and A_0^2 is the center of Q_0^2. By Theorem 2.5, there exists a unique $T \in K_{\delta,B}$ such that $\mathrm{ad}T = \delta$ on B, and this T belongs to $\mathcal{B}_0$. Thus, $\delta' = \delta - \mathrm{ad}T$ takes M into $\mathcal{B}_0$ and vanishes on B. This implies that if $u \in \mathcal{U}(M)$ normalizes B then $\delta'(u)u^* \in \mathcal{B}_0$ commutes with B. So if $s = 1_M - 1_{Q_0^0}$ then $\delta'(u)u^*, u^*\delta'(u)$ commute with the diffuse algebra Bs. By Lemma 1.9, this implies that $\delta'(u)s = 0$. By taking the linear span of such u, then weak closure, and using that B is regular in Q, this shows that $\delta(x)s = [T, x]s$, $s\delta(x) = s[T, x]$, $\forall x \in Q$. But, since $M(1-s) = Q_0^0(1-s) \subset B$, for $x \in Q$, we also have $(1-s)\delta(x)(1-s) = (1-s)\delta(x(1-s)) - (1-s)x\delta(1-s) = (1-s)\mathrm{ad}T(x(1-s)) - (1-s)x\mathrm{ad}T(1-s) = (1-s)\mathrm{ad}T(x)(1-s)$. Altogether, this shows that $\delta = \mathrm{ad}T$ on all Q.

The above argument also shows that any $T' \in \mathcal{B}_0$ that implements δ on Q must satisfy $Ts = T's$, $sT = sT'$, while, by the last part of the proof of 2.5, if $T' \in K_\delta$

implements δ on $M(1-s) = Q_0^0$, then $(1-s)T'(1-s) = (1-s)T(1-s)$. This proves the uniqueness of T. □

Definition 3.6. Let M be a II_1 factor and $N \subset M$ a von Neumann subalgebra. We say that N has *property* (C') in M (or that $N \subset M$ satisfies (C')) if the following holds true: for any finite set $F \subset N$ and any $\varepsilon > 0$, there exist $V = V^* \subset \mathcal{U}(M)$ finite with the following properties:

(a) $\|x - E_{V''}(x)\|_2 \leqslant \varepsilon$, $\forall x \in F$;

(b) there exist diffuse abelian von Neumann subalgebras $A_v \subset \{v\}' \cap M^\omega$, $v \in V$, that mutually commute, i.e., $aa' = a'a$, $\forall a \in A_v, a' \in A_{v'}$, $v, v' \in V$.

If $M \subset M$ has (C'), we simply say that M has property (C'). Note that this property for M is weaker than property (C) of M, as defined in [29].

It is trivial by the definition that any abelian von Neumann subalgebra of M has property (C') in M. More generally, it is easy to see that any amenable von Neumann subalgebra $N \subset M$ has property (C') in M. It has been shown in [29] that, if M either has a Cartan subalgebra or has property (Γ) of [21], then M has property (C), and thus (C') as well. Finally, note that property (C') is hereditary: if $N_0 \subset N \subset M$ and $N \subset M$ has (C'), then so does $N_0 \subset M$.

Theorem 3.7. *Let M be a von Neumann algebra such that any* II_1 *factor summand of M has a diffuse, wq-normal subalgebra with property* (C'). *Let $\mathcal{B}$ be a dual normal operatorial M-bimodule, $\mathcal{B}_0 \subset s^*(\mathcal{B})$ a Banach sub-bimodule, and $\delta : M \to \mathcal{B}_0$ a derivation. Then there exists a unique $T \in K_\delta \cap \mathcal{B}_0$ such that $\delta(x) = \mathrm{ad}T(x)$, $\forall x \in M$, (so $\|T\| \leqslant \|\delta\|$ as well). If in addition M is finite, then T is the only element in K_δ that implements δ on M.*

Proof. We first prove the statement in the case when M is a II_1 factor with a wq-normal diffuse von Neumann subalgebra $N \subset M$ with property (C'). More precisely, we show that, given any $T_0 \in K_\delta \cap \mathcal{B}_0$, there exists $T \in K_\delta(T_0) \cap \mathcal{B}_0$ such that $\delta = \mathrm{ad}T$.

To this end, we first show that, for any finite set $F = F^* \subset N$, there exists $T = T(F) \in K_\delta(T_0)$ such that $\delta(x) = \mathrm{ad}T(x)$, $\forall x \in F$, and thus for all x in von Neumann algebra $M_F \subset M$ generated by F.

By property (C') of $N \subset M$, for any given $\varepsilon > 0$, there exists a finite set $V = V(\varepsilon) \subset \mathcal{U}(M)$ satisfying $V = V^*$, $x \in_\varepsilon V''$, $\forall x \in F$, and such that for any $\alpha > 0$ there exist mutually commuting finite-dimensional abelian subalgebras $A_v \subset M$ with all atoms of trace at most α, such that $\|v - E_{A_v' \cap M}(v)\|_2 \leqslant \alpha$, $\forall v \in V$. Let $A = \vee_v A_v$ be the von Neumann algebra generated by A_v, $v \in V$, and let $S = S(V, \alpha) \in K_{\delta, A}(T_0)$ be an element implementing δ on A (see 2.3(4°)). By 3.5(1°), we have $\|\delta(v - E_{A_v' \cap M}(v))\| \leqslant \alpha'$, with $\alpha' \to 0$ as $\alpha \to 0$. Since $\delta - \mathrm{ad}S$ is A-bimodular, if we denote by $\{e_k\}_k$ the minimal projections of A, it follows that $(\delta - \mathrm{ad}(S))(E_{A' \cap M}(v)) = \Sigma_k e_k(\delta(v) - [S, v])e_k$. Since $\delta(v)$ is fixed and the set $\{[S, v] \mid S \in K_\delta(T_0)\}$ is uniformly smooth (by last part of 3.5(1°)), if we take α sufficiently small then by 1.9(1°) we get $\|(\delta - \mathrm{ad}(S))(E_{A_v' \cap M}(v))\| \leqslant \alpha'$, $\forall v \in V$. Altogether, we get the estimates

$$\|(\delta - \mathrm{ad}S)(v)\| \leqslant \|\delta(v - E_{A_v' \cap M}(v))\| + \|(\delta - \mathrm{ad}(S))(E_{A_v' \cap M}(v))\| \leqslant 2\alpha', \forall v \in V.$$

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Taking α_n so that $\alpha' \leqslant 2^{-n-1}$, and denoting $S_n = S(V, \alpha_n)$, we have thus constructed a sequence $S_n \in K_\delta(T_0)$ with the property that $\|(\delta - \mathrm{ad}S_n)(v)\| \leqslant 2^{-n}$, $\forall v \in V$. Thus, if we let S' be a weak limit point of $\{S_n\}_n$, then $S' \in K_\delta(T_0)$ and $\delta(v) = \mathrm{ad}(S')$, $\forall v \in V$. Since δ and $\mathrm{ad}S'$ are weakly continuous, this shows that $\delta = \mathrm{ad}S'$ on the whole von Neumann algebra generated by V.

Take now $\varepsilon = 2^{-n}$, and let $V_n \subset \mathcal{U}(M)$ be a finite set satisfying $\|x - E_{V_n''}(x)\|_2 \leqslant 2^{-n}$, $\forall x \in F$, and property 3.6(b). Denote by T_n the corresponding element $S'(V_n, 2^{-n}) \in K_\delta(T_0)$ with the property that δ coincides with $\mathrm{ad}T_n$ on the von Neumann subalgebra $Q_n = V_n''$ satisfying $\|x - E_{Q_n}(x)\|_2 \leqslant 2^{-n}$, $\forall x \in F$. Let now T be a weak limit point of $\{T_n\}_n$. Since $T_n \in K_\delta(T_0)$ and $K_\delta(T_0)$ is uniformly smooth, $\delta - \mathrm{ad}T_n$ are uniformly smooth, and thus

$$\|(\delta - \mathrm{ad}T_n)(x)\| = \|(\delta - \mathrm{ad}T_n)(x - E_{Q_n}(x))\| \to 0, \forall x \in F,$$

implying that $\delta = \mathrm{ad}T$ on $F = F^*$. Hence, $\delta = \mathrm{ad}T$ on the whole von Neumann algebra generated by F.

With this at hand, we finally take a weak limit point of the net $\{T(F)\}_F$, indexed by the finite subsets $F = F^* \subset N$, to obtain an element $T \in K_\delta(T_0)$ that implements δ on all N. By 3.5(1°), $K_\delta(T_0) \subset \mathcal{B}_0$, so $T \in \mathcal{B}_0$ as well. By Lemma 3.4, it follows that T actually implements δ on the wq-normalizer of N in M, and thus on M.

Note that the above argument combined with the fact that $K_\delta(T_0) \subset \mathcal{B}_0$ for finite von Neumann algebras (see 3.5(1°)) also shows that, if $M_0 = \oplus_i M(i)$ with each $M(i)$ a II_1 factor with a diffuse wq-normal von Neumann subalgebra $N_i \subset M(i)$ having property (C') in $M(i)$, then there exists $T \in K_\delta(T_0) \subset \mathcal{B}_0$ such that $\delta = \mathrm{ad}T$.

