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# ON $q$-SCALE FUNCTIONS OF SPECTRALLY NEGATIVE LÉVY PROCESSES

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**Abstract**

We obtain series expansions of the $q$-scale functions of arbitrary spectrally negative Lévy processes, including processes with infinite jump activity, and use these to derive various new examples of explicit $q$-scale functions. Moreover, we study smoothness properties of the $q$-scale functions of spectrally negative Lévy processes with infinite jump activity. This complements previous results of Chan *et al.* (*Prob. Theory Relat. Fields* **150**, 2011) for spectrally negative Lévy processes with Gaussian component or bounded variation.

*Keywords:* Laplace transform; $q$-scale functions; series expansions; smoothness; spectrally one-sided Lévy processes; Volterra equations; Bernstein functions; Doney's conjecture

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## 1. Introduction

Spectrally negative Lévy processes, i.e. Lévy processes whose jumps are negative, play a special role in applied mathematics, since many real-life applications in queueing, risk, finance, etc. show jumps or shocks only in one direction. Moreover, the restriction to one-sided jumps also has mathematical advantages, since spectrally negative Lévy processes admit $q$-scale functions. The $q$-scale functions get their name from the related scale functions of regular diffusions, and, just as for diffusions, many fluctuation identities for spectrally one-sided Lévy processes may be expressed in terms of $q$-scale functions; see e.g. [4, 12, 23, 24] or [9, 18], and the references given there.

Let $(L_t)_{t\geq 0}$ be a spectrally negative Lévy process with Laplace exponent $\psi$. For every $q \geq 0$ the $q$-scale function is defined as the unique function $W^{(q)} : \mathbb{R} \to \mathbb{R}_+$ such that $W^{(q)}(x) = 0$ for all $x < 0$ and whose Laplace transform is given by

$$\mathcal{L}\big[W^{(q)}\big](\beta) = \frac{1}{\psi(\beta) - q}, \quad \beta > \Phi(q) := \sup\{y : \psi(y) = q\}. \tag{1.1}$$

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However, it is not often possible to invert the Laplace transform explicitly and to obtain $W^{(q)}$ in a closed-form expression; see e.g. [17] for a collection of known cases. Although there are efficient methods to evaluate $q$-scale functions numerically (cf. [18]) or to approximate them (cf. [11]), it is of considerable interest to expand the set of examples with analytic expressions. Quite often, evaluating the scale function is only an intermediate step, in the sense that one needs functions of $W^{(q)}$, rather than $W^{(q)}$ itself. This shows the limitations of numerical or approximative approaches, e.g. if one has to estimate the jump measure, which is the case in most applications.

Recently, Landriault and Willmot [21] proposed the following method to derive an explicit representation of the $q$-scale function of a spectrally negative Lévy process *with finite Lévy measure*, i.e. of a spectrally negative compound Poisson process: determine suitable functions $f, g : (\Phi(q), \infty) \to \mathbb{R}$ and a constant $b \neq 0$ to rewrite the right-hand side of (1.1) in a form which can be expanded into a geometric series, i.e. such that

$$\frac{\mathcal{L}g(\beta)}{b - \mathcal{L}f(\beta)} = \mathcal{L}g(\beta) \sum_{n=0}^{\infty} \frac{(\mathcal{L}f(\beta))^n}{b^{n+1}} \tag{1.2}$$

on the set $\{\beta \geq 0 : |\mathcal{L}f(\beta)| < b\}$. This Laplace transform can then easily be inverted term by term. The same idea was previously used in [8] in order to derive smoothness properties of the $q$-scale functions of spectrally negative Lévy processes with paths of bounded variation or with Gaussian component.

In Section 3 of the present article we show how one can modify the approach of [21] to obtain series representations of the $q$-scale functions of spectrally negative Lévy processes with infinite jump activity. Moreover, in Section 4 we study the smoothness of scale functions, at least for certain subclasses (that were not treated in [8]). Finally, Section 5 contains several auxiliary technical lemmas which we need for our proofs.

## 2. Preliminaries

Most of our notation is standard or self-explanatory. Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$ are the natural numbers starting from 0, and $\mathbb{R}_+ = [0, \infty)$ is the non-negative half-line. We write $\partial_x := \frac{\partial}{\partial x}$, and we drop the subscript $x$ if no confusion is possible. By $\mathbb{1}$ and id, respectively, we mean the constant function $x \mapsto 1$ and the identity map $x \mapsto x$ on $\mathbb{R}_+$. If needed, we extend a function $f : (0, \infty) \to \mathbb{R}$ onto $[0, \infty)$ by setting $f(0) := f(0+) = \lim_{y \downarrow 0} f(y)$ if the limit exists in $\mathbb{R}$, and $f(0) := 0$ otherwise. In particular, this ensures that $f \in L^1_{\text{loc}}(\mathbb{R}_+)$ implies integrability at 0.

### 2.1. Laplace transforms and convolutions

Throughout, $\mathcal{L}$ denotes the (one-sided) Laplace transform

$$\mathcal{L}\mu(\beta) := \mathcal{L}[\mu](\beta) := \int_{[0,\infty)} \mathrm{e}^{-\beta x}\, \mu(\mathrm{d}x), \tag{2.1}$$

for a measure $\mu$ on $[0, \infty)$ and every $\beta \geq 0$ for which the integral converges. As usual, if $\mu$ is absolutely continuous with respect to Lebesgue measure with a locally integrable density $f$, i.e. $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$, we write $\mu \ll \mathrm{d}x$, identify $\mu$ and $f$, and write $\mathcal{L}f = \mathcal{L}\mu$.

Let $\mu$ and $\nu$ be two measures with support in $\mathbb{R}_+$ such that the Laplace transforms $\mathcal{L}\mu$ and $\mathcal{L}\nu$ converge for some $\beta_0 > 0$. Then they also converge for all $\beta \in [\beta_0, \infty)$, and we have

$$\mathcal{L}[\mu * \nu](\beta) = \mathcal{L}\mu(\beta)\mathcal{L}\nu(\beta) \quad \text{on } [\beta_0, \infty), \tag{2.2}$$

for the convolution of the measures

$$(\mu * \nu)(B) := \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_B(x+y)\, \mu(\mathrm{d}x)\, \nu(\mathrm{d}y), \quad \text{for all Borel sets } B \subset \mathbb{R}. \tag{2.3}$$

If $\mu$ and $\nu$ are absolutely continuous with respect to Lebesgue measure with locally integrable densities $f$ and $g$, then (2.2) becomes

$$\mathcal{L}[f * g](\beta) = \mathcal{L}f(\beta)\mathcal{L}g(\beta),$$

where the convolution is given by

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y = \int_{[0,x]} f(x-y)g(y)\mathrm{d}y, \quad x \geq 0.$$

The second equality is due to the fact that the supports of $f$ and $g$ are contained in $\mathbb{R}_+$. If $f, g \in L^1_{\text{loc}}(\mathbb{R}_+)$, then for all $x \geq 0$

$$\|f * g\|_{L^1([0,x])} \leq \|f\|_{L^1([0,x])}\|g\|_{L^1([0,x])}, \tag{2.4}$$

where $\|f\|_{L^1([0,x])} := \int_{[0,x]} |f(y)|\mathrm{d}y$. Finally, if $f \in AC[0, \infty)$ or $f \in AC(0, \infty)$ is absolutely continuous, integration by parts yields

$$\mathcal{L}[\partial f](\beta) = \beta\mathcal{L}f(\beta) - f(0+). \tag{2.5}$$

## 2.2. Spectrally negative Lévy processes

Throughout this article, $(L_t)_{t\geq 0}$ denotes a spectrally negative Lévy process, $(\mathbb{P}_x)_{x\in\mathbb{R}}$ the law of $(L_t)_{t\geq 0}$, and $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration satisfying the usual assumptions. A spectrally negative Lévy process is a one-dimensional Lévy process with no positive jumps and with Laplace exponent given by

$$\psi(\beta) := \frac{1}{t}\log \mathbb{E}\big[\mathrm{e}^{\beta L_t}\big] = c\beta + \sigma^2\beta^2 + \int_{(-\infty,0)} \big(\mathrm{e}^{\beta x} - 1 - \beta x \mathbb{1}_{(0,1)}(|x|)\big)\, \widetilde{\nu}(\mathrm{d}x), \quad \beta \geq 0,$$

where $\big(c, 2\sigma^2, \widetilde{\nu}\big)$ is the characteristic triplet of $(L_t)_{t\geq 0}$. To simplify notation, we denote by $\nu$ the reflection of $\widetilde{\nu}$, i.e. $\nu(B) = \widetilde{\nu}(-B)$ for all measurable sets $B \subset \mathbb{R}$, which yields

$$\psi(\beta) = c\beta + \sigma^2\beta^2 + \int_{(0,\infty)} \big(\mathrm{e}^{-\beta x} - 1 + \beta x \mathbb{1}_{(0,1)}(x)\big)\, \nu(\mathrm{d}x), \quad \beta \geq 0. \tag{2.6}$$

Depending on the characteristics $\big(c, 2\sigma^2, \widetilde{\nu}\big)$ we may simplify the representation of the Laplace exponent $\psi$; see Lemma 2.1. Although such representations appear frequently in the literature, we provide short arguments for them, which also serves to fix notation and ideas.

**Lemma 2.1.** *Let $(L_t)_{t\geq 0}$ be a spectrally negative Lévy process with characteristic triplet $\big(c, 2\sigma^2, \widetilde{\nu}\big)$ and Laplace exponent $\psi$.*

(i) [*Jumps of bounded variation.*] *If $\int_{(0,1)} y\, \nu(\mathrm{d}y) < \infty$, then*

$$\psi(\beta) = c'\beta + \sigma^2\beta^2 - \beta\mathcal{L}\bar{\nu}(\beta),$$

*with $c' = c + \int_{(0,1)} y\, \nu(\mathrm{d}y)$ and $\bar{\nu}(x) := \nu([x, \infty))$ being the tail function of $\nu$.*

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(ii) [*Finite first moment.*] *If* $\int_{[1,\infty)} y\,\nu(\mathrm{d}y) < \infty$, *then*

$$\psi(\beta) = c''\beta + \sigma^2\beta^2 + \beta^2 \mathcal{L}\bar{\bar{\nu}}(\beta),$$

*with* $c'' = c - \int_{[1,\infty)} y\,\nu(\mathrm{d}y) = \mathbb{E}L_1$ *and* $\bar{\bar{\nu}}(x) := \int_{[x,\infty)} \bar{\nu}(y)\mathrm{d}y$ *being an integrated tail function of* $\nu$.

*Proof.* To see (i), set $c' = c + \int_{(0,1)} y\,\nu(\mathrm{d}y)$ and compute

$$\begin{aligned}
\psi(\beta) &= c'\beta + \sigma^2\beta^2 + \int_{(0,\infty)} (\mathrm{e}^{-\beta x} - 1)\nu(\mathrm{d}x)\\
&= c'\beta + \sigma^2\beta^2 - \beta \int_{(0,\infty)} \int_{[0,x]} \mathrm{e}^{-\beta y}\,\mathrm{d}y\,\nu(\mathrm{d}x)\\
&= c'\beta + \sigma^2\beta^2 - \beta \int_{[0,\infty)} \mathrm{e}^{-\beta y} \int_{[y,\infty)} \nu(\mathrm{d}x)\mathrm{d}y\\
&= c'\beta + \sigma^2\beta^2 - \beta\mathcal{L}\bar{\nu}(\beta).
\end{aligned}$$

Further, (ii) follows in a similar way if we set $c'' = c - \int_{[1,\infty)} y\,\nu(\mathrm{d}y)$ and use

$$\begin{aligned}
\psi(\beta) &= c''\beta + \sigma^2\beta^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-\beta x} - 1 + \beta x\right) \nu(\mathrm{d}x)\\
&= c''\beta + \sigma^2\beta^2 - \beta \int_{(0,\infty)} \int_{[0,x]} \left(\mathrm{e}^{-\beta y} - 1\right)\mathrm{d}y\,\nu(\mathrm{d}x)\\
&= c''\beta + \sigma^2\beta^2 + \beta^2 \int_{(0,\infty)} \int_{[0,x]} \int_{[0,y]} \mathrm{e}^{-\beta z}\,\mathrm{d}z\,\mathrm{d}y\,\nu(\mathrm{d}x)\\
&= c''\beta + \sigma^2\beta^2 - \beta^2 \int_{[0,\infty)} \mathrm{e}^{-\beta z} \int_{[z,\infty)} \int_{[y,\infty)} \nu(\mathrm{d}x)\mathrm{d}y\,\mathrm{d}z\\
&= c''\beta + \sigma^2\beta^2 + \beta^2\mathcal{L}\bar{\bar{\nu}}(\beta).
\end{aligned}$$

This completes the proof. □

From now on we will always use the constant $c \in \mathbb{R}$ to denote the location parameter if $\psi$ is in the form (2.6), and $c', c'' \in \mathbb{R}$ if we write $\psi$ as in Lemma 2.1(i)–(ii).

Note that whenever $\int_{[1,\infty)} y\nu(\mathrm{d}y) < \infty$, the function $\bar{\bar{\nu}}$ is well-defined, and integration by parts implies that in this case

$$\int_{(0,1)} y\nu(\mathrm{d}y) = \infty \quad \Longleftrightarrow \quad \bar{\bar{\nu}}(0+) = \infty.$$

Further background information on (spectrally negative) Lévy processes can be found in [5, 27].

### 2.3. Scale functions

The $q$-scale function of a spectrally negative Lévy process $(L_t)_{t\geq 0}$ is uniquely determined via its Laplace transform given in (1.1). If $\widetilde{\nu} \equiv 0$, i.e. if $(L_t)_{t\geq 0}$ is a Brownian motion with drift, the Laplace transform in (1.1) can be inverted explicitly (cf. [18]), leading to the well-known formula

$$W^{(q)}(x) = \frac{1}{\sqrt{c^2+2q\sigma^2}}\left(\mathrm{e}^{\left(\sqrt{c^2+2q\sigma^2}-c\right)x/\sigma^2} - \mathrm{e}^{-\left(\sqrt{c^2+2q\sigma^2}+c\right)x/\sigma^2}\right), \quad x \geq 0. \tag{2.7}$$

We will therefore restrict ourselves to processes with $\widetilde{\nu} \not\equiv 0$. Moreover, we will exclude subordinators, i.e. monotone Lévy processes. In particular, if $\sigma^2 = 0$ and $\int_{(0,1)} x\,\nu(\mathrm{d}x) < \infty$, we assume $c \geq \int_{(0,1)} x\nu(\mathrm{d}x)$.

