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On the Bernoulli nature of systems with some hyperbolic structure

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Abstract. It is shown that systems with hyperbolic structure have the Bernoulli property. Some new results on smooth cross-sections of hyperbolic Bernoulli flows are also derived. The proofs involve an abstract version of our original methods for showing that the geodesic flow on surfaces of negative curvature are Bernoulli.

1. *Introduction*

It is often stated that one of the most striking and surprising discoveries underlying the new science of chaos is that purely deterministic equations can lead to flows that exhibit random behavior. There is a variety of mixing properties for flows and transformations that culminate in the K-property and then there is the class of Bernoulli flows f_t which display the most random type of behavior possible. It should be pointed out that the Bernoulli class is completely characterized, up to measure theoretic isomorphism, by a single invariant entropy [**O74**]. This means that two Bernoulli flows f_t and g_t differ only by a constant rescaling of time, i.e. $f_t = g_{ct}$ for some constant c. On the other hand, the K-property is enjoyed by a wide variety of quite distinct isomorphism classes.

It has been shown recently that the class of Bernoulli flows is very nicely behaved from several points of view. If one is interested in constructing good approximations to a stochastic process based upon the sequential observations of a single output then a universal scheme exists which is valid for the class of Bernoulli processes. On the other hand, for non-Bernoulli processes one cannot even find universal schemes which will distinguish between two outputs of the same or different process (cf. [**OW90**]). In a more geometric vein, various properties were established for Bernoulli flows that provide a statistical kind of stability even in cases where the topological structural stability is not valid [**OW91**]. These results highlight the need to verify the Bernoullian nature of as many systems as possible.

Our purpose in this paper is to provide some general theorems which show how to deduce the Bernoulli property from the hyperbolic structure that has been established in many cases. The basic proof paradigm that we use is the same as the one that we first

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introduced in [**OW73**] to prove that geodesic flows on surfaces of negative curvature were Bernoulli. Our results are essentially applicable whenever all of the Lyapunov exponents differ from zero. Non-hyperbolic systems like the general toral automorphisms seem to require more delicate methods and are not covered by the results here. Here is a brief overview of what we plan to do.

Let X denote an n-dimensional manifold ($(n+1)$-dimensional for the flow case) and f (f_t) a piecewise smooth transformation (flow) on X. We also have in mind systems with a mechanical origin like billiards, or balls bouncing in a wedge, that are not given by smooth vector fields. The hyperbolic structure takes the form of foliations into unstable leaves $U(x)$ of dimension k and stable leaves $S(x)$ of dimension $(n-k)$. These are not necessarily the complete foliations that one encounters in the uniformly hyperbolic case. They may be only partial foliations like those found in billiards or in the theory developed by Pesin [**P77**]. We make two basic assumptions: (i) the invariant measure that we treat has some smoothness properties; (ii) the foliations U and S are absolutely continuous.

An assumption like (i) is clearly necessary. Markov partitions show that the range of ergodic behavior that is possible if we vary over all invariant measures is restricted only by the finiteness of the entropy. If one wants special properties for a smooth flow one must make special assumptions about the invariant measure. The second assumption is of a technical nature. It is used to obtain the equivalence between the invariant measure and some product measure on hyperbolic blocks. It was established for uniformly hyperbolic systems in the work of Anosov [**A67**], for non-uniformly hyperbolic systems by Pesin [**P77**] and for certain non-smooth systems by Katok *et al* [**KSLP86**].

The main point that we are trying to make in this paper is that this fairly standard hyperbolic structure suffices, not only to establish the K-property on ergodic components, but also to prove that the flows in question are Bernoulli. In their work on ergodic attractors Pugh and Shub [**PS89**] use hyperbolic blocks in a fashion similar to us but they do not go beyond the K-property.

The generality of our methods here enable us to establish some new results concerning smooth cross-sections of hyperbolic Bernoulli flows and also flows built under a function with a Bernoulli base. This latter result is related to some earlier work of Bunimovich [**B74**] and Ratner [**R74**] where they dealt with the case when the base transformation was a symbolic dynamical system. It is also worth pointing out that Ledrappier [**L84**] used the equivalence of the invariant measure with product measure to show that special partitions for certain smooth diffeomorphisms' partitions were *weak Bernoulli* (WB) a property stronger than very weak Bernoulli (VWB). However, the advantage of our method is that it applies also to flows where the weak Bernoulli property is not expected to hold.

In §2 we describe in detail the assumptions that we make concerning the hyperbolic blocks. In §3 we take up some measurability questions which are necessary for the product measure to be comparable to the given invariant measure and show that they are in fact equivalent. The succeeding sections use almost entirely just this property. In §4 we treat transformations, §5 flows and in §6 the cross-sections and the flows built under a function.