Let now M be an arbitrary von Neumann algebra satisfying the hypothesis in 3.7. Thus, $M = Q_0 \oplus Q_1$ with Q_0 having no II_1 factor summand and Q_1 of type II_1 with atomic center such that any of its II_1 factor summands has a diffuse wq-regular von Neumann subalgebra with property (C'). By 3.5(2°), there exists $T_0 \in K_\delta \cap \mathcal{B}_0$ such that $\delta = \mathrm{ad}T_0$ on $Q_0 \oplus \mathbb{C}$. Then apply the above to $M_0 = \mathbb{C}1_{Q_0} \oplus Q_1$ and T_0 to get $T \in K_{\delta, M_0}(T_0) \cap \mathcal{B}_0$ such that $\delta = \mathrm{ad}T$ on M_0. Since any $u \in \mathcal{U}(M_0)$ normalizes $Q_0 \oplus \mathbb{C}1_{Q_1}$, $\mathcal{T}_u(T_0)$ implements δ on $Q_0 \oplus \mathbb{C}1_{Q_1}$ as well; thus any $T' \in K_{\delta, M_0}(T_0)$ implements δ on $Q_0 \oplus \mathbb{C}1_{Q_1}$. Thus T implements δ on both $\mathbb{C}1_{Q_0} \oplus Q_1$ and $Q_0 \oplus \mathbb{C}1_{Q_1}$, and therefore on all M.

The uniqueness of $T \in K_\delta \cap \mathcal{B}_0$ implementing δ on M follows by the same argument as in the proof of the uniqueness in 3.5(2°). □

Theorem 3.8. *Let M be a von Neumann algebra such that any II_1 factor summand of M has a diffuse, wq-normal subalgebra with property (C'). Let $\mathcal{B}_0$ be a smooth operatorial Banach M-bimodule and $\delta : M \to \mathcal{B}_0$ a smooth derivation. Then there exists a unique $T \in K_\delta \cap \mathcal{B}_0$ such that $\delta(x) = \mathrm{ad}T(x)$, $\forall x \in M$, (so $\|T\| \leqslant \|\delta\|$ as well). If in addition M is finite, then T is the only element in K_δ that implements δ on M.*

Proof. We can view δ as a (smooth) derivation of M into the bidual $\mathcal{B} = \mathcal{B}_0^{**}$ of $\mathcal{B}_0$, taking values in the closed sub-bimodule $\mathcal{B}_0$ of $s^*(\mathcal{B})$. Note that, by 1.2(4°), $\mathcal{B}$ is a dual operatorial M-bimodule. By Proposition 2.2, since δ is smooth, $\delta : M \to \mathcal{B}$ is weakly continuous.

Let $A \subset N$ be an abelian von Neumann subalgebra. Thus, by Corollary 2.4, δ is implemented on A by some $T_0 \in K_\delta \subset \mathcal{B}$, where K_δ is the $\sigma(\mathcal{B}, \mathcal{B}_*)$ closure in $\mathcal{B} = \mathcal{B}_0^{**}$ of

$K_\delta^0 \subset \mathcal{B}_0$ (where $\mathcal{B}_* = \mathcal{B}_0^*$). By the smoothness of δ, T_0 is smooth in $\mathcal{B}$ (i.e., it belongs to $s^*(\mathcal{B})$), so, by Theorem 2.6, we actually have $T_0 \in \mathcal{B}_0$.

From this point on, the argument in the proof of 3.7 goes unchanged, to first deduce that there exists $T \in K_\delta \cap \mathcal{B}_0$ such that $\delta = \operatorname{ad} T$, and then show the uniqueness of T as well. □

Remark 3.9. As we mentioned at the beginning of this section, Theorems 3.7, 3.8 can be viewed as vanishing 1-cohomology results, showing that for II_1 factors M the 1-cohomology with values in a closed submodule $\mathcal{B}_0$ of the smooth part of a dual normal operatorial M-bimodule $\mathcal{B}$, $H^1(M, \mathcal{B}_0)$ (respectively, the smooth first cohomology $H_s^1(M, \mathcal{B})$) vanishes as soon as M satisfies some rather weak decomposability properties. In particular, this is the case if M has a Cartan subalgebra, and more generally if M has a diffuse amenable subalgebra whose quasi-normalizer generates M. This holds as well for non-prime factors M (i.e., factors that can be decomposed as a tensor product of two II_1 factors) and for factors having property (Γ) of Murray and von Neumann [21].

The class of factors covered by Theorems 3.7, 3.8 contains many group factors $L(\Gamma)$, with Γ infinite conjugacy class (ICC) groups. For instance, all wreath product groups $\Gamma = H \wr G$, with H non-trivial, are in this class. Indeed, if H is finite, then $L(\Gamma)$ has $L(H^{(\Gamma)})$ as an amenable diffuse regular von Neumann subalgebra. If $|H| = \infty$, then $L(H)$ contains a diffuse abelian von Neumann subalgebra A, and any such algebra is wq-regular in $L(\Gamma)$. Another class of group factors covered by 3.7 is $L(\Gamma_n)$ for $\Gamma_n = PSL(n, \mathbb{Z})$, $n \geqslant 3$. This is because any such Γ_n contains a chain of infinite abelian subgroups $H_1, \ldots, H_m$ that generate Γ_n and are such that H_i commutes with H_{i+1} for all i. Thus, $L(\Gamma_n)$ has diffuse abelian von Neumann subalgebras (e.g., $L(H_1)$) that are wq-regular in $L(\Gamma_n)$. It is interesting to note that in all these cases the L^2-cohomology of the group Γ vanishes as well (see [12]; see also [23]).

Note that $PSL(2, \mathbb{Z})$ and the free groups $\mathbb{F}_t$, $2 \leqslant t \leqslant \infty$, do not have this property. In fact, by results in [10], the free group factors $L(\mathbb{F}_t)$ do not have property (C) of [29]. The same argument in [10] can probably be used to show that they do not satisfy the weaker condition (C') either. But, despite the many in-decomposability properties, one knows to prove for the free group factors (primeness, solidity, absence of diffuse amenable subalgebras with non-amenable normalizing algebra, etc.) the fact that $M = L(\mathbb{F}_t)$ does not have any wq-regular diffuse abelian von Neumann subalgebra is still an open problem. If one could find some smooth $L(\mathbb{F}_t)$-bimodule $\mathcal{B}$ for which $H^1(L(\mathbb{F}_t), \mathcal{B}) \neq 0$, then, by 3.7, 3.8 it would follow that $L(\mathbb{F}_t)$ does not even have any diffuse wq-regular subalgebra with property (C').

4. The case $\mathcal{B} = \mathcal{B}(\mathcal{H})$

We will prove in this section that, if the target M-bimodule is a closed subspace $\mathcal{B}_0$ of the smooth part of the algebra $\mathcal{B}(\mathcal{H})$ of linear bounded operators on the Hilbert space $\mathcal{H}$ on which the von Neumann algebra M acts, then any derivation with values in $\mathcal{B}_0$ is implemented by an element in $\mathcal{B}_0$. In the particular case $\mathcal{B}_0 = \mathcal{K}(\mathcal{H})$, this result

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amounts to Theorem II in [30]. However, as we have seen in § 1, there is a large fauna of non-compact operators in $\mathcal{B}(\mathcal{H})$ that are smooth over M, even when M is diffuse abelian.

One should mention that, while the proof of this result follows closely the ideas and line of proof in [30], the arguments in [30] do not simply extend, as such, to this larger degree of generality, and we will need to resolve additional technical difficulties, notably in the proof of the analog of Lemma 1.1 in [30]. We overcome this through a careful usage of the incremental patching techniques in [31, 33, 34].

Theorem 4.1. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a normal representation of a von Neumann algebra M on a Hilbert space $\mathcal{H}$. Let $\mathcal{B}_0 \subset s^*(\mathcal{B}(\mathcal{H}))$ be a Banach M sub-bimodule and $\delta : M \to \mathcal{B}_0$ a derivation. Then there exists $T \in K_\delta \cap \mathcal{B}_0$ such that $\delta = \operatorname{ad} T$.*

Proof. By Theorem 3.5, we only need to prove 4.1 in the case when M is of type II_1 with atomic center and δ vanishing on $\mathcal{Z}(M)$. This means that we are actually reduced to proving the case when M is a II_1 factor. By writing M as an increasing union of a net of separable II_1 factors (see, e.g., the proof of A.1.2 in [35]), and taking into account that, if a derivation $\delta : M \to \mathcal{B}_0 \subset s^*(\mathcal{B}(\mathcal{H}))$ is implemented by elements $T_1, T_2 \in \mathcal{B}_0$ on subfactors $N_1 \subset N_2 \subset M$, then T_1, T_2 must coincide (by 1.9(2°)), it follows that it is sufficient to prove the statement in the case when M is a separable II_1 factor.

By [28], M contains a hyperfinite II_1 subfactor $R \subset M$ with trivial relative commutant, $R' \cap M = \mathbb{C}1$. By Theorem 3.7, we may assume that δ vanishes on R. We want to prove that δ must then vanish on all M. We will do this by contradiction. If $\delta \neq 0$, then there exists $v \in \mathcal{U}(M)$ such that $\delta(v) \neq 0$. Thus, there exists $\xi_0 \in \mathcal{H}$ such that the orthogonal projection $p' \in M' \cap \mathcal{B}(\mathcal{H})$ of $\mathcal{H}$ onto $\overline{M\xi_0}$ satisfies $p'\delta(v)p' \neq 0$. Since we also have $p's^*(\mathcal{B}(\mathcal{H}))p' \subset s^*(\mathcal{B}(p'(\mathcal{H})))$ (by 1.3(3°)), this shows that it is sufficient to derive the contradiction in the case when $\mathcal{H} = L^2M$, with $M \subset \mathcal{B}(L^2M)$ the standard representation of M. Since $s^*(\mathcal{B}(\mathcal{H}))$ is a M'-bimodule as well, by replacing if necessary δ by a derivation $x \mapsto x_1'\delta(x)x_2'$ for some appropriate $x_1', x_2' \in M'$, it follows that we may actually assume that $\langle \delta(v)(\hat{1}), \hat{v} \rangle = 1$.

