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FACTORIALS OF INFINITE CARDINALS IN ZF PART II: CONSISTENCY RESULTS

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Abstract. For a set x, let $\mathcal{S}(x)$ be the set of all permutations of x. We prove by the method of permutation models that the following statements are consistent with ZF:

(1) There is an infinite set x such that $|\wp(x)| < |\mathcal{S}(x)| < |\mathrm{seq}^{1\text{-}1}(x)| < |\mathrm{seq}(x)|$, where $\wp(x)$ is the power set of x, $\mathrm{seq}(x)$ is the set of all finite sequences of elements of x, and $\mathrm{seq}^{1\text{-}1}(x)$ is the set of all finite sequences of elements of x without repetition.

(2) There is a Dedekind infinite set x such that $|\mathcal{S}(x)| < |[x]^3|$ and such that there exists a surjection from x onto $\mathcal{S}(x)$.

(3) There is an infinite set x such that there is a finite-to-one function from $\mathcal{S}(x)$ into x.

§1. Introduction. Let x be an arbitrary set and let $\mathfrak{a} = |x|$. Let $\mathcal{S}(x)$ denote the set of all permutations of x and let $\mathfrak{a}!$ denote the cardinality of $\mathcal{S}(x)$. In this second part of our work, we continue to investigate the properties of $\mathfrak{a}!$ for infinite cardinals $\mathfrak{a}$. We prove several consistency results concerning this notion by the method of permutation models.

We mainly consider four permutation models. The first one is the basic Fraenkel model in which several cardinals are shown to be incomparable with $\mathfrak{a}!$ where $\mathfrak{a}$ is the cardinality of the set of atoms. The second one is the ordered Mostowski model in which we have $2^{\mathfrak{a}} < \mathfrak{a}! < \mathrm{seq}^{1\text{-}1}(\mathfrak{a}) < \mathrm{seq}(\mathfrak{a})$, where $\mathfrak{a}$ is the cardinality of the set of atoms, $\mathrm{seq}(\mathfrak{a})$ is the cardinality of the set of all finite sequences of atoms, and $\mathrm{seq}^{1\text{-}1}(\mathfrak{a})$ is the cardinality of the set of all finite sequences of atoms without repetition.

The third one is a Shelah-type permutation model. In this model, $|\mathcal{S}(A)| < |[A]^3|$ and there exists a surjection from A onto $\mathcal{S}(A)$, where A is the set of atoms. Since it follows from Cantor's theorem that there are no surjections from A onto $\wp(A)$, we get that there are no surjections from $\mathcal{S}(A)$ onto $\wp(A)$. This answers the Open problem (8) of [17, Section 4].

The last one is a new permutation model. The atoms of this permutation model form an infinite lattice A with a least element such that every initial segment determined by an element of A is finite and such that every permutation of A moves only finitely many elements. Hence the function that maps each permutation u of A to the least upper bound of the elements moved by u is a finite-to-one function from

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$\mathcal{S}(A)$ into A. Thus we establish the relative consistency of the existence of an infinite set x such that there is a finite-to-one function from $\mathcal{S}(x)$ into x. This answers the Open problem (7) of [17, Section 4].

These consistency results show that most of the ZF results proved in Part I are, in a certain sense, optimal, which will be explained in more detail later. All the preliminaries required in this part can be found in the Section 2 of Part I. However, in order to make this part self-contained, we list some of them in the next section.

§2. Preliminaries. For a set x, we use $|x|$ to denote the cardinality of x. We shall use lower case German letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ for cardinals. For a function f, we shall use $\operatorname{dom}(f)$ for the domain of f, $\operatorname{ran}(f)$ for the range of f, $f[x]$ for the image of x under f, $f^{-1}[x]$ for the inverse image of x under f, and $f{\restriction}x$ for the restriction of f to x. For functions f, g, we use $g \circ f$ for the composition of g and f. We use id_x to denote the identity permutation of x.

Definition 2.1. Let x, y be arbitrary sets, let $\mathfrak{a} = |x|$, and let $\mathfrak{b} = |y|$.

(1) $x \preccurlyeq y$ means that there exists an injection from x into y; $\mathfrak{a} \leqslant \mathfrak{b}$ means that $x \preccurlyeq y$.
(2) $x \preccurlyeq^* y$ means that there exists a surjection from a subset of y onto x; $\mathfrak{a} \leqslant^* \mathfrak{b}$ means that $x \preccurlyeq^* y$.
(3) $\mathfrak{a} \nleqslant \mathfrak{b}$ denotes the negation of $\mathfrak{a} \leqslant \mathfrak{b}$; $\mathfrak{a} \nleqslant^* \mathfrak{b}$ denotes the negation of $\mathfrak{a} \leqslant^* \mathfrak{b}$.
(4) $\mathfrak{a} < \mathfrak{b}$ means that $\mathfrak{a} \leqslant \mathfrak{b}$ and $\mathfrak{b} \nleqslant \mathfrak{a}$; $\mathfrak{a} \parallel \mathfrak{b}$ means that $\mathfrak{a} \nleqslant \mathfrak{b}$ and $\mathfrak{b} \nleqslant \mathfrak{a}$.

Clearly, if $\mathfrak{a} \leqslant \mathfrak{b}$ and $\mathfrak{b} \leqslant \mathfrak{c}$ then $\mathfrak{a} \leqslant \mathfrak{c}$. It is the same case when we replace $\leqslant$ by $\leqslant^*$. It is also clear that if $\mathfrak{a} \leqslant \mathfrak{b}$ then $\mathfrak{a} \leqslant^* \mathfrak{b}$, and that if $\mathfrak{a} \leqslant^* \mathfrak{b}$ then $2^{\mathfrak{a}} \leqslant 2^{\mathfrak{b}}$. It follows from the Cantor-Bernstein Theorem that if $\mathfrak{a} \leqslant \mathfrak{b}$ and $\mathfrak{b} \leqslant \mathfrak{a}$ then $\mathfrak{a} = \mathfrak{b}$.

Definition 2.2. Let x be an arbitrary set and let $\mathfrak{a} = |x|$.

(1) x is *Dedekind infinite* if $\omega \preccurlyeq x$; otherwise x is *Dedekind finite*.
(2) x is *power Dedekind infinite* if $\omega \preccurlyeq \wp(x)$; otherwise x is *power Dedekind finite*.
(3) $\mathfrak{a}$ is *Dedekind infinite* if x is Dedekind infinite; otherwise $\mathfrak{a}$ is *Dedekind finite*.
(4) $\mathfrak{a}$ is *power Dedekind infinite* if x is power Dedekind infinite; otherwise $\mathfrak{a}$ is *power Dedekind finite*.

Definition 2.3. Let f be a function.

(1) f is *finite-to-one* if for all $z \in \operatorname{ran}(f)$, $f^{-1}[\{z\}]$ is finite.
(2) f is *Dedekind finite-to-one* if for all $z \in \operatorname{ran}(f)$, $f^{-1}[\{z\}]$ is Dedekind finite.

Definition 2.4. Let x, y be arbitrary sets, let $\mathfrak{a} = |x|$, and let $\mathfrak{b} = |y|$.

(1) $x \preccurlyeq_{\mathrm{fto}} y$ means that there is a finite-to-one function from x into y; $\mathfrak{a} \leqslant_{\mathrm{fto}} \mathfrak{b}$ means that $x \preccurlyeq_{\mathrm{fto}} y$.
(2) $x \preccurlyeq_{\mathrm{dfto}} y$ means that there exists a Dedekind finite-to-one function from x into y; $\mathfrak{a} \leqslant_{\mathrm{dfto}} \mathfrak{b}$ means that $x \preccurlyeq_{\mathrm{dfto}} y$.
(3) $\mathfrak{a} \nleqslant_{\mathrm{fto}} \mathfrak{b}$ denotes the negation of $\mathfrak{a} \leqslant_{\mathrm{fto}} \mathfrak{b}$; $\mathfrak{a} \nleqslant_{\mathrm{dfto}} \mathfrak{b}$ denotes the negation of $\mathfrak{a} \leqslant_{\mathrm{dfto}} \mathfrak{b}$.

Clearly, if $\mathfrak{a} \leqslant_{\mathrm{fto}} \mathfrak{b}$ and $\mathfrak{b} \leqslant_{\mathrm{fto}} \mathfrak{c}$ then $\mathfrak{a} \leqslant_{\mathrm{fto}} \mathfrak{c}$.

2.1. Some special cardinals. For a permutation f of x, we write $\operatorname{mov}(f)$ for the set $\{z \in x \mid f(z) \neq z\}$ (i.e., the elements of x moved by f).

DEFINITION 2.5. Let x be an arbitrary set and let $\mathfrak{a} = |x|$.

(1) $\mathcal{S}(x) = \{f \mid f$ is a permutation of $x\}$; $\mathfrak{a}! = |\mathcal{S}(x)|$.
(2) $\mathcal{S}_{\mathrm{pdfin}}(x) = \{f \in \mathcal{S}(x) \mid \operatorname{mov}(f)$ is power Dedekind finite$\}$; $\mathcal{S}_{\mathrm{pdfin}}(\mathfrak{a}) = |\mathcal{S}_{\mathrm{pdfin}}(x)|$.
(3) $\mathcal{S}_{\mathrm{fin}}(x) = \{f \in \mathcal{S}(x) \mid \operatorname{mov}(f)$ is finite$\}$; $\mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) = |\mathcal{S}_{\mathrm{fin}}(x)|$.
(4) $\operatorname{pdfin}(x) = \{y \subseteq x \mid y$ is power Dedekind finite$\}$; $\operatorname{pdfin}(\mathfrak{a}) = |\operatorname{pdfin}(x)|$.
(5) $\operatorname{fin}(x) = \{y \subseteq x \mid y$ is finite$\}$; $\operatorname{fin}(\mathfrak{a}) = |\operatorname{fin}(x)|$.
(6) $\operatorname{seq}(x) = \{f \mid f$ is a function from an $n \in \omega$ into $x\}$; $\operatorname{seq}(\mathfrak{a}) = |\operatorname{seq}(x)|$.
(7) $\operatorname{seq}^{1\text{-}1}(x) = \{f \mid f$ is an injection from an $n \in \omega$ into $x\}$; $\operatorname{seq}^{1\text{-}1}(\mathfrak{a}) = |\operatorname{seq}^{1\text{-}1}(x)|$.
(8) $[x]^n = \{y \subseteq x \mid |y| = n\}$; $[\mathfrak{a}]^n = |[x]^n|$.
(9) $\mathcal{S}_n(x) = \{f \in \mathcal{S}(x) \mid |\operatorname{mov}(f)| \leqslant n\}$; $\mathcal{S}_n(\mathfrak{a}) = |\mathcal{S}_n(x)|$.

The following five facts are Facts 2.16–2.19 and Lemma 2.21 of [14], respectively.

FACT 2.6. *For all power Dedekind finite cardinals* $\mathfrak{a}$, $\mathfrak{a}!$ *is Dedekind finite.*

FACT 2.7. *For all cardinals* $\mathfrak{a}$, $\mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) \leqslant_{\mathrm{fto}} \operatorname{fin}(\mathfrak{a})$.

FACT 2.8. *For all cardinals* $\mathfrak{a}$, $\mathcal{S}_{\mathrm{pdfin}}(\mathfrak{a}) \leqslant_{\mathrm{dfto}} \operatorname{pdfin}(\mathfrak{a})$.

FACT 2.9. *For all nonzero cardinals* $\mathfrak{a}$, $\operatorname{seq}(\mathfrak{a})$ *is Dedekind infinite.*

FACT 2.10. *For all nonzero cardinals* $\mathfrak{a}$, $\operatorname{seq}(\operatorname{seq}(\mathfrak{a})) = \operatorname{seq}(\mathfrak{a})$.

For $t \in \operatorname{seq}^{1\text{-}1}(x)$, we use $(t(0);\dots;t(n-1))_x$, where $n = \operatorname{dom}(t)$, to denote the permutation of x which moves $t(0)$ to $t(1)$, $t(1)$ to $t(2)$, ..., $t(n-2)$ to $t(n-1)$, and $t(n-1)$ to $t(0)$, and fixes all other elements of x. In particular, for two distinct elements z, v of x, $(z;v)_x$ is the transposition that interchanges z and v. The following lemma is Lemma 2.26 of [14].

LEMMA 2.11. *For all nonzero cardinals* $\mathfrak{a}$, *if there are* x, r *such that* $|x| = \mathfrak{a}$ *and* r *is an ordering of* x, *then* $\mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) \leqslant \operatorname{seq}^{1\text{-}1}(\mathfrak{a}) \leqslant \operatorname{seq}(\mathfrak{a}) = \aleph_0 \cdot \mathcal{S}_{\mathrm{fin}}(\mathfrak{a})$. *Moreover, if in addition* $\mathfrak{a}$ *is Dedekind finite, then* $\mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) < \operatorname{seq}^{1\text{-}1}(\mathfrak{a}) < \operatorname{seq}(\mathfrak{a}) = \aleph_0 \cdot \mathcal{S}_{\mathrm{fin}}(\mathfrak{a})$.

§3. Permutation models.

We refer the readers to [5, Chapter 8] or [10, Chapter 4] for an introduction to the theory of permutation models. Permutation models are not models of ZF; they are models of ZFA (i.e., the Zermelo-Fraenkel set theory with atoms). Let A be the set of atoms and let $\mathcal{G}$ be a group of permutations of A. We shall write $\operatorname{sym}_{\mathcal{G}}(x)$ for the set $\{\pi \in \mathcal{G} \mid \pi(x) = x\}$. For any subset B of A, we shall write $\operatorname{fix}_{\mathcal{G}}(B)$ for the set $\{\pi \in \mathcal{G} \mid \forall a \in B\,(\pi(a) = a)\}$. Let $\mathcal{I} \subseteq \wp(A)$ be a normal ideal. Then x belongs to the permutation model $\mathcal{V}$ determined by $\mathcal{G}$ and $\mathcal{I}$ if and only if $x \subseteq \mathcal{V}$ and there exists a $B \in \mathcal{I}$ such that $\operatorname{fix}_{\mathcal{G}}(B) \subseteq \operatorname{sym}_{\mathcal{G}}(x)$; such a $B \in \mathcal{I}$ is called a *support* of x. Notice that $\mathcal{I} \in \mathcal{V}$.

Although permutation models are not models of ZF, they indirectly give, via the Jech-Sochor theorem (cf. [5, Theorem 17.2] or [10, Theorem 6.1]), models of ZF. The Jech-Sochor theorem provides embeddings of arbitrarily large initial segments of permutation models into ZF models. All statements whose consistency we prove in the present article depend only on a very small initial segment of the permutation model, so they are preserved by the embedding and we thus obtain their consistency with ZF.

3.1. The basic Fraenkel model. Let the set A of atoms be denumerable, let $\mathcal{G} = \mathcal{S}(A)$, and let $\mathcal{I} = \text{fin}(A)$. The permutation model determined by $\mathcal{G}$ and $\mathcal{I}$ is called the *basic Fraenkel model* (cf. [5, pp. 195–196] or [10, Section 4.3]), and is denoted by $\mathcal{V}_\text{F}$ (F for Fraenkel). In $\mathcal{V}_\text{F}$, A is *amorphous* (cf. [5, Lemma 8.2]); that is, A is infinite but every infinite subset of A is co-finite. Since it is obvious that all amorphous sets are power Dedekind finite, it follows that A is power Dedekind finite, and hence, by Fact 2.6, $\mathcal{S}(A)$ is Dedekind finite.

Lemma 3.1. *Let A be the set of atoms of $\mathcal{V}_\text{F}$ and let $\mathfrak{a} = |A|$. In $\mathcal{V}_\text{F}$,*

(i) $[\mathfrak{a}]^2 \not\leqslant_{\text{fto}} \text{seq}(\mathfrak{a})$;
(ii) $\text{seq}^{1\text{-}1}(\mathfrak{a}) \not\leqslant \mathfrak{a}!$;
(iii) $\mathcal{S}_3(\mathfrak{a}) \not\leqslant 2^{\mathfrak{a}+\aleph_0}$;
(iv) $[\mathfrak{a}]^3 \not\leqslant_{\text{fto}} (\mathfrak{a}+\aleph_0)!$;
(v) $([\mathfrak{a}]^2)^2 \not\leqslant (\mathfrak{a}+\aleph_0)!$.

Proof. (i) Assume towards a contradiction that there exists a finite-to-one function $f \in \mathcal{V}_\text{F}$ from $[A]^2$ into $\text{seq}(A)$. Let $B \in \text{fin}(A)$ be a support of f. Let us fix two distinct elements a, b of $A \setminus B$ and consider the sequence $t = f(\{a,b\})$. If there is an $n \in \text{dom}(t)$ such that $t(n) \in A \setminus (B \cup \{a,b\})$, take an arbitrary $c \in A \setminus (B \cup \{a, b, t(n)\})$ and let $\pi = (t(n); c)_A$. Notice that $\pi \in \text{fix}_\mathcal{G}(B \cup \{a,b\})$ but π moves t, contradicting the assumption that B is a support of f. Thus $t \in \text{seq}(B \cup \{a,b\})$. If there is an $m \in \text{dom}(t)$ such that $t(m) \in \{a,b\}$, then $\sigma = (a;b)_A$ is a member of $\text{fix}_\mathcal{G}(B)$ such that $\sigma(\{a,b\}) = \{a,b\}$ and $\sigma(t) \neq t$, which is also a contradiction. Therefore we have

$$\forall a, b \in A \setminus B \big(a \neq b \to f(\{a,b\}) \in \text{seq}(B)\big). \tag{1}$$

Now, for any $p, q \in [A \setminus B]^2$, since it is easy to see that there is a permutation $\tau \in \text{fix}_\mathcal{G}(B)$ such that $\tau(p) = q$, it follows from (1) that $f(p) = f(q)$. Therefore, f maps all elements of $[A \setminus B]^2$ to the same element of $\text{seq}(B)$, contradicting the fact that $[A \setminus B]^2$ is infinite and f is finite-to-one.