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2. *Hyperbolic blocks*

In the following, X is an n-manifold (or at least piecewise an n-manifold) and f, f_t either a transformation of X or a flow on X. It is not necessary to assume global smoothness of f of f_t but we shall suppose that f, f_t are piecewise differentiable. The hyperbolicity that we assume takes the form of two partial foliations of X; the unstable U of dimension k and the stable S of dimension $n-k$ for transformations and $n-1-k$ for flows (to compactify the notation we will suppose that $\dim X = n+1$ in the flow case). There will be a finite invariant measure μ with smoothness properties that we will spell out later. For μ-a.e. x there will be leaves $U(x)$, submanifolds of dimension k, and $S(x)$, submanifolds of dimension $n-k$, that intersect transversally at x and for some fixed metric d on X satisfy:

$$\overline{\lim}_{n\to+\infty}\frac{1}{n}\log d(f^n x, f^n y) \lneqq 0 \quad \text{for } y\in S(x) \tag{2.1}$$

$$\overline{\lim}_{n\to+\infty}\frac{1}{n}\log d(f^{-n} x, f^{-n} y) \lneqq 0 \quad \text{for } y\in U(x), \tag{2.2}$$

in the case of transformations with an analogous condition for flows. The set of points x that have such leaves will be denoted by X_0. For flows, there also are the weakly stable wS and weakly unstable manifolds wU obtained by thickening the leaves of S, U in the flow direction. We shall assume that the foliations both strong and weak are measurable in the following sense. If the foliation is one-dimensional and is given by some vector field that defines a flow g_s then this measurability is equivalent to what is usually called the measurability of the flow.

Measurable coordinates. Denote by B_1^k the unit ball in $\mathbb{R}^k$. Then there are local coordinates for $U(x)$ given by a map $c(x, y)$: $X_0 \times B_1^k \to X$ that satisfies:

(a) $c(x, 0) = x$, $c(x, y) \in U(x)$ for all $y \in B_1^k$;

(b) for fixed x, $c(x, \cdot)$ is a diffeomorphism between B_1^k and $c(x, B_1^k)$;

(c) $c(x, y)$ is a jointly measurable function of the two variables.

A similar assumption is made about the other foliation S whose coordinate function is $\hat{c}(x, z)$. Note that the image $c(x, B_1^k)$ does not exhaust the leaf $U(x)$ and there is also no assumption about its size in X which may be rather small. We shall call the image $c(x, B_1^k)$ a *local leaf* at x.

This structure is what Pesin establishes in [**P77**] for smooth maps all of whose Lyapunov exponents differ from zero. It is also the type of structure that Sinai and others established in billiards and other systems with a physical interpretation. The measurability condition is automatically satisfied whenever the foliations are obtained from a measurable splitting at the tangent bundle by the usual method of pulling back parametrized parallel k-planes in the unstable directions.

It is standard to use the measurability to get uniformity on sets of positive measure. We need three kinds of uniformity: (i) the tangent space does not vary much along a local leaf of U and S: (ii) the local leaves extend a certain fixed distance away from x, i.e. $d(x, c(x, \partial B_1^k))$ and $d(x, \hat{c}(x, \partial B_1^{n-k}))$ are both bounded from below; (iii) the angle between the tangent spaces to $U(x)$ and $S(x)$ is bounded away from zero.

It is easy to see that all three quantities are measurable functions and thus standard

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measure theory gives us some choice for the constants involved such that the set of x with that uniformity has non-zero μ-measure. We really need to do things in two steps. First of all we cut down from B_1^k to a smaller ball δB_1^k so that the tangent space along $c(x, u)$ does not vary much for $u \in \delta B_1^k$. Then we need to make sure that these small local leaves have a minimal size. In what follows, local leaf will mean these small local leaves. If we now let P_0 be a subset of such a set with positive measure and very small diameter we can do the following.

For fixed y, $c(y, \cdot)$ is a differentiable function and thus can be viewed, in particular, as an element of $c(X, \delta B_1^k)$. With the sup norm this is a complete separable metric space and therefore by Lusin's theorem there is a closed subset $P_1 \subset P_0$ of positive measure on which c becomes a continuous function. It follows that these local leaves of U, $\cup_{x \in P_1} c(x, \delta B_1^k)$, form a closed subset of X that has positive μ-measure. Restricting to a further closed subset $P_2 \subset P_1$ of positive μ-measure we can get that $\hat{c}$ is also a continuous function and thus as x varies over P_2 both the stable and the unstable leaves form closed subsets. We shall denote by P the intersection of these sets. Since P contains P_2 it has positive μ-measure. Furthermore, the fact that for $x \in P_0$ the tangent spaces of $U(x)$ and $S(x)$ intersect with positive angle and that the local leaves have some minimal size—but are not too big—guarantee that this closed set P has the abstract structure of a product space. To see this last point, fix one $x_0 \in P_2$ and consider the local leaf in $U(x_0)$. Let Y be the intersection of the stable bundle of leaves above with this local leaf. This is a closed set. Similarly, form Z, a closed subset of $S(x_0)$. Now each $x \in P$ has a unique pair of coordinates (y, z), where y is the intersection of the local leaf we have been considering in $U(x_0)$ with $\hat{c}(x, \delta B_1^{n-k})$ and z is the intersection of $c(x, \delta B_1^k)$ with the local leaf in $S(x_0)$.