At this point, we need some versions of Lemmas 1.1, 1.2 in [30] for $s^*(\mathcal{B}(L^2M))$ in lieu of $\mathcal{K}(L^2M)$. The proof of the first of these lemmas follows the line of proof of 1.1 in [30], using results from [31] and the incremental patching techniques developed in [30, 31, 33, 34]. □

Lemma 4.2. *Assume that $\mathcal{H}$ is a separable Hilbert space and that $R \subset M$ is a hyperfinite II_1 factor with $R' \cap M = \mathbb{C}$. For any countable set $\{T_m\}_m$ in the unit ball of $s^*_M(\mathcal{B}(\mathcal{H}))$, there exist unitary elements $u_n \in R$ such that $\lim_n \|T_m(u_n v)^k(\xi)\|_{\mathcal{H}} = 0$, for all $k \neq 0, \xi \in \mathcal{H}$, $v \in \mathcal{U}(M)$, $m \geqslant 1$.*

Proof. Note first that it is sufficient to prove the statement in the case when $\mathcal{H} = L^2M$. Indeed, the statement only concerns spaces of the form $M(\xi) \subset \mathcal{H}$, which are cyclic representations of M, and are thus included in the left regular representation. Also, if p_ξ is the orthogonal projection of $\mathcal{H}$ onto the closure $\mathcal{H}_\xi$ of this space, then we clearly have $p_\xi T_m p_\xi \in s^*(\mathcal{B}(\mathcal{H}_\xi))$.

Moreover, in order to prove the statement in the case when $\mathcal{H} = L^2M$, it is clearly sufficient to prove that, for any finite sets $F \subset L^2M$, $V \subset \mathcal{U}(M)$, $\mathcal{T} \subset s^*(\mathcal{B}(L^2M))$,

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any $\varepsilon > 0$, and any $n \geqslant 1$, there exists a unitary element $u \in R$ such that $\|T((uv)^k\xi)\|^2 \leqslant \varepsilon$, $\forall \xi \in F$, $v \in V$, $T \in \mathcal{T}$, $1 \leqslant |k| \leqslant n$. Indeed, for, if we have this, then we take $\{\xi_n\}_n$ dense in the unit ball of L^2M, $\{v_n\}_n$ dense in the norm $\| \; \|_2$ in $\mathcal{U}(M)$, and for each $n \geqslant 1$ we apply it to $F = \{\xi_1, \dots, \xi_n\}$, $V = \{v_1, \dots, v_n\}$, $\mathcal{T} = \{T_1, \dots, T_n\}$, $\varepsilon = 2^{-n}$, to get a unitary element u_n that satisfies

$$\|T_l((u_n v_i)^k(\xi_j))\|^2 \leqslant 2^{-n}, \forall i, j, |k|, l \leqslant n. \tag{1}$$

The sequence $\{u_n\}_n$ will then clearly satisfy the condition in the statement of the lemma, by the density of $\{\xi_n\}_n$ in $(L^2M)_1$ and $\{v_n\}_n$ in $\mathcal{U}(M)$.

Denote by $\mathcal{W}$ the set of partial isometries $w \in R$ with the properties that $ww^* = w^*w$ and $\|T((wv)^k\xi)\|^2 \leqslant \varepsilon\tau(ww^*)$, $\forall \xi \in F$, $v \in V$, $T \in \mathcal{T}$, $1 \leqslant |k| \leqslant n$ (where $w^{-k} = (w^*)^k$ for $k > 0$). We endow $\mathcal{W}$ with the order given by $w_1 \leqslant w_2$, $w_2 w_1^* w_1 = w_1$. Then $(\mathcal{W}, \leqslant)$ is clearly inductively ordered, and we let w be a maximal element. All we need is to prove that w is a unitary element. Assume that $p = 1 - ww^* \neq 0$. If $w_0 \in pRp$ is a partial isometry with $w_0 w_0^* = w_0^* w_0 = q$ and we denote $u = w + w_0$, then we have

$$\begin{aligned}\langle T^*T((uv)^k(\xi)), (uv)^k(\xi)\rangle &= \langle T^*T((w+w_0)v)^k(\xi), ((w+w_0)v)^k(\xi)\rangle \\ &= \langle T^*T((wv)^k(\xi)), (wv)^k(\xi)\rangle + \Sigma'_{(i,0)} + \Sigma'_{(0,j)} + \Sigma'_{(i,j)}, \end{aligned} \tag{2}$$

where the last line represents the sum of $2^k \times 2^k$ elements, coming from developing the binomial powers $((w + w_0)v)^k$ in the scalar product of the first line. The summations $\Sigma'_{(i,0)}$, $\Sigma'_{(0,j)}$, $\Sigma'_{(i,j)}$ are indexed over $i, j \geqslant 1$, and they have the following significance.

For each $i \geqslant 1$, $\Sigma'_{(i,0)}$ is the sum of all terms with i appearances of w_0 on the left-hand side of the scalar product and no appearance on the right-hand side, i.e., each such term is of the form

$$\left\langle T^*T(wv)^{m_0}\left(\prod_{r=1}^{i} w_0 v (wv)^{m_r}\right)(\xi), (wv)^k(\xi)\right\rangle, \tag{3}$$

for some $m_0, m_i \geqslant 0$ and $m_1, \dots, m_{i-1} \geqslant 1$. Similarly, $\Sigma'_{(0,j)}$ is the sum of all terms with j appearances of w_0 on the right-hand side of the scalar product and no appearance on the left-hand side, i.e., each such term is of the form

$$\left\langle T^*T(wv)^k(\xi), (wv)^{n_0}\left(\prod_{s=1}^{j} w_0 v (wv)^{n_s}\right)(\xi)\right\rangle, \tag{4}$$

for some $n_0, n_j \geqslant 0$, $n_1, \dots, n_{j-1} \geqslant 1$.

For $i, j \geqslant 1$, $\Sigma'_{(i,j)}$ is the sum of all terms with i appearances of w_0 on the left-hand side and j appearances of w_0 on the right-hand side of the scalar product, i.e., each such term is of the form

$$\left\langle T^*T(wv)^{m_0}\left(\prod_{r=1}^{i} w_0 v (wv)^{m_r}\right)(\xi), (wv)^{n_0}\left(\prod_{s=1}^{j} w_0 v (wv)^{n_s}\right)(\xi)\right\rangle, \tag{5}$$

for some $m_0, m_i, n_0, n_j \geqslant 0$, $m_1, \dots, m_{i-1}$, $n_1, \dots, n_{j-1} \geqslant 1$.

We will show that we can make the choice of the partial isometry $w_0 \neq 0$ so that when estimating all the terms in (3)–(5) they add up to a quantity $\leqslant \varepsilon\tau(w_0^* w_0)$. We construct w_0 by first choosing its support projection $q \in R$ so that all the terms in (5) are small, and then choose a Haar unitary $w_0 \in qRq$ so that the terms in (3), (4) are small. Let $\delta > 0$, which we will take to be sufficiently small, depending on $\varepsilon > 0$ and 2^n.

Estimating $\Sigma'_{(i,j)}$. Each of the terms in (5) has $i \geqslant 1$ appearances of w_0 on the left-hand side and $j \geqslant 1$ on the right-hand side of the scalar product. If either i or j is at least 2, then the corresponding side of the scalar product will be of the form $x_1 w_0 x_2 \dots w_0 x_p \xi$, for some finite number of possible $x_i \in (M)_1$. By using the fact that $(pRp)' \cap pMp = \mathbb{C}p$ and (see Theorem A.1.2 in [35], or Theorem 2.1 in [31]; see also Theorem 0.1 in [33]), it follows that there exists $q \in pRp$ such that for all the x_i that appear this way and all ξ' in a finite subset of $(pL^2Mp)_1$ we have

$$\begin{aligned} &\|qx_i q - \tau(px_i p)q\|_2 \leqslant \delta\|q\|_2;\\ &|\|q\xi'\|_2 - \|q\|_2\|\xi'\|_2| \leqslant \delta\|q\|_2. \end{aligned} \tag{6}$$

By using repeatedly the first inequality in (6) combined with the triangle inequality for $\| \ \|_2$, with the Cauchy–Schwarz inequality and the second inequality in (6), it follows that each element in (5) is at distance no more than $n^2\delta\|q\|_2^2 = n^2\delta\tau(q)$ to an element of the form $\langle qKq(w_0^r\xi'), w_0^s\eta'\rangle$ for some $r, s \geqslant 1$, $K \in s^*(\mathcal{B}(L^2M))$, $\xi', \eta' \in (pL^2Mp)_1$, with the number of possibilities for these terms depending on n.

But the operatorial norm $\|qKq\|$ is small for q having sufficiently small trace, by the smoothness of K. Also, by the second inequality in (6), the 'size' of the vectors $w_0^r\xi'$, $w_0^s\eta'$ is $\|w_0^r\xi'\| \approx \|q\|_2\|\xi'\|$, $\|w_0^s\eta'\| \approx \|q\|_2\|\eta'\|$, with error controlled by $\delta\|q\|_2$. Altogether, this shows that all terms in (6) have absolute value majorized by $n^2\delta\tau(q)$. Since there are at most $2^n \times 2^n = 2^{2n}$ of them, if we take $\delta = \varepsilon n^{-2}2^{-2n}/4$, then the sum of the terms in $\Sigma'_{(i,j)}$, with $i \geqslant 1$, $j \geqslant 1$ and at least one of them $\geqslant 2$, is majorized by $\varepsilon\tau(q)/4$.

We now estimate the summation $\Sigma'_{(1,1)}$, which consists of terms having exactly one occurrence of w_0 on the left and one on the right of the scalar product. But these are of the form $\langle qKqw_0q\xi', w_0q\eta'\rangle$, for which we have, by the Cauchy–Schwarz inequality and the second part of (6),

$$|\langle qKqw_0q\xi', w_0q\eta'\rangle| \leqslant \|qKq\|\|q\xi'\|_2\|q\eta'\|_2 \leqslant 3\|qKq\|\|q\|_2^2 = 3\|qKq\|\tau(q).$$

Since $\|qKq\|$ can be made arbitrarily small for $\tau(q)$ small (by the smoothness of K), it follows that the $\Sigma_{(1,1)}$ can be majorized by $\varepsilon\tau(q)/4$ as well.