(ii) Assume towards a contradiction that there exists an injection $g \in \mathcal{V}_\text{F}$ from $\text{seq}^{1\text{-}1}(A)$ into $\mathcal{S}(A)$. Let $C \in \text{fin}(A)$ be a support of g. Without loss of generality, assume $C \neq \emptyset$. Let us fix an arbitrary $t \in \text{seq}^{1\text{-}1}(C)$ and consider the permutation $u = g(t)$. If there is a $c \in \text{mov}(u) \setminus C$, take an arbitrary $d \in A \setminus (C \cup \{c, u(c)\})$ and let $\pi = (c; d)_A$. Notice that $\pi \in \text{fix}_\mathcal{G}(C \cup \{u(c)\})$ but π moves u, which is a contradiction. Therefore, for all $t \in \text{seq}^{1\text{-}1}(C)$, we have $\text{mov}(g(t)) \subseteq C$. Thus the function f defined on $\text{seq}^{1\text{-}1}(C)$ given by $f(t) = g(t){\restriction}C$ is an injection from $\text{seq}^{1\text{-}1}(C)$ into $\mathcal{S}(C)$. Hence, if we take $n = |C|$, then $n \neq 0$ and $\text{seq}^{1\text{-}1}(n) \leqslant n!$, which is absurd.

(iii) Assume towards a contradiction that there exists an injection $h \in \mathcal{V}_\text{F}$ from $\mathcal{S}_3(A)$ into $\wp(A \cup \omega)$. Let $D \in \text{fin}(A)$ be a support of h. Take three distinct elements a, b, c of $A \setminus D$, let $\pi = (a; b; c)_A$, and let $\sigma = (b; a; c)_A$. Then $\pi, \sigma \in \text{fix}_\mathcal{G}(D)$, and hence $\pi(h) = \sigma(h) = h$. Since $\pi(\pi) = \sigma(\pi) = \pi$, we get $\pi(h(\pi)) = \sigma(h(\pi)) = h(\pi)$. Hence, if $a \in h(\pi)$ then $b = \pi(a) \in h(\pi)$, and if $b \in h(\pi)$ then $a = \sigma(b) \in h(\pi)$; that is, $a \in h(\pi) \leftrightarrow b \in h(\pi)$. Thus, if we set $\tau = (a; b)_A$, then $\tau(h(\pi)) = h(\pi)$. Since

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$\tau \in \mathrm{fix}_{\mathcal{G}}(D)$, it follows that $\tau(h) = h$, and hence $h(\pi) = \tau(h(\pi)) = h(\tau(\pi)) = h(\sigma)$, contradicting that h is injective.

(iv) Assume towards a contradiction that there exists a finite-to-one function $f \in \mathcal{V}_{\mathrm{F}}$ from $[A]^3$ into $\mathcal{S}(A \cup \omega)$. Let $B \in \mathrm{fin}(A)$ be a support of f. Let us fix three distinct elements a, b, c of $A \setminus B$ and consider the permutation $u = f(\{a, b, c\})$. If there is a $d \in \mathrm{mov}(u) \setminus (B \cup \omega \cup \{a, b, c\})$, take an arbitrary $e \in A \setminus (B \cup \{a, b, c, d, u(d)\})$ and let $\pi = (d; e)_A$. Notice that $\pi \in \mathrm{fix}_{\mathcal{G}}(B \cup \{a, b, c\})$, $\pi(d) \neq d$, and $\pi(u(d)) = u(d)$. Hence π moves u, contradicting the assumption that B is a support of f. Therefore, we have that $\mathrm{mov}(u) \subseteq B \cup \omega \cup \{a, b, c\}$. If there is a $v \in \mathrm{mov}(u) \cap \{a, b, c\}$, then, since $\{a, b, c\} \setminus \{v, u(v)\} \neq \emptyset$, take a $w \in \{a, b, c\} \setminus \{v, u(v)\}$ and let $\sigma = (v; w)_A$. Notice that $\sigma \in \mathrm{fix}_{\mathcal{G}}(B)$, $\sigma(\{a, b, c\}) = \{a, b, c\}$, and $\sigma(u) \neq u$, which is also a contradiction. Therefore $\mathrm{mov}(u) \subseteq B \cup \omega$. Thus we have

$$\forall t \in [A \setminus B]^3 \left(\mathrm{mov}(f(t)) \subseteq B \cup \omega\right). \tag{2}$$

Now, for any $p, q \in [A \setminus B]^3$, since it is easy to see that there is a permutation $\tau \in \mathrm{fix}_{\mathcal{G}}(B)$ such that $\tau(p) = q$, it follows from (2) that $f(p) = f(q)$. Therefore, f maps all elements of $[A \setminus B]^3$ to the same element of $\mathcal{S}(A \cup \omega)$, contradicting the fact that $[A \setminus B]^3$ is infinite and f is finite-to-one.

(v) Assume towards a contradiction that there is an injection $g \in \mathcal{V}_{\mathrm{F}}$ from $[A]^2 \times [A]^2$ into $\mathcal{S}(A \cup \omega)$. Let $C \in \mathrm{fin}(A)$ be a support of g. Take four distinct $a_0, a_1, b_0, b_1 \in A \setminus C$, and let $u = g(\{a_0, a_1\}, \{b_0, b_1\})$. If there is a $c \in \mathrm{mov}(u) \setminus (C \cup \omega \cup \{a_0, a_1, b_0, b_1\})$, take an arbitrary $d \in A \setminus (C \cup \{a_0, a_1, b_0, b_1, c, u(c)\})$ and let $\pi = (c; d)_A$. Then we have $\pi \in \mathrm{fix}_{\mathcal{G}}(C \cup \{a_0, a_1, b_0, b_1\})$, $\pi(c) \neq c$, and $\pi(u(c)) = u(c)$. Thus π moves u, contradicting the assumption that C is a support of g. Therefore we have

$$\mathrm{mov}(u) \subseteq C \cup \omega \cup \{a_0, a_1, b_0, b_1\}. \tag{3}$$

We claim that

$$\forall i \leqslant 1 \left(u(a_i) = a_{1-i} \text{ and } u(b_i) = b_{1-i}\right). \tag{4}$$

In fact, if $u(a_i) \notin \{a_0, a_1\}$, then $(a_0; a_1)_A \in \mathrm{fix}_{\mathcal{G}}(C)$ fixes $(\{a_0, a_1\}, \{b_0, b_1\})$ but moves u, contradicting that C is a support of g. Thus $u(a_i) \in \{a_0, a_1\}$. Moreover, $u(a_i) \neq a_i$, since otherwise, if we take an arbitrary $e \in A \setminus (C \cup \{a_0, a_1, b_0, b_1\})$, then, by (3), $u(e) = e$, and therefore $(a_i; e)_A \in \mathrm{fix}_{\mathcal{G}}(C)$ fixes u but moves $(\{a_0, a_1\}, \{b_0, b_1\})$, contradicting that g is injective. Thus $u(a_i) = a_{1-i}$. Similarly $u(b_i) = b_{1-i}$, and therefore (4) is proved. Hence, if we set $\sigma = (a_0; b_0)_A \circ (a_1; b_1)_A$, then it follows from (4) that $\sigma(u) = u$, but $\sigma(\{a_0, a_1\}, \{b_0, b_1\}) = (\{b_0, b_1\}, \{a_0, a_1\}) \neq (\{a_0, a_1\}, \{b_0, b_1\})$, contradicting again the assumption that g is injective. ⊣

Now we derive some consistency results from Lemma 3.1.

PROPOSITION 3.2. *The following statements are consistent with* ZF:

(i) *There is an infinite cardinal* $\mathfrak{a}$ *such that* $\mathfrak{a}! \parallel \mathrm{seq}^{1\text{-}1}(\mathfrak{a})$ *and* $\mathfrak{a}! \parallel \mathrm{seq}(\mathfrak{a})$.

(ii) *There is a Dedekind infinite cardinal* $\mathfrak{b}$ *such that* $\mathfrak{b}! \parallel 2^{\mathfrak{b}}$, $\mathfrak{b}! \parallel [\mathfrak{b}]^3$, *and* $[\mathfrak{b}]^3 \not\leqslant_{\mathrm{fto}} \mathfrak{b}!$.

(iii) *There is a Dedekind infinite cardinal* $\mathfrak{c}$ *such that* $([\mathfrak{c}]^2)^2 \not\leqslant \mathfrak{c}!$.

(iv) *There is a Dedekind infinite cardinal* $\mathfrak{d}$ *such that* $\left[\left[[\mathfrak{d}]^2\right]^2\right]^2 \not\leqslant \mathfrak{d}!$.

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PROOF. By the Jech-Sochor theorem, it suffices to show that there are such cardinals in $\mathcal{V}_{\mathrm{F}}$. Let A be the set of atoms of $\mathcal{V}_{\mathrm{F}}$ and let $\mathfrak{a} = |A|$.

(i) Notice that $\mathrm{seq}^{\text{1-1}}(\mathfrak{a}) \leqslant \mathrm{seq}(\mathfrak{a})$ and $[\mathfrak{a}]^2 \leqslant \mathfrak{a}!$. By Lemma 3.1(i), $[\mathfrak{a}]^2 \nleqslant \mathrm{seq}(\mathfrak{a})$, which implies that $\mathfrak{a}! \nleqslant \mathrm{seq}(\mathfrak{a})$ and $\mathfrak{a}! \nleqslant \mathrm{seq}^{\text{1-1}}(\mathfrak{a})$. By Lemma 3.1(ii), $\mathrm{seq}^{\text{1-1}}(\mathfrak{a}) \nleqslant \mathfrak{a}!$, and hence $\mathrm{seq}(\mathfrak{a}) \nleqslant \mathfrak{a}!$, which completes the proof of (i).

(ii) Let $\mathfrak{b} = \mathfrak{a} + \aleph_0$. Notice that $\mathfrak{b}$ is Dedekind infinite, $[\mathfrak{b}]^3 \leqslant 2^{\mathfrak{b}}$, and $\mathcal{S}_3(\mathfrak{b}) \leqslant \mathfrak{b}!$. By Lemma 3.1(iii), $\mathcal{S}_3(\mathfrak{b}) \nleqslant 2^{\mathfrak{b}}$, and thus $\mathfrak{b}! \nleqslant 2^{\mathfrak{b}}$ and $\mathfrak{b}! \nleqslant [\mathfrak{b}]^3$. By Lemma 3.1(iv), $[\mathfrak{b}]^3 \nleqslant_{\mathrm{fto}} \mathfrak{b}!$, and hence $[\mathfrak{b}]^3 \nleqslant \mathfrak{b}!$ and $2^{\mathfrak{b}} \nleqslant \mathfrak{b}!$, which completes the proof of (ii).

(iii) Let $\mathfrak{c} = \mathfrak{a} + \aleph_0$. By Lemma 3.1(v), $([\mathfrak{c}]^2)^2 \nleqslant \mathfrak{c}!$.

(iv) Let $\mathfrak{d} = \mathfrak{a} + \aleph_0$. For every set x, since all elements of x^2 are 2-element subsets of $2 \times x$, it follows that $x^2 \subseteq [2 \times x]^2$. Since it is easy to verify that $2 \times y \preccurlyeq [y]^2$ for any infinite set y, we have

$$\left([\mathfrak{d}]^2\right)^2 \leqslant \left[2 \cdot [\mathfrak{d}]^2\right]^2 \leqslant \left[\left[[\mathfrak{d}]^2\right]^2\right]^2.$$

Now $\left[\left[[\mathfrak{d}]^2\right]^2\right]^2 \nleqslant \mathfrak{d}!$ follows from (iii). ⊣

REMARK 3.3. It is provable in ZF that for all infinite cardinals $\mathfrak{a}$ and all $n \in \omega$, $\mathfrak{a}^n \leqslant \mathrm{seq}^{\text{1-1}}(\mathfrak{a})$ (cf. [11, Lemma 2.5]). Proposition 3.2(i) shows that, in Corollary 3.29 of [14], we cannot replace $\mathfrak{a}^n$ by $\mathrm{seq}^{\text{1-1}}(\mathfrak{a})$. Proposition 3.2(ii)–(iv) show that, in Corollary 3.26 of [14], we cannot replace $\left[[\mathfrak{a}]^2\right]^2$ by $[\mathfrak{a}]^3$, $([\mathfrak{a}]^2)^2$, or $\left[\left[[\mathfrak{a}]^2\right]^2\right]^2$, even for Dedekind infinite cardinals $\mathfrak{a}$. Proposition 3.2(iii) also shows that, in Theorem 3.14 of [14], we cannot conclude that $\mathrm{seq}(\mathcal{S}_{\mathrm{pdfin}}(\mathfrak{a})) < \mathfrak{a}!$, since $([\mathfrak{a}]^2)^2 \leqslant \mathrm{seq}(\mathcal{S}_{\mathrm{pdfin}}(\mathfrak{a}))$.

PROPOSITION 3.4. *The following statement is consistent with* ZF: *There exists a Dedekind infinite cardinal* $\mathfrak{b}$ *such that* $\mathrm{seq}(\mathfrak{b}) < [\mathfrak{b}]^2$ *and* $[\mathfrak{b}]^2 \nleqslant_{\mathrm{fto}} \mathrm{seq}(\mathfrak{b})$.

PROOF. Let A be the set of atoms of $\mathcal{V}_{\mathrm{F}}$, let $\mathfrak{a} = |A|$, and let $\mathfrak{b} = \mathrm{seq}(\mathfrak{a})$. Then by Fact 2.9, $\mathfrak{b}$ is Dedekind infinite, and by Fact 2.10, $\mathrm{seq}(\mathfrak{b}) = \mathfrak{b}$. By Lemma 3.1(i), $[\mathfrak{a}]^2 \nleqslant_{\mathrm{fto}} \mathfrak{b}$, which implies that $[\mathfrak{b}]^2 \nleqslant_{\mathrm{fto}} \mathfrak{b}$ and thus $\mathfrak{b} < [\mathfrak{b}]^2$. Therefore, we get that $\mathrm{seq}(\mathfrak{b}) = \mathfrak{b} < [\mathfrak{b}]^2$ and $[\mathfrak{b}]^2 \nleqslant_{\mathrm{fto}} \mathfrak{b} = \mathrm{seq}(\mathfrak{b})$. ⊣

3.2. The ordered Mostowski model. Let the set A of atoms be denumerable, and let $<_{\mathrm{M}}$ be an ordering of A with order type that of the rational numbers. Let $\mathcal{G}$ be the group of all automorphisms of $\langle A, <_{\mathrm{M}}\rangle$ and let $\mathcal{I} = \mathrm{fin}(A)$. The permutation model determined by $\mathcal{G}$ and $\mathcal{I}$ is called the *ordered Mostowski model* (cf. [5, pp. 198–202] or [10, Section 4.5]), and is denoted by $\mathcal{V}_{\mathrm{M}}$ (M for Mostowski). Clearly, the relation $<_{\mathrm{M}}$ belongs to the model $\mathcal{V}_{\mathrm{M}}$ (cf. [5, Lemma 8.10]). In $\mathcal{V}_{\mathrm{M}}$, A is infinite but power Dedekind finite (cf. [5, Lemma 8.13]), and therefore, by Fact 2.6, $\mathcal{S}(A)$ is Dedekind finite.

FACT 3.5. *Let A be the set of atoms of $\mathcal{V}_{\mathrm{M}}$. In $\mathcal{V}_{\mathrm{M}}$, $\mathcal{S}_{\mathrm{fin}}(A) = \mathcal{S}(A)$.*

PROOF. Let $f \in \mathcal{V}_{\mathrm{M}}$ be a permutation of A, and let $B \in \mathrm{fin}(A)$ be a support of f. If there exists an $a \in \mathrm{mov}(f) \setminus B$, then take a $\pi \in \mathrm{fix}_{\mathcal{G}}(B \cup \{f(a)\})$ such that $\pi(a) \neq a$. Thus π moves f, contradicting the assumption that B is a support of f. Therefore $\mathrm{mov}(f) \subseteq B$, and hence $f \in \mathcal{S}_{\mathrm{fin}}(A)$. ⊣

PROPOSITION 3.6. *The following statement is consistent with* ZF: *There exists an infinite cardinal* $\mathfrak{a}$ *such that* $2^{\mathfrak{a}} < \mathfrak{a}! < \mathrm{seq}^{\text{1-1}}(\mathfrak{a}) < \mathrm{seq}(\mathfrak{a}) = \aleph_0 \cdot \mathfrak{a}!$.