For flows, one performs the same kind of construction with U and wS or wU and S since we need, of course, complementary dimensions to fill a set of positive measure. At the end of this section when we give a summary description of the hyperbolic blocks we will spell out the case for flows.

Now for the smoothness assumptions on μ. In many cases, especially those originating in conservative mechanical systems where Liouville's theorem holds, the invariant measure is smooth in the sense that it is equivalent to Lebesgue measure. In the case of attractors, the invariant measure is typically singular but nonetheless when disintegrated along the leaves of U its conditional measures are equivalent to the k-dimensional Lebesgue measure on the leaves. We call these measures SBR, after the work of Sinai [**S72**], and Bowen and Ruelle [**BR75**]. Our standing hypothesis is that one of these two hypotheses is valid. For the classical argument of Hopf and its ramifications to be valid we have to relate the measure theory of the two foliations U and S. For this we need the notion of absolute continuity.

First we have to explain the holonomy map. Let $\mathcal{F}$ be a local foliation and let D_0 and D_1 be two disks transversal to $\mathcal{F}$ so that for $x_0 \in D_0$ the leaf $\mathcal{F}(x_0)$ intersects D_1 at a unique point x_1. The map $x_0 \to x_1$ is called the $\mathcal{F}$-holomony map from D_0 to D_1 along $\mathcal{F}$. We can avoid discussing what happens if $\mathcal{F}(x_0)$ intersects D_1 at more than one point since in our situation this will not happen.

The foliation $\mathcal{F}$ is said to be *absolutely continuous* if any such holomony map is

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absolutely continuous with respect to the Lebesgue measures on the disks D_0, D_1. We will assume that our foliations U and S are absolutely continuous in this sense. For the case of SBR it suffices to assume that S is absolutely continuous. Here is a summary of what a hyperbolic block for transformations is.

Hyperbolic block for transformations (HBT).

(1) Product structure—P is identified with a product space $Y \times Z$, $(y, z) \to U(y) \cap S(z)$ and is thus fibered exactly by both foliations.

(2) The invariant measure μ is either smooth or SBR; i.e. $\mu(P) > 0$ for a measure equivalent to Lebesgue measure on P, or the conditional measure of μ on leaves of U is equivalent to Lebesgue measure on the leaves.

(3) Absolute continuity of the foliations.

Note that in (1), since P is the intersection of two bundles of leaves, the U-leaves in P are not *complete* k-manifolds—they are *subsets* of the local leaves.

The definitions for flows are analogous except that one of the foliations should be thickened in the flow directions, either S to wS, the weakly stable leaves or U to wU. The fact that the flow is smooth shows that if S is absolutely continuous so is wS: applying the flow to D sweeps out a disk $\hat{D}$ transversal to S, the foliation of $\hat{D}$ by the flow lines is absolutely continuous and this enables us to pass from the absolute continuity of S to that of wS. The measurability of coordinates is now taken for the weakly stable (or weakly unstable) foliations in addition to the previous assumption.

Hyperbolic block for flows (HBF).

(1) Product structure—P is identified with a product space $Y \times Z$, $(y, z) \leftrightarrow U(y) \cap wS(z)$ and is fibered exactly by both foliations.

(2) The invariant measure μ is either smooth, or SBR; i.e. $\mu(P) > 0$ for a measure μ-equivalent to Lebesgue measure on P, or such that its conditional measure on the leaves of U is equivalent to Lebesgue measure there.

(3) The foliations U and wS are absolutely continuous.

Let us recap what we have done here. Starting from the usual properties that are established for the stable and unstable foliations we have constructed a hyperbolic block P in the case where the measure is nice (smooth or SBR). In the next section we shall see how this leads to the equivalence (on P) of the invariant measure μ with a product measure and this is what will enable us to analyze the ergodic properties of f and f_t.

3. *Measurability and equivalence of invariant measure to product measure*

Recall that when the product structure for the hyperbolic block was obtained we had P represented as the intersection of two closed sets of leaves from the stable and unstable foliations. This representation shows that if we take a Borel subset $B \subset Y$ and look at that part of P that is made up of leaves passing through B, then we get a Borel subset of P. It follows that viewing P as $Y \times Z$, if $B \subset Y$, $A \subset Z$ are Borel then $B \times A$ is a Borel subset of P. By a well-known result in Borel theory (cf. the treatment in [**AR76**]), i.e. that a countably generated sub-σ-algebra of Borel sets that separates points is the full Borel σ-algebra, we can conclude that the product measure structure coincides with the usual Borel structure on P.