For a spectrally negative Lévy process $L = (L_t)_{t\geq 0}$ with Laplace exponent $\psi$ and characteristic triplet $\left(c, 2\sigma^2, \widetilde{\nu}\right)$ it is well known (cf. [18, Section 3.3], [8], and [19, Chapter 8.2]) that under the exponential change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\Phi(q)}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathrm{e}^{\Phi(q)(L_t-x)-qt}, \quad q \geq 0,$$

the process $\left(L, \mathbb{P}^{\Phi(q)}\right)$ is again a spectrally negative Lévy process whose Laplace exponent is given by $\psi_{\Phi(q)}(\beta) = \psi(\beta + \Phi(q)) - q$ and whose Lévy measure is $\widetilde{\nu}_{\Phi(q)}(\mathrm{d}x) = \mathrm{e}^{\Phi(q)x}\widetilde{\nu}(\mathrm{d}x)$. The shape of $\psi_{\Phi(q)}$, together with (1.1), yields $W^{(q)}(x) = \mathrm{e}^{\Phi(q)x}W_{\Phi(q)}(x)$, where $W_{\Phi(q)}$ denotes the 0-scale function of $\left(L, \mathbb{P}^{\Phi(q)}\right)$. This relation often allows us to assume without loss of generality that $q = 0$, and we will use this observation below in Section 4.1 for the proof of Theorem 4.1.

The surveys [2] and [18] contain a thorough introduction to scale functions and some of their applications.

## 3. Series expansions of $q$-scale functions

As before, $(L_t)_{t\geq 0}$ is a spectrally negative Lévy process with Laplace exponent $\psi$ and Lévy triplet $\left(c, 2\sigma^2, \widetilde{\nu}\right)$; $\nu(B) := \widetilde{\nu}(-B)$; we set $\bar{\nu}(x) = \nu([x, \infty))$ and $\bar{\bar{\nu}}(x) = \int_{[x,\infty)} \bar{\nu}(y)\mathrm{d}y$; $c'$ and $c''$ are the constants from Lemma 2.1.

We will extend the approach of [21] to derive series expansions for the $q$-scale function of $(L_t)_{t\geq 0}$, which will be presented in Theorems 3.1, 3.2, and 3.3 below. These three theorems cover the following three different situations:

(i) $\sigma^2 > 0$ (Section 3.1, Theorem 3.1): In this case, our result extends [21, Theorem 2.4] from finite to arbitrary jump measures.

(ii) $\sigma^2 = 0$ and $\int_{(0,1)} y\,\nu(\mathrm{d}y) < \infty$ (Section 3.2, Theorem 3.2): We recover results which have previously been obtained in [10] and, in special cases, also in [8, 13, 21].

(iii) $\sigma^2 = 0$ and $\int_{(0,1)} y\,\nu(\mathrm{d}y) = \infty$ (Section 3.3, Theorem 3.3): Our results in this situation seem to be new.

The proofs of all three theorems use the same strategy: find a suitable expansion of the Laplace transform, which can be inverted term by term.

### 3.1. Lévy processes with a Gaussian component

We start by presenting a series expansion for scale functions of spectrally negative Lévy processes that have a Gaussian component. Results of this kind have a long history. For example, if we combine in [8] the proof of Theorem 1 with Corollary 9, we arrive at a series representation for $\partial W^{(0)}$, and this leads to an (implicit) expansion for $\partial W^{(q)}$ using the relations between $W^{(0)}$

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and $W^{(q)}$. Also note that, setting $\nu \equiv 0$ in Theorem 3.1 below, we recover (2.7). At this point we note that the moment condition $\int_{[1,\infty)} y\,\nu(\mathrm{d}y) < \infty$ appearing in Theorem 3.1 is for notational convenience only, and can be discarded at the expense of heavier notation; see also Remark 3.1 below.

**Theorem 3.1.** *Let $(L_t)_{t\geq 0}$ be a spectrally negative Lévy process with triplet $\left(c, 2\sigma^2, \widetilde{\nu}\right)$ such that $\sigma^2 > 0$ and $\int_{[1,\infty)} y\,\nu(\mathrm{d}y) < \infty$. For $q \geq 0$ set*

$$f(x) := -c'' + qx - \bar{\bar{\nu}}(x) = \int_{[1,\infty)} y\,\nu(\mathrm{d}y) - c + qx - \bar{\bar{\nu}}(x), \quad x \geq 0;$$

*then the q-scale function of $(L_t)_{t\geq 0}$ is given by*

$$W^{(q)} = \mathrm{id} * \sum_{n=0}^{\infty} \frac{f^{*n}}{(\sigma^2)^{n+1}}, \tag{3.1}$$

*and the series on the right-hand side converges uniformly on compact subsets of $\mathbb{R}_+$ to a limit in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$.*

*Proof.* By Lemma 2.1(ii), the Laplace exponent of $L$ is given by $\psi(\beta) = c''\beta + \sigma^2\beta^2 + \beta^2\mathcal{L}\bar{\bar{\nu}}(\beta)$ with $c'' = c - \int_{[1,\infty)} y\,\nu(\mathrm{d}y)$. From the definition of the scale function we thus get for all $q \geq 0$

$$\mathcal{L}\big[W^{(q)}\big](\beta) = \frac{1}{\beta^2\left(c''\beta^{-1} + \sigma^2 + \mathcal{L}\bar{\bar{\nu}}(\beta) - q\beta^{-2}\right)}.$$

For $\beta > 1$ large enough we have $\sigma^2 > \left|c''\beta^{-1} + \mathcal{L}\bar{\bar{\nu}}(\beta) - q\beta^{-2}\right|$, and so we may expand the fraction appearing on the right-hand side into a geometric series:

$$\mathcal{L}\big[W^{(q)}\big](\beta) = \frac{1}{\beta^2}\sum_{n=0}^{\infty} \frac{\left(-c''\beta^{-1} - \mathcal{L}\bar{\bar{\nu}}(\beta) + q\beta^{-2}\right)^n}{(\sigma^2)^{n+1}} = \frac{1}{\beta^2}\sum_{n=0}^{\infty} \frac{(\mathcal{L}f(\beta))^n}{(\sigma^2)^{n+1}} \tag{3.2}$$

with $f(x) = -c'' - \bar{\bar{\nu}}(x) + qx$ as in Theorem 3.1, and for sufficiently large $\beta > 1$.

Note that it is enough to know the (existence of the) Laplace transform for large values of $\beta \in [\Phi(q), \infty)$ to characterize $W^{(q)}$; see e.g. [30, Theorem 6.3]. Thus, to prove the representation for $W^{(q)}$, it remains to show that the summation and the Laplace transform in (3.2) can be interchanged. To do so, let us write for a moment $g(x) := \sigma^{-2} f(x)$. Then for every $m \in \mathbb{N}$

$$\begin{aligned}\left|\mathcal{L}\left(\sum_{n=0}^{\infty} g^{*n}\right) - \sum_{n=0}^{m}\mathcal{L}\left(g^{*n}\right)\right| &= \left|\mathcal{L}\left(\sum_{n=0}^{\infty} g^{*n}\right) - \mathcal{L}\left(\sum_{n=0}^{m} g^{*n}\right)\right| \\ &\leq \mathcal{L}\left(\sum_{n=m+1}^{\infty} |g|^{*n}\right) = \sum_{n=m+1}^{\infty} \mathcal{L}[|g|]^n.\end{aligned}$$

For sufficiently large arguments $\beta > 1$ we know that $\mathcal{L}[|g|](\beta) < 1$, and so the series on the right-hand side converges, which allows us to interchange the summation and the Laplace transform. Moreover, the above calculation shows that $\sum_{n=1}^{\infty} g^{*n}$ converges in the weighted space $L^1\left(\mathrm{e}^{-\beta y}\,\mathrm{d}y\right)$, and so it is well-defined in, say, $L^1_{\mathrm{loc}}(\mathbb{R}_+)$. Further, from Hölder's inequality it follows that

$$\left|\left(\mathrm{id} * \sum_{n=1}^{\infty} g^{*n}\right)(x)\right| \leq x \left\|\sum_{n=1}^{\infty} g^{*n}\right\|_{L^1[0,x]}, \quad x \geq 0.$$

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By Lemma 5.3, the series converges uniformly on compact subsets of $\mathbb{R}_+$. Finally, since $\mathcal{L}[\mathrm{id}](\beta) = \beta^{-2}$ and $\mathrm{id} * g^{*0} = \mathrm{id} * \delta_0 = \mathrm{id}$, we get

$$W^{(q)} = \mathrm{id} * \sum_{n=0}^{\infty} \frac{f^{*n}}{(\sigma^2)^{n+1}},$$

as stated. □

**Remark 3.1.** The condition $\int_{[1,\infty)} y\, \nu(\mathrm{d}y) < \infty$ ensures that the integrated tail $\bar{\bar{\nu}}$ is finite. We can obtain similar results without this condition, if we split the Lévy measure into $\nu^{(0)} = \nu|_{[0,z)}$ and $\nu^{(\infty)} = \nu|_{[z,\infty)}$ for some fixed $z \geq 1$, and replace $\bar{\bar{\nu}}$ by $\bar{\bar{\nu}}^{(0)}(x) = \int_{[x,z)} \bar{\nu}(y)\mathrm{d}y$. The Laplace exponent $\psi$ is then given by

$$\psi(\beta) = c_z''\beta + \sigma^2\beta^2 + \beta^2\mathcal{L}\big[\bar{\bar{\nu}}^{(0)}\big](\beta) + \mathcal{L}\big[\nu^{(\infty)}\big](\beta) - \big\|\nu^{(\infty)}\big\|,$$

where $c_z'' := c - \int_{[1,\infty)} x\nu^{(0)}(\mathrm{d}x)$ and $\big\|\nu^{(\infty)}\big\| = \nu^{(\infty)}([0,\infty))$ is the total mass. Thus, with the same argument as in the proof of Theorem 3.1,

$$W^{(q)} = \mathrm{id} * \sum_{n=0}^{\infty} \frac{f^{*n}}{(\sigma^2)^{n+1}}, \tag{3.3}$$

with $f$ such that

$$\mathcal{L}f(\beta) = -c_z''\beta^{-1} - \mathcal{L}\big[\bar{\bar{\nu}}^{(0)}\big](\beta) - \beta^{-2}\mathcal{L}\big[\nu^{(\infty)}\big](\beta) + \beta^{-2}\big(q + \big\|\nu^{(\infty)}\big\|\big),$$

i.e. with $f$ given by

$$f(x) = -c_z'' - \bar{\bar{\nu}}^{(0)}(x) - \int_{[z,x]} (x-y)\nu^{(\infty)}(\mathrm{d}y) + \big(q + \big\|\nu^{(\infty)}\big\|\big)x, \qquad x \geq 0.$$

Note that the integral term in $f$ vanishes on $[0, z)$; therefore, it can be omitted when evaluating $W^{(q)}$ on $[0, z)$. The above procedure thus yields a series expansion of the same form as (3.1) for bounded subsets of $\mathbb{R}_+$, if we pick $z \geq 0$ sufficiently large and adjust $c''$ and $\bar{\bar{\nu}}^{(0)}$ accordingly.

**Remark 3.2.** A different construction of the series expansions yields further useful formulae for the $q$-scale function. Consider the setting of Theorem 3.1. Instead of (3.2) we can use

$$\mathcal{L}\big[W^{(q)}\big](\beta) = \frac{1}{c''\beta + \sigma^2\beta^2 + \beta^2\mathcal{L}\bar{\bar{\nu}}(\beta) - q} = \sum_{n=0}^{\infty} \frac{(-1)^n\big(\beta^2\mathcal{L}\bar{\bar{\nu}}(\beta)\big)^n}{\big(c''\beta + \sigma^2\beta^2 - q\big)^{n+1}}.$$

Denote by

$$a_{1,2} = -\frac{c''}{2\sigma^2} \pm \sqrt{\left(\frac{c''}{2\sigma^2}\right)^2 + \frac{q}{\sigma^2}}$$

the roots of the polynomial in the denominator. This allows us to invert the Laplace transform, leading to

$$W^{(q)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(\sigma^2)^{n+1}} \mathcal{L}^{-1}\left(\frac{\beta^n}{(\beta + a_1)^{n+1}}\right) * \mathcal{L}^{-1}\left(\frac{\beta^n}{(\beta + a_2)^{n+1}}\right) * \bar{\bar{\nu}}^{*n},$$

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with the inverse Laplace transforms of the rational functions given by

$$\mathcal{L}^{-1}\left(\frac{\beta^n}{(\beta+a)^{n+1}}\right)=\frac{1}{n!}\partial^n\left(x^n \mathrm{e}^{-ax}\right) \quad \text{for all } n\in\mathbb{N},\ a\in\mathbb{R};$$

cf. [30]. If, moreover, the Lévy measure $\nu$ has finite total mass $\|\nu\|<\infty$, we get

$$\mathcal{L}\big[W^{(q)}\big](\beta)=\frac{1}{c'\beta+\sigma^2\beta^2+\mathcal{L}\nu(\beta)-(q+\|\nu\|)}=\sum_{n=0}^{\infty}\frac{(-1)^n(\mathcal{L}\nu(\beta))^n}{\left(c'\beta+\sigma^2\beta^2-(q+\|\nu\|)\right)^{n+1}}$$

with $c'=c+\int_{(0,1)}x\nu(\mathrm{d}x)$. Thus, we have

$$W^{(q)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(\sigma^2)^{n+1}}\left(\frac{x^n}{n!}\mathrm{e}^{-a_3x}\right)*\left(\frac{x^n}{n!}\mathrm{e}^{-a_4x}\right)*\nu^{*n} \tag{3.4}$$

with

$$a_{3,4}=-\frac{c'}{2\sigma^2}\pm\sqrt{\left(\frac{c'}{2\sigma^2}\right)^2+\frac{q+\|\nu\|}{\sigma^2}}.$$

The convolution

$$\left(\frac{1}{n!}x^n\mathrm{e}^{-a_3x}\right)*\left(\frac{1}{n!}x^n\mathrm{e}^{-a_4x}\right)$$

can be explicitly worked out. Rather than stating the lengthy expression, we refer to [21], where the case of finite Lévy measures is thoroughly studied for both $\sigma^2>0$ and $\sigma^2=0$.

