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AVERAGES OF EXPONENTIAL TWISTS OF THE VON MANGOLDT FUNCTION

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Abstract

We obtain some improved results for the exponential sum $\sum_{x<n\leq 2x}\Lambda(n)e(\alpha kn^{\theta})$ with $\theta\in(0,5/12)$, where $\Lambda(n)$ is the von Mangoldt function. Such exponential sums have relations with the so-called quasi-Riemann hypothesis and were considered by Murty and Srinivas ['On the uniform distribution of certain sequences', *Ramanujan J.* **7** (2003), 185–192].

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1. Introduction

We are interested in the exponential sum

$$S(k,x,\theta):=\sum_{x<n\leq 2x}\Lambda(n)e(k\alpha n^{\theta}),$$

where $x\geq 2$ and $k\in\mathbb{Z}^{+}$ are the main parameters, $\alpha\neq 0$ and $0<\theta<1$ are fixed, $\Lambda(n)$ is the von Mangoldt function and $e(z)=e^{2\pi iz}$. In [4], Iwaniec *et al.* showed that such exponential sums are connected to the quasi-Riemann hypothesis (or the existence of a zero-free region) for $L(s,f)$, where f is any holomorphic cusp form of integral weight for $\mathrm{SL}(2,\mathbb{Z})$.

We refer to $S(k,x,\theta)$ as Vinogradov's exponential sum, since it was first considered by Vinogradov [8] in the special case $\theta=1/2$. He proved in [8] that, for $k\leq x^{1/10}$,

$$S(k,x,1/2)\ll k^{1/4}x^{7/8+\varepsilon},$$

where $\varepsilon>0$, and the implied constant may depend on α and ε. Iwaniec and Kowalski (see [3, formula (13.55)]) remarked that the stronger inequality

$$S(1,x,1/2)\ll x^{5/6}\log^4 x$$

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follows from an application of Vaughan's identity. For general θ and k, Murty and Srinivas [5] proved that

$$S(k,x,\theta) \ll k^{1/8}x^{(7+\theta)/8}\log(xk^3),$$

where the implied constant may depend on α and θ. In 2006, Ren [6] proved that

$$S(k,x,\theta) \ll (k^{1/2}x^{(1+\theta)/2} + x^{4/5} + k^{-1/2}x^{1-\theta/2})\log^A x,$$

for arbitrary $A > 0$, and that for $\theta \le 1/2$ and $k < x^{1/2-\theta}$,

$$S(k,x,\theta) \ll (k^{1/10}x^{3/4+\theta/10} + k^{-1/2}x^{1-\theta/2})\log^{11} x. \tag{1.1}$$

We prove the following result, which is new for $\theta \in (0, 5/12)$.

THEOREM 1.1. *For $0 < \theta < 5/12$, $\varepsilon > 0$ and $1 \le k < x^{5/12-\theta-\varepsilon}$, there exists an absolute constant $c_0 > 0$ such that*

$$S(k,x,\theta) \ll k^{-1/2}x^{1-\theta/2}\exp(-c_0(\log x)^{1/3-\varepsilon}),$$

where the implied constant may depend on α, θ and ε.

Obviously, when $\theta < 5/12$ and $k < x^{5/12-\theta-\varepsilon}$, Theorem 1.1 improves (1.1). Some much sharper estimates can be obtained if one assumes the zero-density hypothesis,

$$N(\sigma,T) \ll T^{2(1-\sigma)}\log^B T, \quad \text{for all } \sigma \ge 1/2, \tag{1.2}$$

where $N(\sigma,T)$ is the number of zeros of $\zeta(s)$ in the region $\{\sigma \le \Re s \le 1, |t| \le T\}$ and B is some positive constant. Under (1.2), it is proved in [6] that

$$S(k,x,\theta) \ll (k^{1/2}x^{(1+\theta)/2} + k^{-1/2}x^{1-\theta/2})\log^{B+2} x, \tag{1.3}$$

where the implied constant may depend on α, θ and B. Our idea can also be used to improve (1.3).

THEOREM 1.2. *Under (1.2), for $0 < \theta < 1/2$, $\varepsilon > 0$ and $1 \le k < x^{1/2-\theta-\varepsilon}$, there exists an absolute constant c_0 such that*

$$S(k,x,\theta) \ll k^{-1/2}x^{1-\theta/2}\exp(-c_0(\log x)^{1/3-\varepsilon}),$$

where the implied constant may depend on α, ε and θ.

It is worth pointing out that, compared with Theorem 1.1, the ranges of θ and k have been extended in Theorem 1.2.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we will borrow the idea in [6] and use results related to zeros of the Riemann zeta function. The following lemma will be used in the proofs of Theorems 1.1 and 1.2.