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We shall need a criterion which will enable us to say when is a measure μ on $X \times Y$ equivalent to a product measure. Disintegrate μ along the fibers $X \times \{y\}$ and in that disintegration denote the conditional μ-measure on $X \times \{y\}$ by μ_y, thought of as a measure on X. Let ν denote the projection of μ on Y, i.e. $\nu(C) = \mu(X \times C)$, and if δ_y denotes the point mass at $\{y\}$ we have

$$\mu = \int_Y \mu_y \times \delta_y \, d\nu(y).$$

PROPOSITION 3.1. *A necessary and sufficient condition that μ be equivalent to a product measure is that for ν-a.e. y_1, y_2 the measures μ_{y_1}, μ_{y_2} are equivalent measures.*

Proof. If μ is equivalent to some product measure it is not hard to see that μ is equivalent to $p \times \nu$, where $p(\nu)$ is the projection of μ on $X(Y)$. This gives the necessity of the condition. For the sufficiency observe that one can find a y_0 so that for ν-a.e. y, μ_y is equivalent to μ_{y_0} and thus μ is equivalent to $\mu_{y_0} \times \nu$ by Fubini's theorem. □

THEOREM 3.1. *If μ is an invariant measure which is smooth or SBR on a hyperbolic block P, then μ is equivalent to a product measure.*

Proof. If μ is smooth let us fix one of the foliations, say U, and consider a nice smooth foliation $\mathcal{F}$ into $(n-k)$-manifolds transversal to U. By using local coordinates on P (we could divide into a finite number of local coordinate charts if necessary) we can think of these as being parallel and then it is clear that the disintegration of Lebesgue measure along these $(n-k)$-spaces yields $(n-k)$-dimensional Lebesgue measure. Now the U foliation is assumed to be absolutely continuous and thus we are exactly in the situation of the Proposition 3.1 and we can conclude that Lebesgue measure λ is equivalent to product measure when the product structure is now coming from U and $\mathcal{F}$. From this it easily follows now using the absolute continuity of the $\mathcal{F}$-foliation that the conditional measure of λ along leaves of U is equivalent to the k-dimensional Lebesgue measure there. Thus we have arrived at the situation of the SBR measure since λ and μ are equivalent.

To conclude the proof we repeat the above argument with U playing the role of $\mathcal{F}$ and S playing the role of U, and obtain that μ is equivalent to a product measure on the product structure coming from U and S on P. □

The proof for the flow case is completely analogous using the absolute continuity of the two strict foliations which implies the absolute continuity of the weak versions.

4. *Ergodic properties of transformations*

From this point on the reader can forget about the absolute continuity of the fibers and keep in mind only the main conclusion of the preceding section: *for our hyperbolic block the invariant measure is equivalent to a product measure*. To begin let us recall the Hopf argument. Take some bounded continuous function ϕ on X and consider $\hat{\phi}$ as its projection onto the space of functions that are invariant μ-a.e. By the ergodic theorem

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we have two distinct representations for $\hat{\phi}$:

$$\hat{\phi}(x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{1}^{n} \phi(f^j x) \quad \mu\text{-a.e.} \tag{4.1}$$

$$\hat{\phi}(x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{1}^{n} \phi(f^{-j} x) \quad \mu\text{-a.e.} \tag{4.2}$$

From the first representation we see that $\hat{\phi}$ is constant μ-a.e. on leaves of S and from the second that it is constant μ-a.e. along leaves of U and then the fact that μ is equivalent to product measure allows us to conclude that $\hat{\phi}$ is constant μ-a.e. on P. Thus $\cup_{-\infty}^{\infty} f^n(P)$ consists of a single ergodic component in the ergodic decomposition of μ and henceforth we restrict μ to that component and assume that (X, μ, f) is an ergodic system. Note that the ergodicity of the original measure μ requires further arguments and is a major source of difficulty in proving ergodicity of billiards and other mechanical systems.

For the K-property we replace continuous functions by smooth partitions. Actually, all that we really need is a good estimate for the set of points that comes within ε of the boundary of the sets in the partition. Here is a simple lemma that enables us to construct such partitions.

LEMMA 4.1. *If (X, d) is a metric space and μ is a probability measure on X, and if $x_0 \in X$ and an interval $[a, b]$ is specified, then there is some $r \in [a, b]$ such that for all $\varepsilon > 0$*

$$\mu\{x : |d(x, x_0) - r| \leq \varepsilon\} \leq \frac{6}{|b-a|} \cdot \varepsilon.$$

Proof. If there is no such r, then any $r \in [a, b]$ is contained in an interval J_r satisfying

$$\mu\{x : d(x, x_0) \in J_r\} \geq \frac{3}{|b-a|} |J_r|.$$

One can find a finite subcollection of the J_r's that covers $[a, b]$ and such that every point is in at most two such intervals. Summing over all these intervals we get the contradiction that $2 \geq 3$ which proves the lemma. □

With the lemma it is easy to construct finite partitions Q of arbitrarily small diameter such that there is a constant C and for any a,

$$\mu(\{x : d(x, \partial Q) \leq a\}) \leq Ca. \tag{4.3}$$

For the next step, namely that of establishing the K-property, we must recall the basic Pinsker–Rohlin–Sinai (PRS) theorem.