Estimating $\Sigma'_{(i,0)}, \Sigma'_{(0,j)}$. Note that in the above estimates we only used the properties of the support projection q of w_0. We will choose now $w_0 \in \mathcal{U}(qRq)$ to be a Haar unitary so that the sum over i of all the summations $\Sigma'_{(i,0)}$ is majorized by $\varepsilon\tau(q)/4$ (similarly for $\Sigma'_{(0,j)}$). Indeed, each one of the terms in these sums is either of the form $\langle(\Pi_{r=1}^i w_0 x_r)\xi', q(K^*\eta')\rangle$ or $\langle q(K\xi'), (\Pi_{s=1}^j w_0 x_s)\eta'\rangle$, for some finite number of operators K in the unit ball of $\mathcal{B}(L^2M)$ and vectors $\xi', \eta' \in (L^2M)_1$.

By the second inequality in (6), if $i, j \geqslant 2$, then, combining with the first inequality in (6) and the Cauchy–Schwarz inequality, we get that each one of these terms can be perturbed by $2\delta\|q\|_2^2 = 2\delta\tau(q)$ to an element of the form $\langle w_0^i\xi', \eta'\rangle$ or $\langle\xi', w_0^j\eta'\rangle$. Moreover, each of the terms in $\Sigma'_{(1,0)}, \Sigma'_{(0,1)}$ is also of this form, but with $i, j = 1$. We can now take any Haar unitary $u_0 \in qRq$ and use the fact that $\lim_n\langle u^n\xi', \eta'\rangle = 0$, for any vectors ξ', η', to obtain that for any finite set $F' \subset (L^2M)_1$ and any $\delta' > 0$ there exists n_0 such that $|\langle u^n\xi', \eta'\rangle| \leqslant \delta'\tau(q), \forall\xi', \eta' \in F', |n| \geqslant n_0$. Thus, if we define $w_0 = u_0^{n_0}$ then for sufficiently small δ' we get that $\Sigma'_{(i,0)} \leqslant \varepsilon\tau(q)/4$ and $\Sigma'_{(0,j)} \leqslant \varepsilon\tau(q)/4$.

Putting now together the two sets of estimates, one gets that $\Sigma'_{(i,j)} + \Sigma'_{(i,0)} + \Sigma'_{(0,j)} \leqslant \varepsilon\tau(q) = \varepsilon\tau(w_0^*w_0)$, which, combined with $\langle T^*T((wv)^k(\xi)), (wv)^k(\xi)\rangle \leqslant \varepsilon\tau(w^*w)$ (due to w being in $\mathcal{W}$), shows that the last line in (2) is further majorized by $\varepsilon\tau(u^*u)$. Thus, $u = w + w_0 \in \mathcal{W}$, with $w \leqslant u$, and, since $w_0 \neq 0$, we have $u \neq w$, contradicting the maximality of w in $\mathcal{W}$. Thus, w is actually a unitary, and the lemma is proved. □

The second technical lemma corresponds to Lemma 1.2 in [30], but for $s^*(\mathcal{B})$ instead of $\mathcal{K}(\mathcal{H})$. It shows that once 4.2 holds true for a sequence of unitaries $u_n \in M_\delta$ then, as $n \to \infty$, the restriction of δ to the abelian von Neumann algebra generated by the unitary u_nv tends to behave 'virtually' like the derivation of $L^\infty(\mathbb{T})$ into $\mathcal{B}(L^2(\mathbb{T}))$ implemented by ad$P$, where P is the projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Lemma 4.3. *Let $\delta : M \to s^*(\mathcal{B}(L^2M))$ be a derivation. Let $v \in \mathcal{U}(M)$, and assume that $\{u_n\}_n \subset \mathcal{U}(M)$ is a sequence of unitary elements such that $\delta(u_n) = 0$, $\forall n$, and $\lim_n\|T((u_nv)^k\xi)\| = 0$, $\forall\xi \in L^2M$, $k \neq 0$, $T \in \{\delta(v), \delta(v)^*, \delta(v^{-1}), \delta(v^{-1})^*\}$. Then the sequence $\{\langle\delta((u_nv)^r)\hat{1}, (u_nv)^s\hat{1}\rangle\}_n$ tends to $\langle\delta(v)\hat{1}, \hat{v}\rangle$ if $r = s > 0$ and to 0 in all other cases.*

Proof. Let $t > 0$ be a positive integer. Since $\delta(u_n) = 0$, we have

$$\delta((u_nv)^t) = \sum_{i=0}^{t-1}(u_nv)^iu_n\delta(v)(u_nv)^{t-i-1}; \tag{1}$$

$$\delta((u_nv)^{-t}) = \sum_{i=0}^{t-1}(u_nv)^{-i}\delta(v^{-1})u_n^{-1}(u_nv)^{-t+i+1}. \tag{2}$$

But by the assumptions in the hypothesis, applied to the case $T = \delta(v)$ and $\xi = \hat{1}$, it follows that, whenever $t - i - 1 \neq 0$, we have

$$\|(u_nv)^iu_n\delta(v)(u_nv)^{t-i-1}\hat{1}\| = \|\delta(v)(u_nv)^{t-i-1}\hat{1}\| \to 0. \tag{3}$$

Similarly, by applying the hypothesis to $T = \delta(v^{-1})$ and $\xi = \hat{v}$, for any $t - i - 1 \geqslant 0$, we have

$$\|(u_nv)^{-i}\delta(v^{-1})u_n^{-1}(u_nv)^{-t+i+1}\hat{1}\| = \|\delta(v^{-1})u_n^{-1}(u_nv)^{-t+i+1}\hat{1}\| \to 0. \tag{4}$$

Thus, from (1) and (3), for any integer $t \geqslant 1$, we get

$$\lim_n\|\delta((u_nv)^t)\hat{1} - (u_nv)^{t-1}u_n\delta(v)\hat{1}\| = 0, \tag{5}$$

while from (2) and (4) we get

$$\lim_n\|\delta((u_nv)^{-t})\hat{1}\| = 0. \tag{6}$$

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Now, if $r < 0$ and s is arbitrary, then the statement follows immediately from (6). If in turn $r > 0$, then by (5) we have, for any integer s,

$$\lim_n(\langle\delta((u_n v)^r)\hat{1}, (u_n v)^s\hat{1}\rangle - \langle\delta(v)\hat{1}, u_n^{-1}(u_n v)^{s-r+1}\hat{1}\rangle) = 0. \tag{7}$$

But, if $s-r+1 \neq 1$, then, by applying the hypothesis to $T = \delta(v)^*$, $\xi = \hat{v}$, when $s-r+1 \leqslant 0$, and to $T = \delta(v^{-1})^*$, $\xi = \hat{1}$, when $s-r+1 \geqslant 2$, we obtain $\lim_n \|\delta(v)^* u_n^{-1}(u_n v)^{s-r+1}\hat{1}\| = 0$. So the second term in (7) tends to 0. Thus, if $s \neq r$, then $\lim_n \langle\delta((u_n v)^r)\hat{1}, (u_n v)^s\hat{1}\rangle = 0$. In addition, if $s = r > 0$, then, by (7) again, we get $\lim_n \langle\delta((u_n v)^r)\hat{1}, (u_n v)^s\hat{1}\rangle = \langle\delta(v)\hat{1}, v\hat{1}\rangle)$.

The case $r = 0$ is trivial, because then we have $(u_n v)^r = 1$ and $\delta(1) = 0$. □

End of the proof of 4.1. From this point on, the rest of the proof of the theorem goes exactly the same way as the proof of Theorem II in [30], by using Lemmas 4.2, 4.3, in lieu of Lemmas 1.1, 1.2 in [30], as well as Lemma 1.3 in [30], which can be used unchanged. We include this last part of the argument, for completeness.

Recall that we have reduced the proof to the case when M is a separable II_1 factor acting on $\mathcal{H} = L^2M$ and $\delta : M \to \mathcal{B}_0 \subset s^*(\mathcal{B}(L^2M))$ a derivation vanishing on a hyperfinite subfactor $R \subset M$ with $R' \cap M = \mathbb{C}1$. We want to prove that $\delta = 0$ on all M. We assumed by contradiction that there exists $v \in \mathcal{U}(M)$ with $\delta(v) \neq 0$, and we have seen that we may assume that $\langle\delta(v)\hat{1}, \hat{v}\rangle = 1$.

By 4.2, there exists a sequence of unitaries $u_n \in \mathcal{U}(R)$ such that $\lim_n \|T(u_n v)^k\xi\| = 0$, for all $k \neq 0$, all $\xi \in L^2M$, and $T \in \{\delta(v), \delta(v)^*, \delta(v^{-1}), \delta(v^{-1})^*\}$, or $T \in \{p_m\}_m$, for some sequence of finite-rank projections $p_m \to 1$. Note that

$$\lim_n \|p_m(u_n v)^k\xi\| = 0, \forall m, \xi,$$

implies that $\{(u_n v)^k\}_n$ tends weakly to 0, for all $k \neq 0$. By Lemma 1.3 in [30], this implies there exist Haar unitaries $v_n \in M$ such that $\lim_n \|v_n - u_n v\| = 0$. Since $\langle\delta(v)\hat{1}, v\hat{1}\rangle = 1$ and δ is norm continuous, by Lemma 4.3 this implies that $\lim_n \langle\delta(v_n^r)\hat{1}, v_n^s\hat{1}\rangle$ is equal to 1 if $r = s > 0$ and to 0 in all other cases.