PROOF. Let A be the set of atoms of $\mathcal{V}_{\mathrm{M}}$ and let $\mathfrak{a} = |A|$. In $\mathcal{V}_{\mathrm{M}}$, $<_{\mathrm{M}}$ is an ordering of A. Since A is Dedekind finite, by Lemma 2.11, $\mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) < \mathrm{seq}^{\text{1-1}}(\mathfrak{a}) < \mathrm{seq}(\mathfrak{a}) =$

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$\aleph_0 \cdot \mathcal{S}_{\text{fin}}(\mathfrak{a})$. By Fact 3.5, we have $\mathcal{S}_{\text{fin}}(\mathfrak{a}) = \mathfrak{a}!$, which implies that $\mathfrak{a}! < \text{seq}^{1\text{-}1}(\mathfrak{a}) < \text{seq}(\mathfrak{a}) = \aleph_0 \cdot \mathfrak{a}!$. Finally, $2^{\mathfrak{a}} < \mathfrak{a}!$ was proved in [2]. ⊣

REMARK 3.7. Fact 3.5 shows that, in Corollary 3.21 of [14], the requirement $\mathcal{S}_{\text{fin}}(x) \neq \mathcal{S}(x)$ cannot be replaced by the requirement that x is infinite. Proposition 3.6 shows that, in Corollary 3.16 of [14], the requirement that $\mathfrak{a}!$ is Dedekind infinite cannot be replaced by the requirement that $\mathfrak{a}$ is infinite. Propositions 3.2(i) and 3.6 show that, for an arbitrary infinite cardinal $\mathfrak{a}$, we cannot conclude any relationship between $\mathfrak{a}!$ and $\text{seq}(\mathfrak{a})$ except for Corollary 3.17 of [14].

For more cardinal relations that hold in $\mathcal{V}_{\text{M}}$, see [8, p. 249].

3.3. A Shelah-type permutation model. In [7, Section 1], Shelah constructed a permutation model in which there is an infinite cardinal $\mathfrak{a}$ such that $\text{seq}(\mathfrak{a}) < \text{fin}(\mathfrak{a})$. Later, in [8, Section 7.3], a similar model was constructed in order to show that the existence of an infinite cardinal $\mathfrak{a}$ such that $\mathfrak{a}^2 < [\mathfrak{a}]^2$ is consistent with ZF. Recently, in [6], Halbeisen generalized these two results by proving that the existence of an infinite cardinal $\mathfrak{a}$ such that $\text{seq}(\mathfrak{a}) < [\mathfrak{a}]^2$ and $[\mathfrak{a}]^2 \not\leqslant_{\text{fto}} \text{seq}(\mathfrak{a})$ is consistent with ZF. These permutation models are called *Shelah-type permutation models* (cf. [5, pp. 209–211]). The atoms of Shelah-type permutation models are always constructed by recursion, where every atom encodes certain sets of atoms on a lower level.

Here we construct a Shelah-type permutation model in which there exists a Dedekind infinite cardinal $\mathfrak{a}$ such that $\mathfrak{a}! < [\mathfrak{a}]^3$, $[\mathfrak{a}]^3 \not\leqslant_{\text{dfto}} \mathfrak{a}!$, and $\mathfrak{a}! \leqslant^* \mathfrak{a}$. Proposition 3.4 shows that, in the basic Fraenkel model, there already exists a Dedekind infinite cardinal $\mathfrak{b}$ such that $\text{seq}(\mathfrak{b}) < [\mathfrak{b}]^2$ and $[\mathfrak{b}]^2 \not\leqslant_{\text{fto}} \text{seq}(\mathfrak{b})$. Hence, in such a case, we do not really need to construct new models. However, for our purpose here, the proof of Proposition 3.4 does not work, because, unlike the case for $\text{seq}(\mathfrak{a})$, $(\mathfrak{a}!)! = \mathfrak{a}!$ does not hold; in fact, $\mathfrak{a}! < (\mathfrak{a}!)!$ for any infinite cardinal $\mathfrak{a}$.

In this subsection, we shall work in ZFA + AC. For a set x, let $\mathcal{S}_{\text{ctbl}}(x)$ be the set of all permutations of x which move only countably many elements. The atoms of this Shelah-type permutation model are constructed as follows:

(i) A_0 is an arbitrary uncountable set of atoms.
(ii) $\mathcal{G}_0 = \mathcal{S}(A_0)$.
(iii) $A_{n+1} = A_n \cup \{(n, u, i) \mid u \in \mathcal{S}_{\text{ctbl}}(A_n) \text{ and } i < 3\}$.
(iv) $\mathcal{G}_{n+1}$ is the subgroup of $\mathcal{S}(A_{n+1})$ such that for all $h \in \mathcal{S}(A_{n+1})$, $h \in \mathcal{G}_{n+1}$ if and only if there exists a $g \in \mathcal{G}_n$ such that
- $g = h{\restriction}A_n$;
- for all $u \in \mathcal{S}_{\text{ctbl}}(A_n)$, there is a permutation p of $\{0, 1, 2\}$ such that $h(n, u, i) = (n, g \circ u \circ g^{-1}, p(i))$ for any $i < 3$.

Let $A = \bigcup_{n\in\omega} A_n$. For each triple $(n, u, i) \in A$ we assign a new atom $a_{n,u,i}$ and define the set of atoms by stipulating $\tilde{A} = A_0 \cup \{a_{n,u,i} \mid (n, u, i) \in A\}$. However, for the sake of simplicity, we shall work with A as the set of atoms rather than with $\tilde{A}$. Now, let

$$\mathcal{G} = \{\pi \in \mathcal{S}(A) \mid \forall n \in \omega \, (\pi{\restriction}A_n \in \mathcal{G}_n)\},$$

and let

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$$\mathcal{I} = \big\{ B \subseteq A \bigm| \exists n \in \omega \, \big(B \text{ is a countable subset of } A_n \big) \big\}.$$

Clearly, $\mathcal{G}$ is a group of permutations of A and $\mathcal{I}$ is a normal ideal. The permutation model determined by $\mathcal{G}$ and $\mathcal{I}$ is denoted by $\mathcal{V}_{\mathrm{S}}$ (S for Shelah).

We say that a subset C of A is *closed* if for all triples $(n, u, i) \in C$, $\mathrm{mov}(u) \subseteq C$ and $\{(n, u, j) \mid j < 3\} \subseteq C$. The *closure* of $B \subseteq A$ is the least closed set that includes B. Since we are working in ZFA + AC, it is easy to verify that the closure of a countable subset of A is also countable, and therefore for all $B \in \mathcal{I}$, the closure of B belongs to $\mathcal{I}$.

LEMMA 3.8. *For all closed subsets C of A and all $m \in \omega$, every $g \in \mathcal{G}_m$ fixing $C \cap A_m$ pointwise extends to a permutation $\pi \in \mathrm{fix}_{\mathcal{G}}(C)$.*

PROOF. Define $h_n \in \mathcal{G}_{m+n}$ by recursion on n as follows: $h_0 = g$; h_{n+1} is the permutation of A_{m+n+1} such that $h_n = h_{n+1} \restriction A_{m+n}$ and such that for all $u \in \mathcal{S}_{\mathrm{ctbl}}(A_{m+n})$, we have $h_{n+1}(m + n, u, i) = (m + n, h_n \circ u \circ h_n^{-1}, i)$ for any $i < 3$. Now we prove by induction on n that h_n fixes $C \cap A_{m+n}$ pointwise. By the assumption, h_0 fixes $C \cap A_m$ pointwise. Assume, as an induction hypothesis, that h_n fixes $C \cap A_{m+n}$ pointwise. Then h_{n+1} fixes $C \cap A_{m+n}$ pointwise, since h_{n+1} extends h_n. For any $(m+n, u, i) \in C$, since C is closed, we have $\mathrm{mov}(u) \subseteq C \cap A_{m+n}$, and therefore $h_n \circ u \circ h_n^{-1} = u$, which implies that $h_{n+1}(m + n, u, i) = (m + n, u, i)$. Hence h_{n+1} fixes $C \cap A_{m+n+1}$ pointwise. Let $\pi = \bigcup_{n \in \omega} h_n$. Then $\pi \in \mathcal{G}$ extends g and fixes C pointwise. ⊣

LEMMA 3.9. *For all closed subsets C of A and all $n \in \omega$, if a, b are two distinct elements of A such that $a \in A_{n+1} \setminus (A_n \cup C)$ and $b \in A_{n+1} \cup C$, then there exists a permutation $\pi \in \mathrm{fix}_{\mathcal{G}}(C \cup A_n \cup \{b\})$ such that $\pi(a) \neq a$.*

PROOF. Let $a = (n, t, j)$, where $t \in \mathcal{S}_{\mathrm{ctbl}}(A_n)$ and $j < 3$. Let $l < 3$ be the least natural number such that $(n, t, l) \notin \{a, b\}$ and let $p = (j; l)_3$. Since $a \notin C$ and C is closed, $(n, t, l) \notin C$. Let g be the permutation of A_{n+1} such that g fixes A_n pointwise and for all $u \in \mathcal{S}_{\mathrm{ctbl}}(A_n)$ and all $i < 3$, $g(n, u, i) = (n, u, p(i))$, if $u = t$, and $g(n, u, i) = (n, u, i)$, otherwise. Then $g \in \mathcal{G}_{n+1}$ fixes $A_{n+1} \setminus \{a, (n, t, l)\}$ pointwise. By Lemma 3.8, g extends to some $\pi \in \mathrm{fix}_{\mathcal{G}}(C)$. Then $\pi \in \mathrm{fix}_{\mathcal{G}}(C \cup A_n \cup \{b\})$ and $\pi(a) = (n, t, l) \neq a$. ⊣

LEMMA 3.10. *In $\mathcal{V}_{\mathrm{S}}$, $\mathcal{S}(A) = \{u \in \mathcal{S}(A) \mid \mathrm{mov}(u) \in \mathcal{I}\}$.*

PROOF. Let $u \in \mathcal{V}_{\mathrm{S}}$ be a permutation of A, and let $B \in \mathcal{I}$ be a support of u. Let C be the closure of B. Then we have $C \in \mathcal{I}$. Assume towards a contradiction that there exists an $a \in \mathrm{mov}(u) \setminus C$. Let $b = u(a) \neq a$.

If $a \in A_0$ and $b \in A_0 \cup C$, take an arbitrary $c \in A_0 \setminus (C \cup \{a, b\})$ and let $g = (a; c)_{A_0}$. By Lemma 3.8, g extends to some $\pi \in \mathrm{fix}_{\mathcal{G}}(C)$. Then $\pi(a) = c \neq a$ and $\pi(b) = b$. Hence π moves u, contradicting the assumption that B is a support of u.

If there is an $n \in \omega$ such that $a \in A_{n+1} \setminus A_n$ and $b \in A_{n+1} \cup C$, then by Lemma 3.9, there is a permutation $\sigma \in \mathrm{fix}_{\mathcal{G}}(C \cup \{b\})$ such that $\sigma(a) \neq a$. Therefore σ moves u, contradicting the assumption that B is a support of u.

Thus, $b \notin C$ and there exists an $m \in \omega$ such that $b \in A_{m+1} \setminus A_m$ and $a \in A_m$. Again by Lemma 3.9, there is a permutation $\tau \in \mathrm{fix}_{\mathcal{G}}(C \cup \{a\})$ such that $\tau(b) \neq b$. Hence τ moves u, which is also a contradiction.

Therefore, we have $\mathrm{mov}(u) \subseteq C$, and hence $\mathrm{mov}(u) \in \mathcal{I}$. ⊣

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For all $n \in \omega$, since $\pi[A_n] = A_n$ for any $\pi \in \mathcal{G}$, it follows that $A_n \in \mathcal{V}_{\mathrm{S}}$, and therefore the function that maps each $n \in \omega$ to A_n belongs to $\mathcal{V}_{\mathrm{S}}$. For every $B \in \mathcal{I}$, let k_B be the least $n \in \omega$ such that $B \subseteq A_n$. Since $k_B = k_{\pi[B]}$ for all $B \in \mathcal{I}$ and all $\pi \in \mathcal{G}$, it follows that the function that maps each $B \in \mathcal{I}$ to k_B belongs to $\mathcal{V}_{\mathrm{S}}$.

LEMMA 3.11. *Let A be the set of atoms of $\mathcal{V}_{\mathrm{S}}$ and let $\mathfrak{a} = |A|$. In $\mathcal{V}_{\mathrm{S}}$,*

(i) $\mathfrak{a}$ *is Dedekind infinite;*
(ii) $\mathfrak{a}! \leqslant [\mathfrak{a}]^3$ *and* $\mathfrak{a}! \leqslant^* \mathfrak{a}$;
(iii) $[\mathfrak{a}]^3 \nleqslant_{\mathrm{dfto}} \mathfrak{a}!$.

PROOF. (i) Let q be an injection from ω into A_0. Then $\operatorname{ran}(q) \in \mathcal{I}$, and therefore $q \in \mathcal{V}_{\mathrm{S}}$. Hence, in $\mathcal{V}_{\mathrm{S}}$, A is Dedekind infinite.

(ii) Let Φ be the function defined on $\{u \in \mathcal{S}(A) \mid \operatorname{mov}(u) \in \mathcal{I}\}$ given by

$$\Phi(u) = \bigl\{(k_{\operatorname{mov}(u)}, u{\restriction}A_{k_{\operatorname{mov}(u)}}, i) \bigm| i < 3\bigr\}.$$

Then Φ is an injection from $\{u \in \mathcal{S}(A) \mid \operatorname{mov}(u) \in \mathcal{I}\}$ into $[A]^3$ and the sets in the range of Φ are pairwise disjoint. It is easy to verify that $\Phi \in \mathcal{V}_{\mathrm{S}}$. In $\mathcal{V}_{\mathrm{S}}$, by Lemma 3.10, $\mathcal{S}(A) = \{u \in \mathcal{S}(A) \mid \operatorname{mov}(u) \in \mathcal{I}\}$, and hence Φ is an injection from $\mathcal{S}(A)$ into $[A]^3$, which implies that $\mathfrak{a}! \leqslant [\mathfrak{a}]^3$. Since the sets in the range of Φ are pairwise disjoint, it follows that $\mathfrak{a}! \leqslant^* \mathfrak{a}$.

(iii) Assume towards a contradiction that there is a function $f \in \mathcal{V}_{\mathrm{S}}$ from $[A]^3$ into $\mathcal{S}(A)$ such that in $\mathcal{V}_{\mathrm{S}}$,

$$f \text{ is a Dedekind finite-to-one function.} \tag{5}$$

Let $B \in \mathcal{I}$ be a support of f, and let C be the closure of B. Then $C \in \mathcal{I}$.

Let us now fix three distinct elements a, b, c of $A_0 \setminus C$ and consider the permutation $u = f(\{a, b, c\})$. We claim that

$$\operatorname{mov}(u) \subseteq C \cup A_0. \tag{6}$$

Assume towards a contradiction that there exists a $d \in \operatorname{mov}(u) \setminus (C \cup A_0)$.

If there is an $n \in \omega$ such that $d \in A_{n+1} \setminus A_n$ and $u(d) \in A_{n+1} \cup C$, then by Lemma 3.9, there exists a permutation $\pi_0 \in \operatorname{fix}_{\mathcal{G}}(C \cup A_0 \cup \{u(d)\})$ such that $\pi_0(d) \neq d$. Hence π_0 fixes $\{a, b, c\}$ but moves u, contradicting the assumption that B is a support of f.

Thus, $u(d) \notin C$ and there exists an $m \in \omega$ such that $u(d) \in A_{m+1} \setminus A_m$ and $d \in A_m$. By Lemma 3.9, there is a permutation $\pi_1 \in \operatorname{fix}_{\mathcal{G}}(C \cup A_0 \cup \{d\})$ such that $\pi_1(u(d)) \neq u(d)$. Thus π_1 fixes $\{a, b, c\}$ but moves u, contradicting again the assumption that B is a support of f. Hence (6) is proved.

If there exists an $e \in \operatorname{mov}(u) \setminus (C \cup \{a, b, c\})$, then $u(e) \in \operatorname{mov}(u)$, and therefore it follows from (6) that $e \in A_0 \setminus (C \cup \{a, b, c\})$ and $u(e) \in C \cup A_0$. Take an arbitrary $v \in A_0 \setminus (C \cup \{a, b, c, e, u(e)\})$ and let $g_0 = (e; v)_{A_0}$. By Lemma 3.8, g_0 extends to some $\sigma_0 \in \operatorname{fix}_{\mathcal{G}}(C)$. Then $\sigma_0 \in \operatorname{fix}_{\mathcal{G}}(C \cup \{a, b, c\})$, $\sigma_0(e) = v \neq e$, and $\sigma_0(u(e)) = u(e)$. Hence σ_0 fixes $\{a, b, c\}$ but moves u, contradicting that B is a support of f. Therefore $\operatorname{mov}(u) \subseteq C \cup \{a, b, c\}$.