For a finite partition Q, define the remote past and remote future by

$$Q_{-\infty} = \cap_{n=1}^{\infty} \vee_{-\infty}^{-n} f^{-j} Q$$
$$Q_{+\infty} = \cap_{n=1}^{\infty} \vee_{+n}^{+\infty} f^{-j} Q.$$

Note that since $\{x : f^j x \in Q_\alpha\} = f^{-j} Q_\alpha$, the 'future' from the point of view of the transformation f corresponds to negative powers of f acting on the partition. A process (X, μ, Q, f) is K if $Q_{-\infty}$ contains only sets of measure zero or one. The PRS theorem

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asserts, inter alia, that in all cases $Q_{-\infty}$ and $Q_{+\infty}$ coincide as σ-algebras modulo μ-null sets.

A system (X, μ, f) is said to have the *K-property* if for any finite partition Q, the process (X, μ, Q, f) is K. It is known that it suffices to verify this for one refining sequence of partitions Q_n. For the sake of completeness we sketch a quick proof of this last fact. We need the alternative characterization of K-processes which is the main assertion of the PRS theorem, namely that a process is K if and only if only trivial partitions give rise to zero entropy processes. Assume, therefore, that Q_n is a refining sequence of partitions and that (Q_n, f) is K. If R is any partition giving a zero entropy process, and $\varepsilon > 0$, choose a large n so that one can find $\hat{R}$, measurable with respect to $\vee_{-\infty}^{\infty} f^{-j} Q_r$ and satisfying

$$H(R|\hat{R}) - H(\hat{R}|R) < \varepsilon, \tag{4.4}$$

where $H(\cdot|\cdot)$ denotes conditional entropy. Now, since $h(R, f) = 0$ one also has for any N, that $h(R, f^N) = 0$ and then from (4.4) one deduces that $h(\hat{R}, f^N) < \varepsilon$.

By letting N tend to infinity we see that $\hat{R}$ is ε-constrained in the remote past of Q_n and thus R is 2ε-contained there. Since that past is trivial, as (Q_n, f) is a K-process we conclude that R is within 2ε of the trivial partition and since ε is arbitrary R is trivial. Thus (X, μ, f) has completely positive entropy and is therefore a K-process.

The proof we have just outlined establishes a more general result. The *Pinsker algebra* $\mathcal{P}$ of a process (or system) is the maximal σ-algebra all of whose sets A have the property that the entropy of the process defined by $(A, X \backslash A)$ is zero. The general result is that if the Q_n are a refining sequence of partitions then the Pinsker algebras $\mathcal{P}_n$ of the processes (Q_n, f) converge to the Pinsker algebra of the system (X, f). Now we have to check that, indeed, if Q is a smooth partition (Q, f) is a K-process.

Here, the simple observation is that if a stable leaf $S(x)$ in P never gets too close to the boundary of Q under arbitrarily high positive iterates of f, then any set measurable with respect to $Q_{+\infty}$ contains the entire leaf, or no part of it. Because of the exponential contraction along S it follows from (4.3) that a set of leaves of $S \cap P$ of full μ-measure have this property. Thus any set measurable with respect to $Q_{+\infty}$ is foliated by entire S-leaves, μ-a.e. Since $Q_{-\infty}$ and $Q_{+\infty}$ coincide μ-a.e. any such set is also foliated by U-leaves and since μ is equivalent to product measure we conclude that if such a set intersects P in positive measure it fills up P μ-a.e. This does not suffice to establish the K-property for the process (X, μ, Q, f) but it does show that the Pinsker algebra $(Q_{-\infty} = Q_{+\infty})$ is atomic and thus since f is ergodic it is in fact finite.

Notice that we have a fixed lower bound for the size of a minimal atom in the Pinsker algebra of (x, μ, Q, f) which is given by the measure of the hyperbolic block. As we repeat the argument for a refining sequence of partitions this fixed lower bound does not change and thus by our earlier observation about the convergence of Pinsker algebras we can conclude that the Pinsker algebra of (X, μ, f) is atomic. This proves the following theorem.

THEOREM 4.2. *Either f has the K-property or f permutes M sets periodically, $f A_i = A_{i+1}$, $f^M A_0 = A_0$, $\cup_0^{M-1} A_i = X$, μ-a.e., and on each A_i, f^M has the K-property, i.e. f is a K-automorphism times a finite rotation.*

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We shall continue the discussion assuming that f itself is K. The general case is readily obtained from this one by replacing f by f^M. To go from K to Bernoulli we will use the method we first introduced in [**OW73**].

In order to keep the exposition self-contained we shall give a rapid review of what we need from the theory of Bernoulli systems. A finite measure-preserving system $(X, \mathcal{B}, \mu, f)$ is *Bernoulli* if there is a partition Q of X with the properties:

(1) the $\{T^{-i}Q\}_{i\in\mathbb{Z}}$ are independent;

(2) $\vee_{-\infty}^{\infty} T^{-i}Q = \mathcal{B}$, μ-a.e.

In the language of isomorphism theory a system is Bernoulli if it is isomorphic to a stationary process consisting of independent random variables. A flow f_t is a *Bernoulli flow* if for each d, the transformation f_d is Bernoulli. The zero-one law for independent random variables shows that the Bernoulli property implies K.