Take A_n to be the von Neumann subalgebra of M generated by the Haar unitary v_n, and denote by e_n the orthogonal projection onto L^2A_n. Since $s_M^*(\mathcal{B}(L^2M)) \subset s_{A_n}^*(\mathcal{B}(L^2M))$ and $s_{A_n}^*(\mathcal{B}(L^2M))$ is an A_n'-bimodule with $e_n \in A_n'$, it follows that $e_n\delta(A_n)e_n \subset e_n s_{A_n}^*(\mathcal{B}(L^2M))e_n = s_{A_n}^*(L^2A_n)$. Thus

$$A_n \ni a \mapsto \delta_n(a) = e_n\delta(a)e_n \in s_{A_n}^*(L^2A_n)$$

defines derivations with the property that $\|\delta_n\| \leqslant \|\delta\|$, $\forall n$. Moreover, since all δ_n are restrictions of δ, which is smooth, the derivations $\{\delta_n\}_n$ are uniformly smooth; i.e., $\forall\varepsilon > 0$, $\exists\alpha > 0$ such that, for any given n, if $a \in (A_n)_1$ is such that $\|a\| \leqslant \alpha$, then $\|\delta_n(a)\| \leqslant \varepsilon$.

Since $A_n \subset \mathcal{B}(L^2A_n)$ are all spatially isomorphic to $L^\infty(\mathbb{T}) \subset \mathcal{B}(L^2(\mathbb{T}))$, with the identification sending the Haar unitary generating A_n to the operator M_z, acting on $f \in L^2(\mathbb{T})$ by multiplication with the function $g(z) = z$, we may view all δ_n as derivations from $L^\infty(\mathbb{T})$ into $s_{L^\infty(\mathbb{T})}^*(\mathcal{B}(L^2(\mathbb{T}))$, which are uniformly bounded in norm by $\|\delta\|$ and

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uniformly smooth. Moreover, by spatiality, $\lim_n\langle\delta_n(M_{z^r})\hat{1}, \hat{z^s}\rangle$ is equal to 1 if $r = s > 0$ and to 0 in all other cases.

We now fix a free ultrafilter ω on $\mathbb{N}$, and define $\Delta : L^\infty(\mathbb{T}) \to \mathcal{B}(L^2(\mathbb{T}))$ to be the weak limit over ω of δ_n; then it is easy to see that Δ is still a derivation, which satisfies $\|\Delta\| \leqslant \|\delta\|$ and is smooth (because δ_n are uniformly smooth). Moreover, we have that

$$\langle\Delta(M_{z^r})\hat{1}, z^s\hat{1}\rangle = \lim_n\langle\delta(v_n^r)\hat{1}, v_n^s\hat{1}\rangle$$

is equal to 1 if $r = s > 0$ and to 0 in all other cases. By using the derivation properties, this implies that $\langle\Delta(M_z)z^r, z^s\rangle$ is equal to 1 if $r = 0$, $s = 1$, and to 0 otherwise. But this means that $\Delta(M_z)$ coincides with the commutator $[P, M_z]$, where P is the orthogonal projection of $L^2(\mathbb{T})$ onto the subspace $H^2(\mathbb{T}) = \overline{\mathrm{sp}}\{z^k \mid k > 0\}$. Since both Δ and $\mathrm{ad}P$ are derivations and are weakly continuous (Δ is even smooth), and they coincide on the generator M_z of $L^\infty(\mathbb{T})$, it follows that $\Delta = \mathrm{ad}P$ on all $L^\infty(\mathbb{T})$.

Since Δ is smooth, this implies that $\mathrm{ad}P$ is smooth. Moreover, since $\mathrm{ad}P$ takes compact values on the dense *-subalgebra $\mathrm{sp}\{M_{z^r} \mid r \in \mathbb{Z}\}$ and is smooth, it takes compact values on all $L^\infty(\mathbb{T})$. By 2.5 or 3.7, $\mathrm{ad}P = \mathrm{ad}K$ for some $K \in \mathcal{K}(L^2(\mathbb{T}))$. Thus $P - K$ commutes with $L^\infty(\mathbb{T})$, which is maximal abelian in $\mathcal{B}(L^2(\mathbb{T}))$. It follows that $P - K = M_f$ for some $f \in L^\infty(\mathbb{T})$. But then

$$\langle(P - K)z^k, z^k\rangle = \langle M_f z^k, z^k\rangle = \int f(z)\,d\nu(z)$$

for all $k \in \mathbb{Z}$, with the left-hand side tending to 1 as $k \to \infty$ and to 0 as $k \to -\infty$, while the right-hand side is constant. This final contradiction finishes the proof. □

Corollary 4.4. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a normal representation of a von Neumann algebra M on a Hilbert space $\mathcal{H}$. If $\delta : M \to \mathcal{B}(\mathcal{H})$ is a smooth derivation, then there exists $T \in s_M^*(\mathcal{B}(\mathcal{H}))$, with $\|T\| \leqslant \|\delta\|$, such that $\delta = \mathrm{ad}T$.*

Corollary 4.5. *Let M_0 be a C^*-algebra with a faithful trace τ and $M_0 \subset \mathcal{B}(\mathcal{H})$ a faithful representation of M_0. Let $\delta : M_0 \to \mathcal{B}(\mathcal{H})$ be a derivation. Assume that δ is continuous from the unit ball of M_0 with the topology given by the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M_0$, to $\mathcal{B}(\mathcal{H})$ with the operator norm topology. Then there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\delta = \mathrm{ad}T$ and $\|T\| \leqslant \|\delta\|$. More precisely, if p_0 denotes the projection onto the closure $\mathcal{H}_0$ of $\mathrm{sp}\{\delta(x)\xi \mid \xi \in \mathcal{H}, x \in M_0\}$, then $[p_0, M_0] = 0$, $[p_0, \delta(M_0)] = 0$, the weak closure M of $M_0p_0 \subset \mathcal{B}(\mathcal{H}_0)$ is a finite von Neumann algebra, and δ extends to a smooth derivation of M into $\mathcal{B}(\mathcal{H}_0)$ and is implemented by an element $T \in s_M^*(\mathcal{B}(\mathcal{H}_0))$, with $\|T\| \leqslant \|\delta\|$.*

Proof. Since $y\delta(x)\xi = (\delta(yx) - \delta(y)x)\xi$, we have $M_0\mathcal{H}_0 \subset \mathcal{H}_0$. The projection p_0 onto $\mathcal{H}_0$ satisfies $(1 - p_0)\delta(x) = 0$, $\forall x \in M_0$, and it is the smallest projection with this property. For each $T = \delta(y)p_0 \in \mathcal{B}(\mathcal{H}_0)$ and $\xi \in \mathcal{H}_0$, we have $\|xT(\xi)\|$ small if $x \in (M_0)_1$, $\|x\|_2$ small.

Thus, if we denote by $(\pi_\tau, \mathcal{H}_\tau)$ the GNS representation of M_0 corresponding to the trace τ, then the map $\theta : \pi(M_0) \to M_0p_0 \subset \mathcal{B}(\mathcal{H}_0)$ defined by $\theta(\pi(x)) = xp_0$ is a *-algebra morphism which is continuous from the unit ball of $\pi(M_0)$ with the $\|\ \|_\tau$-topology to $(M_0p_0)_1$ with the so-topology. This implies that θ extends to a *-morphism, still denoted θ, from $\pi(M_0)''$ to $M = (M_0p_0)'' \subset \mathcal{B}(\mathcal{H}_0)$. This means that $\ker\theta$ is given by a central projection z_0 in $\pi(M_0)''$, i.e., $M \simeq \pi_\tau(M_0)''z_0$, and the rest of the statement follows

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from 4.1, once we notice that the extension by continuity of δ to all $M = Mp_0$ is smooth (and is thus smooth valued). □

5. Further comments

5.1. Generalized smooth cohomology

Note that the proof of 2.5 still works (and thus the conclusion in 2.5 still holds true) if we merely assume that $\mathcal{B}/\mathcal{B}_0$ is operatorial, instead of the (stronger) condition that $\mathcal{B}$ is operatorial.

But an even greater degree of generality for which the arguments in the proof of 2.5 work exactly the same way is the following framework, inspired by the work in [36]. Let M be a von Neumann algebra, and let $\mathcal{B}$ be a dual normal M-bimodule. Assume that $L \subset \mathcal{B}_*$ is a subset of the unit ball of $\mathcal{B}_*$ with the property that if $\varphi \in L$ then $\varphi(x \cdot y) \in L$ for any $x, y \in (M)_1$ (so in particular $\lambda L \subset L$ for any $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$). Moreover, we assume the set L is 'separating' for $\mathcal{B}$, i.e., $\forall T \in \mathcal{B}$, $\exists \varphi \in L$ such that $\varphi(T) \neq 0$.

For $T \in \mathcal{B}$, we denote $\|T\|_L = \sup\{|\varphi(T)| \mid \varphi \in L\}$. Note that $\| \ \|_L$ is a norm on $\mathcal{B}$ which is majorized by $\| \ \|$ and is complete on the unit ball of $(\mathcal{B}, \| \ \|)$. Moreover, $(\mathcal{B}, \| \ \|_L)$ is a normed M-bimodule, and we denote by $s^*(\mathcal{B}, \| \ \|_L)$ its smooth part. As in 1.2(1°), this space is easily seen to be a Banach M-bimodule with respect to the usual norm $\| \ \|$, and its unit ball is complete in the norm $\| \ \|_L$.