We close this section with two explicit examples in the spirit of the previous remark. Suppose that the Lévy measure $\nu$ is bounded away from zero, i.e. there exists some $\varepsilon>0$ such that $\operatorname{supp}(\nu)\cap[0,\varepsilon)=\emptyset$; in particular, $\nu$ is a finite measure. In this case, the series expansion in (3.4) becomes a closed-form expression, since $\operatorname{supp}(\nu^{*n})\cap[0,n\varepsilon)=\emptyset$ for any $n\in\mathbb{N}$. As usual, we use $\lfloor x\rfloor:=\max\{y\in\mathbb{Z}:y\le x\}$ to denote the largest integer below $x\in\mathbb{R}$.

**Example 3.1.** (*Geometrically distributed jumps.*) Let $\nu=\sum_{k=1}^{\infty}(1-p)^{k-1}p\,\delta_k$ be a geometric distribution with success probability $p\in(0,1)$. It is well known that the $n$-fold convolution $\nu^{*n}$ is a negative binomial distribution, i.e.

$$\nu^{*n}=\sum_{k=n}^{\infty}\binom{k-1}{k-n}(1-p)^{k-n}p^k\,\delta_k,\qquad n\in\mathbb{N}.$$

Hence for $q\ge 0$ the $q$-scale function of the spectrally negative Lévy process with Laplace exponent

$$\psi(\beta)=c'\beta+\sigma^2\beta^2+\int_{(0,\infty)}\left(\mathrm{e}^{-\beta x}-1\right)\nu(\mathrm{d}x)$$

and $\sigma^2>0$ is given by

$$W^{(q)}(x)=\sum_{n=0}^{\lfloor x\rfloor}\sum_{k=n}^{\lfloor x\rfloor}(-1)^n\frac{\binom{k-1}{k-n}(1-p)^{k-n}p^k}{(\sigma^2)^{n+1}}\left(\frac{x^n}{n!}\mathrm{e}^{-a_3x}\right)*\left(\frac{x^n}{n!}\mathrm{e}^{-a_4x}\right)(x-k),\qquad x\ge 0,$$

with $a_3$, $a_4$ as in Remark 3.2.

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**Example 3.2.** (*Zero-truncated Poisson distributed jumps.*) Let $\nu$ denote the distribution of a Poisson random variable $\xi$ with parameter $\mu > 0$. In order to determine the $q$-scale functions using Remark 3.2, we need to know the probability mass functions $f_n$ of $\nu^{*n}$, $n \in \mathbb{N}$. For $k \neq 0$ they are given by

$$f_n(k) = \frac{\mathbb{P}\big(\sum_{i=1}^n \xi_i = k \,\&\, \forall i \leq n : \xi_i > 0\big)}{\mathbb{P}(\forall i \leq n : \xi_i > 0)} =: \frac{z(k, n)}{(1 - \mathrm{e}^{-\mu})^n}, \quad k \in \mathbb{N} \setminus \{0\}, \tag{3.5}$$

where $\xi_i$, $i \in \mathbb{N} \setminus \{0\}$ are independent and identically distributed copies of $\xi$. For $n = 0$ we have, by definition, $\nu^{*0} = \delta_0$. The numerator $z(k, n)$ in (3.5) can be computed recursively: obviously, we have $z(k, n) = 0$ if $k < n \in \mathbb{N} \setminus \{0\}$, and $z(k, 1) = \frac{\mu^s}{s!}\mathrm{e}^{-\mu}$ if $k \in \mathbb{N} \setminus \{0\}$. Otherwise, we have

$$\begin{aligned} z(k, n) &= \mathbb{P}\left(\sum_{i=1}^n \xi_i = k\right) - \sum_{\ell=1}^{n-k-1} \binom{n}{\ell} \mathbb{P}\left(\sum_{i=1}^{\ell} \xi_i = 0\right) z(k, n - \ell) \\ &= \frac{(n\mu)^k}{k!}\mathrm{e}^{-n\mu} - \sum_{\ell=1}^{n-k-1} \binom{n}{\ell} \mathrm{e}^{-\ell\mu} z(k, n - \ell). \end{aligned}$$

Setting $z(0, 0) = 1$ and $z(k, 0) = 0$, $k \in \mathbb{N} \setminus \{0\}$, the $q$-scale function of the spectrally negative Lévy process with Laplace exponent

$$\psi(\beta) = c'\beta + \sigma^2\beta^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-\beta x} - 1\right) \nu(\mathrm{d}x)$$

and $\sigma^2 > 0$ is given by

$$W^{(q)}(x) = \frac{1}{c} \sum_{n=0}^{\lfloor x \rfloor} \sum_{k=n}^{\lfloor x \rfloor} \frac{(-1)^n z(n, k)}{(\sigma^2)^{n+1} \left(1 - \mathrm{e}^{-\mu}\right)^k} \left(\frac{x^n}{n!}\mathrm{e}^{-a_3 x}\right) * \left(\frac{x^n}{n!}\mathrm{e}^{-a_4 x}\right)(x - k), \qquad x \geq 0,$$

with $a_3$, $a_4$ as in Remark 3.2.

### 3.2. Lévy processes with paths of bounded variation

If $(L_t)_{t\geq 0}$ has paths of bounded variation, the following series expansions for the $q$-scale functions hold true.

**Theorem 3.2.** *Let $(L_t)_{t\geq 0}$ be a spectrally negative Lévy process with triplet $\left(c, 0, \widetilde{\nu}\right)$ such that $\int_{(0,1)} y\, \nu(\mathrm{d}y) < \infty$. Recall $c' = c + \int_{(0,1)} y\, \nu(\mathrm{d}y)$. For $q \geq 0$ set*

$$f(x) := q + \bar{\nu}(x), \quad x \geq 0;$$

*then the q-scale function of $(L_t)_{t\geq 0}$ is given by*

$$W^{(q)} = \mathbb{1} * \sum_{n=0}^{\infty} \frac{f^{*n}}{(c')^{n+1}}, \tag{3.6}$$

*and the series on the right-hand side converges uniformly on compact subsets of $\mathbb{R}_+$.*

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*Proof.* By Lemma 2.1(i), the Laplace exponent of $L_t$ is given by $\psi(\beta) = c'\beta - \beta\mathcal{L}\bar{\nu}(\beta)$ with $c' = c + \int_{(0,1)} y\,\nu(\mathrm{d}y)$, and we obtain for $q \geq 0$

$$\mathcal{L}\big[W^{(q)}\big](\beta) = \frac{1}{c'\beta - \beta\mathcal{L}\bar{\nu}(\beta) - q} = \frac{1}{\beta\big(c' - \mathcal{L}\bar{\nu}(\beta) - q\beta^{-1}\big)}.$$

As in the proof of Theorem 3.1, we have $c' > \big|\mathcal{L}\bar{\nu}(\beta) - q\beta^{-1}\big|$ for sufficiently large $\beta > 1$, and thus

$$\mathcal{L}\big[W^{(q)}\big](\beta) = \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{\big(\mathcal{L}\bar{\nu}(\beta) + q\beta^{-1}\big)^n}{(c')^{n+1}} = \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(\mathcal{L}f(\beta))^n}{(c')^{n+1}}$$

with $f(x) = q + \bar{\nu}(x)$. The rest of the proof coincides with the proof of Theorem 3.1, with the slight difference that we now use

$$\left|\left(\mathbb{1} * \sum_{n=1}^{\infty}\frac{f^{*n}}{(c')^{n+1}}\right)(x)\right| \leq \left\|\sum_{n=1}^{\infty}\frac{f^{*n}}{(c')^{n+1}}\right\|_{L^1[0,x]}, \quad x \geq 0,$$

to obtain uniform convergence on compact subsets of $\mathbb{R}_+$. □

**Remark 3.3.** The formula (3.6) may be regarded as a generalization of the Pollaczek–Khinchin representation of ruin probabilities in the Cramér–Lundberg compound Poisson model. Let $(L_t)_{t\geq 0}$ be a spectrally negative compound Poisson process with positive drift; i.e. its Laplace exponent is of the form

$$\psi(\beta) = c\beta + \lambda\int_{(0,\infty)}\big(\mathrm{e}^{-\beta x} - 1\big)\Pi(\mathrm{d}x),$$

where $c, \lambda > 0$ and $\Pi((0, \infty)) = 1$. Under the net-profit condition

$$\psi'(0+) = c - \lambda\int_{(0,\infty)} x\Pi(\mathrm{d}x) > 0,$$

the ruin probability $r(x) := \mathbb{P}(\exists t \geq 0 : x + L_t < 0)$ is given by the Pollaczek–Khinchin formula

$$r(x) = 1 - \psi'(0+)\int_{(0,x)}\sum_{n=0}^{\infty}\frac{\lambda^n\bar{\Pi}^{*n}(y)}{c^{n+1}}\,\mathrm{d}y, \quad x \geq 0, \tag{3.7}$$

with $\bar{\Pi}(y) = \int_{[y,\infty)}\Pi(\mathrm{d}x)$; see e.g. [1, Eq. IV(2.2)] or [19, Eq. (1.15)]. Moreover, we have (cf. [19, Theorem 8.1])

$$r(x) = 1 - \psi'(0+)W^{(0)}(x),$$

which yields the following representation of the 0-scale function:

$$W^{(0)}(x) = \int_{(0,x)}\sum_{n=0}^{\infty}\frac{\lambda^n\bar{\Pi}^{*n}(y)}{c^{n+1}}\,\mathrm{d}y = \mathbb{1} * \sum_{n=0}^{\infty}\frac{\lambda^n\bar{\Pi}^{*n}(x)}{c^{n+1}}.$$

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Theorem 3.6 works, however, both for all $q \geq 0$ and for a larger class of jump measures, and it does not require any assumption on $\psi'(0+)$.

In the actuarial context, and in renewal theory, one often uses the following definition of the convolution:

$$f \circledast g(x) := \int_{[0,x]} f(x-y)\mathrm{d}g(y),$$
$$f^{\circledast n}(x) := \int_{[0,x]} f^{\circledast(n-1)}(x-y)\mathrm{d}f(y), \quad \text{and} \quad f^{\circledast 0} := \mathbb{1},$$

for any $n \in \mathbb{N}$ and whenever these integrals are defined. In particular, $f \circledast g = f * g'$ if $g$ is absolutely continuous with derivative $g'$. With this notation, (3.7) can be rewritten in the more familiar form

$$r(x) = 1 - \psi'(0+)\sum_{n=0}^{\infty} \frac{(\lambda\mu)^n \Pi_I^{\circledast n}(x)}{c^{n+1}} = 1 - \left(1 - \frac{\lambda\mu}{c}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda\mu}{c}\right)^n \Pi_I^{\circledast n}(x)$$

with $\Pi_I(x) := \mu^{-1}\int_{(0,x)} \bar{\Pi}(y)\mathrm{d}y = \mu^{-1}\mathbb{1} * \bar{\Pi}(x)$ and the first moment $\mu = \int_{(0,\infty)} x\Pi(\mathrm{d}x)$ of $\Pi$.

### 3.3. Lévy processes with paths of unbounded variation without a Gaussian component

It remains to consider spectrally negative Lévy processes with paths of unbounded variation and $\sigma^2 = 0$. The following theorem provides a series expansion of the $q$-scale involving an auxiliary function $h$ such that $h * \bar{\bar{\nu}}(0+) = 1$. The existence of $h$ as well as its properties and some examples will be discussed below. Since we are only interested in the behaviour of $h$ near $x = 0$ $h$ need not be unique. Indeed, any $h_a := h \cdot \mathbb{1}_{[0,a]}$, with $a > 0$, will be as good as $h$.

**Theorem 3.3.** *Let $(L_t)_{t\geq 0}$ be a spectrally negative Lévy process with triplet $(c, 0, \widetilde{\nu})$ such that $\int_{(0,1]} y\,\nu(\mathrm{d}y) = \infty$ and $\int_{[1,\infty)} y\,\nu(\mathrm{d}y) < \infty$. Recall that $c'' = c - \int_{[1,\infty)} y\,\nu(\mathrm{d}y)$. If there exists some $h \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that $\partial\left(h * \bar{\bar{\nu}}\right) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $h * \bar{\bar{\nu}}(0+) = 1$, then the $q$-scale function of $(L_t)_{t\geq 0}$ is given by*

$$W^{(q)} = H * \sum_{n=0}^{\infty} f^{*n}, \quad q \geq 0. \tag{3.8}$$

*Here, $H(x) := \int_{(0,x)} h(y)\mathrm{d}y = \mathbb{1} * h(x)$ is the primitive of $h$, and*

$$f(x) := qH(x) - c''h(x) - \partial\left(h * \bar{\bar{\nu}}\right)(x), \quad x \geq 0.$$

*The series on the right-hand side of (3.8) converges uniformly on compact subsets of $\mathbb{R}_+$.*

*Proof.* Let $h$ be the function from the statement of the theorem, and assume in addition that $h \in L^1(\mathbb{R}_+)$, which ensures that the Laplace transform exists. The identity (2.5) for the Laplace transforms shows that

$$\mathcal{L}\left[\partial\left(h * \bar{\bar{\nu}}\right)\right](\beta) = \beta\mathcal{L}h(\beta)\mathcal{L}\bar{\bar{\nu}}(\beta) - h * \bar{\bar{\nu}}(0+) = \beta\mathcal{L}h(\beta)\mathcal{L}\bar{\bar{\nu}}(\beta) - 1.$$

By assumption, $\nu_h := \partial\left(h * \bar{\bar{\nu}}\right)$ exists. Thus, the Laplace exponent $\psi$ can be written as

$$\psi(\beta) = c''\beta + \frac{\beta}{\mathcal{L}h(\beta)}\left(\mathcal{L}\nu_h(\beta) + 1\right),$$

and from the definition of the scale function, we get for all $q \geq 0$

$$\mathcal{L}\big[W^{(q)}\big](\beta) = \frac{\mathcal{L}h(\beta)}{\beta} \frac{1}{\big(c''\mathcal{L}h(\beta) + \mathcal{L}\nu_h(\beta) + 1 - q\beta^{-1}\mathcal{L}h(\beta)\big)}.$$

Since $\nu_h$, $h \in L^1_{\text{loc}}(\mathbb{R}_+)$ we obtain

$$\mathcal{L}\big[W^{(q)}\big](\beta) = \frac{\mathcal{L}h(\beta)}{\beta} \sum_{n=0}^{\infty} (\mathcal{L}f(\beta))^n ,$$

for large enough $\beta > 1$ and with $f(x) = q \int_{(0,x)} h(y)\mathrm{d}y - c''h(x) - \partial\big(h * \bar{\bar{\nu}}\big)(x)$ as in Theorem 3.3. Using the same arguments as in the proof of Theorem 3.1, we can invert the Laplace transform and obtain the formula (3.8).