Without going into a detailed discussion of the various other characterizations and properties of Bernoulli systems we shall explain what very weak Bernoulli (VWB) is and cite a result that says that this property implies Bernoulli. It will be convenient to use the following terminology: a property holds for ε*-almost every* point of a set E in a measure space if the set of points of E for which the property fails to hold has relative measure less than ε, i.e. if μ denotes the measure

$$\mu(\text{exceptional points in } E)/\mu(E) \le \varepsilon.$$

A mapping θ between two measure spaces (E, μ), (F, ν) is ε-measure preserving if there are subsets $E_1 \subset E$, $F_1 \subset F$ satisfying

$$\mu(E_1)/\mu(E) \le \varepsilon, \quad \nu(E_1)/\nu(E) \le \varepsilon$$

and for all $A \subset E \backslash E_1$

$$|\nu(\theta(A))/\mu(A) - 1| < \varepsilon.$$

If Q is a finite partition of X, then (Q, f) will have the *very weak Bernoulli* property if given $\varepsilon > 0$ there is an m_0 and for all $m \ge m_0$ for ε-almost every atom E of $\vee_{-m}^{-m_0} f^{-j}Q$ there is an ε-measure-preserving map θ from $E \times [0, 1]$ with the measure $\mu/\mu(E) \times$ Lebesgue measure to (X, μ) such that for ε-almost every point $(x, u) \in E \times [0, 1]$ we have here

$$\overline{\lim_{n\to\infty}} \sum_{j=1}^{n} |Q(f^j x) - Q(f^j \theta(x, u))| \le \varepsilon. \tag{4.5}$$

Here $Q(x)$ denotes the index α such that $x \in Q_\alpha$. This condition (4.5) means that on average the future behavior of x and $\theta(x, u)$ are very similar. The reason for crossing E with $[0, 1]$ is to give us greater flexibility in constructing the mapping θ. It gives us the possibility of dividing the points $x \in E$ into many small pieces and mapping them to different parts of the space. The usual definition of VWB involves a more general notion—that of the $\bar{d}$-distance between processes—which is not needed here. If the process is independent then the future is independent of the conditioning on the past atom E and then this condition is satisfied with $\varepsilon = 0$. The VWB is a weakening of this independence. It is proven in [**O74**] that VWB implies the Bernoulli property.

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We shall construct θ having property (4.5) by mapping points along stable manifolds, so that θx and x will behave in an almost identical fashion under positive iterates of f. The mapping will be carried out separately on small pieces that we call mapping boxes. In our situation we will construct them out of small rectangles in hyperbolic blocks where the Radon–Nikodym derivative of μ with respect to product measure is approximately constant. First, for a general result we return to the setup of §3. Given a product space $Y \times Z$ and a measure μ on $Y \times Z$ equivalent to product measure $\rho \times \nu$ we let $r(y, z)$ denote the Radon–Nikodym derivative of λ with respect to $\rho \times \nu$. In all that follows, some $\varepsilon > 0$ will be fixed. We now use the fact that $r(y, z)$ can be approximated by simple functions based on measurable rectangles to find sets Y_0, Z_0 and a constant r_0 such that for $r_0 \gg \varepsilon$,

$$\{(y, z) : |r_0 - r(y, z)| \geq \varepsilon^2\} \cap Y_0 \times Z_0$$

has *relative* measure at most ε^2 in $Y_0 \times Z_0$ both with respect to $\mu | Y_0 \times Z_0$ and with respect to $\rho \times \nu | Y_0 \times Z_0$.

Let us call P_0 the hyperbolic block that we get when we restrict to $Y_0 \times Z_0$ as above. Disintegrate μ along the fibers $Y_0 \times \{Z\}$ to get μ_z and observe that since we have an almost constant derivative with respect to product measure for ε-almost every z we can map $Y_0 \times \{z\}$ equipped with μ_z in an ε-measure-preserving way along wS-leaves to all of P_0. We denote this mapping by θ_z. This new almost uniform hyperbolic block will serve as a prototype for our mapping boxes—but of course we need to cover the space with them.

Instead of trying to do this in a precise way let us observe that if we map P_0 by f^n then we get a new block $P_n = f^n P_0$ which is again foliated by U and by S and f^n carries θ_z to mappings $\theta^{(n)}$ between ε-almost every leaf U of P_n to all of P_n along leaves of S, in an ε-measure-preserving fashion. According to the ergodic theorem the sets $P_0, P_1, \ldots, P_{N-1}$ for large N give us an almost even covering of the space—in the sense that most points of the space are covered approximately the same number of times.