Let now $\mathcal{B}_0 \subset s^*(\mathcal{B}, \| \ \|_L)$ be a sub-bimodule with the property that the unit ball of $(\mathcal{B}_0, \| \ \|)$ is complete in the norm $\| \ \|_L$. In addition, we will require the norm implemented by $\| \ \|_L$ on the quotient space $\mathcal{B}/\mathcal{B}_0$ by $\|T/\mathcal{B}_0\|_{L,ess} = \sup\{\|T - T_0\|_L \mid T_0 \in \mathcal{B}_0\}, T \in \mathcal{B}$, to be *operatorial*, in the sense that, for any $T \in \mathcal{B}$ and any $p \in \mathcal{P}(M)$, we have

$$\|pTp + (1-p)T(1-p)\|_{L,ess} = \max\{\|pTp\|_{L,ess}, \|(1-p)T(1-p)\|_{L,ess}\}.$$

Theorem 5.1.1. *With the above assumptions and notation, let $\delta : M \to \mathcal{B}_0$ be a derivation. Given any amenable von Neumann subalgebra $B \subset M$, δ is implemented on B by some $T \in K_{\delta,B} \cap \mathcal{B}_0$. Moreover, if B is diffuse and wq-regular in M, then T implements δ on all M. And if $\mathcal{B}$ is a von Neumann algebra that contains M as a von Neumann subalgebra, then T implements δ on all M even if A is not wq-regular in M.*

Example 5.1.2. (1°) If in the above we take L to be the whole unit ball of $\mathcal{B}_*$, then $\| \ \|_L$ coincides with the norm $\| \ \|$.

(2°) Let $\mathcal{M}$ be a semifinite von Neumann factor, and denote by $\mathcal{J} \subset \mathcal{M}$ the *compact ideal space* of $\mathcal{M}$, i.e., the set of all elements $T \in \mathcal{M}$ with the property that all spectral projections of $|T|$ corresponding to intervals $[c, \infty)$ for $c > 0$ are finite projections in $\mathcal{M}$. Let Tr be a normal semifinite trace on $\mathcal{M}$, and take $L \subset \mathcal{M}_*$ to be the set of all normal functionals on $\mathcal{M}$ of the form $\varphi_{x,y}(T) = Tr(yTx)$, with $x, y \in \mathcal{M}$, $\|x\| \leqslant 1$, $Tr(x^*x) \leqslant 1$, $Tr(y^*y) \leqslant 1$. Then the corresponding norm $\| \ \|_L$ satisfies all the above conditions. Moreover, $\mathcal{J}$ is contained in $s^*(\mathcal{M}, \| \ \|_L)$, and the quotient norm $\| \ \|_{L,ess}$ on $\mathcal{M}/\mathcal{J}$ is operatorial. Thus, we recover in this case the framework and results obtained in [36], for which Theorem 5.1.1 provides a more abstract setting, with a higher degree of generalization.

Note that, if $\mathcal{M} = \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, then $\mathcal{J} = \mathcal{K}(\mathcal{H})$ and $\| \|_L$ coincides with the operatorial norm on $\mathcal{B}(\mathcal{H})$. But for type II_∞ factors $\mathcal{M}$, there are elements T in $\mathcal{M}$ for which $\|T\|_L < \|T\|$. Moreover, if $M \subset \mathcal{M}$ is a von Neumann algebra and we view $\mathcal{M}$ as a dual M-bimodule by left/right multiplication of the elements in $\mathcal{M}$ by elements in M, then the norm $\| \|$ is not smooth on $\mathcal{J}$, and is not operatorial on $\mathcal{M}/\mathcal{J}$. In addition, from the above we see that $\| \|_L$ is both smooth on $\mathcal{J}$ and operatorial on the quotient $\mathcal{M}/\mathcal{J}$. Thus, Theorem 5.1.1 is more general than Theorem 2.5.

5.2. Smooth n-cohomology

We propose here a definition of higher Hochschild cohomology groups with smooth coefficients, $H_s^n(M, \mathcal{B}_0)$, $n \geqslant 2$, which extends the 1-cohomology in § 3 to all n. This is done in the spirit of the classical operatorial approach to Hochschild cohomology in [15, 17, 18] (see also [41]), but with additional continuity requirements.

Thus, let $\mathcal{B}$ be a dual operatorial M-bimodule and $\mathcal{B}_0 \subset s^*(\mathcal{B})$ a Banach M sub-bimodule. Denote by $\mathcal{L}_s^n(M, \mathcal{B}_0)$ the space of n-linear maps $\Phi : M^n = M \times M \times \cdots \times M \to \mathcal{B}_0$, which are separately s^*-norm continuous in each variable, i.e., for each i and each fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, the map $M \ni x \mapsto \Phi(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in \mathcal{B}_0$ is continuous from the unit ball of M with the s^*-topology to $\mathcal{B}_0$ with its norm topology. We view $\{\mathcal{L}_s^n(M, \mathcal{B}_0)\}_n$ as a *chain complex* with *boundary operators* $\partial_n : \mathcal{L}_s^n \to \mathcal{L}_s^{n+1}$ defined by

$$\begin{aligned}\partial_n(\Phi)(x_1, \ldots, x_{n+1}) &= x_1\Phi(x_2, x_3, \ldots, x_{n+1}) - \Phi(x_1x_2, x_3, \ldots, x_{n+1}) \\ &\quad + \cdots + (-1)^n\Phi(x_1, x_2, \ldots, x_nx_{n+1}) + (-1)^{n+1}\Phi(x_1, \ldots, x_n)x_{n+1}.\end{aligned}$$

Denote by $Z_s^n(M, \mathcal{B}_0)$ the kernel of ∂_{n+1}, by $B_s^n(M, \mathcal{B}_0)$ the image of $\mathcal{L}_s^n$ under ∂_n, and by $H_s^n(M, \mathcal{B}_0) = Z_s^n(M, \mathcal{B}_0)/B_s^n(M, \mathcal{B}_0)$ the corresponding (vector) quotient space.

As we mentioned in § 3, Theorems 3.7, 3.8, 4.1, can be formulated as vanishing 1-cohomology results, in the form $H^1(M, \mathcal{B}_0) = H_s^1(M, \mathcal{B}_0) = 0$. Note that smoothness is automatic for 1-cocycles (=derivations). Such ‘automatic continuity’ phenomena for 1-cocycles are often present in cohomology theories involving operator algebras. But that is no longer the case for the higher cohomology. We have thus required smoothness in each variable, as part of the definition of n-cocycles.

Note that in the case when the target smooth bimodule $\mathcal{B}_0$ is the ideal of compact operators on the Hilbert space $\mathcal{H}$ on which M acts, $\mathcal{B}_0 = \mathcal{K}(\mathcal{H})$, smoothness in each variable is in fact automatic, once separate weak continuity is assumed, a fact that was pointed out in Proposition 3 of [40]. We recall that statement here, and include a proof, for the reader's convenience, but also in order to emphasize the crucial way in which the assumption that the target bimodule is the space of compact operators is being used. This shows the difficulty of dealing with higher cohomology with coefficients in arbitrary smooth bimodules (even when restricting to cocycles that are smooth in each variable).

Lemma 5.2.1 [39]. *Let (M, τ) be a finite von Neumann algebra, acting normally on the Hilbert space $\mathcal{H}$. If $\Phi : M^n \to \mathcal{K}(\mathcal{H})$ is n-linear, separately norm continuous, and weakly continuous in each variable, then Φ is separately s^*-norm continuous on the unit ball of M, in each variable. In particular, $Z_s^n(M, \mathcal{K}(\mathcal{H})) = Z_w^n(M, \mathcal{K}(\mathcal{H}))$, $\forall n \geqslant 1$.*

Proof. By A.1, it is sufficient to prove that, if $F : M \to \mathcal{K}(\mathcal{H})$ is a bounded linear map which is weakly continuous (i.e., continuous from $(M)_1$ with the $\sigma(M, M_*)$-topology to $\mathcal{K}(\mathcal{H})$ with the $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})_*)$-topology), then for any mutually orthogonal projections $e_n \in \mathcal{P}(M)$ we have $\lim_n \|F(e_n)\| = 0$.

Assume by contradiction that there exist mutually orthogonal projections e_n with $\|F(e_k)\| \geqslant c > 0$, $\forall n$. Since the $F(e_k)$ are compact operators and they tend weakly to 0, it follows that we can choose recursively a fast growing sequence of integers $k_1 \ll k_2 \ll \cdots$ such that $F(e_{k_{n+1}})$ is ε_{n+1}-supported by a finite-rank projection p_{n+1} with $p_k p_{n+1} = 0$, $1 \leqslant k \leqslant n$. If the ε_n are taken sufficiently small, then by weak continuity of F we have that $F(e) = F(\Sigma_n e_{k_n}) = \Sigma_n F(e_{k_n})$ must be compact, while being norm close to $\Sigma_n p_n F(e_{k_n}) p_n$, which is a non-compact operator (because under each p_n there exist unit vectors on which $p_n F(e_{k_n}) p_n$ has norm $\geqslant c/2$). This contradiction completes the proof. □

5.3. Toward a 'good' cohomology theory for II_1 factors

The work in this paper can be viewed as a revisitation of techniques and results in [16, 29, 30] in light of the recent efforts ([8, 22, 43], among others) to find a 'good' 1-cohomology theory for II_1 factors M. By a 'good cohomology' we mean one that does not always vanish and can therefore detect properties of II_1 factors such as primeness, absence of Cartan subalgebras, and infinite generation. An ideal such theory should also be calculable, and in the case of group II_1 factors $M = L(\Gamma)$ should reflect the cohomology theory of the group Γ.

One can try to define such a cohomology theory on weakly dense *-subalgebras of M, as in one of the venues proposed in [8], or as in [22]. But this would require showing that the definition of the resulting invariant (the cohomology group, or at least its 'dimension') does not depend on the choice of the dense subalgebra of M. Solving this type of problem has run into difficulties that seem insurmountable at the moment, similar to the problem of showing that Voiculescu's free entropy dimension of a set of generators of M is independent on the choice of generators.