If there exists a $z \in \operatorname{mov}(u) \cap \{a, b, c\}$, then $u(z) \in \operatorname{mov}(u)$, and hence, by (6), $u(z) \in C \cup A_0$. Take a $w \in \{a, b, c\} \setminus \{z, u(z)\}$ and let $g_1 = (z; w)_{A_0}$. Again by Lemma 3.8, g_1 extends to some $\sigma_1 \in \operatorname{fix}_{\mathcal{G}}(C)$. Then $\sigma_1(z) = w \neq z$ and

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$\sigma_1(u(z)) = u(z)$. Hence $\sigma_1(\{a,b,c\}) = \{a,b,c\}$ but $\sigma_1(u) \neq u$, which is also a contradiction. Therefore $\mathrm{mov}(u) \subseteq C$. Thus we have

$$\forall t \in [A_0 \setminus C]^3 \left(\mathrm{mov}(f(t)) \subseteq C\right). \tag{7}$$

For any $t_0, t_1 \in [A_0 \setminus C]^3$, it is easy to see that there is an $h \in \mathcal{G}_0$ such that h fixes $C \cap A_0$ pointwise and $h[t_0] = t_1$. By Lemma 3.8, h extends to some $\tau \in \mathrm{fix}_{\mathcal{G}}(C)$. Then $\tau(f) = f$ and $\tau(t_0) = t_1$, and hence, by (7), $f(t_0) = f(t_1)$. Therefore, f maps all elements of $[A_0 \setminus C]^3$ to the same element of $\mathcal{S}(A)$. Since A_0 is uncountable and C is countable, there is an injection p from ω into $A_0 \setminus C$. Then $\mathrm{ran}(p) \in \mathcal{I}$, which implies that $p \in \mathcal{V}_{\mathrm{S}}$. Thus, in $\mathcal{V}_{\mathrm{S}}$, $A_0 \setminus C$ is Dedekind infinite, and hence $[A_0 \setminus C]^3$ is Dedekind infinite, contradicting (5). ⊣

THEOREM 3.12. *The following statement is consistent with* ZF: *There exists a Dedekind infinite cardinal* $\mathfrak{a}$ *such that* $\mathfrak{a}! < [\mathfrak{a}]^3$, $[\mathfrak{a}]^3 \not\leqslant_{\mathrm{dfto}} \mathfrak{a}!$, *and* $\mathfrak{a}! \leqslant^* \mathfrak{a}$.

PROOF. Let A be the set of atoms of $\mathcal{V}_{\mathrm{S}}$ and let $\mathfrak{a} = |A|$. Then by Lemma 3.11, $\mathfrak{a}$ is Dedekind infinite, $\mathfrak{a}! \leqslant [\mathfrak{a}]^3$, $\mathfrak{a}! \leqslant^* \mathfrak{a}$, and $[\mathfrak{a}]^3 \not\leqslant_{\mathrm{dfto}} \mathfrak{a}!$. Since $[\mathfrak{a}]^3 \not\leqslant_{\mathrm{dfto}} \mathfrak{a}!$, we have that $[\mathfrak{a}]^3 \not\leqslant \mathfrak{a}!$, and therefore $\mathfrak{a}! < [\mathfrak{a}]^3$. ⊣

§4. A new permutation model.

In this section, we construct a new permutation model in which there exists an infinite cardinal $\mathfrak{a}$ such that $\mathfrak{a}! \leqslant_{\mathrm{fto}} \mathfrak{a}$. The strategy of our construction is as follows:

We construct step-by-step an infinite lattice A with a least element such that every initial segment determined by an element of A is finite. The permutation model will then be determined by the group of all automorphisms of A and the normal ideal $\mathrm{fin}(A)$. The lattice A is constructed in a way such that it has enough automorphisms (but not too much) to guarantee that every permutation of A which has a finite support moves only finitely many elements. Since the function that maps each finite subset of A to its least upper bound is a finite-to-one function from $\mathrm{fin}(A)$ into A, it follows from Fact 2.7 that in the permutation model we have $\mathcal{S}(A) = \mathcal{S}_{\mathrm{fin}}(A) \preccurlyeq_{\mathrm{fto}} \mathrm{fin}(A) \preccurlyeq_{\mathrm{fto}} A$.

This section is arranged as follows: In 4.1 and 4.2, we study two purely lattice-theoretic notions. In 4.1, we define a covering condition for partially ordered sets and prove some basic properties of it. In 4.2, we impose an additional quantitative condition on a nonvoid finite lattice satisfying this covering condition and obtain the notion of a building block. Then we prove a key property of building blocks that allows us to extend automorphisms of building blocks. In 4.3, we define by recursion a certain sequence of building blocks and then define the set of atoms of the permutation model to be the union of this sequence. The key property of building blocks guarantees that every permutation of the set of atoms which has a finite support moves only finitely many elements.

4.1. A covering condition.

Let $\langle P, < \rangle$ be a partially ordered set. For all $a, b \in P$, $a \leqslant b$ means that $a < b$ or $a = b$; $a \nleqslant b$ denotes the negation of $a \leqslant b$.

DEFINITION 4.1. Let $a, b \in P$.

(1) The (*closed*) *interval* from a to b is the set $[a, b] = \{c \in P \mid a \leqslant c \leqslant b\}$.
(2) a is *covered* by b (or b *covers* a), denoted by $a \lessdot b$, if $a < b$ but $a < c < b$ for no $c \in P$.

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(3) $\operatorname{cov}(b)$ is the set $\{c \in P \mid c \lessdot b\}$ (i.e., the elements of P covered by b).
(4) A *saturated chain* in $[a, b]$ is a sequence $t \in \operatorname{seq}(P)$ of length (i.e., the domain of t) $n > 0$ such that $t(0) = a$, $t(n-1) = b$, and $t(i) \lessdot t(i+1)$ for any $i < n-1$.
(5) $\langle P, < \rangle$ is *locally finite* if for all $a, b \in P$, $[a, b]$ is finite.

For all subsets M of P, the least upper bound and the greatest lower bound of M, if they exist, are denoted by $\sup M$ and $\inf M$, respectively. Note that if $\langle P, < \rangle$ has a least element, then the least upper bound of $\emptyset$ exists and is the least element of $\langle P, < \rangle$. We say that $\langle P, < \rangle$ is a *lattice* if any two elements of P have a least upper bound and a greatest lower bound. Notice that if $\langle P, < \rangle$ is a lattice, then any nonvoid finite subset M of P has a least upper bound and a greatest lower bound.

FACT 4.2. *Let $\langle P, < \rangle$ be a locally finite lattice with a least element and let $\mathfrak{a} = |P|$. Then* $\operatorname{fin}(\mathfrak{a}) \leqslant_{\text{fto}} \mathfrak{a}$.

PROOF. The function that maps each $M \in \operatorname{fin}(P)$ to $\sup M$ is a finite-to-one function from $\operatorname{fin}(P)$ into P, and hence $\operatorname{fin}(\mathfrak{a}) \leqslant_{\text{fto}} \mathfrak{a}$. ⊣

DEFINITION 4.3. A partially ordered set $\langle P, < \rangle$ satisfies the *finitary lower covering condition* if for all $M \in \operatorname{fin}(P)$ containing at least two elements,

$$\big(\exists b \in P\, \forall a \in M\, (a \lessdot b)\big) \rightarrow \big(\exists c \in P\, \forall a \in M\, (c \lessdot a)\big). \tag{8}$$

REMARK 4.4. Let $\langle P, < \rangle$ be a lattice. Then the statement that (8) holds for all $M \in [P]^2$ is equivalent to the condition (ξ') of [1, p. 14], which is in turn equivalent to the usual *lower covering condition* (cf. [4, p. 213]) if $\langle P, < \rangle$ is locally finite. Locally finite lattices satisfying the lower covering condition are often called *Birkhoff lattices*.

LEMMA 4.5. *Let $\langle P, < \rangle$ be a locally finite partially ordered set with a least element. If $\langle P, < \rangle$ satisfies the finitary lower covering condition, then the Jordan-Dedekind chain condition holds in $\langle P, < \rangle$; that is, for any $a, b \in P$ such that $a \leqslant b$, all saturated chains in $[a, b]$ have the same length.*

PROOF. Cf. [1, p. 40, Theorem 14]. ⊣

DEFINITION 4.6. Let $\langle P, < \rangle$ be a locally finite partially ordered set with a least element o, and assume that $\langle P, < \rangle$ satisfies the finitary lower covering condition. By Lemma 4.5, for any $b \in P$, all saturated chains in $[o, b]$ have the same length $n > 0$; the *height* of b, denoted by $\operatorname{ht}(b)$, is defined to be $n - 1$. Notice that for all $a, b \in P$, $a \lessdot b$ if and only if $a < b$ and $\operatorname{ht}(a) + 1 = \operatorname{ht}(b)$. If $\langle P, < \rangle$ has a greatest element e, the height of e is also called the *height* of $\langle P, < \rangle$.

DEFINITION 4.7. Let $\langle P, < \rangle$ be a locally finite lattice with a least element, and assume that $\langle P, < \rangle$ satisfies the finitary lower covering condition. For any $b \in P$, the *reflection point* of b, denoted by $b^\star$, is defined to be $\inf \operatorname{cov}(b)$. It follows from (8) that for all $b \in P$ covering at least two elements, we have

$$\forall a \lessdot b\, (b^\star \lessdot a). \tag{9}$$

LEMMA 4.8. *Let $\langle P, < \rangle$ be a locally finite lattice with a least element. If $\langle P, < \rangle$ satisfies the finitary lower covering condition, then for all $a, b \in P$ such that $a < b$ and $a \nleqslant b^\star$,*

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(i) *there exists a* $c \leqslant b^\star$ *such that* $c \lessdot a$ *and for all* $d \leqslant b^\star$, *if* $d < a$ *then* $d \leqslant c$;
(ii) $a^\star \leqslant b^\star$;
(iii) *there exists a unique saturated chain in* $[a, b]$.

Proof. (i) Fix an arbitrary $b \in P$. We prove by induction on $n < \mathrm{ht}(b)$ that for all $a \in P$ such that $a < b$ and $a \nleqslant b^\star$, if $\mathrm{ht}(a) = \mathrm{ht}(b) - n - 1$, then there exists a $c \leqslant b^\star$ such that $c \lessdot a$ and for all $d \leqslant b^\star$, if $d < a$ then $d \leqslant c$. If $n = 0$, then $a \lessdot b$ and, by (9), it suffices to take $c = b^\star$. Now, let $\mathrm{ht}(a) = \mathrm{ht}(b) - n - 2$, where $n + 1 < \mathrm{ht}(b)$. Let $v \in P$ be such that $v < b$ and $a \lessdot v$. By the induction hypothesis, there exists a $w \leqslant b^\star$ such that $w \lessdot v$ and for all $d \leqslant b^\star$, if $d < v$ then $d \leqslant w$. Take $c = \inf\{a, w\}$. Then $c \leqslant w \leqslant b^\star$ and, since $a, w \lessdot v$, it follows from (8) that $c \lessdot a$. Moreover, for all $d \leqslant b^\star$, if $d < a$, then, since $d < a \leqslant v$, we have $d \leqslant w$, and hence $d \leqslant c$, which completes the proof of (i).

(ii) By (i), there exists a $c \leqslant b^\star$ such that $c \lessdot a$, and therefore $a^\star \leqslant c \leqslant b^\star$.

(iii) Assume towards a contradiction that there are two distinct saturated chains t, u in $[a, b]$. Then it follows from Lemma 4.5 that t and u have the same length $n > 0$. Let $m = \max\{i < n \mid t(i) \neq u(i)\}$. Since $b = t(n-1) = u(n-1)$, it follows that $m < n - 1$. Let $c = t(m+1) = u(m+1)$. Then c covers both $t(m)$ and $u(m)$. By (9), $c^\star$ is covered by $t(m)$ and $u(m)$. Thus $c^\star = \inf\{t(m), u(m)\}$, and hence $a \leqslant c^\star$. If $c = b$ or $c \leqslant b^\star$, then $a \leqslant b^\star$, which is a contradiction. Otherwise, $c < b$ and $c \nleqslant b^\star$, and thus it follows from (ii) that $a \leqslant c^\star \leqslant b^\star$, which is also a contradiction. ⊣

4.2. Building blocks. We define the notion of a building block as follows:

Definition 4.9. A *building block* is a nonvoid finite lattice $\langle P, < \rangle$ satisfying the finitary lower covering condition and such that for all $b \in P$, if $\mathrm{ht}(b) = 2$ then $|\mathrm{cov}(b)| = 4$, and if $\mathrm{ht}(b) > 2$ then

$$\forall c \lessdot b^\star \Big(\big| \{a \in \mathrm{cov}(b) \mid a^\star = c\} \big| = 4 \Big). \tag{10}$$

Now we investigate what does a building block look like. Let $\langle P, < \rangle$ be a building block, o the least element of $\langle P, < \rangle$, and e the greatest element of $\langle P, < \rangle$. We do not, in general, know how to draw the Hasse diagram of $\langle P, < \rangle$, but if we already have the Hasse diagram of $\langle Q, < \rangle$ at hand, where $Q = \{c \in P \mid c \leqslant e^\star\}$ is a building block of lower height than $\langle P, < \rangle$, then we can draw the Hasse diagram of $\langle P, < \rangle$ in the following way:

(i) Draw the Hasse diagram of $\langle Q, < \rangle$ and add a point e above it (cf. Figure 1(a)).
(ii) For each $d \lessdot e^\star$, add four points covered by e such that they all cover $e^\star$ and associate d with them as their future reflection point (cf. Figure 1(b)). By letting d range over elements of $\mathrm{cov}(e^\star)$, we obtain all points of height $\mathrm{ht}(e) - 1$ in $\langle P, < \rangle$.
(iii) For each b of height $\mathrm{ht}(e) - 1$ in $\langle P, < \rangle$, if d is the associated reflection point of b, then for each $c \lessdot d$, if c is not the reflection point of $e^\star$, then add four points covered by b such that they all cover d and associate c with them as their future reflection point, and for $e^{\star\star}$ (i.e., the reflection point of $e^\star$), only add three points covered by b such that they all cover d and associate $e^{\star\star}$ with them as their future reflection point (cf. Figure 1(c)). For $e^{\star\star}$ we only

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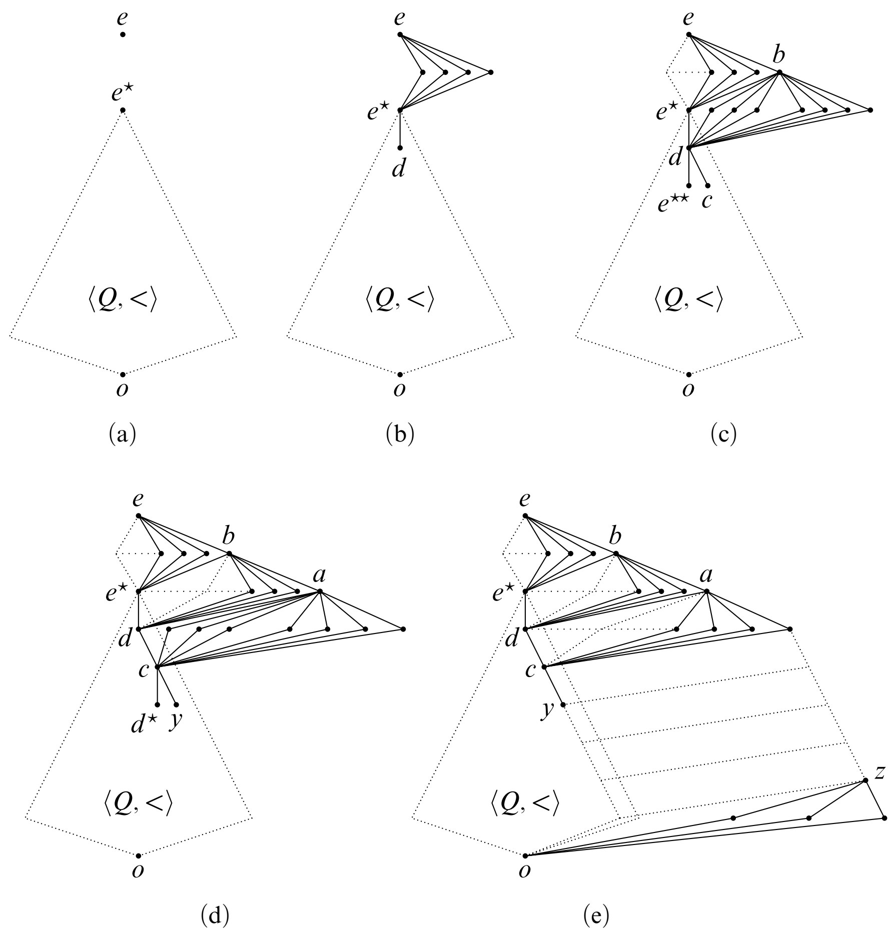

FIGURE 1. Drawing the Hasse diagram of a building block.

add three points since $e^\star$ is already a point covered by b such that it covers d and its reflection point is $e^{\star\star}$. By letting b range over elements of height $\mathrm{ht}(e) - 1$, we obtain all points of height $\mathrm{ht}(e) - 2$ in $\langle P, <\rangle$.

(iv) Let a be an element of $P \setminus Q$ of height $\mathrm{ht}(e) - 2$ in $\langle P, <\rangle$, let b be the unique point covering a, let d be the reflection point of b, and let c be the associated reflection point of a. For each $y \lessdot c$, if y is not the reflection point of d, then add four points covered by a such that they all cover c and associate y with them as their future reflection point, and for $d^\star$, only add three points covered by a such that they all cover c and associate $d^\star$ with them as their future reflection point (cf. Figure 1(d)). For $d^\star$ we only add three points since d is already a point covered by a such that it covers c and its reflection point is $d^\star$. By letting a range over elements of $P \setminus Q$ of height $\mathrm{ht}(e) - 2$, we obtain all points of height $\mathrm{ht}(e) - 3$ in $\langle P, <\rangle$.