If $h \in L^1_{\text{loc}}(\mathbb{R}+) \setminus L^1(\mathbb{R}_+)$, we set $h_a = h \cdot \mathbb{1}_{[0,a]}$ for some $a > 0$ and observe that $h_a$ meets the requirements stated in the theorem and that $\mathcal{L}(h_a)$ is well-defined. Thus, $W^{(q)}$ is given by the series expansion (3.8) with respect to $h_a$. Replacing $h$ by $h_a$ does not affect $f(x)$; hence $W^{(q)}(x)$ if $x \in [0, a]$. Since $a > 0$ is arbitrary, the proof is complete. □

**Remark 3.4.** Note that the same reasoning as explained in Remark 3.1 allows us to omit the moment condition $\int_{[1,\infty)} y\, \nu(\mathrm{d}y) < \infty$ in Theorem 3.3.

*Resolvents, Bernstein functions, and renewal equations*

In order to find a suitable function $h$ in the setting of Theorem 3.3, i.e. $h \in L^1_{\text{loc}}(\mathbb{R}_+)$ such that $\partial\big(h * \bar{\bar{\nu}}\big) \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $h * \bar{\bar{\nu}}(0+) = 1$, let us consider the auxiliary problem

$$\rho * \bar{\bar{\nu}} \equiv 1, \tag{3.9}$$

where $\rho = \rho(\mathrm{d}x)$ is a measure on $[0, \infty)$ and $\bar{\bar{\nu}}$ is the integrated tail function. Equations of this type are also used in the theory of Volterra integral equations, where the measure $\rho$ appearing in (3.9) is called the resolvent of the function $\bar{\bar{\nu}}$; cf. [16]. Taking Laplace transforms on both sides of (3.9) yields

$$\mathcal{L}[\rho](\beta) = \frac{1}{\beta\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta)}, \quad \beta > 0. \tag{3.10}$$

The calculations in the proof of Lemma 2.1 in our current setting show that $\beta\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta) = \int_{(0,\infty)} \big(1 - \mathrm{e}^{-x\beta}\big)\, \bar{\nu}(x)\mathrm{d}x$. Hence $\beta\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta)$ is a Bernstein function (BF) with Lévy triplet $(0, 0, \bar{\nu}(x)\mathrm{d}x)$; cf. [28, Theorem 3.2]. Thus, its reciprocal $1/\big(\beta\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta)\big)$ is a potential (cf. [28, Definition 5.24]), and in particular it is a completely monotone function. From Bernstein's theorem (cf. [28, Theorem 1.4]), we thus see that (3.10), and hence (3.9), *always* has a *measure-valued* solution $\rho$; see also [15]. These observations also allow for an important probabilistic interpretation of (3.9): $\rho * \bar{\bar{\nu}} = 1$ is the probability that a subordinator $(S_t)_{t\geq 0}$ with BF $\beta\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta)$ and Lévy triplet $(0, 0, \bar{\nu}(x)\mathrm{d}x)$ crosses any level $x > 0$; see Bertoin [5, Chapter III.2]. The measure $\rho$ is then the potential or zero-resolvent of $(S_t)_{t\geq 0}$.

For our purpose, however, only $L^1_{\text{loc}}(\mathbb{R}_+)$-solutions of (3.9) are of interest. The following lemma constructs such solutions in the setting of Theorem 3.3.

**Lemma 3.1.** *Let $\nu$ be as in Theorem 3.3. Then the convolution equation $\rho * \bar{\bar{\nu}}(x) = 1$, $x \geq 0$, has a unique strictly positive solution $\rho \in L^1_{\text{loc}}(\mathbb{R}_+)$.*

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From the probabilistic interpretation of $\rho * \bar{\bar{\nu}}$, the fact that $\rho$ is absolutely continuous is not surprising. One can argue as follows: since the Lévy measure $\bar{\nu}(x)\mathrm{d}x$ of the subordinator $(S_t)_{t\geq 0}$ is absolutely continuous, this property is inherited by the transition densities $\mathbb{P}(S_t \in dx)$ (cf. Sato [27, Theorem 27.7]) and all $q$-resolvent measures [27, p. 242], in particular by the zero-resolvent measure $\rho$. The following proof, based on the renewal equation, gives a short and direct approach to this, complementing also Neveu's renewal theorem; cf. [6, Proposition 1.7].

*Proof of Lemma* 3.1. Set $\phi(\beta) := \int_{(0,\infty)} \left(1 - e^{-x\beta}\right) \bar{\nu}(x)\mathrm{d}x$ and $\bar{N}(x) = x\bar{\nu}(x)$. Under the assumptions of Theorem 3.3 we know that $\bar{\bar{\nu}}(0+) = \infty$. We have already seen that there exists a measure $\rho$ on $[0, \infty)$ solving (3.9). Since $\bar{\bar{\nu}}(0+) = \infty$, the measure $\rho$ cannot have atoms: indeed, if $\rho \geq c\delta_y$ for some $c > 0$ and $y \in [0, \infty)$, this would lead to the contradiction

$$1 = \rho * \bar{\bar{\nu}}(x) \geq c\delta_y * \bar{\bar{\nu}}(x) = c\bar{\bar{\nu}}(x-y) \xrightarrow{x\downarrow y} c\bar{\bar{\nu}}(0+) = \infty.$$

Moreover, $\rho$ satisfies $\mathcal{L}[\rho](\beta) = \left(\beta\mathcal{L}\left[\bar{\bar{\nu}}\right](\beta)\right)^{-1} = 1/\phi(\beta)$. Therefore,

$$\partial\mathcal{L}[\rho] = \partial\frac{1}{\phi} = -\frac{1}{\phi^2}\partial\phi = -\mathcal{L}[\rho]\mathcal{L}[\rho]\mathcal{L}[y\bar{\nu}(y)] = -\mathcal{L}[\rho * \rho * \bar{N}].$$

On the other hand, $\partial\mathcal{L}[\rho] = -\mathcal{L}[y\rho]$, and the uniqueness of the Laplace transform shows that

$$x\rho(\mathrm{d}x) = \rho * \rho * \bar{N}(x)\mathrm{d}x. \tag{3.11}$$

This implies $\rho(\mathrm{d}x) = \rho\{0\}\delta_0(\mathrm{d}x) + r(x)\mathrm{d}x$ with the density function $r(x) = x^{-1}(\rho * \rho * \bar{N})(x)$ on $(0, \infty)$. Because of $\bar{\bar{\nu}}(0+) = \infty$ we know that $\rho\{0\} = 0$.

Further, recall that $\mathrm{supp}(\rho * \sigma) = \overline{\mathrm{supp}\ \rho + \mathrm{supp}\ \sigma}$ holds for any two measures $\rho, \sigma$. Thus we see from $\rho * \bar{\bar{\nu}} = 1$ that $0 \in \mathrm{supp}\ \bar{\bar{\nu}} \cap \mathrm{supp}\ \rho$; in $\overline{\text{particular, supp } \rho \text{ and supp}}\ \bar{\bar{\nu}}$ contain some neighbourhood of 0. Using again (3.11), $\mathrm{supp}\ \rho = \mathrm{supp}\ \rho + \mathrm{supp}\ \rho + \mathrm{supp}\ \bar{N}$, and $0 \in \mathrm{supp}\ \bar{N}$ (this follows from $0 \in \mathrm{supp}\ \bar{\bar{\nu}}$), we conclude that $\mathrm{supp}\rho$ is unbounded and even that $\mathrm{supp}\rho = [0, \infty)$, since it contains a neighbourhood of 0.

Finally, by the very definition of the convolution,

$$r(x) = \frac{1}{x}\int_{(0,x)}\int_{(0,x-y)} \bar{N}(x-y-z)\,\rho(\mathrm{d}z)\,\rho(\mathrm{d}y),$$

proving that $r(x) > 0$ since $\mathrm{supp}\ \rho = [0, \infty)$ and $0 \leq \bar{N} \not\equiv 0$. Thus $\rho$ is positive as well. □

The proof of further regularity properties of solutions to (3.9) seems to be quite difficult, with the exception of complete monotonicity (CM), which is treated in the following corollary. Although this result is known (cf. [22] or [29, Rem 2.2]), we provide a short alternative proof for the reader's convenience.

**Corollary 3.1.** *Let $\nu$ be as in Theorem 3.3 and assume that $\nu(\mathrm{d}z) = n(z)\mathrm{d}z$ with a completely monotone function $n \in \mathrm{CM}$. Then the convolution equation $\rho * \bar{\bar{\nu}}(x) = 1$, $x \geq 0$, has a unique solution $\rho \in \mathrm{CM}$. Conversely, if the solution of $\rho * \bar{\bar{\nu}} = 1$ is completely monotone, so is $\bar{\bar{\nu}}$, and hence $n$.*

*Proof.* From $\bar{\bar{\nu}}(x) = \int_{[x,\infty)}\int_{[y,\infty)} n(z)\mathrm{d}z\,\mathrm{d}y$ we see that $n \in \mathrm{CM}$ if, and only if, $\bar{\bar{\nu}} \in \mathrm{CM}$. Moreover, $\mathcal{L}\bar{\bar{\nu}}$ is a Stieltjes transform (i.e. a double Laplace transform) and $\beta\mathcal{L}\bar{\bar{\nu}}(\beta)$ a complete

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Bernstein function; see [28, Definition 2.1, Theorem 6.2]. Further, the reciprocals of complete Bernstein functions are Stieltjes functions [28, Theorem 7.3]; i.e. $\rho(\mathrm{d}y) = \rho\{0\}\delta_0 + r(y)\mathrm{d}y$ for some completely monotone function $r$. As before, the condition $\bar{\bar{\nu}}(0+) = \infty$ ensures that $\rho\{0\} = 0$. The converse statement follows from the symmetric roles played by $\rho$ and $\bar{\bar{\nu}}$ in the equation $\rho * \bar{\bar{\nu}} = 1$. □

**Remark 3.5.** With the exception of Corollary 3.1, higher-order regularity properties of the solution to $\rho * \bar{\bar{\nu}} \equiv 1$ seem to be quite difficult to prove; see also [14] for a related study. The following observation points in an interesting direction. Recall that a Bernstein function $f \in \mathrm{BF}$ is a special Bernstein function (see [28, Chapter 11]) if the conjugate function $\beta/f(\beta)$ is also a Bernstein function; iterating this argument, it becomes 'self-improving' and shows that $\beta/f(\beta)$ is even a special Bernstein function. A subordinator whose Laplace exponent is a special Bernstein function is a special subordinator. Note that complete Bernstein functions are special, and we have used this (implicitly) in Corollary 3.1.

Assume, for a moment, that we know $\beta\mathcal{L}[\rho](\beta) \in \mathrm{BF}$. Then $\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta) = 1/(\beta\mathcal{L}[\rho](\beta))$ is the potential of a special subordinator; hence $\beta\mathcal{L}[\rho](\beta)$ is a special Bernstein function. Thus $\beta/\beta\mathcal{L}[\rho](\beta) = 1/\mathcal{L}[\rho](\beta) = \beta\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta)$ is also a special Bernstein function. This, in turn, means that $\mathcal{L}[\rho](\beta) = 1/\beta\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta)$ is the potential of a special subordinator, and according to [28, Theorem 11.3], $\rho$ is of the form $\rho(\mathrm{d}x) = c\delta_0(\mathrm{d}x) + r(x)\mathrm{d}x$ for some constant $c \ge 0$ and a decreasing function $r(x)$. If we assume, as before, that $\nu(0, \infty) = \infty$, $\rho$ has no atoms, so $c = 0$ and $\rho(\mathrm{d}x) = r(x)\,\mathrm{d}x$.

From $\int_{(0,\infty)} \big(1 - e^{-\beta x}\big)\mu(\mathrm{d}x) = \beta\mathcal{L}[\bar{\mu}](\beta)$ we see that $\beta\mathcal{L}[\rho](\beta)$ is a Bernstein function only if $\rho$ has a decreasing density. This means the above argument becomes a vicious circle, unless we can come up with an independent criterion: let us consider the case that $x \mapsto \bar{\nu}(x)$ is logarithmically convex, i.e.

$$\bar{\nu}(\lambda x + (1-\lambda)y) \le \bar{\nu}(x)^{\lambda}\bar{\nu}(y)^{1-\lambda} \quad \text{for all } x, y > 0 \text{ and } \lambda \in (0, 1).$$

Since $\bar{\bar{\nu}}(x)$ has a negative derivative, $\beta\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta) = \int_{(0,\infty)} \big(1 - e^{-\beta x}\big)\,\bar{\nu}(x)\mathrm{d}x$ is a Bernstein function and, because of log-convexity, even a special Bernstein function (see [28, Theorem 11.11]); hence the conjugate function $1/\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta) = \beta\mathcal{L}[\rho](\beta)$ is a Bernstein function. Together with the above consideration, this proves the following:

*If $\nu(0, \infty) = \infty$ and $x \mapsto \bar{\nu}(x)$ is logarithmically convex, the solution to $\rho * \bar{\bar{\nu}} \equiv 1$ is of the form $\rho(\mathrm{d}x) = r(x)\mathrm{d}x$ with a decreasing density $r(x) > 0$ on $(0, \infty)$.*

See also [20, Lemma 2.2] for a similar result stated in terms of potential measures of subordinators.