We have thus taken the point of view that a 'good cohomology' for M should be everywhere defined. All the weight for constructing such a cohomology is thus left on the choice of the target M-bimodules $\mathcal{B}$. The M-bimodules may be purely algebraic, with no topology on them (as in [8]). But the danger in this case is that the resulting 1-cohomology may be non-zero for any II_1 factor. Also, in order for a cohomology to detect finite/infinite generation of M, a derivation $\delta : M \to \mathcal{B}$ should be uniquely determined by its values on a set of generators of M, thus imposing that δ satisfies some continuity properties.

A topological version of the Connes–Shlyakhtenko's cohomology [8] has been proposed by Thom in [43], where the target M-bimodule is the space $\mathcal{B} = \mathrm{Aff}(M\overline{\otimes}M)$ of all densely defined closable operators affiliated with $M\overline{\otimes}M$, and the derivations $\delta : M \to \mathcal{B}$ are taken to be continuous from M with the norm topology to $\mathcal{B}$ with the topology of convergence in measure. By using the non-commutative Lusin-type theorem, such derivations are actually shown to be continuous from the unit ball of M with the Hilbert norm $\| \ \|_2$ given by the trace to $\mathcal{B}$ with the topology of convergence in measure [3, 43]. Thus, they are indeed uniquely determined by their values on a set of von Neumann generators of M. But this 1-cohomology was shown to always vanish in [37] (after prior work in

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this direction in [2, 3]). Moreover, it was proved in [37] that any derivation $\delta : M \to \mathcal{B}$ that is continuous from M with the norm-$\| \, \|-2$ topology to $\mathcal{B}$ with any 'reasonable' weak topology is inner. So the approach in [8, 43], among others, is at a stalemate at the moment.

But we retain from the above that, in order for the derivations to be uniquely determined by their value on a set of generators of M, the class of M-bimodules $\mathcal{B}$ and derivations $\delta : M \to \mathcal{B}$ considered should satisfy some topological/continuity conditions. We restrict our attention to the case when $\mathcal{B}$ are Banach M-bimodules. Note that, by [40], any derivation of M into a Banach M-bimodule $\mathcal{B}$ is automatically norm continuous. Two typical classes of Banach M-bimodules are the 'L^∞-type' (operatorial) and the 'L^1-type', where the former come from representations $M \subset \mathcal{B}(\mathcal{H})$, with $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ a norm closed subspace satisfying $M\mathcal{B}M \subset \mathcal{B}$, while the latter appear as M-bivariant subspaces of the dual of L^∞-type Banach bimodules, $\mathcal{B} \subset \mathcal{B}(\mathcal{H})^*$. The L^1-type M-bimodules seem to always give rise to 1-cohomologies that are either vanishing (like in the case $\mathcal{B} = M_*$ in [13], or due to the Ryll–Nardzewski theorem as in [4]), or are very large (like in the case of the dual normal M bimodule $\mathcal{B} \subset \mathcal{B}(L^2M)^*$ that detects the non-amenability of M in [7]). So the 'L^∞-type' bimodules appear to be more suitable for our purpose.

Another important 'wishful' feature is that derivations should be uniquely determined by their values on wq-regular diffuse von Neumann subalgebras of M and, if possible, to be 'integrable' on abelian subalgebras. The first of this requirements is by analogy to properties in L^2-cohomology of groups (see, e.g., [23]) and countable equivalence relations [12]. It is a property that ensures that the 1-cohomology is 'small' and calculable, and that it is vanishing once M contains wq-regular abelian diffuse subalgebras. This last property is best ensured when any δ considered is inner (i.e., implemented by an element in $\mathcal{B}$) on any abelian subalgebra $A \subset M$ (δ is 'integrable' on A). This is not the case in the initial Cheeger–Gromov approach to L^2-cohomology, where the Γ-module is $\mathcal{B} = \ell^2\Gamma$, but it is the case in the Peterson–Thom recent approach in [23], where $\mathcal{B}$ if taken to be the bigger module $\mathrm{Aff}(L(\Gamma))$.

We will call a bimodule $\mathcal{B}$ *thin* if there are no non-zero elements in $\mathcal{B}$ that commute with a diffuse von Neumann subalgebra of M. Thin bimodules $\mathcal{B}$ do have the property that any derivation $\delta : M \to \mathcal{B}$ is uniquely determined by its values on a wq-regular diffuse subalgebra of M.

The smooth operatorial bimodules and cohomology that we propose in this paper satisfy all the above 'wishful' properties: (a) smooth derivations are uniquely determined by their values on sets of generators of the von Neumann algebra; (b) they are integrable on abelian subalgebras; (c) they are uniquely determined by their values on wq-regular diffuse subalgebras: in fact they take values into smooth bimodules, which are thin. The problem is, of course, that any such cohomology may be vanishing. In the (optimistic!) case that they do not, one needs to find a class of smooth bimodules for which the general results in Theorems 2.5, 3.7, 3.8 (or 0.1 in the introduction) hold true, but for which there do exist II_1 factors M (e.g., $M = L(\mathbb{F}_n)$) that have non-inner derivations.

The proof of Theorem 4.1 gives ideas and techniques that could be used to show that all 1-cohomologies into operatorial smooth bimodules vanish (see also the problem formulated at the end of Remark 1.4). Alternatively, 4.1 provides 'boundary conditions'

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toward the search for a 'good class' of smooth bimodules $\mathcal{B}$, by indicating the type of conditions they should NOT satisfy: if $\mathcal{B}$ is 'too operatorial' (e.g., $\mathcal{B}$ is a sub-bimodule in $\mathcal{B}(\mathcal{H})$ like in 4.1), then all derivations into the smooth part of $\mathcal{B}$ will be inner. The room to maneuver is slim, but there are ways to generalize even more the class of 'smooth-type' bimodules considered, while still retaining 'thinness', integrability over abelian subalgebras, and vanishing cohomology for wq-regular II_1 factors, as in 2.5, 3.7, and 3.8 (e.g., in the spirit of 5.1).

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Appendix A

We recall here, for the reader's convenience, a result from [29]; see also A.1 in the Appendix of [30]), showing that, if a weakly continuous linear map from a finite von Neumann algebra into a dual Banach space has the property that it is s^*-norm continuous on the unit ball of any copy of $\ell^\infty\mathbb{N}$ in M, then it follows s^*-norm continuous on the unit ball of M. We include the proof from [29], for the sake of completeness.

Lemma A.1. *Let (M, τ) be a finite von Neumann algebra with a faithful normal trace and $\mathcal{B}$ a dual Banach space. Let $F : M \to \mathcal{B}$ be a weakly continuous linear map. Assume that $\lim_n \|F(e_n)\| = 0$ for any sequence of mutually orthogonal projections in M. Then $\lim_n \|F(x_n)\| = 0$ for any sequence $\{x_n\}_n$ in the unit ball of M that tends to 0 in the strong operator topology.*

Proof. Let us prove first that $\lim_n \|F(f_n)\| = 0$ for any sequence $f_n \in \mathcal{P}(M)$ with $\tau(f_n) \to 0$. Suppose by contradiction that there exists such a sequence with $\|F(f_n)\| \geqslant c > 0$, $\forall n$. By taking a subsequence, if necessary, we may assume that $\Sigma_n \tau(f_n) < \infty$. Let $g_n = \vee_{k \geqslant n} f_k$, and note that g_n is a decreasing sequence of projections with $\tau(g_n) \leqslant \Sigma_{k \geqslant n} \tau(f_k) \to_n 0$.

Denote by s_{nm} the support projection of $f_m g_n f_m$. Then $s_{nm} \leqslant f_m$ and s_{nm} is majorized (in the sense of comparison of projections) by g_n. Thus, since τ is a trace, we have $\tau(s_{nm}) \leqslant \tau(g_n) \to_n 0$ for each m. Since $\{g_n\}_n$ is decreasing, $\{f_m g_n f_m\}_n$ is decreasing, and thus $\{s_{nm}\}_n$ is decreasing in n, for each m. It follows that $\{f_m - s_{nm}\}_n$ increases to f_m, implying that $\{F(f_m - s_{nm})\}_n$ is weakly convergent to $F(f_m)$. By the inferior semicontinuity of the norm on $\mathcal{B}$ with respect to the w*-topology, it follows that, for each fixed m and large enough n_m, we have $\|F(f_m - s_{n_m,m})\| \geqslant c/2$.

This shows that we can construct recursively an increasing sequence of integers $n_1 < n_2 < \cdots$ such that the projections $h_k = f_{n_k} - s_{n_{k+1},n_k}$ satisfy $\|F(h_k)\| \geqslant c/2$, $\forall k$. These projections also satisfy $\tau(h_k) \leqslant \tau(f_{n_k}) \to_k 0$. Moreover, by the definitions of h_k and of s_{n_{k+1},n_k}, it follows that $h_k g_{n_{k+1}} h_k = 0$, so in particular $h_k f_{n_l} = 0$ for $l \geqslant k+1$. Thus, $h_k h_l = 0$, $\forall l \geqslant k+1$, i.e., $\{h_k\}_k$ are mutually orthogonal projections. Since $\|F(h_k)\| \geqslant c/2$, $\forall k$, this contradicts the fact that F is s^*-norm continuous on atomic abelian von Neumann subalgebras.