(v) Continue this process to get elements of $P \setminus Q$ of lower and lower height. Suppose that all elements of $P \setminus Q$ of height $\geqslant 2$ are obtained. For each

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$z \in P \setminus Q$ of height 2 in $\langle P, < \rangle$, add three points covered by z such that they all cover o (cf. Figure 1(e)). We only add three points since there must be a unique point in Q covered by z such that it covers o. By letting z range over elements of $P \setminus Q$ of height 2, we obtain all points of height 1 in $\langle P, < \rangle$. Since o is the unique element of P of height 0, we finish drawing the Hasse diagram of $\langle P, < \rangle$.

Next we want to prove a key property which allows us to extend automorphisms of building blocks. Let $\langle P, < \rangle$ be a building block, o the least element of $\langle P, < \rangle$, and e the greatest element of $\langle P, < \rangle$. Notice that for all $b \in P$ such that $\text{ht}(b) \geqslant 2$, $|\text{cov}(b)| \geqslant 4$, and hence it follows from (9) that $b^\star \lessdot a$ for any $a \lessdot b$. We first formalize some of the ideas embodied in the above process.

Let $Q = \{c \in P \mid c \leqslant e^\star\}$. Let $a \in P \setminus (Q \cup \{e\})$. By Lemma 4.8(iii), there exists a unique saturated chain in $[a, e]$, and therefore there exists a unique $c \in P$ such that $a \lessdot c$; we use $\text{succ}(a)$ to denote the unique $c \in P$ such that $a \lessdot c$. Clearly,

$$\text{succ}(a) \in P \setminus Q \wedge a \lessdot \text{succ}(a) \wedge \forall b \in P \left(a < b \leftrightarrow \text{succ}(a) \leqslant b\right). \tag{11}$$

Let $\text{pred}(a) = (\text{succ}(a))^\star$. We claim that

$$\text{pred}(a) \in Q \wedge \text{pred}(a) \lessdot a \wedge \forall d \in Q \left(d < a \leftrightarrow d \leqslant \text{pred}(a)\right). \tag{12}$$

In fact, by Lemma 4.8(ii), $\text{pred}(a) \in Q$. Since $a \notin Q$, we have $a \neq o$, and thus $\text{ht}(a) \geqslant 1$ and $\text{ht}(\text{succ}(a)) \geqslant 2$, which implies that $\text{pred}(a) \lessdot a$. On the other hand, by Lemma 4.8(i), there is a $c \in Q$ such that $c \lessdot a$ and for all $d \in Q$, if $d < a$ then $d \leqslant c$. Since $\text{pred}(a) \in Q$ and $\text{pred}(a) \lessdot a$, it follows that $\text{pred}(a) = c$, and hence for all $d \in Q$, $d < a$ if and only if $d \leqslant \text{pred}(a)$. Thus (12) is proved. Hence $\text{pred}(a)$ is the unique $c \in Q$ such that $c \lessdot a$. Notice that if $\text{ht}(a) \geqslant 2$ then $a^\star \lessdot \text{pred}(a)$, and hence if $\text{ht}(a) > 2$ then $(\text{pred}(a))^\star \lessdot a^\star$. For example, in Figure 1(d), $\text{succ}(b) = e$, $\text{pred}(b) = e^\star$, $\text{succ}(a) = b$, and $\text{pred}(a) = d$.

Let $C = \{b \in P \setminus Q \mid \text{ht}(b) = 2\}$, and let

$$D = \left\{(b, c) \in (P \setminus Q) \times P \;\middle|\; \text{ht}(b) > 2 \text{ and } c \lessdot b^\star\right\}.$$

For any $b \in C$, let $k_b = |\text{cov}(b) \setminus Q|$, and for any $(b, c) \in D$, let

$$l_{b,c} = \left|\{a \in \text{cov}(b) \mid a^\star = c\} \setminus Q\right|.$$

Then it is easy to verify that for all $b \in C$,

$$k_b = \begin{cases} 3, & \text{if } b \neq e; \\ 4, & \text{if } b = e, \end{cases} \tag{13}$$

and that for all $(b, c) \in D$,

$$l_{b,c} = \begin{cases} 3, & \text{if } b \neq e \text{ and } (\text{pred}(b))^\star = c; \\ 4, & \text{if } b \neq e \text{ and } (\text{pred}(b))^\star \neq c; \\ 4, & \text{if } b = e. \end{cases} \tag{14}$$

For example, in Figure 1(c), $(e, d), (b, c), (b, e^{\star\star}) \in D$, $l_{e,d} = l_{b,c} = 4$, and $l_{b,e^{\star\star}} = 3$.

Now, let f be an automorphism of $\langle Q, < \rangle$. Our purpose here is to show that f can be extended to an automorphism of $\langle P, < \rangle$ in a very flexible way. The key point is that, for any $(b, c) \in D$, we have at least three elements of $P \setminus Q$ covered by b

such that c is their common reflection point and hence $\langle P, <\rangle$ cannot distinguish between them, and similarly, for any $b \in C$, we have at least three elements of $P \setminus Q$ covered by b such that they all cover o and again $\langle P, <\rangle$ cannot distinguish between them.

We use the following parameters to define an automorphism g of $\langle P, <\rangle$ extending f:

- The building block $\langle P, <\rangle$, which determines Q, C, D, and the functions succ, pred, $b \mapsto k_b$, and $(b, c) \mapsto l_{b,c}$.
- A function σ on C such that for all $b \in C$, $\sigma(b)$ is a bijection from $\operatorname{cov}(b) \setminus Q$ onto k_b, and a function τ on D such that for all $(b, c) \in D$, $\tau(b, c)$ is a bijection from $\{a \in \operatorname{cov}(b) \mid a^\star = c\} \setminus Q$ onto $l_{b,c}$. Such functions σ and τ exist since P is finite.
- A function p on C such that for all $b \in C$, $p(b)$ is a permutation of k_b, and a function q on D such that for all $(b, c) \in D$, $q(b, c)$ is a permutation of $l_{b,c}$.
- The automorphism f of $\langle Q, <\rangle$.

Functions σ, τ, p, q will be used to prescribe, at each stage, how we move the elements between which $\langle P, <\rangle$ cannot distinguish. For better readability, we write $\tau_{b,c}$ for $\tau(b, c)$, p_b for $p(b)$, and so on. The automorphism g is defined as follows:

For each $d \in Q$, take $g(d) = f(d)$. We define $g(a)$ for $a \in P \setminus Q$ by recursion on $\operatorname{ht}(e) - \operatorname{ht}(a)$ as follows. Take $g(e) = e$. Now, let $a \in P \setminus (Q \cup \{e\})$ and assume that for all $b \in P \setminus Q$ such that $\operatorname{ht}(b) = \operatorname{ht}(a) + 1$, $g(b)$ is already defined and we have:

$$g(b) \in P \setminus Q \wedge \operatorname{ht}(g(b)) = \operatorname{ht}(b); \tag{15}$$

$$(g(b))^\star = f(b^\star); \tag{16}$$

$$b \neq e \to \operatorname{pred}(g(b)) = f(\operatorname{pred}(b)). \tag{17}$$

Take $b = \operatorname{succ}(a)$. Then, by (11), $a \lessdot b \in P \setminus Q$, and thus $\operatorname{ht}(b) = \operatorname{ht}(a) + 1$, which implies that $g(b)$ is defined and (15)–(17) hold. We consider the following two cases:

Case 1. $\operatorname{ht}(a) = 1$. Then $\operatorname{ht}(b) = 2$ and hence $b \in C$. By (15), $g(b) \in P \setminus Q$ and $\operatorname{ht}(g(b)) = \operatorname{ht}(b) = 2$, which implies that $g(b) \in C$. Since $g(b) = e$ if and only if $b = e$, it follows from (13) that $k_{g(b)} = k_b$. Now we define

$$g(a) = \left(\sigma_{g(b)}\right)^{-1}\left(p_b(\sigma_b(a))\right). \tag{18}$$

Then $g(a) \in \operatorname{cov}(g(b)) \setminus Q$, and hence $\operatorname{ht}(g(a)) = \operatorname{ht}(g(b)) - 1 = 1 = \operatorname{ht}(a)$. Since $\operatorname{cov}(g(a)) = \operatorname{cov}(a) = \{o\}$ and $\operatorname{pred}(g(a)) = \operatorname{pred}(a) = o$, we get that (15)–(17) hold with b replaced by a. Notice that

$$\operatorname{succ}(g(a)) = g(\operatorname{succ}(a)). \tag{19}$$

Case 2. $\operatorname{ht}(a) > 1$. Then $\operatorname{ht}(b) > 2$. Let $c = a^\star$. Then $c \lessdot \operatorname{pred}(a) \in Q$, which implies that $c \in Q$ and $(b, c) \in D$. By (15), $g(b) \in P \setminus Q$ and $\operatorname{ht}(g(b)) = \operatorname{ht}(b) > 2$. Since f is an automorphism of $\langle Q, <\rangle$, we have $f(c) \lessdot f(b^\star)$, and thus, by (16), $f(c) \lessdot (g(b))^\star$, which implies that $(g(b), f(c)) \in D$. Since $\operatorname{ht}(g(b)) = \operatorname{ht}(b)$, it follows that $g(b) = e$ if and only if $b = e$. By (17) and the assumption that f is an automorphism of $\langle Q, <\rangle$, if $b \neq e$, then $(\operatorname{pred}(g(b)))^\star = f(c)$ if and only

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if $f((\mathrm{pred}(b))^\star) = f(c)$ if and only if $(\mathrm{pred}(b))^\star = c$. Hence, by (14), we have $l_{g(b),f(c)} = l_{b,c}$. Now we define

$$g(a) = \left(\tau_{g(b),f(c)}\right)^{-1}\left(q_{b,c}(\tau_{b,c}(a))\right). \tag{20}$$

Then $g(a) \in \{v \in \mathrm{cov}(g(b)) \mid v^\star = f(c)\} \setminus Q$; that is, $g(a) \in P \setminus Q$, $g(a) \lessdot g(b)$, and $(g(a))^\star = f(a^\star)$. Hence $\mathrm{ht}(g(a)) = \mathrm{ht}(g(b)) - 1 = \mathrm{ht}(b) - 1 = \mathrm{ht}(a)$ and

$$\mathrm{succ}(g(a)) = g(\mathrm{succ}(a)). \tag{21}$$

Thus, by (16), $\mathrm{pred}(g(a)) = (\mathrm{succ}(g(a)))^\star = (g(b))^\star = f(b^\star) = f(\mathrm{pred}(a))$, and therefore we get that (15)–(17) hold with b replaced by a.

Therefore, for all $b \in P \setminus Q$, $g(b)$ is defined and (15)–(17) hold. Also it follows from (19) and (21) that for all $a \in P \setminus (Q \cup \{e\})$ we have

$$\mathrm{succ}(g(a)) = g(\mathrm{succ}(a)). \tag{22}$$

We still have to prove that g is an automorphism of $\langle P, < \rangle$. For this, we first prove that g is injective. Since $g \restriction Q = f$ is injective, by (15), it suffices to show that $g \restriction (P \setminus Q)$ is injective. We prove by induction on $n < \mathrm{ht}(e)$ that for all $a_0, a_1 \in P \setminus Q$ such that $\mathrm{ht}(a_0) = \mathrm{ht}(e) - n$, if $g(a_0) = g(a_1)$ then $a_0 = a_1$. The case $n = 0$ is obvious. Now, let $n < \mathrm{ht}(e) - 1$ and let $a_0, a_1 \in P \setminus Q$ be such that $\mathrm{ht}(a_0) = \mathrm{ht}(e) - n - 1$ and $g(a_0) = g(a_1)$. By (15), $\mathrm{ht}(a_1) = \mathrm{ht}(g(a_1)) = \mathrm{ht}(g(a_0)) = \mathrm{ht}(a_0) < \mathrm{ht}(e)$, and hence $a_0, a_1 \in P \setminus (Q \cup \{e\})$. Let $b_0 = \mathrm{succ}(a_0)$ and let $b_1 = \mathrm{succ}(a_1)$. Then, by (11), we have $b_0, b_1 \in P \setminus Q$ and $\mathrm{ht}(b_0) = \mathrm{ht}(a_0) + 1 = \mathrm{ht}(e) - n$, and thus, by the induction hypothesis, if $g(b_0) = g(b_1)$ then $b_0 = b_1$. By (22), $g(b_0) = \mathrm{succ}(g(a_0)) = \mathrm{succ}(g(a_1)) = g(b_1)$, which implies that $b_0 = b_1$. We consider the following two cases:

Case 1. $\mathrm{ht}(a_0) = 1$. Then $\mathrm{ht}(a_1) = 1$. Since $g(a_0) = g(a_1)$ and $b_0 = b_1$, by (18), $p_{b_0}(\sigma_{b_0}(a_0)) = p_{b_1}(\sigma_{b_1}(a_1))$, and thus $\sigma_{b_0}(a_0) = \sigma_{b_1}(a_1)$, which implies that $a_0 = a_1$.

Case 2. $\mathrm{ht}(a_0) > 1$. Then we have $\mathrm{ht}(a_1) > 1$. Let $c_0 = a_0^\star$ and let $c_1 = a_1^\star$. By (16), $f(c_0) = (g(a_0))^\star = (g(a_1))^\star = f(c_1)$, and therefore $c_0 = c_1$. Since $g(a_0) = g(a_1)$, $b_0 = b_1$, and $c_0 = c_1$, it follows from (20) that $q_{b_0,c_0}(\tau_{b_0,c_0}(a_0)) = q_{b_1,c_1}(\tau_{b_1,c_1}(a_1))$, and therefore $\tau_{b_0,c_0}(a_0) = \tau_{b_1,c_1}(a_1)$, which implies that $a_0 = a_1$.

Hence g is injective, which implies that g is a permutation of P since P is finite. It remains to show that for all $a, b \in P$,

$$a < b \leftrightarrow g(a) < g(b). \tag{23}$$

Let $a, b \in P$. If $b \in Q \cup \{e\}$, then obviously (23) holds. Suppose that $b \in P \setminus (Q \cup \{e\})$. Then, by (15), we have $g(b) \in P \setminus (Q \cup \{e\})$. If $a \in Q$, then $g(a) = f(a) \in Q$, and therefore, by (12) and (17),

$$a < b \leftrightarrow a \leqslant \mathrm{pred}(b) \leftrightarrow g(a) \leqslant \mathrm{pred}(g(b)) \leftrightarrow g(a) < g(b).$$

Thus if $a \in Q$ then (23) holds. Also, if $a = e$, then (23) holds trivially. Assume that $a \in P \setminus (Q \cup \{e\})$ and that for all $c \in P \setminus Q$ such that $\mathrm{ht}(c) = \mathrm{ht}(a) + 1$, $c < b$ if and only if $g(c) < g(b)$. Then, by (11) and the injectivity of g, $\mathrm{succ}(a) \leqslant b$ if and only if $g(\mathrm{succ}(a)) \leqslant g(b)$. By (15), we have $g(a) \in P \setminus (Q \cup \{e\})$, and hence, by (11) and (22),

$$a < b \leftrightarrow \mathrm{succ}(a) \leqslant b \leftrightarrow \mathrm{succ}(g(a)) \leqslant g(b) \leftrightarrow g(a) < g(b).$$

Thus (23) is proved. We use $\Phi(P,<,\sigma,\tau,p,q,f)$ to denote the function g. Hence we have proved that

$$\Phi(P,<,\sigma,\tau,p,q,f) \text{ is an automorphism of } \langle P,<\rangle \text{ extending } f. \tag{24}$$

Now, let p' be the function defined on C such that for all $b \in C$, $p'(b) = \mathrm{id}_{k_b}$, and let q' be the function defined on D such that for all $(b,c) \in D$, $q'(b,c) = \mathrm{id}_{l_{b,c}}$. Let $\Psi(P,<,\sigma,\tau,f) = \Phi(P,<,\sigma,\tau,p',q',f)$. Then, by (24), we have that

$$\Psi(P,<,\sigma,\tau,f) \text{ is an automorphism of } \langle P,<\rangle \text{ extending } f. \tag{25}$$

Lemma 4.10. *Let $\langle P,<\rangle$ be a building block, let e be the greatest element of $\langle P,<\rangle$, and let $Q = \{c \in P \mid c \leqslant e^\star\}$. For all $a \in P \setminus (Q \cup \{e\})$ and all $d \in P \setminus \{a\}$ such that either $\mathrm{ht}(d) \geqslant \mathrm{ht}(a)$ or $d \in Q$, there exists an automorphism g of $\langle P,<\rangle$ fixing $Q \cup \{d\}$ pointwise and such that $g(a) \neq a$.*

Proof. Let σ and τ be functions as above. Let $b_0 = \mathrm{succ}(a)$. We consider the following two cases:

Case 1. $\mathrm{ht}(a) = 1$. Then, let $i = \sigma_{b_0}(a) < k_{b_0}$, and let $j < k_{b_0}$ be the least natural number such that $(\sigma_{b_0})^{-1}(j) \notin \{a,d\}$. Let p be the function defined on C such that for all $b \in C$,

$$p(b) = \begin{cases} (i;j)_{k_b}, & \text{if } b = b_0; \\ \mathrm{id}_{k_b}, & \text{otherwise,} \end{cases}$$

and let q be the function defined on D such that for all $(b,c) \in D$, $q(b,c) = \mathrm{id}_{l_{b,c}}$. Let $g = \Phi(P,<,\sigma,\tau,p,q,\mathrm{id}_Q)$. Then, by (24), g is an automorphism of $\langle P,<\rangle$ fixing Q pointwise. By (20) and a routine induction, for all $v \in P \setminus Q$ such that $\mathrm{ht}(v) > 1$, we have $g(v) = v$. Therefore, by (18), $g(a) = (\sigma_{b_0})^{-1}(j) \neq a$ and for all $w \in P \setminus Q$ such that $\mathrm{ht}(w) = 1$, if $w \notin \{a, (\sigma_{b_0})^{-1}(j)\}$, then $g(w) = w$. Hence $g(d) = d$.