The following lemma provides a particular representation of the resolvent of $\bar{\bar{\nu}}$ as a continuous function; in fact, it is the Radon–Nikodym derivative of the scale function of some Lévy process.

**Lemma 3.2.** *Let $(L_t)_{t\ge0}$ be a spectrally negative Lévy process with triplet $\big(c, 0, \widetilde{\nu}\big)$ such that $\int_{(0,1)} y\,\nu(\mathrm{d}y) = \infty$ and $\int_{[1,\infty)} y\,\nu(\mathrm{d}y) < \infty$. Denote by $\widetilde{L}_t := L_t - \mathbb{E}[L_t]$ the compensated process, and write $\widetilde{W}$ for the 0-scale function of $(\widetilde{L}_t)_{t\ge0}$. Then $\widetilde{W}' * \bar{\bar{\nu}} = \mathbb{1}$, i.e. $\rho := \widetilde{W}'$ is a resolvent of $\bar{\bar{\nu}}$. In particular, $\rho \in \mathcal{C}(0, \infty)$.*

*Proof.* From Lemma 2.1(ii) we know that the Laplace exponent of $(\widetilde{L}_t)_{t\ge0}$ is given by

$$\widetilde{\psi}(\beta) = \psi(\beta) - \mathbb{E}[L_1]\beta = c''\beta + \beta^2\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta) - c''\beta = \beta^2\mathcal{L}\big[\bar{\bar{\nu}}\big](\beta).$$

By the definition of the scale function, we see $\mathcal{L}\widetilde{W} = \left(\beta^2 \mathcal{L}\bar{\bar{\nu}}\right)^{-1}$, and with (2.5) it follows that

$$\mathcal{L}[\widetilde{W}'](\beta) = \frac{1}{\beta \mathcal{L}\left[\bar{\bar{\nu}}\right](\beta)} \iff \mathcal{L}[\widetilde{W}'](\beta) \cdot \mathcal{L}\left[\bar{\bar{\nu}}\right](\beta) = \frac{1}{\beta},$$

since $\widetilde{W}(0+) = 0$. Finally, by [18, Lemma 2.4] we have $\widetilde{W} \in \mathcal{C}^1(0, \infty)$, and thus $\rho \in \mathcal{C}(0, \infty)$. □

The above considerations prove that in the setting of Theorem 3.3 there always exists a resolvent of the integrated tail function $\bar{\bar{\nu}}$. This immediately leads to the following simple corollary of Theorem 3.3.

**Corollary 3.2.** *Let $(L_t)_{t\geq 0}$ be as in Theorem 3.3 and let $\rho \in L^1_{\text{loc}}(\mathbb{R}_+)$ be a resolvent of $\bar{\bar{\nu}}$. Then the q-scale function of $(L_t)_{t\geq 0}$ is given by*

$$W^{(q)} = R * \sum_{n=0}^{\infty} (qR - c''\rho)^{*n}, \qquad q \geq 0, \tag{3.12}$$

*where $R(x) := \int_0^x \rho(y)\mathrm{d}y = \mathbb{1} * \rho(x)$ for all $x \in \mathbb{R}_+$. The series on the right-hand side of (3.12) converges uniformly on compact subsets of $\mathbb{R}_+$.*

The deep relationship between scale functions and renewal equations has been explained in detail in [8]. From the definition of scale functions (1.1) one immediately obtains that for a spectrally negative Lévy process $(L_t)_{t\geq 0}$ with triplet $\left(c, 2\sigma^2, \widetilde{\nu}\right)$ and $\int_{[1,\infty)} y\nu(\mathrm{d}y) < \infty$,

$$c''W^{(0)} + \sigma^2 \partial W^{(0)} + \bar{\bar{\nu}} * \partial W^{(0)} = 1. \tag{3.13}$$

In the case $\sigma^2 > 0$ and with the notation of Remark 3.3, this equation is a renewal equation on $\mathbb{R}_+$ of the type

$$f = 1 + f \circledast g \tag{3.14}$$

with $f(x) = \sigma^2 \partial W^{(0)}(x)$ and $g(x) = -2\sigma^{-2} \int_0^x \bar{\bar{\nu}}(y)\mathrm{d}y - 2c''\sigma^{-2}x$. This has been discussed in [8], where (3.14) is solved for $\partial^{(0)}$ to infer smoothness properties of $W^{(0)}$. However, if $\sigma^2 = 0$ and $\int_{(0,1)} y\nu(\mathrm{d}y) = \infty$, then (3.13) yields

$$c''W^{(0)} = 1 - \frac{\bar{\bar{\nu}}}{c''} * \partial\left(c''W^{(0)}\right) = 1 - \frac{\bar{\bar{\nu}}}{c''} \circledast \left(c''W^{(0)}\right),$$

which is not a renewal equation in the sense of [8], since $\bar{\bar{\nu}}(0+) \neq 0$. Nevertheless, our approach to obtaining series expansions for $W^{(0)}$ in such cases still provides a technique for solving this kind of renewal equation on $\mathbb{R}_+$, namely

$$f = 1 + g * \partial f = 1 + g \circledast f, \tag{3.15}$$

where we do not necessarily assume that $g(0+) = 0$. This is illustrated in the next theorem, whose proof follows the lines of the proof of Theorem 3.3.

**Theorem 3.4.** *Let $g \in L^1_{\text{loc}}(\mathbb{R}_+)$.*

(i) *If $g$ admits a resolvent $\rho$, then (3.15) is solved by $f = \mathbb{1} * \sum_{n=1}^{\infty} (-1)^n \rho^{*n}$.*

(ii) *If there exists a (not necessarily unique) function $h \in L^1_{loc}(\mathbb{R}_+)$ with $g * h(0+) = 1$ and such that $g_h := \partial(g * h)$ is well-defined, then (3.15) is solved by*

$$f = \mathbb{1} * h * \sum_{n=1}^{\infty} (-1)^n (h + g_h)^{*(n-1)}.$$

*Proof.* It is enough to prove (ii). First note that $f(0+) = 0$, since $f(x) = \int_{[0,x]} u(y)\mathrm{d}y$ with locally integrable integrand

$$u(y) := h * \sum_{n=1}^{\infty} (-1)^n (h + g_h)^{*(n-1)}.$$

Convolving both sides of (3.15) with $h$ and applying Lemma 5.1(i) yields

$$h * f = \mathbb{1} * h + h * g * \partial f = \mathbb{1} * h + \partial(h * g * f) = \mathbb{1} * h + \partial(h * g) * f + f.$$

Applying the Laplace transform and solving for $\mathcal{L}[f](\beta)$ we get

$$\begin{aligned}\mathcal{L}[f](\beta) = \frac{\mathcal{L}[\mathbb{1} * h](\beta)}{\mathcal{L}[h](\beta) - \mathcal{L}[g_h](\beta) - 1} &= -\mathcal{L}[\mathbb{1} * h](\beta) \sum_{n=0}^{\infty} (\mathcal{L}[h - g_h](\beta))^n \\ &= -\mathcal{L}[\mathbb{1} * h](\beta) \sum_{n=0}^{\infty} \mathcal{L}\big[(h - g_h)^{*n}\big](\beta),\end{aligned}$$

from which the assertion follows by the uniqueness of the Laplace transform. □

*Examples*

Although the existence of a resolvent $\rho$ in Corollary 3.2 is clear, there seems to be no general method of constructing $\rho$ explicitly. Fortunately, there are quite a few cases where a suitable function $h$ (as in Theorem 3.3) can be constructed. In particular, this is true if $\bar{\bar{\nu}}$ is regularly varying at 0. Recall that a measurable function $f : (0, \infty) \to \mathbb{R}$ is regularly varying at 0 with index $\gamma \in \mathbb{R}$ if

$$\lim_{x \to 0+} \frac{f(\lambda x)}{f(x)} = \lambda^{-\gamma} \quad \text{exists for all } \lambda > 0.$$

We write $\mathcal{R}_\gamma$ for the space of regularly varying functions with index $\gamma$. Our standard reference is the monograph [7].

Let $(L_t)_{t \geq 0}$ be a spectrally negative Lévy process as in Theorem 3.3, and assume that $\bar{\bar{\nu}}$ is in $\mathcal{R}_\gamma$ for some $\gamma \in (0, 1)$. From [7, Theorem 1.8.3] and its proof we know that there exists a completely monotone function $k$ such that $\lim_{x \to 0} \bar{\bar{\nu}}(x)/k(x) = 1$. With the arguments of the previous subsection, this implies that there exists a unique completely monotone function $h$ for which $h * k = 1$. Set $k_\varepsilon := \bar{\bar{\nu}} - k$. Since $\lim_{x \to 0+} k_\varepsilon(x)/k(x) = 0$, we obtain

$$\begin{aligned}|h * k_\varepsilon(x)| &= \left| \int_{(0,x)} h(x - y) k_\varepsilon(y)\, \mathrm{d}y \right| \\ &\leq \int_{(0,x)} h(x - y) \left| \frac{k_\varepsilon(y)}{k(y)} \right| k(y) \mathrm{d}y \leq \sup_{y \leq x} \left| \frac{k_\varepsilon(y)}{k(y)} \right| \xrightarrow[x \to 0+]{} 0;\end{aligned} \tag{3.16}$$

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hence,

$$h * \bar{\bar{\nu}}(0+) = 1. \tag{3.17}$$

We may thus use this function $h$ in Theorem 3.3 to obtain the representation of the $q$-scale function given in the following corollary.

**Corollary 3.3.** *Let $(L_t)_{t\geq 0}$ be a spectrally negative Lévy process as in Theorem 3.3, and assume that the Lévy measure $\nu$ is such that $\bar{\bar{\nu}} \in \mathcal{R}_\gamma \cap \mathcal{C}^1(0, \infty)$ for some $\gamma \in (0, 1)$. Let $k_\varepsilon$ and $h$ be the functions in (3.16). Then*

$$W^{(q)} = H * \sum_{n=0}^{\infty} \big(qH - c''h - \partial(h * k_\varepsilon)\big)^{*n}$$

*where $H(x) := \int_{(0,x)} h(y)\mathrm{d}y = \mathbb{1} * h(x)$ for all $x \in \mathbb{R}_+$.*

*Proof.* We check the conditions stated in Theorem 3.3. First of all, $h \in L^1_{\text{loc}}(\mathbb{R}_+)$ as it is the resolvent of $k \in L^1_{\text{loc}}(\mathbb{R}_+)$. Furthermore, we note that

$$h * \bar{\bar{\nu}} = h * (k + k_\varepsilon) = 1 + h * k_\varepsilon,$$

which implies $\partial\big(h * \bar{\bar{\nu}}\big) = \partial(h * k_\varepsilon)$. By assumption, $k_\varepsilon \in \mathcal{C}^1(0, \infty)$. Thus, $h * k_\varepsilon \in \mathcal{C}^1(0, \infty)$ by Lemma 5.1(ii), and as $h * k_\varepsilon(0+) = 0$ we obtain $\partial(h * k_\varepsilon) \in L^1_{\text{loc}}(\mathbb{R}_+)$. Finally, in view of (3.16), $h * \bar{\bar{\nu}}(0+) = 1 + h * k_\varepsilon(0+) = 1$, finishing the proof. □

In the following, we restrict ourselves to jump measures $\nu$ satisfying $\bar{\bar{\nu}} \in \mathcal{R}_\gamma$, $\gamma \in (0, 1)$, and for which, in addition, there exists a constant $C > 0$ such that

$$\lim_{x \to 0+} \frac{Cx^{-\gamma}}{\bar{\bar{\nu}}(x)} = 1. \tag{3.18}$$

Clearly, these are special cases of the jump measures treated in Corollary 3.3, but we can now use techniques from fractional calculus which will simplify our calculations. Recall that the Riemann–Liouville fractional integral $I^\alpha_{0+}f$ of order $\alpha \in (0, 1)$ of a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\big(I^\alpha_{0+}f\big)(x) := \frac{1}{\Gamma(\alpha)} \int_{(0,x)} \frac{f(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y = \frac{1}{\Gamma(\alpha)}((\,\cdot\,)^{\alpha-1} * f)(x), \qquad x \geq 0,$$

and the Riemann–Liouville fractional derivative $D^\alpha_{0+}$ of order $\alpha \in (0, 1)$ of $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\big(D^\alpha_{0+}f\big)(x) := \partial\big(I^{1-\alpha}_{0+}f\big)(x).$$

A thorough introduction to fractional calculus is found in the monograph [26].