By the K-property, if m_0 is large enough then ε-almost every atom E of $\vee_{-m}^{-m_0} f^{-j} Q$ will be intersecting each of the mapping boxes $P_0, \ldots, P_{N-1}$ in a set whose relative measure is almost equal to the μ-measure of the mapping box. In addition, for ε-almost every $x \in E$ the sum $\sum_0^{N-1} 1_{P_i}(x)$ is almost equal to $\mu(P_0) \cdot N$. Now if for each P_i we take a separate interval $J_i \subset \mathbb{R}$ of length $1/N\mu(P_0)$ and consider $\cup P_i \times J_i$, we have a good approximation to $X \times [0, 1]$. Restricting to E we get a good approximation to $E \times [0, 1]$ and we can map this to all of X using $\theta^{(i)}$ on $(E \cap P_i) \times J_i$. Taking into account the approximation, both on $X \times [0, 1]$ and on $E \times [0, 1]$, we still get an ε-measure-preserving map of $E \times [0, 1]$ to X along leaves of S as required.

5. *Ergodic properties of flows*

The pattern of the proof for flows is the same as it is for transformations, so we will concentrate here on the new points that come up. We will work with a hyperbolic block P built from the unstable U and weakly stable wS foliations. For Hopf's argument no change is required since the limits

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$$\hat{\phi} = \lim_{n\to+\infty} \frac{1}{n} \int_0^n \phi(f_t x)\, dt \quad \mu\text{-a.e.} \tag{5.1}$$

$$\hat{\phi} = \lim_{n\to+\infty} \frac{1}{n} \int_0^n \phi(f_{-t} x)\, dt \quad \nu\text{-a.e.} \tag{5.2}$$

are clearly invariant under the flow and therefore the constancy of (5.1) along leaves of S implies that $\hat{\phi}$ is also constant along leaves of wS. Thus, as before, the entire block P lies in a single ergodic component to which we will restrict μ.

For the K-property we first define what it means for a flow to be K. We shall adopt the definition that a flow is a K-flow if for any d, the transformation f_d has the K-property. For fixed d, f_t commutes with f_d and therefore the Pinsker algebra of f_d is invariant—as a σ-algebra—under the action of the flow. To establish the K-property fix a d and consider $f = f_d$. As before, it suffices to verify that (Q, f) is K for a smooth partition Q. Unlike what happened in ergodicity proof, now the argument of §4 only gives us that sets in the Pinsker algebra are μ-a.e. fibered by leaves of U and S—not wS. We focus on showing that if the Pinsker algebra $\mathcal{P}$ is not trivial then it is at most a circle rotation under the flow f_t. To this end recall the Ambrose–Kakutani representation of an arbitrary flow as a flow built under a function. In that representation pure rotations correspond to the base being atomic. If the base is *not* atomic then one easily sees that for any t_0 and any $\delta > 0$ there are sets of positive measure E in the Pinsker algebra that cover the space such that $\cup_{|t|\le t_0} f_t E$ is also measurable and

$$0 < \mu\Big(\bigcup_{|t|\le t_0} f_t E\Big) < \delta. \tag{5.3}$$

We shall show that this cannot happen. Indeed, if E is such a set, measurable with respect to $\mathcal{P}$, the Pinsker algebra, and E intersects our hyperbolic block P in a set of positive measure, then if t_0 is large enough $(\cup_{|t|\le t_0} f_t(E)) \cap P$ would be completely foliated μ-a.e. by both U and wS and therefore using the fact that μ is equivalent to product measure this latter set coincides with P, μ-a.e. If δ is chosen sufficiently small this would contradict (5.3).

This argument shows that either the Pinsker algebra for f_d is trivial or f_t is a pure rotation on the Pinsker algebra.

In the latter case, (X, f_t) has a circle rotation as a factor, and if $\pi: X \to S'$ is the factor map, $\pi^{-1}(\zeta)$ gives rise to a measurable partition of X and, clearly, the fibers of this partition are foliated by U- and S-leaves. Thus the foliations U and S 'commute' in a measurable sense. Since the measure μ is trivially well behaved in the flow direction this means that for the return map to these fibers, $\pi^{-1}(\zeta)$, we have the hyperbolic block structure for transformations.

Indeed, by what we have just said, these fibers $\pi^{-1}(\zeta)$ are foliated by both U and S. The invariant measure μ decomposes along these fibers to μ_ζ say. If t_0 represents the period, then f_{t_0} fixes these fibers and the μ_ζ are invariant under f_{t_0}. Furthermore, the fact that μ was equivalent to a product measure with respect to the product structure given by U and wS on P implies that μ_ζ will be equivalent to a product measure given by U and S on $\pi^{-1}(\zeta)$. Indeed, the conditional μ-measures on the U-leaves are equivalent under the holonomy map defined by wS since we have already seen that μ is equivalent

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to product measure. Now on the $\pi^{-1}(\zeta)$, the holonomy map between U-leaves defined by S is exactly the same as the one defined by wS in X. Therefore we can apply Proposition 3.1 to conclude that μ_ζ is equivalent to a product measure. The general discussion of §5 will be applied to f_{t_0} and one fixed $\pi^{-1}(\zeta_0)$. The original flow will be represented as a flow built over a K-transformation with constant height function.