Case 2. $\mathrm{ht}(a) > 1$. Then, let $c_0 = a^\star$, let $i = \tau_{b_0,c_0}(a) < l_{b_0,c_0}$, and let $j < l_{b_0,c_0}$ be the least natural number such that $(\tau_{b_0,c_0})^{-1}(j) \notin \{a,d\}$. Let p be the function defined on C such that for all $b \in C$, $p(b) = \mathrm{id}_{k_b}$, and let q be the function defined on D such that for all $(b,c) \in D$,

$$q(b,c) = \begin{cases} (i;j)_{l_{b,c}}, & \text{if } b = b_0 \text{ and } c = c_0; \\ \mathrm{id}_{l_{b,c}}, & \text{otherwise.} \end{cases}$$

Let $g = \Phi(P,<,\sigma,\tau,p,q,\mathrm{id}_Q)$. Then it follows from (24) that g is an automorphism of $\langle P,<\rangle$ fixing Q pointwise. By (20) and a routine induction, for all $v \in P \setminus Q$ such that $\mathrm{ht}(v) > \mathrm{ht}(a)$, we have $g(v) = v$. Hence, again by (20), $g(a) = (\tau_{b_0,c_0})^{-1}(j) \neq a$ and for all $w \in P \setminus Q$ such that $\mathrm{ht}(w) = \mathrm{ht}(a)$, if $w \notin \{a, (\tau_{b_0,c_0})^{-1}(j)\}$, then $g(w) = w$. Since $d \notin \{a, (\tau_{b_0,c_0})^{-1}(j)\}$ and either $\mathrm{ht}(d) \geqslant \mathrm{ht}(a)$ or $d \in Q$, we have $g(d) = d$, which completes the proof. ⊣

4.3. Construction of the permutation model. For any quintuple (x_0,x_1,x_2,x_3,x_4) and for any $j < 5$, let $\mathrm{pr}_j(x_0,x_1,x_2,x_3,x_4) = x_j$. Let o be an arbitrary atom. The atoms of the permutation model are quintuples or o, which are constructed by recursion as follows:

(i) $e_0 = o$, $A_0 = \{o\}$, and $\lessdot_0 = \emptyset$.
(ii) $e_1 = (0,0,\emptyset,o,3)$, $A_1 = \{o, e_1\}$, and $\lessdot_1 = \{(o,e_1)\}$.

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(iii) For any $n \geqslant 1$, $e_{n+1} = (n, n, \emptyset, e_{n-1}, 3)$ and $A_{n+1} = A_n \cup \bigcup_{i \leqslant n} B_{n,i}$, where $B_{n,i}$ is defined by recursion on $i \leqslant n$ as follows:
- $B_{n,0} = \{e_{n+1}\}$;
- $B_{n,n} = \{(n, 0, b, o, j) \mid b \in B_{n,n-1} \wedge j < 3\}$;
- $B_{n,i} = \{(n, n-i, b, c, j) \mid b \in B_{n,i-1} \wedge c \lessdot_n \operatorname{pr}_3 b \wedge j < L_{b,c}\}$, where $0 < i < n$ and

$$L_{b,c} = \begin{cases} 3, & \text{if } b \neq e_{n+1} \text{ and } \operatorname{pr}_3 \operatorname{pr}_3 \operatorname{pr}_2 b = c; \\ 4, & \text{if } b \neq e_{n+1} \text{ and } \operatorname{pr}_3 \operatorname{pr}_3 \operatorname{pr}_2 b \neq c; \\ 3, & \text{if } b = e_{n+1} \text{ and } c = e_{n-2}; \\ 4, & \text{if } b = e_{n+1} \text{ and } c \neq e_{n-2}. \end{cases}$$

(iv) For any $n \geqslant 1$, $\lessdot_{n+1}$ is defined as follows:

$$\begin{aligned} \lessdot_{n+1} = \lessdot_n &\cup \big\{(e_n, e_{n+1})\big\} \\ &\cup \big\{(\operatorname{pr}_3 \operatorname{pr}_2 a, a) \bigm| a \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})\big\} \\ &\cup \big\{(a, \operatorname{pr}_2 a) \bigm| a \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})\big\}. \end{aligned}$$

(v) For any $n \in \omega$, $<_n$ is the transitive closure of $\lessdot_n$; that is, for all a, b, $a <_n b$ if and only if there exists a sequence t of length $m > 1$ such that $t(0) = a$, $t(m-1) = b$, and $t(j) \lessdot_n t(j+1)$ for any $j < m-1$. Such a t is called a $\lessdot_n$-chain from a to b.

Let $A = \bigcup_{n \in \omega} A_n$ and let $< = \bigcup_{n \in \omega} <_n$. For the sake of simplicity we shall work with A as the set of atoms. Let $\mathcal{G}$ be the group of all automorphisms of $\langle A, < \rangle$ and let $\mathcal{I} = \operatorname{fin}(A)$. The permutation model determined by $\mathcal{G}$ and $\mathcal{I}$ is denoted by $\mathcal{V}_{\mathcal{S}}$ ($\mathcal{S}$ for the operator $\mathcal{S}$).

The main idea of the construction is as follows. For each $n \in \omega$, we use A_n to encode a building block of height n, and for each quintuple a in A, we use $\operatorname{pr}_0(a)$ to encode the unique $n \in \omega$ such that $a \in A_{n+1} \setminus A_n$, $\operatorname{pr}_1(a) + 1$ the height of a, $\operatorname{pr}_2(a)$ the successor of a in $A_{n+1} \setminus A_n$ ($\emptyset$ if a has no successors in $A_{n+1} \setminus A_n$), $\operatorname{pr}_3(a)$ the reflection point of a, and $\operatorname{pr}_4(a) < 4$ the position of a in the elements between which $\langle A, < \rangle$ cannot distinguish. Hence, for all $n, i \in \omega$ such that $n \geqslant 1$ and $i \leqslant n$, $B_{n,i}$ is the set of elements of $A_{n+1} \setminus A_n$ of height $n - i + 1$. Notice that A_{n+1} is constructed from A_n in the same way as described after Definition 4.9, but there is a slight difference: In (i) of the process described after Definition 4.9, we add a point e which is two level higher than the greatest element of $\langle Q, < \rangle$, but the greatest element e_{n+1} of A_{n+1} is only one level higher than the greatest element e_n of A_n. So the process from A_n to A_{n+1} is just (iii)–(v) of the process described after Definition 4.9, where in (iii) we only consider one b, namely e_{n+1}, and the associated reflection point d of b is e_{n-1}. We draw the Hasse diagram of $\langle A_4, <_4 \rangle$ in Figure 2 (with some points of height 1 omitted), which illustrates the construction.

We still need to give a formal proof that $\langle A_n, <_n \rangle$ is a building block for every $n \in \omega$. We first note that, for all $n \in \omega$, we have $e_n \in A_n$ and for all $a \in A_{n+1} \setminus A_n$, $\operatorname{pr}_0 a = n$ and if $n \geqslant 1$ then $n - \operatorname{pr}_1 a$ is the unique $i \leqslant n$ such that $a \in B_{n,i}$. Thus, for all $n \geqslant 1$, A_n and $\bigcup_{i \leqslant n} B_{n,i}$ are disjoint, and the sets $B_{n,i}$ ($i \leqslant n$) are pairwise

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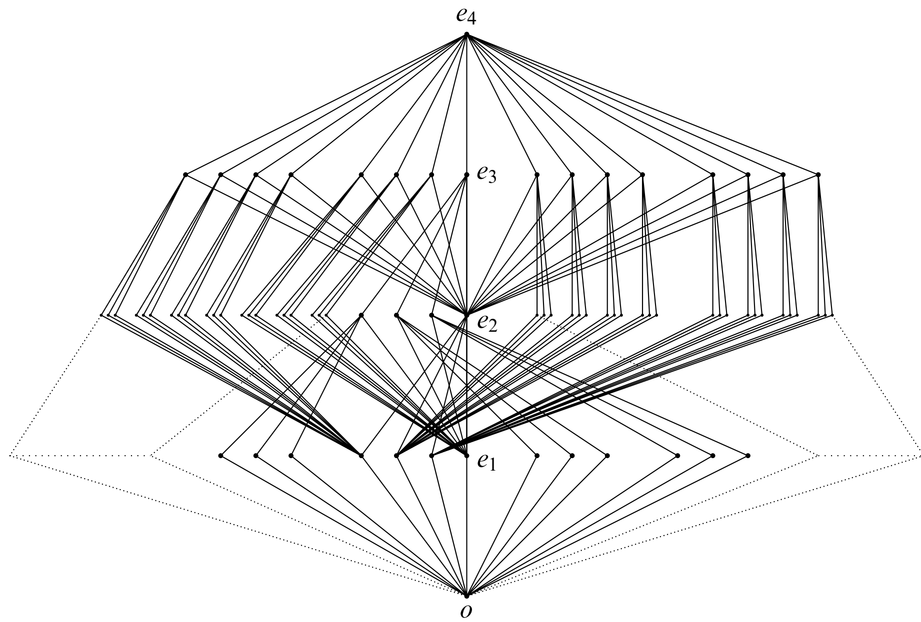

FIGURE 2. The Hasse diagram of $\langle A_4, <_4\rangle$.

disjoint. Notice that for all $n \geqslant 1$ and all $a \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$,

$$\mathrm{pr}_2\, a \in A_{n+1} \setminus A_n \wedge \mathrm{pr}_1\, \mathrm{pr}_2\, a = \mathrm{pr}_1\, a + 1; \tag{26}$$

$$\mathrm{pr}_1\, a > 0 \rightarrow \mathrm{pr}_3\, a \lessdot_n \mathrm{pr}_3\, \mathrm{pr}_2\, a. \tag{27}$$

LEMMA 4.11. *For every $n \in \omega$, $\langle A_n, <_n\rangle$ is a building block, $\lessdot_n$ is the covering relation of $<_n$, o is the least element of $\langle A_n, <_n\rangle$, e_n is the greatest element of $\langle A_n, <_n\rangle$, and for all $a \in A_n \setminus \{o\}$, we have* $\mathrm{ht}(a) = \mathrm{pr}_1\, a + 1$ *and* $a^\star = \mathrm{pr}_3\, a$.

PROOF. We prove this lemma by induction on n. The cases $n = 0$ and $n = 1$ are obvious. Next, for the inductive step, let $n \geqslant 1$ and assume that the assertion holds for n. The proof that the assertion holds for $n + 1$ proceeds in the following eight steps.

STEP 1. We prove some results about the relation $<_{n+1}$:

$$a <_{n+1} b \wedge b \in A_n \rightarrow a \in A_n \wedge a <_n b; \tag{28}$$

$$a <_{n+1} b \wedge a \in A_{n+1} \setminus A_n \rightarrow \mathrm{pr}_1\, a < \mathrm{pr}_1\, b; \tag{29}$$

$$a <_{n+1} b \wedge a \in A_{n+1} \setminus A_n \rightarrow \exists! t (t \text{ is a } \lessdot_{n+1}\text{-chain from } a \text{ to } b); \tag{30}$$

$$a <_{n+1} b \wedge a, b \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\}) \rightarrow \mathrm{pr}_3\, \mathrm{pr}_2\, a <_n \mathrm{pr}_3\, \mathrm{pr}_2\, b; \tag{31}$$

$$a <_{n+1} b \wedge a \in A_n \wedge b \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\}) \rightarrow a \leqslant_n \mathrm{pr}_3\, \mathrm{pr}_2\, b. \tag{32}$$

Let $a <_{n+1} b$ and let t be a $\lessdot_{n+1}$-chain of length $m > 1$ from a to b. By (26) and the definition of $\lessdot_{n+1}$, if $b \in A_n$, then $\mathrm{ran}(t) \subseteq A_n$ and t is a $\lessdot_n$-chain from a to b. Thus (28) is proved. If $a \in A_{n+1} \setminus A_n$, then $t[m-1] \subseteq A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$, $t(m-1) \in A_{n+1} \setminus A_n$, and for all $j < m-1$, $t(j+1) = \mathrm{pr}_2\, t(j)$ and thus $\mathrm{pr}_1\, t(j+1) = \mathrm{pr}_1\, t(j) + 1$ by (26), which implies that $m = \mathrm{pr}_1\, b - \mathrm{pr}_1\, a + 1$ and thus t is uniquely determined by a and b. Therefore (29) and (30) are proved. If $a, b \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$, then we have that $\operatorname{ran}(t) \subseteq A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$ and $t(j+1) = \mathrm{pr}_2\, t(j)$ for any $j < m-1$, and hence, by (26) and (27), $\mathrm{pr}_3\,\mathrm{pr}_2\, t(j) \lessdot_n \mathrm{pr}_3\,\mathrm{pr}_2\, t(j+1)$ for any $j < m-1$. Thus (31) is proved. Finally, suppose that $a \in A_n$ and that $b \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$. Let i be the least $j < m$ such that $t(j) \in A_{n+1} \setminus A_n$. Clearly, $i > 0$, $t[i] \subseteq A_n$, $t[m \setminus i] \subseteq A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$, $a \leqslant_n t(i-1)$, $t(i-1) = \mathrm{pr}_3\,\mathrm{pr}_2\, t(i)$, and $t(i) \leqslant_{n+1} b$, which implies that, by (31), $\mathrm{pr}_3\,\mathrm{pr}_2\, t(i) \leqslant_n \mathrm{pr}_3\,\mathrm{pr}_2\, b$. Therefore we have $a \leqslant_n t(i-1) = \mathrm{pr}_3\,\mathrm{pr}_2\, t(i) \leqslant_n \mathrm{pr}_3\,\mathrm{pr}_2\, b$. Thus (32) is proved.

STEP 2. Now we prove that $\langle A_{n+1}, <_{n+1}\rangle$ is a partially ordered set. Since $<_{n+1}$ is the transitive closure of $\lessdot_{n+1}$, it suffices to prove that $<_{n+1}$ is irreflexive. Assume towards a contradiction that there is a $b \in A_{n+1}$ such that $b <_{n+1} b$. If $b \in A_n$, then, by (28), $b <_n b$, contradicting the assumption that $<_n$ is irreflexive. Otherwise, by (29), $\mathrm{pr}_1\, b < \mathrm{pr}_1\, b$, which is also a contradiction.

STEP 3. Now we prove that o and e_{n+1} are the least and greatest elements of $\langle A_{n+1}, <_{n+1}\rangle$. Since o is the least element of $\langle A_n, <_n\rangle$, it follows that $o \leqslant_n e_n \lessdot_{n+1} e_{n+1}$ and for all $a \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$ we have $o \leqslant_n \mathrm{pr}_3\,\mathrm{pr}_2\, a \lessdot_{n+1} a$, which implies that o is also the least element of $\langle A_{n+1}, <_{n+1}\rangle$. Since e_n is the greatest element of $\langle A_n, <_n\rangle$, we have $d \leqslant_n e_n \lessdot_{n+1} e_{n+1}$ for any $d \in A_n$. For all $a \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$, the sequence t of length $n - \mathrm{pr}_1\, a + 1$ such that $t(0) = a$ and $t(j+1) = \mathrm{pr}_2\, t(j)$ for any $j < n - \mathrm{pr}_1\, a$ is a $\lessdot_{n+1}$-chain from a to e_{n+1}, and therefore $a <_{n+1} e_{n+1}$, which implies that e_{n+1} is the greatest element of $\langle A_{n+1}, <_{n+1}\rangle$.

STEP 4. We prove that $\lessdot_{n+1}$ is the covering relation of $<_{n+1}$; that is, for all a, b,

$$a \lessdot_{n+1} b \leftrightarrow a <_{n+1} b \wedge \neg\exists c\big(a <_{n+1} c <_{n+1} b\big). \tag{33}$$

Clearly, if $a <_{n+1} b$ but $a <_{n+1} c <_{n+1} b$ for no $c \in A_{n+1}$, then $a \lessdot_{n+1} b$. For the other direction, assume towards a contradiction that $a \lessdot_{n+1} b$ and $a <_{n+1} c <_{n+1} b$ for some $c \in A_{n+1}$. We consider the following four cases:

CASE 1. $a \lessdot_n b$. Then $b \in A_n$ and hence $a <_n c <_n b$ by (28), contradicting the assumption that $\lessdot_n$ is the covering relation of $<_n$.

CASE 2. $a = e_n$ and $b = e_{n+1}$. Then we have $c \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$ and the sequence t of length $n - \mathrm{pr}_1\, c + 1$ such that $t(0) = c$ and $t(j+1) = \mathrm{pr}_2\, t(j)$ for any $j < n - \mathrm{pr}_1\, c$ is a $\lessdot_{n+1}$-chain from c to e_{n+1}, and hence it follows from (32) and (31) that $e_n \leqslant_n \mathrm{pr}_3\,\mathrm{pr}_2\, c \leqslant_n \mathrm{pr}_3\,\mathrm{pr}_2\, t(n - \mathrm{pr}_1\, c - 1) = \mathrm{pr}_3\, e_{n+1} = e_{n-1}$, which is absurd.