LEMMA 2.1 [7, page 71]. *Let $F(u)$ and $G(u)$ be real functions on $[a,b]$, such that $G(u)$ and $1/F'(u)$ are monotone and $|G(u)| \le M$.*

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(1) *If* $F'(u) \geq m > 0$ *or* $F'(u) \leq -m < 0$, *then*

$$\int_a^b G(u)e(F(u))\,du \ll \frac{M}{m}.$$

(2) *If* $F''(u) \geq r > 0$ *or* $F''(u) \leq -r < 0$, *then*

$$\int_a^b G(u)e(F(u))\,du \ll \frac{M}{\sqrt{r}}.$$

PROOF OF THEOREM 1.1. Denote the zeros of $\zeta(s)$ in the critical strip by $\rho = \beta + i\gamma$, where $0 < \beta < 1$, $|\gamma| \leq T$. Using partial summation and the explicit formula in [3, formula (5.53)],

$$\sum_{n\leq x} \Lambda(n) = x - \sum_{|\gamma|\leq T} \frac{x^\rho}{\rho} + O\Big(\frac{x}{T}(\log xT)^2\Big), \quad \text{for } 1 \leq T \leq x.$$

From this formula,

$$\begin{aligned}\sum_{x<n\leq 2x} \Lambda(n)e(k\alpha n^\theta) &= \int_x^{2x} e(k\alpha u^\theta)\,d\sum_{n\leq u}\Lambda(n)\\ &= \int_x^{2x} e(k\alpha u^\theta)\,du - \sum_{|\gamma|\leq T}\int_x^{2x} u^{\rho-1}e(k\alpha u^\theta)\,du\\ &\quad + O\Big((1+k|\alpha|x^\theta)\frac{x\log^2 x}{T}\Big). \end{aligned} \tag{2.1}$$

Setting

$$T = T_0 = x,$$

the error-term is $O((1 + k|\alpha|x^\theta)\log^2 x) = O_\alpha(kx^\theta \log^2 x)$. Moreover,

$$\int_x^{2x} e(k\alpha u^\theta)\,du = \frac{1}{\theta}\int_{x^\theta}^{(2x)^\theta} u^{1/\theta-1}e(k\alpha u)\,du \ll_{\alpha,\theta} k^{-1}x^{1-\theta}. \tag{2.2}$$

Making the change of variable $u^\theta = v$,

$$\int_x^{2x} u^{\rho-1}e(k\alpha u^\theta)\,du = \frac{1}{\theta}\int_{x^\theta}^{(2x)^\theta} v^{\beta/\theta-1}e(f(v))\,dv,$$

where

$$f(v) = k\alpha v + \frac{\gamma}{2\pi\theta}\log v.$$

Trivially,

$$\int_x^{2x} u^{\rho-1}e(k\alpha u^\theta)\,du \ll x^\beta. \tag{2.3}$$

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However,

$$|f'(v)| = \left|k\alpha + \frac{\gamma}{2\pi\theta v}\right| \geq \frac{\min_{v\in[x^\theta,(2x)^\theta]} |\gamma + 2\theta\pi k\alpha v|}{2\pi\theta|v|},$$
$$|f''(v)| = \frac{|\gamma|}{2\pi\theta v^2}.$$

By Lemma 2.1 and (2.3),

$$\int_{x^\theta}^{(2x)^\theta} v^{\beta/\theta-1} e(f(v))\,dv \ll \begin{cases} \dfrac{x^\beta}{\sqrt{1+\theta k|\alpha|x^\theta}} & \text{for } |\gamma| \leq 4(1+\theta\pi k|\alpha|(2x)^\theta), \\ \dfrac{x^\beta}{1+|\gamma|} & \text{for } 4(1+\theta\pi k|\alpha|(2x)^\theta) < |\gamma| \leq T_0. \end{cases}$$

Therefore,

$$\sum_{|\gamma|\leq T} \int_x^{2x} u^{\rho-1} e(k\alpha u^\theta)\,du$$
$$\ll \frac{1}{\sqrt{1+\theta k|\alpha|x^\theta}} \sum_{|\gamma|\leq 4(1+\theta\pi k|\alpha|(2x)^\theta)} x^\beta + \sum_{4(1+\theta\pi k|\alpha|(2x)^\theta)<|\gamma|\leq T_0} \frac{x^\beta}{1+|\gamma|}.$$

Assume that, for some positive constant C,

$$N(\sigma, T) \ll T^{A(\sigma)(1-\sigma)} \log^C T.$$

Then by the Riemann–von Mangoldt formula, for $2 \leq U \leq T_0$,

$$\sum_{|\gamma|\leq U} x^\beta = -\int_0^1 x^\sigma\,dN(\sigma, U)$$
$$\ll x^{1/2} U \log U + (\log U)^C \log x \max_{1/2\leq\sigma\leq\sigma_0} U^{A(\sigma)(1-\sigma)} x^\sigma,$$

where

$$\sigma_0 = 1 - c_0(\log T)^{-2/3}(\log\log T)^{-1/3}$$

with c_0 an absolute positive constant. Here we have used the well-known zero-free region results (for example, see [3, 7]) which state that $\zeta(s) \neq 0$ for $\sigma > \sigma_0$.