To prove that, if $\{x_n\}_n$ is an arbitrary sequence in $(M)_1$ with $\|x_n\|_2 \to 0$, then $\|F(x_n)\| \to 0$, it is clearly sufficient to show this for $x_n = x_n^*$. Moreover, since $\||x_n|\|_2 = \|x_n\|_2$, it follows that, if $\|x_n\|_2 \to 0$, then $\|(x_n)_+\|_2 \to 0$, $\|(x_n)_-\|_2 \to 0$, showing that it is sufficient to prove the implication for sequences $x_n \in (M_+)_1$, with $\tau(x_n) \to 0$. Let $x_n = \Sigma_{m \geqslant 1} 2^{-m} e_{nm}$, $e_{nm} \in \mathcal{P}(M)$, be the dyadic decomposition of x_n. It follows that $\tau(e_{nm}) \to 0$, $\forall m$. Let $\varepsilon > 0$. Let m_0 be such that $2^{-m_0}(\|F\|+1) \leqslant \varepsilon/2$. By the first part of the proof, it follows that there exists n_0 such that $\|F(e_{nk})\| \leqslant \varepsilon/2$, $\forall n \geqslant n_0$ and $k \leqslant m_0$. Thus, if $n \geqslant n_0$ we have

$$\begin{aligned}\|F(x_n)\| &\leqslant \left(\sum_{k=1}^{m_0} 2^{-k}\|F(e_{nm})\|\right) + \left(\sum_{k>m_0} 2^{-k}\right)\|F(e_{nk})\| \\ &\leqslant \left(\sum_{k=1}^{m_0} 2^{-k}\right)\varepsilon/2 + \left(\sum_{k>m_0} 2^{-k}\right)\|F\| \leqslant \varepsilon. \qquad \square\end{aligned}$$

Remark A.2. If a von Neumann algebra M is normally represented on a Hilbert space $\mathcal{H}$ and we endow $\mathcal{B} = \mathcal{B}(\mathcal{H})$ with the M-bimodule structure given by left–right multiplication by elements in M, and we let $\mathcal{B}_0 \subset s^*(\mathcal{B})$ be the ideal of compact operators, $\mathcal{B}_0 = \mathcal{K}(\mathcal{H})$, then the automatic smoothness of any derivation $\delta : M \to \mathcal{B}_0$ was proved in [29]. The argument consists in reducing the problem to the case when $M \simeq \ell^\infty\mathbb{N}$ (via Lemma A.1), which in turn follows from the innerness of compact-valued derivations on atomic abelian von Neumann algebras, established in [16].

It was then noticed in [39] that in fact ANY weakly continuous linear map $F : M \to \mathcal{K}(\mathcal{H})$ is s^*-norm continuous on atomic abelian von Neumann subalgebras of M, and thus, by p. 224 in [29] (i.e., A.1 above), all such weakly continuous linear compact-valued maps are s^*-norm continuous on countable decomposable finite von Neumann algebras.

This general 'principle' is of course no longer true if we replace $\mathcal{K}(\mathcal{H})$ by the space of all smooth elements $s^*(\mathcal{B}(\mathcal{H}))$. Indeed, we have seen in 1.8(2°) that if M is a finite diffuse von Neumann algebra then the smooth part of $\mathcal{B}(L^2M)$ contains infinite-dimensional projections, and, since $\mathcal{B}_0 = s^*(\mathcal{B}(L^2M))$ is a hereditary C*-algebra, any such projection p satisfies $p\mathcal{B}_0 p \simeq \mathcal{B}(L^2M)$. But there are of course plenty of weakly continuous linear maps from M into $\mathcal{B}(L^2M)$ which are not s^*-norm continuous on all their atomic abelian subalgebras; for instance, the inclusion map $M \subset \mathcal{B}(L^2M)$ does not satisfy this continuity property on any $\ell^\infty\mathbb{N} \hookrightarrow M$.

References

1. C. E. Akemann and P. A. Ostrand, Computing norms in group C^*-algebras, *Amer. J. Math.* **98** (1976), 1015–1047.
2. V. Alekseev, On the first continuous L^2-cohomology of free group factors. arXiv:1312.5647.
3. V. Alekseev and D. Kyed, Measure continuous derivations on von Neumann algebras and applications to L^2-cohomology. arXiv:1110.6155.

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4. U. Bader, T. Gelander and N. Monod, A fixed point theorem for L^1 spaces, *Invent. Math.* **189** (2012), 143–148.
5. M. Bozejko, On $\Lambda(p)$ sets with minimal constant in discrete noncommutative groups, *Proc. Amer. Math. Soc.* **51** (1975), 407–412.
6. E. Christensen, *Derivations and Their Extensions to Perturbations of Operator Algebras*, Proceedings of Symposia in Pure Mathematics, Volume 38, pp. 261–273 (American Mathematical Society, Providence, RI, 1982).
7. A. Connes, *On the Classification of von Neumann Algebras and Their Automorphisms*, Symposia Mathematica, Volume XX, pp. 435–478 (Academic Press, London, 1976).
8. A. Connes and D. Shlyakhtenko, L^2-homology for von Neumann algebras, *J. Reine Angew. Math.* **586** (2005), 125–168.
9. J. Dixmier, *Les algebres d'operateurs dans l'espace hilbertien* (Gauthier-Villars, Paris, 1957), 1996.
10. K. Dykema, Two applications of free entropy, *Math. Ann.* **308** (1997), 545–558.
11. J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras I, II, *Trans. Amer. Math. Soc.* **234** (1977), 289–324. 325–359.
12. D. Gaboriau, Invariants ℓ^2 de rélations d'équivalence et de groupes, *Publ. Math. Inst. Hautes Études Sci.* **95** (2002), 93–150.
13. U. Haagerup, All nuclear C^*-algebras are amenable, *Invent. Math.* **74** (1983), 305–319.
14. A. Ioana, J. Peterson and S. Popa, Amalgamated free products of w-rigid factors and calculation of their symmetry groups, *Acta Math.* **200** (2008), 85–153. arXiv:math.OA/0505589.
15. B. E. Johnson, R. V. Kadison and J. R. Ringrose, Cohomology of operator algebras III: reduction to normal cohomology, *Bull. Soc. Math. France* **100** (1972), 73–96.
16. B. Johnson and S. Parrott, Operators commuting with a von Neumann algebra modulo the set of compact operators, *J. Funct. Anal.* **11** (1972), 39–61.
17. R. V. Kadison and J. R. Ringrose, Cohomology of operator algebras I: type I von Neumann algebras, *Acta Math.* **126** (1971), 227–243.
18. R. V. Kadison and J. R. Ringrose, Cohomology of operator algebras II: extended cobounding and the hyperfinite case, *Ark. Mat.* **9** (1971), 55–63.
19. H. Kesten, Symmetric random walks on groups, *Trans. Amer. Math. Soc.* **92** (1959), 336–354.
20. E. Kirchberg, The derivation problem and the similarity problem are equivalent, *J. Operator Theory* **36** (1996), 59–62.
21. F. Murray and J. von Neumann, Rings of operators IV, *Ann. of Math.* (2) **44** (1943), 716–808.
22. J. Peterson, L^2-rigidity in von Neumann algebras, *Invent. Math.* **175** (2009), 417–433.
23. J. Peterson and A. Thom, Group cocycles and the ring of affiliated operators, *Invent. Math.* **185** (2011), 561–592.
24. G. Pisier, *Factorizations of Linear Operators and the Geometry of Banach Spaces*, CBMS Lecture Notes, Volume 60 (1986).
25. G. Pisier, *Introduction to Operator Space Theory*, London Mathematical Society Lecture Notes, Volume 294 (2013).
26. G. Pisier, *Similarity Problems and Completely Bounded Maps*, Springer Lecture Notes, Volume 1618 (1995).
27. G. Pisier, Private communication.
28. S. Popa, On a problem of RV Kadison on maximal abelian $*$-subalgebras in factors, *Invent. Math.* **65** (1981), 269–281.
29. S. Popa, On derivations into the compacts and some properties of type II factors, in *Proceedings of XIV-th Conference in Operator Theory, Herculane–Timisoara 1983* (ed. I. Gohberg), Volume 14, pp. 221–227 (Birkhäuser Verlag, 1984).

30. S. Popa, The commutant modulo the set of compact operators of a von Neumann algebra, *J. Funct. Anal.* **71** (1987), 393–408.
31. S. Popa, Free independent sequences in type II_1 factors and related problems, *Astérisque* **232** (1995), 187–202.
32. S. Popa, Some computations of 1-cohomology groups and construction of non orbit equivalent actions, *J. Inst. Math. Jussieu* **5** (2006), 309–332. arXiv:mathOA/0407199.
33. S. Popa, A II_1 factor approach to the Kadison–Singer problem, *Comm. Math. Phys.* (2014). arXiv:1303.1424.
34. S. Popa, Independence properties in subalgebras of ultraproduct II_1 factors, *J. Funct. Anal.* **266** (2014), 5818–5846. arXiv:mathOA/1308.3982.
35. S. Popa, Classification of amenable subfactors of type II, *Acta Math.* **172** (1994), 163–255.
36. S. Popa and F. Radulescu, Derivations of von Neumann factors into the compact ideal space of a semifinite algebra, *Duke Math. J.* **57** (1988), 485–518.
37. S. Popa and S. Vaes, Vanishing of the continuous first L^2-cohomology for II_1 factors, *Int. Math. Res. Not. IMRN* (2014). arXiv:mathOA/1401.1186.
38. R. Powers and E. Størmer, Free states of the canonical anticommutation relations, *Comm. Math. Phys.* **16** (1970), 1–33.
39. F. Radulescu, Vanishing of $H^2_w(M, \mathcal{K}(\mathcal{H}))$ for certain finite von Neumann algebras, *Trans. Amer. Math. Soc.* **326** (1991), 569–584.
40. J. Ringrose, Automatic continuity of derivations of operator algebras, *J. Lond. Math. Soc.* **5** (1972), 432–438.
41. A. M. Sinclair and R. R. Smith, *Hochschild Cohomology of von Neumann Algebras*, London Mathematical Society Lecture Note Series, Volume 203 (Cambridge University Press, Cambridge, 1995).
42. M. Takesaki, *Theory of Operator Algebras, I*, Encyclopaedia of Mathematical Sciences, Volume 124 (Springer-Verlag, Berlin, 2002). Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.
43. A. Thom, L^2-cohomology for von Neumann algebras, *GAFA* **18** (2008), 51–70.