CASE 3. $b \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$ and $a = \mathrm{pr}_3\,\mathrm{pr}_2\, b$. If $c \in A_n$, then, by (32), $c \leqslant_n \mathrm{pr}_3\,\mathrm{pr}_2\, b = a$, contradicting that $a <_{n+1} c$. Therefore $c \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$, and hence it follows from (32) and (31) that $a \leqslant_n \mathrm{pr}_3\,\mathrm{pr}_2\, c <_n \mathrm{pr}_3\,\mathrm{pr}_2\, b = a$, which is absurd.

CASE 4. $a \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$ and $b = \mathrm{pr}_2\, a$. Then it follows from (29) and (26) that $\mathrm{pr}_1\, a < \mathrm{pr}_1\, c < \mathrm{pr}_1\, b = \mathrm{pr}_1\, a + 1$, which is also a contradiction. Thus (33) is proved.

STEP 5. Now we prove that $\langle A_{n+1}, <_{n+1}\rangle$ is a finite lattice. Since A_n is finite, it follows that A_{n+1} is finite. Since A_{n+1} is finite and $\langle A_{n+1}, <_{n+1}\rangle$ has a greatest element, we only need to prove that any two elements of A_{n+1} have a greatest lower bound. Let $a, b \in A_{n+1}$. If $a \leqslant_{n+1} b$ or $b \leqslant_{n+1} a$, then obviously a and b have a greatest lower bound. Suppose that a and b are incomparable. If $a, b \in A_n$, then, by (28), the greatest lower bound of a and b in $\langle A_n, <_n\rangle$ is also their greatest

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lower bound in $\langle A_{n+1}, <_{n+1}\rangle$. If $a \in A_n$ and $b \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$, then, by (28) and (32), the greatest lower bound of a and $\mathrm{pr}_3\,\mathrm{pr}_2\,b$ in $\langle A_n, <_n\rangle$ is also the greatest lower bound of a and b in $\langle A_{n+1}, <_{n+1}\rangle$. Finally, we claim that if $a, b \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$, then the greatest lower bound of $\mathrm{pr}_3\,\mathrm{pr}_2\,a$ and $\mathrm{pr}_3\,\mathrm{pr}_2\,b$ in $\langle A_n, <_n\rangle$ is the greatest lower bound of a and b in $\langle A_{n+1}, <_{n+1}\rangle$. By (32), it suffices to show that for all $d \in A_{n+1}$, if $d \leqslant_{n+1} a$ and $d \leqslant_{n+1} b$, then $d \in A_n$. In fact, for all $c \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$, by (30), there is a unique $\lessdot_{n+1}$-chain from c to e_{n+1}, and thus, since a and b are incomparable, it cannot happen that $c \leqslant_{n+1} a$ and $c \leqslant_{n+1} b$ simultaneously.

STEP 6. We prove that $\langle A_{n+1}, <_{n+1}\rangle$ satisfies the finitary lower covering condition. Since $\langle A_n, <_n\rangle$ satisfies the finitary lower covering condition, by (28), it suffices to prove that for all $b \in A_{n+1} \setminus A_n$ covering at least two elements, (9) holds. Let b be an element of $A_{n+1} \setminus A_n$ covering at least two elements. Then, by the definition of $\lessdot_{n+1}$, b covers some element of $A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$. Let a be an arbitrary element of $A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$ covered by b. Then $b = \mathrm{pr}_2\,a$ and thus $\mathrm{pr}_3\,b = \mathrm{pr}_3\,\mathrm{pr}_2\,a \lessdot_{n+1} a$. Notice that if $b = e_{n+1}$ then $\mathrm{cov}(b) \cap A_n = \{e_n\}$ and $\mathrm{pr}_3\,b = e_{n-1} \lessdot_n e_n$, and if $b \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$ then $\mathrm{cov}(b) \cap A_n = \{\mathrm{pr}_3\,\mathrm{pr}_2\,b\}$ and, by (26) and (27), $\mathrm{pr}_3\,b \lessdot_n \mathrm{pr}_3\,\mathrm{pr}_2\,b$. Thus $\mathrm{pr}_3\,b = b^\star$ and (9) holds. Hence $\langle A_{n+1}, <_{n+1}\rangle$ satisfies the finitary lower covering condition and

$$\forall b \in A_{n+1} \setminus A_n \big(|\mathrm{cov}(b)| \geqslant 2 \rightarrow b^\star = \mathrm{pr}_3\,b\big). \tag{34}$$

STEP 7. Now, by Lemma 4.5, in $\langle A_{n+1}, <_{n+1}\rangle$, the height of b is well-defined for any $b \in A_{n+1}$. Notice that for all $d \in A_n$, by (28), the height of d in $\langle A_{n+1}, <_{n+1}\rangle$ is the same as its height in $\langle A_n, <_n\rangle$. We claim that

$$\forall a \in A_{n+1} \setminus \{o\}\big(\mathrm{ht}(a) = \mathrm{pr}_1\,a + 1\big). \tag{35}$$

Since in $\langle A_n, <_n\rangle$ we have $\mathrm{ht}(a) = \mathrm{pr}_1\,a + 1$ for any $a \in A_n \setminus \{o\}$, it suffices to prove that $\mathrm{ht}(b) = \mathrm{pr}_1\,b + 1$ for any $b \in A_{n+1} \setminus A_n$. Let $b \in A_{n+1} \setminus A_n$. If $b = e_{n+1}$, then $\mathrm{ht}(b) = \mathrm{ht}(e_n) + 1 = \mathrm{pr}_1\,e_n + 2 = n + 1 = \mathrm{pr}_1\,b + 1$. Otherwise, the sequence t of length $n - \mathrm{pr}_1\,b + 1$ such that $t(0) = b$ and $t(j+1) = \mathrm{pr}_2\,t(j)$ for any $j < n - \mathrm{pr}_1\,b$ is a $\lessdot_{n+1}$-chain from b to e_{n+1}, which implies that $\mathrm{ht}(b) + n - \mathrm{pr}_1\,b = \mathrm{ht}(e_{n+1})$ and hence $\mathrm{ht}(b) = \mathrm{pr}_1\,b + 1$. Thus (35) is proved.

STEP 8. Finally, we prove that $\langle A_{n+1}, <_{n+1}\rangle$ is a building block and $a^\star = \mathrm{pr}_3\,a$ for any $a \in A_{n+1} \setminus \{o\}$. Since $\langle A_n, <_n\rangle$ is a building block in which we have $a^\star = \mathrm{pr}_3\,a$ for any $a \in A_n \setminus \{o\}$, by (28), it suffices to prove that for all $b \in A_{n+1} \setminus A_n$, $b^\star = \mathrm{pr}_3\,b$, if $\mathrm{ht}(b) = 2$ then $|\mathrm{cov}(b)| = 4$, and if $\mathrm{ht}(b) > 2$ then (10) holds. Let $b \in A_{n+1} \setminus A_n$. We consider the following three cases:

CASE 1. $\mathrm{ht}(b) = 1$. Then $\mathrm{pr}_1\,b = 0$ by (35), and hence $b^\star = o = \mathrm{pr}_3\,b$.

CASE 2. $\mathrm{ht}(b) = 2$, Then $\mathrm{pr}_1\,b = 1$ by (35), which implies that $b \in B_{n,n-1}$ and $\mathrm{cov}(b) \cap (A_{n+1} \setminus A_n) = \{(n, 0, b, o, j) \mid j < 3\}$. Since $\mathrm{cov}(b) \cap A_n$ is a singleton, it follows that $|\mathrm{cov}(b)| = 4$, which implies that $b^\star = \mathrm{pr}_3\,b$ by (34).

CASE 3. $\mathrm{ht}(b) > 2$. Then $\mathrm{pr}_1\,b > 1$ by (35). We further consider two subcases:

CASE 3A. $b = e_{n+1}$. Then we have $n = \mathrm{pr}_1\,b > 1$ and thus $\mathrm{pr}_3\,b = e_{n-1} \neq o$. For all $c \lessdot_n \mathrm{pr}_3\,b$, if $c = e_{n-2}$ then $L_{b,c} = 3$ and hence

$$\big|\{a \in \mathrm{cov}(b) \mid \mathrm{pr}_3\,a = c\}\big| = \big|\{e_n\} \cup \{(n, n-1, b, c, j) \mid j < L_{b,c}\}\big| = 4,$$

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and if $c \neq e_{n-2}$ then $L_{b,c} = 4$ and hence

$$\big|\{a \in \mathrm{cov}(b) \mid \mathrm{pr}_3\, a = c\}\big| = \big|\{(n, n-1, b, c, j) \mid j < L_{b,c}\}\big| = 4.$$

Thus we have $|\mathrm{cov}(b)| \geqslant 4$, which implies that $b^\star = \mathrm{pr}_3\, b$ by (34).

CASE 3B. $b \in A_{n+1} \setminus (A_n \cup \{e_{n+1}\})$. Then, by (27), we have $\mathrm{pr}_3\, b \lessdot_n \mathrm{pr}_3\, \mathrm{pr}_2\, b \lessdot_{n+1} b$, which implies that $\mathrm{ht}(\mathrm{pr}_3\, b) = \mathrm{ht}(b) - 2 > 0$ and therefore $\mathrm{pr}_3\, b \neq o$. For all $c \lessdot_n \mathrm{pr}_3\, b$, if $\mathrm{pr}_3\, \mathrm{pr}_3\, \mathrm{pr}_2\, b = c$ then $L_{b,c} = 3$ and hence

$$\big|\{a \in \mathrm{cov}(b) \mid \mathrm{pr}_3\, a = c\}\big| = \big|\{\mathrm{pr}_3\, \mathrm{pr}_2\, b\} \cup \{(n, \mathrm{pr}_1\, b - 1, b, c, j) \mid j < L_{b,c}\}\big| = 4,$$

and if $\mathrm{pr}_3\, \mathrm{pr}_3\, \mathrm{pr}_2\, b \neq c$ then $L_{b,c} = 4$ and hence

$$\big|\{a \in \mathrm{cov}(b) \mid \mathrm{pr}_3\, a = c\}\big| = \big|\{(n, \mathrm{pr}_1\, b - 1, b, c, j) \mid j < L_{b,c}\}\big| = 4.$$

Thus we have $|\mathrm{cov}(b)| \geqslant 4$, which implies that $b^\star = \mathrm{pr}_3\, b$ by (34).

Now, since in all cases we have that $b^\star = \mathrm{pr}_3\, b$, we can replace $\mathrm{pr}_3\, b$ by $b^\star$ and $\mathrm{pr}_3\, a$ by $a^\star$ in the above two subcases, and hence (10) holds in both subcases, which completes the proof. ⊣

COROLLARY 4.12. *$\langle A, < \rangle$ is a locally finite lattice with a least element.*

PROOF. By Lemma 4.11, for all $n \in \omega$, $\langle A_n, <_n \rangle$ is a finite lattice and o is the least element of $\langle A_n, <_n \rangle$. Hence it follows from (28) that $\langle A, < \rangle$ is a locally finite lattice and o is the least element of $\langle A, < \rangle$. ⊣

LEMMA 4.13. *For all $m \in \omega$, every automorphism of $\langle A_m, <_m \rangle$ extends to an automorphism of $\langle A, < \rangle$.*

PROOF. Let $m \in \omega$ and let g be an automorphism of $\langle A_m, <_m \rangle$. We define an automorphism π of $\langle A, < \rangle$ extending g as follows:

We want to use the function Ψ defined before Lemma 4.10 to extend g step by step. For each $n \in \omega$, let

$$C_n = \big\{b \in A_{m+2n+2} \setminus A_{m+2n} \,\big|\, \mathrm{pr}_1\, b = 1\big\},$$

let

$$D_n = \big\{(b, c) \,\big|\, b \in A_{m+2n+2} \setminus A_{m+2n} \wedge \mathrm{pr}_1\, b > 1 \wedge c \lessdot_{m+2n} \mathrm{pr}_3\, b\big\},$$

let σ_n be the function defined on C_n such that for all $b \in C_n$, $\sigma_n(b)$ is the function defined on $\{a \in A_{m+2n+2} \setminus A_{m+2n} \mid a \lessdot_{m+2n+2} b\}$ given by $\sigma_n(b)(a) = \mathrm{pr}_4\, a$, and let τ_n be the function defined on D_n such that for all $(b, c) \in D_n$, $\tau_n(b, c)$ is the function defined on $\{a \in A_{m+2n+2} \setminus A_{m+2n} \mid a \lessdot_{m+2n+2} b \wedge \mathrm{pr}_3\, a = c\}$ given by $\tau_n(b, c)(a) = \mathrm{pr}_4\, a$. Notice that, by Lemma 4.11, for each $n \in \omega$, C_n and D_n correspond to the sets C and D in Section 4.2 respectively if we take $\langle P, < \rangle$ to be $\langle A_{m+2n+2}, <_{m+2n+2} \rangle$, and σ_n, τ_n satisfy the corresponding conditions satisfied by σ, τ respectively. Now we define h_n by recursion on n as follows:

$$h_0 = g;$$
$$h_{n+1} = \Psi(A_{m+2n+2}, <_{m+2n+2}, \sigma_n, \tau_n, h_n).$$

Then, by Lemma 4.11 and (25), it follows from a routine induction that for all $n \in \omega$, h_{n+1} is an automorphism of $\langle A_{m+2n+2}, <_{m+2n+2} \rangle$ extending h_n. Now it suffices to take $\pi = \bigcup_{n \in \omega} h_n$. ⊣

Now we are ready to prove our main lemma.

LEMMA 4.14. *In $\mathcal{V}_{\mathcal{S}}$, $\mathcal{S}(A) = \mathcal{S}_{\text{fin}}(A)$.*

PROOF. Let $u \in \mathcal{V}_{\mathcal{S}}$ be a permutation of A, and let $B \in \text{fin}(A)$ be a support of u. Let k be the least natural number such that $B \subseteq A_k$. We claim that

$$\text{mov}(u) \subseteq A_k.$$

In fact, assume towards a contradiction that there is an $a \in \text{mov}(u) \setminus A_k$. Let $n = \text{pr}_0\, a$ and let $b = u(a) \neq a$. Then $a \in A_{n+1} \setminus A_n$ and hence $k \leqslant n$.

If $b \in A_n$, or if $b \in A_{n+1} \setminus A_n$ and $\text{pr}_1\, b \geqslant \text{pr}_1\, a$, then, by Lemmas 4.11 and 4.10, there exists an automorphism g of $\langle A_{n+2}, <_{n+2} \rangle$ fixing $A_n \cup \{b\}$ pointwise and such that $g(a) \neq a$. By Lemma 4.13, g extends to an automorphism π of $\langle A, < \rangle$. Then we have $\pi \in \text{fix}_{\mathcal{G}}(B \cup \{b\})$ and $\pi(a) \neq a$. Hence π moves u, contradicting the assumption that B is a support of u.

Therefore, $b \notin A_n$, and if $b \in A_{n+1} \setminus A_n$ then $\text{pr}_1\, b < \text{pr}_1\, a$. Let $m = \text{pr}_0\, b$. Then $b \in A_{m+1} \setminus A_m$ and hence $n \leqslant m$, which implies that either $a \in A_m$ or $a \in A_{m+1} \setminus A_m$ and $\text{pr}_1\, a > \text{pr}_1\, b$. Thus, by Lemmas 4.11 and 4.10, there exists an automorphism h of $\langle A_{m+2}, <_{m+2} \rangle$ fixing $A_m \cup \{a\}$ pointwise and such that $h(b) \neq b$. By Lemma 4.13, h extends to an automorphism σ of $\langle A, < \rangle$. Then $\sigma \in \text{fix}_{\mathcal{G}}(B \cup \{a\})$ and $\sigma(b) \neq b$. Hence σ moves u, contradicting again that B is a support of u.

Thus $\text{mov}(u) \subseteq A_k$. Since A_k is finite, it follows that $u \in \mathcal{S}_{\text{fin}}(A)$. ⊣

COROLLARY 4.15. *Let A be the set of atoms of $\mathcal{V}_{\mathcal{S}}$ and let $\mathfrak{a} = |A|$. In $\mathcal{V}_{\mathcal{S}}$, $\mathfrak{a}! \leqslant_{\text{fto}} \mathfrak{a}$.*

PROOF. By Lemma 4.14, we have $\mathfrak{a}! = \mathcal{S}_{\text{fin}}(\mathfrak{a})$, and by Fact 2.7, $\mathcal{S}_{\text{fin}}(\mathfrak{a}) \leqslant_{\text{fto}} \text{fin}(\mathfrak{a})$. Also, it follows from Corollary 4.12 and Fact 4.2 that $\text{fin}(\mathfrak{a}) \leqslant_{\text{fto}} \mathfrak{a}$. Therefore, we have $\mathfrak{a}! = \mathcal{S}_{\text{fin}}(\mathfrak{a}) \leqslant_{\text{fto}} \text{fin}(\mathfrak{a}) \leqslant_{\text{fto}} \mathfrak{a}$. ⊣

Now the following theorem immediately follows from Corollary 4.15 and the Jech-Sochor theorem.

THEOREM 4.16. *The following statement is consistent with* ZF: *There exists an infinite cardinal $\mathfrak{a}$ such that $\mathfrak{a}! \leqslant_{\text{fto}} \mathfrak{a}$.*

§5. Conclusion.