**Corollary 3.4.** *Let $(L_t)_{t\geq 0}$ be a spectrally negative Lévy process as in Theorem 3.3, assume that the Lévy measure $\nu$ satisfies $\bar{\bar{\nu}} \in \mathcal{C}^1(0, \infty)$ and (3.18) for some $C > 0$, and let $\gamma \in (0, 1)$. Set $h(x) := x^{\gamma-1}$ and $H(x) := \mathbb{1} * h(x) = \gamma^{-1}x^\gamma$ for $x > 0$. The $q$-scale function of $(L_t)_{t\geq 0}$ is given by*

$$W^{(q)} = H * \sum_{n=0}^{\infty} \left(\frac{\sin(\gamma\pi)}{C\pi}\right)^{n+1} f^{*n}, \qquad q \geq 0, \tag{3.19}$$

*with $f(x) := qH(x) - c''h(x) - \Gamma(\gamma) D^{1-\gamma}_{0+}\bar{\bar{\nu}}(x)$ for $x > 0$.*

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*Proof.* Set $k(x) = Cx^{-\gamma}$ such that $\lim_{x\to 0} \bar{\bar{\nu}}(x)/k(x) = 1$, and define $k_\varepsilon := \bar{\bar{\nu}} - k$. Clearly, $k_\varepsilon = o(x^{-\gamma})$. Set $h(x) = x^{\gamma-1}$. The same computation as in (3.16) yields

$$|h * k_\varepsilon(x)| \to 0 \quad \text{as } x \to 0.$$

Moreover,

$$\begin{aligned} h * k(x) &= \int_{(0,x)} h(x-y)k(y)\,\mathrm{d}y = \int_{(0,1)} h(x(1-z))k(xz)x\,\mathrm{d}z \\ &= C \cdot \int_{(0,1)} (1-z)^{\gamma-1} z^{-\gamma}\,\mathrm{d}z = C \cdot B(1-\gamma, \gamma), \end{aligned}$$

where $B(x, y)$, $x, y > 0$, denotes the beta function. Since $B(1-\gamma, \gamma) = \pi/\sin(\gamma\pi)$, we have

$$h * \bar{\bar{\nu}}(0+) = h * k_\varepsilon(0+) + h * k(0+) = C\frac{\pi}{\sin(\gamma\pi)}.$$

We may thus apply Theorem 3.3 with $\widetilde{h}(x) := \frac{\sin(\gamma\pi)}{C\pi} h(x)$. Clearly, $\widetilde{h} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, and the calculations above show $\widetilde{h} * \bar{\bar{\nu}}(0+) = 1$. Furthermore, $\partial\big(\widetilde{h} * \bar{\bar{\nu}}\big) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, since

$$\partial\big(h * \bar{\bar{\nu}}\big)(x) = \partial\Big(\Gamma(\gamma) I^\gamma_{0+} \bar{\bar{\nu}}(x)\Big) = \Gamma(\gamma) D^{1-\gamma}_{0+} \bar{\bar{\nu}}(x),$$

with $D^{1-\gamma}_{0+}\bar{\bar{\nu}} \in \mathcal{C}^\gamma$ as $\bar{\bar{\nu}} \in \mathcal{C}^1$ by assumption; cf. [26, Lemma 2.2]. This yields the representation of $W^{(q)}$. □

**Example 3.3.** (*Spectrally negative stable processes.*) Consider a spectrally negative stable Lévy process with stability index $\alpha \in (1, 2)$ and Laplace exponent $\psi(\beta) = \beta^\alpha$. Then as shown in [3] (see also [17]), for all $q \geq 0$,

$$W^{(q)} = \alpha x^{\alpha-1} E'_\alpha\big(qx^\alpha\big), \qquad x \geq 0, \tag{3.20}$$

where $E_\alpha(x) = \sum_{k=0}^\infty \Gamma(1+\alpha k)^{-1} x^k$, $x \geq 0$, is the Mittag-Leffler function.

A lengthy but otherwise straightforward calculation shows that this formula agrees with (3.19): use that $\bar{\bar{\nu}}$ fulfils the limit relation (3.18) with $C = 1/\Gamma(2-\alpha)$ and $\gamma = \alpha - 1$, and moreover $\int_{0+}^x (x-y)^{\alpha-1} y^{-\alpha}\,\mathrm{d}y = \pi/\sin(\alpha\pi)$ for any $x > 0$ and $\alpha \in (0, 1)$.

The following observation allows us to construct further explicit examples of scale functions. Consider a spectrally negative Lévy process $(L_t)_{t\geq 0}$ with Laplace exponent $\psi_L$ and suppose that we know its scale functions $W^{(q)}_L$. Further, let $(C_t)_{t\geq 0}$ be an independent compound Poisson process with positive jumps and without drift; we assume that the jumps have intensity $\lambda > 0$ and are distributed according to the probability measure $\Pi$. Then $(L_t - C_t)_{t\geq 0}$ is a spectrally negative Lévy process whose Laplace exponent is given by

$$\psi_{L-C}(\beta) = \psi_L(\beta) + \lambda\mathcal{L}[\Pi](\beta) - \lambda.$$

As for any $\lambda$ we have $|\lambda\mathcal{L}[\Pi](\beta)\mathcal{L}\big[W^{(q+\lambda)}_L(\beta)| < 1$ for $\beta$ large enough, the $q$-scale function of $(L_t - C_t)_{t\geq 0}$ is thus defined by

$$\mathcal{L}\Big[W^{(q)}_{L-C}\Big] = \frac{1}{\psi_{L-C} - q} = \frac{\mathcal{L}\Big[W^{(q+\lambda)}_L\Big]}{1 + \lambda\mathcal{L}[\Pi]\mathcal{L}\Big[W^{(q+\lambda)}_L\Big]} = \mathcal{L}\Big[W^{(q+\lambda)}_L\Big] \sum_{k=0}^\infty \Big(-\lambda\mathcal{L}[\Pi]\mathcal{L}\Big[W^{(q+\lambda)}_L\Big]\Big)^k.$$

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This immediately implies

$$W_{L-C}^{(q)} = W_L^{(q+\lambda)} * \sum_{k=0}^{\infty} \lambda^k \left(-\Pi * W_L^{(q+\lambda)}\right)^{*k}. \tag{3.21}$$

**Example 3.4.** (*Stable process plus Poisson process.*) Let $(L_t)_{t\geq 0}$ be a spectrally negative $\alpha$-stable process with $\alpha \in (1, 2)$ and let $\Pi = \delta_1$. For each $n \in \mathbb{N}$ and $q \geq 0$ we then have from (3.21) that

$$W_{L-C}^{(q)}(x) = W_L^{(q+\lambda)} * \sum_{k=0}^{\infty} (-\lambda)^k \left(W_L^{(q+\lambda)}(\cdot - 1)\right)^{*k},$$

where $W_L^{(q+\lambda)}$ is given by (3.20). In fact, the sum on the right-hand side is finite, as $W_L^{(q+\lambda)}(x-1) = 0$ for $x \in [0, 1]$, and thus,

$$\left(W_L^{(q+\lambda)}(\cdot - 1)\right)^{*k}(x) = 0 \quad \text{for all } k > x.$$

## 4. Smoothness properties

The question of smoothness of $q$-scale functions is of considerable practical and theoretical interest. On the one hand, several fluctuation identities require the evaluation of derivatives of the $q$-scale functions; see e.g. [18, Section 1]. On the other hand (cf. [8, 20] and [18, Section 3.5]), one can interpret $W^{(q)}$ as an eigenfunction of (a suitable extension of) the infinitesimal generator $\mathcal{A}$ of $(L_t)_{t\geq 0}$, i.e.

$$(\mathcal{A} - q)W^{(q)} = 0.$$

However, this equation has to be treated with caution, since the domain of $\mathcal{A}$ has to be made explicit.

It has been conjectured by Doney (see [18, Conjecture 3.13]) that the smoothness of the scale functions depends on the smoothness of the jump measure. In the three different situations considered in Section 3, Doney's conjecture reads as follows:

(i) If $\sigma^2 > 0$, then $W^{(q)} \in \mathcal{C}^{n+3}(0, \infty) \iff \bar{\nu} \in \mathcal{C}^n(0, \infty)$ for all $n \in \mathbb{N}$, $q \geq 0$.

(ii) If $\sigma^2 = 0$ and $\int_{(0,1)} y\,\nu(\mathrm{d}y) < \infty$, then $W^{(q)} \in \mathcal{C}^{n+1}(0, \infty) \iff \bar{\nu} \in \mathcal{C}^n(0, \infty)$ for all $n \in \mathbb{N}$, $q \geq 0$.

(iii) If $\sigma^2 = 0$ and $\int_{(0,1)} y\,\nu(\mathrm{d}y) = \infty$, then $W^{(q)} \in \mathcal{C}^{n+2}(0, \infty) \iff \bar{\nu} \in \mathcal{C}^n(0, \infty)$ for all $n \in \mathbb{N}$, $q \geq 0$.

Under additional assumptions, there are some partial answers. In [8, Theorem 2], the equivalence in (i) is shown to be true if one assumes, in addition, that the Blumenthal–Getoor index of the pure-jump part satisfies

$$\inf\left\{\alpha \geq 0 : \int_{(0,1)} x^\alpha\,\nu(\mathrm{d}x) < \infty\right\} < 2.$$

The equivalence in (ii) is shown in [8, Theorem 3] under the additional assumption that $\nu(\mathrm{d}x) = \pi(x)\mathrm{d}x$ such that $\pi(x) \leq C|x|^{-1-\alpha}$ for $\alpha < 1$, $C > 0$, in a neighbourhood of the origin. To the

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best of our knowledge, the only case where (iii) is known to be true is that of a process whose Lévy measure has a completely monotone density; cf. [8]. In this section we use the series expansion of Theorem 3.3 to prove more general smoothness properties for the scale functions of processes in the case (iii).

As one would expect, we have to impose constraints on the behaviour of the function $f$ appearing in Theorem 3.3 near $x = 0$, which resemble those used in the cases (i) and (ii). More precisely, in order to derive smoothness properties of a convolution power series $\sum_{n=1}^{\infty} f^{*n}$, one first has to ensure the convergence of the series. This requires, however, that $f$ behaves nicely near the origin. In Theorem 3.3, the function $f$ is rather implicit as compared to those in Theorems 3.1 and 3.2 that refer to cases (i) and (ii), respectively. This is the origin of the comparatively technical conditions in Theorem 4.1 below. The nevertheless considerable theoretical and practical use of our result will be demonstrated in Section 4.2.

A key ingredient in the proof is the asymptotic behaviour of a function near the origin. To simplify notation, let us introduce for a function $f \in L^1_{\text{loc}}(\mathbb{R}_+)$ the parameter $\kappa(f) \in \{1, 2\}$ defined by

$$\kappa(f) := \begin{cases} 2 & \text{if there are constants } C, \alpha > 0 \text{ such that } |f(x)| < Cx^{\alpha - 0.5} \text{ for } x \in (0, 1), \\ 1 & \text{otherwise.} \end{cases}$$

**Theorem 4.1.** *Let $(L_t)_{t \geq 0}$ be a spectrally negative Lévy process with triplet $(c, 0, \widetilde{\nu})$ such that $\int_{(0,1)} y\, \nu(\mathrm{d}y) = \infty$ and $\int_{[1,\infty)} y\, \nu(\mathrm{d}y) < \infty$. Let $\bar{\nu} \in \mathcal{C}^{k_0}(0, \infty)$ for some $k_0 \in \mathbb{N}$. Further, assume that there exist a decomposition $\bar{\bar{\nu}} = N_1 + N_2$ and some constants $C_1, C_2 > 0$, $\alpha \in (0, 1)$ such that*

(i) *$N_1 \in L^1_{\text{loc}}(\mathbb{R}_+)$ admits a resolvent $\rho_1 \in \mathcal{C}^1(0, \infty)$ such that $|\rho_1(x)| < C_1 x^{\alpha-1}$ on (0, 1), and*

(ii) *$N_2 \in \mathcal{C}^{k_0+1}(0, \infty)$ with $N_2(0+) \in \mathbb{R}$ and $|\partial N_2(x)| < C_2 x^{-\alpha}$ for $x \in (0, 1)$.*

*Then $W^{(q)} \in \mathcal{C}^{k_0 + \kappa(\rho_1)}(0, \infty)$ for all $q \geq 0$.*

**Remark 4.1.** In the following we will, without loss of generality, assume $N_2(0+) = 0$. Otherwise, if $N_2(0+) =: c_0 \neq 0$, we set $d = c'' + c_0$ and $\bar{\bar{\mu}} := \bar{\bar{\nu}} - c_0$ such that

$$\psi(\beta) = c''\beta + \beta^2 \mathcal{L}\big[\bar{\bar{\nu}}\big] = (c'' + c_0)\beta + \beta^2 \mathcal{L}[\bar{\bar{\nu}} - c_0] = d\beta + \beta^2 \mathcal{L}\big[\bar{\bar{\mu}}\big] \tag{4.1}$$

yields a representation of the Laplace exponent for which the conditions (i) and (ii) of Theorem 4.1 are met. Note that, in general, $\bar{\bar{\mu}}$ is not an integrated tail function of any spectrally positive Lévy measure. However, this has no consequences for the proof.

### 4.1. Proof of Theorem 4.1

We will use the series expansion of Theorem 3.3 in order to prove Theorem 4.1. At first sight, this seems rather circumstantial in comparison with the representation in Corollary 3.2. The trouble is that for the convolution equation $\rho * f = 1$ rather little is known about how the smoothness of $\rho$ and $f$ are related to each other; see Section 3.3 above.

In order to apply Theorem 3.3, we need to construct a suitable function $h$. We are going to show that, in general, a suitable approximation of the resolvent of $N_1$ is good enough for our purposes. By assumption, the resolvent $\rho_1$ of $N_1$ is continuously differentiable. The following

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lemma will help us to approximate $\rho_1'$ and then construct a suitable function $h$ in the subsequent Lemma 4.2.

**Lemma 4.1.** *Let $f \in \mathcal{C}(0, \infty)$. For any strictly positive function $g \in \mathcal{C}(0, \infty)$ there exists some $u \in \mathcal{C}^\infty(0, \infty)$ such that $|f(x) - u(x)| < g(x)$ for all $x \in (0, \infty)$.*

*Proof.* Let $\bigcup_\lambda U_\lambda = (0, \infty)$ be any cover of $(0, \infty)$ with open intervals. There is a locally finite, smooth subordinate partition of unity, i.e. a sequence $(\chi_n)_{n\geq 1} \subset \mathcal{C}_c^\infty(0, \infty)$ such that $\operatorname{supp}\chi_n \subset U_{\lambda(n)}$ for some $\lambda(n)$, $\chi_n \geq 0$, and $\sum_{n=1}^\infty \chi_n(x) = \mathbb{1}_{(0,\infty)}(x)$, such that for every compact set $K$ and $x \in K$ the sum only consists of finitely many summands; see e.g. [25, Theorem 6.20].

Choose $U_n := \left(2^{n-1}, 2^{n+1}\right)$, $n \in \mathbb{Z}$. Since $g$ is continuous and strictly positive, $\varepsilon_n := \inf_{\overline{U}_n} g > 0$. By Weierstrass's theorem, we can approximate any continuous function $f$ on $\overline{U}_n$ uniformly by a polynomial $p_n$ such that $\sup_{\overline{U}_n} |f - p_n| \leq \varepsilon_n$. Therefore, we get

$$\left| f - \sum_{n\in\mathbb{Z}} p_n \chi_n \right| = \left| \sum_{n\in\mathbb{Z}} (f - p_n)\chi_n \right| \leq \sum_{n\in\mathbb{Z}} |f - p_n|\chi_n \leq \sum_{n\in\mathbb{Z}} \varepsilon_n \chi_n \leq \sum_{n\in\mathbb{Z}} g\chi_n = g.$$

Since the sum $\sum_{n\in\mathbb{Z}} \chi_n = \mathbb{1}_{(0,\infty)}$ is locally finite, we see that $\sum_{n\in\mathbb{Z}} p_n\chi_n \in \mathcal{C}_c^\infty(0, \infty)$. □

Lemma 4.1 now leads to a smooth approximation $h$ of $\rho_1$ as follows.