Let x be sufficiently large such that $\theta\pi k|\alpha|(2x)^\theta \gg 1$. Then

$$\frac{1}{\sqrt{1+\theta k|\alpha|x^\theta}} \sum_{|\gamma|\leq 4(1+\theta\pi k|\alpha|(2x)^\theta)} x^\beta$$
$$\ll (\log x)^{C+1}(k^{1/2}x^{(1+\theta)/2} + \max_{1/2\leq\sigma\leq\sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{\sigma+\theta A(\sigma)(1-\sigma)-\theta/2}),$$

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and

$$\sum_{4(1+\theta\pi k|\alpha|(2x)^{\theta})\le|\gamma|\le T_0} \frac{x^{\beta}}{1+|\gamma|} \ll (\log x) \max_{4(1+\theta\pi k|\alpha|(2x)^{\theta})<T_1\le T} T_1^{-1} \sum_{T_1\le|\gamma|\le 2T_1} x^{\beta}$$
$$\ll (\log x)^{C+2}(x^{1/2} + \max_{1/2\le\sigma\le\sigma_0} k^{A(\sigma)(1-\sigma)-1} x^{\sigma+\theta A(\sigma)(1-\sigma)-\theta}).$$

Writing

$$g(\sigma) = \sigma + \theta A(\sigma)(1-\sigma) - \frac{\theta}{2}$$

and collecting the above estimates,

$$\sum_{|\gamma|\le T} \int_x^{2x} u^{\rho-1} e(k\alpha u^{\theta})\, du \ll (\log x)^{C+2}(k^{1/2}x^{(1+\theta)/2} + \max_{1/2\le\sigma\le\sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{g(\sigma)}).$$

By the well-known result of Ingham [2] and Huxley [1], we can choose $A(\sigma) = 12/5$. Thus we have

$$\max_{1/2\le\sigma\le\sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{g(\sigma)} \ll (\log x)^{C_1} \sup_{1/2\le\sigma\le\sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{\sigma+12\theta(1-\sigma)/5-\theta/2}$$
$$\ll k^{-1/2}x^{1-\theta/2}(\log x)^{C_1} \sup_{1/2\le\sigma\le\sigma_0} (k^{12/5}x^{12\theta/5-1})^{1-\sigma}.$$

Thus for $\theta < 5/12$ and $k < x^{5/12-\theta-\varepsilon}$,

$$\max_{1/2\le\sigma\le\sigma_0} k^{A(\sigma)(1-\sigma)-1/2} x^{g(\sigma)} \ll k^{-1/2}x^{1-\theta/2}(\log x)^{C_1} \sup_{1/2\le\sigma\le\sigma_0} x^{-c_0(\log x)^{-2/3}(\log\log x)^{-1/3}}$$
$$\ll k^{-1/2}x^{1-\theta/2} \exp(-c_0(\log x)^{1/3}(\log x \log x)^{-1/3})$$
$$\ll k^{-1/2}x^{1-\theta/2} \exp(-c_0(\log x)^{1/3-\varepsilon}).$$

This together with (2.1) and (2.2) shows that, for $\theta \in (0, 5/12)$ and $1 \le k < x^{5/12-\theta-\varepsilon}$,

$$\sum_{x<n\le 2x} \Lambda(n)e(\alpha n^{\theta})$$
$$\ll k^{1/2}x^{(1+\theta)/2}(\log x)^{C} + k^{-1/2}x^{1-\theta/2}\exp(-c_0(\log x)^{1/3-\varepsilon}) + k^{-1}x^{1-\theta} + kx^{\theta}$$
$$\ll k^{-1/2}x^{1-\theta/2}\exp(-c_0(\log x)^{1/3-\varepsilon}).$$

This finishes the proof of Theorem 1.1. □

References

[1] M. N. Huxley, 'On the difference between consecutive primes', *Invent. Math.* **15** (1972), 164–170.

[2] A. E. Ingham, 'On the estimation of $N(\sigma, T)$', *Q. J. Math.* **15** (1940), 291–292.

[3] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, 53 (American Mathematical Society, Providence, RI, 2004).

[4] H. Iwaniec, W. Z. Luo and P. Sarnak, 'Low lying zeros of families of L-functions', *Publ. Math. Inst. Hautes Études Sci.* **91** (2000), 55–131.

[5] M. R. Murty and K. Srinivas, 'On the uniform distribution of certain sequences', *Ramanujan J.* **7** (2003), 185–192.

[6] X. M. Ren, 'Vinogradov's exponential sum over primes', *Acta Arith.* **124** (2006), 269–285.
[7] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd edn (Oxford University Press, Oxford, 1986).
[8] I. M. Vinogradov, *Special Variants of the Method of Trigonometric Sums* (Nauka, Moscow, 1976) (in Russian); English translation I. M. Vinogradov, *Selected Works* (Springer, Berlin, 1985).

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