In what follows, we first list some open problems which are of interest for future work, and then summarize the relationships between $\mathfrak{a}!$ and some other cardinals considered in the two parts of this work. Finally, we make a comparison of these relationships with those between $2^{\mathfrak{a}}$ and some other cardinals.

5.1. Open problems. Now we propose three open problems as follows:

QUESTION 5.1. *Is it consistent with* ZF *that there exists an infinite cardinal $\mathfrak{a}$ such that $\mathfrak{a}! < \aleph_0 \cdot \mathfrak{a}$?*

By Lemma 3.31 of [14], an affirmative answer to this question would yield a generalization of Theorem 4.16.

QUESTION 5.2. *Is it consistent with* ZF *that there exists an infinite cardinal $\mathfrak{a} \leqslant 2^{\aleph_0}$ such that $\mathfrak{a}! \leqslant_{\text{fto}} \mathfrak{a}$?*

By Theorem 3.33 of [14], an affirmative answer to this question would give an affirmative answer to Question 5.1.

QUESTION 5.3. *Is it consistent with* ZF *that there exists a cardinal $\mathfrak{a}$ such that $\mathfrak{a}! = [\mathfrak{a}]^3$?*

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Note that, by Theorem 3.12, the existence of an infinite cardinal $\mathfrak{a}$ such that $\mathfrak{a}! < [\mathfrak{a}]^3$ is consistent with ZF.

5.2. Summary. Now we summarize the ZF results proved in Part I and the consistency results obtained in this part. For all cardinals $\mathfrak{a}$, if $\mathfrak{a}!$ is Dedekind infinite, then $\mathfrak{a}!$ cannot be too small, in the following sense:

- $\mathfrak{a}! \not\leqslant_{\mathrm{dfto}} \mathrm{seq}(\mathcal{S}_{\mathrm{pdfin}}(\mathfrak{a}))$ (cf. [14, Theorem 3.14]);
- $2^{\aleph_0} \cdot \mathfrak{a} \leqslant 2^{\aleph_0} \cdot \mathrm{seq}(\mathfrak{a}) \leqslant 2^{\aleph_0} \cdot \mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) \leqslant 2^{\aleph_0} \cdot \mathcal{S}_{\mathrm{pdfin}}(\mathfrak{a}) \leqslant \mathfrak{a}!$ (cf. [14, Theorem 3.9]);
- $\aleph_0 \cdot \mathfrak{a} \leqslant \mathrm{seq}(\mathfrak{a}) \leqslant \aleph_0 \cdot \mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) \leqslant \aleph_0 \cdot \mathcal{S}_{\mathrm{pdfin}}(\mathfrak{a}) < \mathfrak{a}!$ (cf. [14, Corollary 3.15]).

However, if we replace the requirement that $\mathfrak{a}!$ is Dedekind infinite by the requirement that $\mathfrak{a}$ is infinite, then it may consistently happen that $\mathfrak{a}! \leqslant_{\mathrm{fto}} \mathfrak{a}$ (cf. Theorem 4.16) and that $\mathfrak{a}! < \mathrm{seq}^{1\text{-}1}(\mathfrak{a}) < \mathrm{seq}(\mathfrak{a})$ (cf. Proposition 3.6). It is an open problem whether or not it may consistently happen that $\mathfrak{a}! < \aleph_0 \cdot \mathfrak{a}$ (cf. Question 5.1). Nevertheless, for all infinite cardinals $\mathfrak{a}$, we have:

- $\mathfrak{a}! \neq \mathrm{seq}^{1\text{-}1}(\mathfrak{a})$ (cf. [15, Theorem 2.2]);
- $\mathfrak{a}! \neq \mathrm{seq}(\mathfrak{a})$ (cf. [14, Corollary 3.17]);
- $\mathfrak{a}! \neq \aleph_0 \cdot \mathfrak{a}$ (cf. [14, Corollary 3.19]);
- $[\mathfrak{a}]^2 \leqslant \left[[\mathfrak{a}]^2\right]^2 < \mathfrak{a}!$ (cf. [14, Corollary 3.26]);
- $\mathfrak{a}^n < \mathfrak{a}!$ (cf. [14, Corollary 3.29]).

We also proved that for all infinite cardinals $\mathfrak{a}$, if there is a permutation without fixed points on a set which is of cardinality $\mathfrak{a}$, then $\mathfrak{a}! \not\leqslant_{\mathrm{fto}} \mathfrak{a}^n$ for any $n \in \omega$ (cf. [14, Corollary 3.30]). Theorem 4.16 shows that, in this result, we cannot remove the assumption that there is a permutation without fixed points on a set which is of cardinality $\mathfrak{a}$.

Even for Dedekind infinite cardinals $\mathfrak{a}$, it is not provable that $\left[\left[[\mathfrak{a}]^2\right]^2\right]^2 \leqslant \mathfrak{a}!$ or that $([\mathfrak{a}]^2)^2 \leqslant \mathfrak{a}!$ (cf. Proposition 3.2), and it may consistently happen that $\mathfrak{a}! < [\mathfrak{a}]^3$ and $\mathfrak{a}! \leqslant^* \mathfrak{a}$ (cf. Theorem 3.12). It is an open problem whether or not it may consistently happen that $\mathfrak{a}! = [\mathfrak{a}]^3$ (cf. Question 5.3).

For infinite cardinals $\mathfrak{a}$, it may consistently happen that $\mathfrak{a}! \parallel \mathrm{seq}^{1\text{-}1}(\mathfrak{a})$, $\mathfrak{a}! \parallel \mathrm{seq}(\mathfrak{a})$, $\mathfrak{a}! \parallel [\mathfrak{a}]^3$, and $\mathfrak{a}! \parallel 2^{\mathfrak{a}}$ (cf. Proposition 3.2). Also, it is easy to verify that $\mathfrak{a}! \parallel \aleph_0 \cdot \mathfrak{a}$ for any infinite but power Dedekind finite cardinal $\mathfrak{a}$, and therefore it may consistently happen that $\mathfrak{a}! \parallel \aleph_0 \cdot \mathfrak{a}$.

Now, for infinite cardinals $\mathfrak{a}$, we list all the possible relationships between $\mathfrak{a}!$ and $\mathfrak{a}$, $\aleph_0 \cdot \mathfrak{a}$, $\mathrm{seq}^{1\text{-}1}(\mathfrak{a})$, $\mathrm{seq}(\mathfrak{a})$, $[\mathfrak{a}]^3$, or $2^{\mathfrak{a}}$ in the following table.

We should also mention that it is consistent with ZF that there exists a power Dedekind infinite cardinal $\mathfrak{a}$ such that $\mathfrak{a}! \leqslant_{\mathrm{fto}} \aleph_0$. The sketch of the proof is as follows: Consider the permutation model $\mathcal{N}2(3)$ in [9]. In this permutation model, the set A of atoms is the union of a denumerable set B of pairwise disjoint 3-element sets, $\mathcal{G}$ is the group of all permutations of A that leave B pointwise fixed, and $\mathcal{I}$ is the normal ideal $\mathrm{fin}(A)$. It is easy to verify that in $\mathcal{N}2(3)$, we have $\mathcal{S}(A) = \mathcal{S}_{\mathrm{fin}}(A)$ and there is a three-to-one surjection from A onto ω. Hence, if $\mathfrak{a} = |A|$, then $\mathfrak{a}$ is power Dedekind infinite, $\mathfrak{a} \leqslant_{\mathrm{fto}} \aleph_0$, and $\mathfrak{a}! = \mathcal{S}_{\mathrm{fin}}(\mathfrak{a})$, which implies that, by Fact 2.7, $\mathfrak{a}! = \mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) \leqslant_{\mathrm{fto}} \mathrm{fin}(\mathfrak{a}) \leqslant_{\mathrm{fto}} \mathrm{fin}(\aleph_0) = \aleph_0$.

5.3. Comparison with powers. The relationships between $2^{\mathfrak{a}}$ and some other cardinals are studied in [3, 7, 8, 12, 13, 16, 19]. In [12, Proposition 3.13], the first author

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	$\mathfrak{a}$	$\aleph_0 \cdot \mathfrak{a}$	$\mathrm{seq}^{1\text{-}1}(\mathfrak{a})$	$\mathrm{seq}(\mathfrak{a})$	$[\mathfrak{a}]^3$	$2^{\mathfrak{a}}$
$\mathfrak{a}! >$	✓	✓	✓	✓	✓	✓
$\mathfrak{a}! =$	X	X	X	X	?	✓
$\mathfrak{a}! <$	X	?	✓	✓	✓	✓
$\mathfrak{a}! \parallel$	X	✓	✓	✓	✓	✓
$\mathfrak{a}! \leqslant_{\mathrm{fto}}$	✓	✓	✓	✓	✓	✓
$\mathfrak{a}! \leqslant^*$	✓	✓	✓	✓	✓	✓

proved that $2^{\mathfrak{a}} \nleqslant_{\mathrm{dfto}} \mathrm{seq}^{1\text{-}1}(\mathrm{pdfin}(\mathfrak{a}))$ for any power Dedekind infinite cardinal $\mathfrak{a}$. In fact, for power Dedekind infinite cardinals $\mathfrak{a}$, we have:

- $2^{\mathfrak{a}} \nleqslant_{\mathrm{dfto}} \mathrm{seq}(\mathrm{pdfin}(\mathfrak{a})), \mathrm{pdfin}(\mathrm{seq}(\mathfrak{a})), \mathrm{fin}(\mathrm{pdfin}(\mathfrak{a})), \mathrm{pdfin}(\mathrm{fin}(\mathfrak{a}))$;
- $2^{\aleph_0} \cdot \mathfrak{a} \leqslant 2^{\aleph_0} \cdot \mathrm{fin}(\mathfrak{a}) \leqslant 2^{\aleph_0} \cdot \mathrm{pdfin}(\mathfrak{a}) \leqslant 2^{\mathfrak{a}}$ (cf. [12, Lemma 3.18]);
- $\aleph_0 \cdot \mathfrak{a} \leqslant \aleph_0 \cdot \mathrm{fin}(\mathfrak{a}) \leqslant \aleph_0 \cdot \mathrm{pdfin}(\mathfrak{a}) < 2^{\mathfrak{a}}$ (cf. [12, Proposition 3.19]).

We shall omit the proof here. It is an open problem whether or not it is provable in ZF that $2^{\mathfrak{a}} \nleqslant \mathrm{pdfin}(\mathrm{pdfin}(\mathfrak{a}))$ for any power Dedekind infinite cardinal $\mathfrak{a}$. Hence, $2^{\mathfrak{a}}$ has stronger properties than $\mathfrak{a}!$, in the sense that the requirement that $\mathfrak{a}$ is power Dedekind infinite is weaker than the requirement that $\mathfrak{a}!$ is Dedekind infinite (cf. Fact 2.6), and $2^{\mathfrak{a}} \nleqslant_{\mathrm{dfto}} \mathrm{seq}(\mathrm{pdfin}(\mathfrak{a}))$ is stronger than $2^{\mathfrak{a}} \nleqslant_{\mathrm{dfto}} \mathrm{seq}(\mathcal{S}_{\mathrm{pdfin}}(\mathfrak{a}))$ (cf. Fact 2.8). Also, for infinite cardinals $\mathfrak{a}$, we have:

- $2^{\mathfrak{a}} \nleqslant_{\mathrm{fto}} [\mathfrak{a}]^n$ (cf. [13, Corollary 3.7]);
- $2^{\mathfrak{a}} \nleqslant_{\mathrm{fto}} \mathfrak{a}^n$ (cf. [12, Proposition 3.11]);
- $2^{\mathfrak{a}} \nleqslant_{\mathrm{fto}} \aleph_0 \cdot \mathfrak{a}$;
- $\mathrm{fin}(\mathfrak{a}) < 2^{\mathfrak{a}}$ (cf. [7, Theorem 3]);
- $2^{\mathfrak{a}} \neq \mathrm{seq}^{1\text{-}1}(\mathfrak{a})$ (cf. [7, Theorem 4]);
- $2^{\mathfrak{a}} \neq \mathrm{seq}(\mathfrak{a})$ (cf. [7, Theorem 5]).

We also omit the proof here. Notice that, even for infinite cardinals $\mathfrak{a}$, $2^{\mathfrak{a}}$ has stronger properties than $\mathfrak{a}!$, in the sense that it may consistently happen that $\mathfrak{a}! \leqslant_{\mathrm{fto}} \mathfrak{a}$ (cf. Theorem 4.16) and that $\mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) = \mathfrak{a}!$ (cf. Fact 3.5). Nevertheless, it may consistently happen that $2^{\mathfrak{a}} < \mathcal{S}_{\mathrm{fin}}(\mathfrak{a}) = \mathfrak{a}! < \mathrm{seq}^{1\text{-}1}(\mathfrak{a}) < \mathrm{seq}(\mathfrak{a})$ (cf. Fact 3.5 and Proposition 3.6) and hence, by Fact 2.7, $2^{\mathfrak{a}} \leqslant_{\mathrm{fto}} \mathrm{fin}(\mathfrak{a})$.

For the relation $\leqslant^*$, on the one hand, it may consistently happen that $\mathfrak{a}! \leqslant^* \mathfrak{a}$ (cf. Theorem 3.12), and on the other hand, by Cantor's theorem, we have that $2^{\mathfrak{a}} \nleqslant^* \mathfrak{a}$ for any cardinal $\mathfrak{a}$. Moreover, for all infinite cardinals $\mathfrak{a}$ and all cardinals $\mathfrak{b} \leqslant_{\mathrm{fto}} \mathfrak{a}$, we have that $2^{\mathfrak{a}} \nleqslant^* \mathfrak{b}$ (cf. [12, Theorem 5.3]). Notice that, in [7, Theorem 1], Halbeisen and Shelah proved that the existence of an infinite cardinal $\mathfrak{a}$ such that $2^{\mathfrak{a}} \leqslant^* \mathrm{fin}(\mathfrak{a})$ is consistent with ZF. Now we propose three open problems concerning the relation $\leqslant^*$ as follows:

QUESTION 5.4. *Is it consistent with* ZF *that there exists an infinite cardinal* $\mathfrak{a}$ *such that* $2^{\mathfrak{a}} \leqslant^* \mathfrak{a}^2$?

This question is known as the dual Specker problem and is asked in [18] (cf. also [5, p. 133] or [12, Problem 5.8]).

QUESTION 5.5. *Is it consistent with* ZF *that there exists an infinite cardinal* $\mathfrak{a}$ *such that* $2^{\mathfrak{a}} \leqslant^* [\mathfrak{a}]^2$?

This question is asked in [6]. Notice that an affirmative answer to this question would give an affirmative answer to Question 5.4.

QUESTION 5.6. *Is it consistent with* ZF *that there exists an infinite cardinal* $\mathfrak{a}$ *such that* $2^{\mathfrak{a}} \leqslant^* \aleph_0 \cdot \mathfrak{a}$?

In fact, an affirmative answer to this question would yield an affirmative answer to Question 5.5. The sketch of the proof is as follows: Notice that for all power Dedekind infinite cardinals $\mathfrak{a}$ we have $\aleph_0 \cdot \mathfrak{a} \leqslant^* [\mathfrak{a}]^2$. Hence, we only need to prove that for all infinite cardinals $\mathfrak{a}$, if $2^{\mathfrak{a}} \leqslant^* \aleph_0 \cdot \mathfrak{a}$, then $\mathfrak{a}$ is power Dedekind infinite. Let x be a set such that $|x| = \mathfrak{a}$, and let f be a surjection from $\omega \times x$ onto $\wp(x)$. Then we can explicitly define a surjection $g \subseteq f$ from a subset of $\omega \times x$ onto $\wp(x)$ such that for all $z \in x$, $g{\restriction}(\omega \times \{z\})$ is injective. If $\wp(x)$ is Dedekind finite, then for all $z \in x$, $\mathrm{dom}(g) \cap (\omega \times \{z\})$ is finite, and hence there exists a finite-to-one function from $\mathrm{dom}(g)$ into x, contradicting Theorem 5.3 of [12]. Therefore, we get that x is power Dedekind infinite, and hence $\mathfrak{a}$ is power Dedekind infinite.

Finally, for infinite cardinals $\mathfrak{a}$, we list all the possible relationships between $2^{\mathfrak{a}}$ and $\mathfrak{a}$, $\aleph_0 \cdot \mathfrak{a}$, $\mathfrak{a}^2$, $\mathrm{fin}(\mathfrak{a})$, $\mathrm{seq}^{1\text{-}1}(\mathfrak{a})$, or $\mathrm{seq}(\mathfrak{a})$ in the following table.

	$\mathfrak{a}$	$\aleph_0 \cdot \mathfrak{a}$	$\mathfrak{a}^2$	$\mathrm{fin}(\mathfrak{a})$	$\mathrm{seq}^{1\text{-}1}(\mathfrak{a})$	$\mathrm{seq}(\mathfrak{a})$
$2^{\mathfrak{a}} >$	✓	✓	✓	✓	✓	✓
$2^{\mathfrak{a}} =$	X	X	X	X	X	X
$2^{\mathfrak{a}} <$	X	X	X	X	✓	✓
$2^{\mathfrak{a}} \parallel$	X	✓	✓	X	✓	✓
$2^{\mathfrak{a}} \leqslant_{\mathrm{fto}}$	X	X	X	✓	✓	✓
$2^{\mathfrak{a}} \leqslant^*$	X	?	?	✓	✓	✓

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