**Lemma 4.2.** *Let $\bar{\bar{\nu}} = N_1 + N_2$ be an integrated tail function as in Theorem 4.1 and Remark 4.1, and let $\rho_1$ be the resolvent of $N_1$. In particular, we require that the assumptions (i), (ii) of Theorem 4.1 hold. Then there exists a function $h \in \mathcal{C}^\infty(0, \infty)$ such that*

(i) $\lim_{x\downarrow 0} h(x)/\rho_1(x) = 1$,

(ii) $h * \bar{\bar{\nu}}(0+) = 1$,

(iii) $\partial\left(h * \bar{\bar{\nu}}\right)$ *is bounded on (0, 1), and*

(iv) $h * \partial\left(h * \bar{\bar{\nu}}\right)(0+) = 0$.

*Proof.* By assumption, $\rho_1'$ exists and is continuous. Using Lemma 4.1 we can find a smooth function $u \in \mathcal{C}^\infty(0, \infty)$ such that $|u(x) - \rho_1'(x)| \leq x$ for all $x \in (0, 1)$.

(i) By construction, $u - \rho_1' \in L^1_{\text{loc}}(\mathbb{R}_+)$, and so we may define

$$h(x) := -\int_{(x,1)} u(y)\mathrm{d}y + \int_{(0,1)} (u(y) - \rho_1'(y))\mathrm{d}y + \rho_1(1).$$

This choice results in

$$|h(x) - \rho_1(x)| = \left| \int_{(0,x)} \left(u(y) - \rho_1'(y)\right) \mathrm{d}y \right| \leq \int_{(0,x)} y\, \mathrm{d}y = \frac{1}{2}x^2, \qquad x \in (0, 1),$$

which implies that $h$ and $\rho_1$ are asymptotically equivalent at zero, since $\rho_1(0+) \neq 0$.

(ii) Note that

$$\bar{\bar{\nu}} * h = N_1 * h + N_2 * h = N_1 * (h - \rho_1) + N_1 * \rho_1 + N_2 * h = N_1 * (h - \rho_1) + \mathbb{1} + N_2 * h. \tag{4.2}$$

Thus the claim follows from Lemma 5.2(ii) together with the assumption $N_2(0+) = 0$.

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(iii) Using dominated convergence we have for the right derivative

$$
\begin{aligned}
&|\partial_+(N_1 * h)(x)| = |\partial_+(N_1 * (h-\rho_1))(x)| \\
&\leq \left|\lim_{t\to 0}\int_{(0,x)} \frac{h(x+t-y)-\rho_1(x+t-y)-h(x-y)+\rho_1(x-y)}{t} N_1(y)\mathrm{d}y\right| \\
&\quad + \left|\lim_{t\to 0}\frac{1}{t}\int_{[x,x+t)} h(x+t-y)-\rho_1(x+t-y)N_1(y)\mathrm{d}y\right| \\
&\leq \int_{(0,x)} \left|h'(x-y)-\rho_1'(x-y)\right| |N_1(y)|\,\mathrm{d}y \\
&\quad + \lim_{t\to 0}\frac{1}{t}\int_{(0,t]} |h(y)-\rho_1(y)|\,|N_1(x+t-y)|\mathrm{d}y. \\
&\leq x\int_{(0,x)} |N_1(y)|\mathrm{d}y + \lim_{t\to 0}\frac{t}{2}\int_{(0,t]} |N_1(x+t-y)|\mathrm{d}y \\
&= x\int_{(0,x)} |N_1(y)|\mathrm{d}y + \lim_{t\to 0}\frac{t}{2}\int_{(0,t]} |N_1(x+z)|\mathrm{d}z.
\end{aligned}
$$

Since $N_1 \in L^1_{\text{loc}}(\mathbb{R}_+)$, the right-hand side tends to zero as $x \to 0$, and thus

$$\lim_{x\to 0} \partial_+(N_1 * h)(x) = 0;$$

a similar calculation for the left derivative shows that $\partial_-(N_1 * h) = 0$. Further, combining Lemma 4.2(i) with the assumptions on $\rho_1$ shows that $|h(x)| < Cx^{\alpha-1}$ for $x \in (0, 1)$. Thus the assumptions on $N_2$ and Lemma 5.2(ii) ensure that

$$
\begin{aligned}
|\partial(N_2 * h)(x)| &= |(\partial N_2) * h(x)| \\
&\leq C_1C_2\int_{(0,x)} (x-y)^{-\alpha}y^{\alpha-1}\,\mathrm{d}y \leq C_1C_2B(1-\alpha,\alpha)
\end{aligned}
\tag{4.3}
$$

for $x \in (0, 1)$ and with $C_1, C_2, \alpha > 0$ as in Theorem 4.1.

(iv) This follows from (iii) and the fact that $h \in L^1_{\text{loc}}(\mathbb{R}_+)$. Indeed,

$$\lim_{x\to 0+} |h * \partial\big(h * \bar{\bar{\nu}}\big)(x)| \leq \lim_{x\to 0+}\left(\sup_{z\in(0,1)} |\partial\big(h*\bar{\bar{\nu}}\big)(z)|\right)\int_{(0,x)} |h(y)|\mathrm{d}y = 0.$$

This completes the proof. □

We are now ready to prove the main result of this section.

*Proof of Theorem* 4.1. It suffices to consider the case $q = 0$, as otherwise an exponential change of measure allows us to represent $W^{(q)}$ in terms of $W^{(0)}$ as stated in Section 2.3.

The function $h$ from Lemma 4.2 satisfies the requirements of Theorem 3.3. Therefore, we have the representation

$$W := W^{(0)} = \mathbb{1} * h * \sum_{n=0}^{\infty} f^{*n}, \quad q \geq 0,$$

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with $f(x) := c''h(x) - \partial\big(h * \bar{\bar{\nu}}\big)(x)$ for $x \in (0, \infty)$; moreover,

$$W' = h * \sum_{n=0}^{\infty} f^{*n}. \tag{4.4}$$

By assumption, $\bar{\nu} \in \mathcal{C}^{k_0}(0, \infty)$. Hence, $\bar{\bar{\nu}} \in \mathcal{C}^{k_0+1}(0, \infty)$, and, since $h \in L^1_{\text{loc}}(\mathbb{R}_+)$, Lemma 5 shows that $h * \bar{\bar{\nu}} \in \mathcal{C}^{k_0+1}(0, \infty)$. Again with Lemma 5.1 we see that $f^{*n} \in \mathcal{C}^{k_0}(0, \infty)$ for all $n \in \mathbb{N} \setminus \{0\}$. Thus, $W \in \mathcal{C}^{k_0+1}(0, \infty)$ follows if we can show that the series (4.4) converges in the space $\mathcal{C}^{k_0}(0, \infty)$, i.e. locally uniformly with all derivatives up to order $k_0$.

Lemma 4.2(i), together with the assumptions on $\rho_1$, shows that there exist constants $C, \alpha > 0$ such that $|h(x)| < Cx^{\alpha-1}$ for $x \in (0, 1)$. As $\partial(\bar{\bar{\nu}} * h)$ is bounded on $(0, 1)$ (see Lemma 4.2(iii)), there exist constants $\bar{C}, \bar{\alpha} > 0$ such that $|f(x)| < \bar{C}x^{\bar{\alpha}-1}$ for $x \in (0, 1)$.

Hence, the assumptions of Corollary 5.1 are satisfied, and setting $m := \lfloor 2/\bar{\alpha} \rfloor + 1$ we have for all $k \le k_0$ and $n \ge k_0 m$

$$\partial^k f^{*n} = \big(\partial f^{*m}\big)^{*k} * f^{*(n-km)}.$$

By Lemma 5.2(ii) we have $\partial^k f^{*(n+m)}(0) = 0$ for all $k \le k_0$. Thus, we can use Lemma 5.1(i) to obtain

$$\partial^k\big(h * f^{*(n+m)}\big) = h * \big(\partial f^{*m}\big)^{*k} * f^{*(n-(k-1)m)}.$$

Returning to the series (4.4), this means that for $N > (k_0 + 1)m$

$$\partial^k\left(h * \sum_{n=0}^{N} f^{*n}\right) = \partial^k\left(h * \sum_{n=0}^{(k_0+1)m} f^{*n}\right) + h * \big(\partial f^{*m}\big)^{*k} * \sum_{n=(k_0+1)m+1}^{N} f^{*(n-km)}. \tag{4.5}$$

The first term is independent of $N$. If we pull out $f^{*m}$ from the sum appearing in the second term, we see that

$$h * \big(\partial f^{*m}\big)^{*k} * \sum_{n=(k_0+1)m+1}^{N} f^{*(n-km)} = h * \big(\partial f^{*m}\big)^{*k} * f^{*m} * \sum_{n=k_0m+1}^{N-m} f^{*(n-km)}.$$

The first three convolution factors can be bounded by

$$S := \sup_{x \le z} \left|h * \big(\partial f^{*m}\big)^{*k} * f^{*m}(x)\right| < \infty \quad \text{for all } z > 0;$$

here we use Lemma 5.2 and the continuity of $h$, $\big(\partial f^{*m}\big)^{*k}$, and $f^{*m}$. Thus, with Hölder's inequality,

$$\left|h * \big(\partial f^{*m}\big)^{*k} * f^{*m} * \sum_{n=k_0m+1}^{N-m} f^{*(n-km)}(x)\right| \le S \sum_{n=k_0m+1}^{N-m} \left\|f^{*(n-km)}\right\|_{L^1[0,z]}$$

for all $x \in [0, z]$. Lemma 5.3 implies that the sum on the right-hand side converges uniformly for all $x \in [0, z]$, and so the series

$$\partial^k\left(h * \sum_{n=0}^{\infty} f^{*n}\right)$$

converges locally uniformly in $(0, \infty)$. This proves $W \in \mathcal{C}^{k_0+1}(0, \infty)$.

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Assume now that $\kappa(\rho_1)=2$. Lemma 4.2(i) implies that $\kappa(h)=2$, as well. Thus, Lemma 5.2 gives $h^{*2}(0+)=0$. Plugging the definition of $f$ into the series (4.4) shows that for $x>0$

$$W'(x)=\lim_{N\to\infty} h*\sum_{n=0}^{N}\left(c''h-\partial\left(h*\bar{\bar{\nu}}\right)\right)^{*n}(x)$$
$$=\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}(c''h)^{*(n-\ell)}*h*\left(\partial\left(h*\bar{\bar{\nu}}\right)\right)^{*\ell}.$$

Each summand contains a factor of the form $h*\left(\partial\left(h*\bar{\bar{\nu}}\right)\right)^{*\ell}$ for some $\ell\in\mathbb{N}$, and we will now see how smooth it is. The case $\ell=0$ is trivial, as $h\in\mathcal{C}^{\infty}(0,\infty)$. For $\ell=1$ we have, by Lemma 5.1,

$$h*\partial\left(h*\bar{\bar{\nu}}\right)=\partial\left(h^{*2}*\bar{\bar{\nu}}\right)-h=\partial\left(h^{*2}\right)*\bar{\bar{\nu}}-h\in\mathcal{C}^{k_0+1}(0,\infty). \tag{4.6}$$

Finally, if $\ell\geq 2$, we use

$$h*\left(\partial\left(h*\bar{\bar{\nu}}\right)\right)^{*\ell}=h*\partial\left(h*\bar{\bar{\nu}}\right)*\left(\partial\left(h*\bar{\bar{\nu}}\right)\right)^{*(\ell-1)}.$$

With (4.6), Lemma 4.2(iv), and Lemma 5.1 we obtain

$$\partial\left(h*\left(\partial\left(h*\bar{\bar{\nu}}\right)\right)^{*\ell}\right)=\partial\left(h*\partial\left(h*\bar{\bar{\nu}}\right)\right)*\left(\partial\left(h*\bar{\bar{\nu}}\right)\right)^{*(\ell-1)}\in\mathcal{C}^{k_0}(0,\infty).$$

This shows that each summand of

$$\sum_{n=0}^{N}\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}(c''h)^{*(n-\ell)}*h*\left(\partial\left(h*\bar{\bar{\nu}}\right)\right)^{*\ell}$$

is $(k_0+1)$ times continuously differentiable on $(0,\infty)$. As in the first case we see that this smoothness in inherited by the series (4.4). Consequently, $W\in\mathcal{C}^{k_0+2}(0,\infty)$, and the proof is complete. □

### 4.2. Discussion

We begin the section with an example that illustrates Theorem 4.1.

**Example 4.1.** Denote by $\left(L_t^{(i)}\right)_{t\geq 0}$, $i=1,2$, two independent spectrally negative Lévy processes with characteristic triplets $\left(c^{(i)},0,\widetilde{\nu}^{(i)}\right)$, $i=1,2$. The characteristic triplet of the sum $L_t:=L_t^{(1)}+L_t^{(2)}$ is then $\left(c^{(1)}+c^{(2)},\widetilde{\nu}^{(1)}+\widetilde{\nu}^{(2)}\right)$. For $i=1,2$, denote by $N_i$ the integrated tail function of the reflection $\nu^{(i)}$ of the Lévy measure of $L_t^{(i)}$.

Let $N_1$ be completely monotone, and assume that there exist constants $C>0$, $\alpha\in(0,1)$ such that $N_1(x)\geq Cx^{-\alpha}$ for $x\in(0,1)$. By Corollary 3.1, $N_1$ admits a completely monotone resolvent $\rho_1\in\mathcal{C}^{\infty}(0,\infty)$ for which

$$\rho_1(x)\frac{C}{1-\alpha}=x^{\alpha-1}\rho_1(x)\int_{(0,x)}Cy^{-\alpha}\,\mathrm{d}y\leq\int_{(0,x)}\rho_1(x-y)N_1(y)\mathrm{d}y=1$$

for $x\in(0,1)$. Thus, $\rho_1(x)<\frac{1-\alpha}{C}x^{\alpha-1}$ on $(0,1)$, and the condition (i) in Theorem 4.1 is met.

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This shows that the smoothness of the scale functions $W^{(q)}$ of the sum $L_t$ depends on the smoothness of $N_2$ only, i.e. on the Lévy measure of $L_t^{(2)}$. For example, $L_t^{(2)}$ could be such that $-L_t^{(2)}$ is a subordinator, and with Lévy measure $\nu^{(2)}$ such that $\bar{\nu}^{(2)} < C_2 x^{-\alpha}$ on (0, 1) for some constant $C_2 > 0$. The condition (ii) of Theorem 4.1 is then easily verified.

We conclude that in this case, if $\bar{\nu}^{(2)} \in \mathcal{C}^{k_0}(0, \infty)$ for $k_0 \in \mathbb{N}$, then $W^{(q)} \in \mathcal{C}^{k_0+1}(0, \infty)$, and even $W^{(q)} \in \mathcal{C}^{k_0+2}(0, \infty)$, if additionally $\alpha \geq \frac{1}{2}$.

The following simple corollary of Theorem 4.1 shows that the implication

$$\bar{\nu} \in \mathcal{C}^n(0, \infty) \implies W^{(q)} \in \mathcal{C}^{n+\kappa(\rho_1)}(0, \infty)$$

holds, if the 0-scale function of the compensated process is at least twice continuously differentiable and of bounded growth.

**Corollary 4.1.** *Let $(L_t)_{t\geq 0}$ be a spectrally negative Lévy process with triplet $(c, 0, \widetilde{\nu})$ such that $\int_{(0,1)} y\,\nu(\mathrm{d}y) = \infty$ and $\int_{[1,\infty)} y\,\nu(\mathrm{d}y) < \infty$. Let $\bar{\nu} \in \mathcal{C}^{k_0}(0, \infty)$ for some $k_0 \in \mathbb{N}$. Further, denote by $\widetilde{W}$ the 0-scale function of the compensated process $\widetilde{L}_t := L_t - \mathbb{E}[L_t]$. If $\widetilde{W} \in \mathcal{C}^2(0, \infty)$ and $|\widetilde{W}'(x)| < Cx^{\alpha-1}$ for some constants $C, \alpha > 0$, then $W^{(q)} \in \mathcal{C}^{k_0+\kappa(\widetilde{W}')}(0, \infty)$.*

*Proof.* By Lemma 3.2, $\widetilde{W}'$ is the resolvent of $\bar{\bar{\nu}}$. Thus, the assertion follows immediately from Theorem 4.1 upon setting $N_2 \equiv 0$. □

Both Theorem 4.1, and the two previously mentioned results from [8] on the smoothness of scale functions of spectrally negative Lévy processes with either a Gaussian component or paths of bounded variation, require a growth condition of the type

$$|f(x)| < Cx^{\alpha-1}, \quad x \in (0, 1),$$

for some constants $C, \alpha > 0$ on one of the occurring terms. Getting rid of this requirement seems to be the biggest challenge in a possible proof of Doney's conjecture.

In this regard, the next corollary gives a hint as to how the growth condition on $\rho_1$ in Theorem 4.1 can be avoided by imposing stronger conditions on the component $N_2$.

**Corollary 4.2.** *Let $(L_t)_{t\geq 0}$ be a spectrally negative Lévy process with triplet $(c, 0, \widetilde{\nu})$ such that $\int_{(0,1)} y\,\nu(\mathrm{d}y) = \infty$ and $\int_{[1,\infty)} y\,\nu(\mathrm{d}y) < \infty$. Let $\bar{\nu} \in \mathcal{C}^{k_0}(0, \infty)$ for some $k_0 \in \mathbb{N}$, and assume that there exist a decomposition $\bar{\bar{\nu}} = N_1 + N_2$ and a constant $C_2 > 0$ such that*

(i) *$N_1 \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ admits a resolvent $\rho_1 \in \mathcal{C}^1(0, \infty)$, and*

(ii) *$N_2 \in \mathcal{C}^{k_0+1}(0, \infty)$ with $N_2(0+) = -c''$ and $|\partial N_2(x)| < C_2$ for $x \in (0, 1)$.*

*Then $W^{(q)} \in \mathcal{C}^{k_0+\kappa(\rho_1)}(0, \infty)$ for all $q \geq 0$.*

*Proof.* Without loss of generality we assume that $q = 0$ and consider $W := W^{(0)}$. By Remark 4.1 we can assume $N_2(0+) = 0$, which then yields that the function $f$ in the series expansion (4.4) of $W'$ has to be replaced by

$$f(x) = -\partial\left(h * \left(\bar{\bar{\nu}} + c''\right)\right)(x). \tag{4.7}$$

Thus, the term $c''h(x)$ vanishes owing to the assumptions on $N_2$.

We can now follow the lines of the proof of Theorem 4.1 and observe that the growth condition on $\rho_1$ appears at exactly two places.

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Firstly, in the proof of Lemma 4.2(iii) it appears in the argument (4.3), which can now be replaced by

$$|\partial(N_2 * h)(x)| = |\partial N_2 * h(x)| \le C$$

by Lemma 5.2. Thus, Lemma 4.2(iii) still holds true, and therefore, $\partial\big(h * \big(\bar{\bar{\nu}} + c''\big)\big)(x)$ is bounded on (0, 1). Note that $\bar{\bar{\nu}} + c''$ plays the role of $\bar{\bar{\nu}}$; see Remark 4.1.

Secondly, it is needed to apply Corollary 5.1 to the function $f$ in the series expansion of $W$. But the requirements of Corollary 5.1 are already met, as $f$ stays bounded in (0, 1) by the argument above. □

Finally, we mention here that the close connection between scale functions and Volterra resolvents of the first kind as discussed in Section 3.3 allows for a smoothness result on the latter as well.

**Corollary 4.3.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be a positive, decreasing, and convex function with $f(0+) = \infty$ such that $\partial^2 f(x)\mathrm{d}x$ is a Lévy measure. Denote by $\rho$ the resolvent of $f$.*

(i) *If $\rho \in \mathcal{C}^1(0, \infty)$ and if there are constants $C, \alpha > 0$ such that $|\rho(x)| < Cx^{\alpha-1}$, then*

$$f \in \mathcal{C}^n(0, \infty) \Rightarrow \rho \in \mathcal{C}^{n-2+\kappa(\rho)}(0, \infty)$$

*for all $n \ge 2$.*

(ii) *If Doney's conjecture (iii) (see the beginning of Section 4) is true, then*

$$f \in \mathcal{C}^n(0, \infty) \iff \rho \in \mathcal{C}^n(0, \infty)$$

*for all $n \in \mathbb{N}$.*

*Proof.* The assumptions on $f$ ensure that $\partial^2 f(x)\mathrm{d}x$ is the reflection of the Lévy measure of a spectrally negative Lévy process with paths of unbounded variation, while the requirements for $\rho$ in (i) imply that the spectrally negative Lévy process with Laplace exponent

$$\psi(\beta) = \beta^2 \mathcal{L}[f](\beta)$$

fulfils the assumptions of Theorem 4.1. The assertion (i) now follows directly as $\rho = \partial W^{(0)}$, while (ii) follows directly from Doney's conjecture without further assumptions on $\rho$. □

## 5. Auxiliary results

In this section we collect a few technical results and their proofs. We believe that this material is well known, but we could not pinpoint suitable references, and so we decided to include the arguments for the reader's convenience.

**Lemma 5.1.** *Let $k \in \mathbb{N}$.*

(i) *If $f \in \mathcal{C}^{k+1}(0, \infty)$ be such that $\partial^j f(0+) \in \mathbb{R}$ exists for all $j = 0, 1, \ldots, k$, and $g \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \cap \mathcal{C}^k(0, \infty)$, then $f * g \in \mathcal{C}^{k+1}(0, \infty)$ and*

$$\partial^\ell (f * g)(x) = (\partial^\ell f) * g(x) + \sum_{j=0}^{\ell-1} \partial^j f(0+) \partial^{\ell-1-j} g(x), \ \ \ell \in \{1, \ldots, k+1\}. \tag{5.1}$$

(ii) *If $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \cap \mathcal{C}^k(0, \infty)$, then $f * g \in \mathcal{C}^k(0, \infty)$.*

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*Proof.* (i) The formula (5.1) follows from a straightforward calculation using integration by parts; since the terms on the right-hand side are finite and continuous, we see that $f * g$ is in $\mathcal{C}^{k+1}(0, \infty)$.

(ii) Note that $f$ and $\partial^j f$ may have singularities at $x = 0$, so that the formula (5.1) need not hold. We can, however, argue as follows: let $x \geq 0$ and fix $\delta < x/2$. We have

$$f * g(x) = \int_0^\delta f(x-y)g(y)\mathrm{d}y + \int_\delta^{x-\delta} f(x-y)g(y)\mathrm{d}y + \int_{x-\delta}^x f(x-y)g(y)\mathrm{d}y.$$

The first and the third term are $k$ times continuously differentiable, while the second term can be written as

$$\int_\delta^{x-\delta} f(x-y)g(y)\mathrm{d}y = \int_0^{x-2\delta} f(x-2\delta-y+\delta)g(y+\delta)\mathrm{d}y = f_\delta * g_\delta(x-2\delta)$$

for the shifted functions $f_\delta(\,\cdot\,) := f(\,\cdot\,+\delta)$ and $g_\delta(\,\cdot\,) := g(\,\cdot\,+\delta)$. By assumption, the functions $f_\delta, g_\delta \in \mathcal{C}^k(\mathbb{R}_+)$ are such that we can apply Part (i), showing that $f_\delta * g_\delta \in \mathcal{C}^k(0, \infty)$, and the claim follows. □

**Lemma 5.2.** *Let $f_1, f_2 \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be two functions such that $|f_1(x)| < C_1 x^{\alpha_1 - 1}$ for $x$ in a neighbourhood of zero and some constants $C_1, \alpha_1 > 0$.*

(i) *If, additionally, $|f_2(x)| < C_2 x^{\alpha_2 - 1}$ for $x$ in a neighbourhood of zero and some constants $C_2, \alpha_2 > 0$, then $f_1 * f_2$ exists and satisfies $|f_1 * f_2(x)| < C x^{\alpha_1 + \alpha_2 - 1}$ for $x$ in a neighbourhood of zero and some constant $C > 0$.*

(ii) *If $\alpha_1 \geq 1$, then the convolution $f_1 * f_2$ exists, and there exists a constant $C > 0$ such that $|f_1 * f_2(x)| < C x^{\alpha_1 - 1}$ for all $x$ in a neighbourhood of zero.*

*Proof.* (i) By assumption, we find that for all $x$ in a neighbourhood of zero

$$|f_1 * f_2(x)| \leq \int_0^x |f_1(x-y)||f_2(y)|\mathrm{d}y \leq C_1 C_2 \int_0^x (x-y)^{\alpha_1 - 1} y^{\alpha_2 - 1}\, \mathrm{d}y.$$

As $\alpha_1, \alpha_2 > 0$, the assertion follows after an obvious change of variables.

(ii) The second assertion follows from Hölder's inequality, as

$$|f_1 * f_2(x)| \leq \int_0^x |f_2(x-y)||y^{\alpha_1 - 1}\, \mathrm{d}y \leq \|f_2\|_{L^1([0,x])} \sup_{y \in [0,x]} |y^{\alpha_1 - 1}| = \|f_2\|_{L^1([0,x])} x^{\alpha_1 - 1},$$

for $x \geq 0$ small enough. □

**Corollary 5.1.** *Let $f \in \mathcal{C}^1(0, \infty)$, and assume that $|f(x)| < Cx^{\alpha - 1}$ for $x \in (0, 1)$ and some constants $C, \alpha > 0$. For all $k \in \mathbb{N}$ and $n \geq k(\lfloor 2/\alpha \rfloor + 1)$, one has $f^{*n} \in \mathcal{C}^k(0, \infty)$. In particular,*

$$\partial^k f^{*n} = \left(\partial f^{*m}\right)^{*k} * f^{*(n-km)} \quad \text{with } m = \lfloor 2/\alpha \rfloor + 1.$$

*Proof.* The case $k = 0$ is trivial. Let $k = 1$. Lemma 5.2(i) shows that there is some constant $C' > 0$ such that $|f^{*m}(x)| < C'x$ for all sufficiently small $x < 1$. Therefore, by Lemma 5.1(i), $\partial f^{*n} = \left(\partial f^{*m}\right) * f^{*(n-m)}$ for $n \geq m$. If $k > 1$, the assertion follows by iteration. □

The following lemma is essential for our main results as it ensures uniform convergence on compact subsets of $\mathbb{R}_+$ of the series expansions of the $q$-scale functions.

**Lemma 5.3.** *Let* $f \in L^1_{\text{loc}}(\mathbb{R}_+)$, $z > 0$, *and* $\varepsilon > 0$. *Then the limit*

$$\lim_{n\to\infty} \frac{\|f^{*n}\|_{L^1([0,x])}}{\varepsilon^n} = 0 \tag{5.2}$$

*exists uniformly for all* $x \in [0, z]$.

*Proof.* Fix $\varepsilon, x > 0$. Since for $y \in [0, 1]$ the convolution satisfies

$$\int_0^y f(y-z)f(z)\mathrm{d}z = f * f(y) = (f\mathbb{1}_{[0,x]}) * (f\mathbb{1}_{[0,x]})(y),$$

we may assume that $f(y) = 0$ for $y > x$ and for fixed $x \in \mathbb{R}_+$. This ensures, in particular, that the Laplace transform of $f$ exists. For all $\beta \geq 0$ we have

$$\|f^{*n}\|_{L^1([0,x])} \leq \mathrm{e}^{\beta x}\mathcal{L}[|f^{*n}|](\beta) \leq \mathrm{e}^{\beta x}\left(\mathcal{L}[|f|](\beta)\right)^n,$$

and the assertion follows since there exists some $\beta_0 \geq 0$ such that $\mathcal{L}[|f|](\beta_0) < \frac{1}{2}\varepsilon$. □

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