Continuing with the analysis of §5, which is not hampered by the fact that we do not know that $\pi^{-1}(\zeta)$ has the structure of an n-manifold we obtain the result that f_{t_0} is a Bernoulli transformation. Thus f_t is the constant suspension of a Bernoulli transformation. Here we can give a nicer interpretation of this fact in terms of flows. Since we know that a Bernoulli flow exists, we can imbed f_{t_0} in a flow, say $\hat{f}_t$ that takes $\pi^{-1}(\zeta_0)$ to itself. Using the imbedded flow it is easy to see that f_t itself is isomorphic to a Bernoulli flow $\hat{f}_t$ times (direct product) a rotation. We were unable to do this for suspensions of K-transformations since not every K has a square root [**O73**] let alone can it be imbedded in a flow. We shall continue the discussion with the new elements that appear when f_t is a K-flow.

Fix some discretization of the flow $\{f_{nd}\}_{n\in\mathbb{Z}}$ and some smooth partition Q. We want to verify the VWB condition. Assume that the hyperbolic block is of the form $U \times wS$. As before, by localizing to some sub-block P_0, we can assume that the invariant measure μ is not merely equivalent to product measure on P_0 but that most fibers have nearly the same conditional measure. We would like to make mapping boxes as before, using P_0, and mapping along leaves of wS. However, since the leaf is only weakly stable for $y \in S(x)$ it is no longer the case that $d(f_{nd}x, f_{nd}y) \to 0$ as $n \to +\infty$. What is true is that the positive orbits of x and y never diverge by much since by moving a little bit on the orbit of y we can bring y to be on the same strong stable leaf as x. This distance can be made quite small, given Q, and then the ergodic theorem will guarantee that for most points x, any $y \in P_0$ in the same weakly stable leaf as w will have $Q(f_{nd}x) \neq Q(f_{nd}y)$ for a set of low density. This amount is controlled by the size of the block P_0 in the flow direction. Thus condition (4.5) can still be satisfied even when mapping along weakly stable leaves.

Having made this basic observation the rest of the argument of §4 can be carried out in exactly the same way. We apply many iterates of f_d to P_0 to get an almost even covering of X by good blocks which can serve as mapping boxes, and then verify the VWB condition as in §4.

6. *Cross-sections and flows built under functions*

In the previous section we had occasion to recall the Ambrose–Kakutani theorem that gives measurable cross-sections for any flow and shows how to view the flow as being built up from the return time map to the cross-section. The general study of the relationship between the ergodic properties of these return maps and the ergodic properties of the flows involves the notion of Kakutani equivalence and was studied in detail in [**ORW82**]. That theory teaches us that a Bernoulli flow has return maps on *measurable cross-sections* with a variety of different properties and, in particular, they need not be Bernoulli or Bernoulli times a periodic transformation. The purpose of this section is to investigate what happens for smooth cross-sections when the Bernoulli property is

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established via the hyperbolic blocks.

Suppose then that D is a smooth disk of dimension n transversal to the flow f_t defined on X with invariant measure μ. One defines the measure μ_D invariant under the return map f_τ by imbedding D in a family of disjoint disks $f_t D$, $|t| \leq \delta$, and then disintegrating μ along these leaves. Since the measure in the t direction is simply the Lebesgue measure along the flow lines we can say this is a more explicit way: for small t_0 and $A \subset D$, calculate $(1/t_0)\mu(\cup_0^{t_0} f_t A)$ and take a limit as t_0 tends to zero. A key property of the smoothness is that the μ_D measure of the points in D that are within ε of the boundary may be bounded by some constant times ε. This is clear if the original measure μ is smooth in the sense that it is equivalent to Lebesgue measure.

Now we have to see what happens to the stable and unstable manifolds from the point of view of D. We can project a piece of $S(x)$ to D by flowing back with f_t until the first time D is reached. This is easily done by forming $wS(x) \cap D$ and gives a set $\hat{S}(x_0)(x_0 = f_{-t}x, x_0 \in D)$ in D. If $y_0 \in \hat{S}(x_0)$ then we know that the flow lines $f_t x_0$, $f_t y_0$ approach each other exponentially fast as $t \to +\infty$. The same will be true of the iterates of $f_\tau^n x_0$, $f_\tau^n y_0$ as long as the number of times that x_0 and y_0 return to D in the time interval $[0, t]$ is the same. Clearly, the only time that a problem can occur is when $f_\tau^n x_0$ is very near the boundary. Our assumption on the smoothness of the boundary of D shows that for a.e. point x_0 of D, a sufficiently small piece of $\hat{S}(x_0)$ is indeed a stable manifold. In a similar fashion, one can see that U projects down to a local foliation $\hat{U}$ of D.

The last point that has to be checked now is the equivalence of μ_D with a product measure on a hyperbolic block that is built up from $\hat{U}$ and $\hat{S}$. Here the argument is similar (but not identical) to the one we presented when dealing with the possibility that the Pinsker algebra of the flow is a circle rotation. Consider a leaf of wS. The flow lines of f_t are absolutely continuous and, therefore, one can see that the conditional μ_D measure on $\hat{S}$ is identified with the conditional μ measure on leaves of S in a fixed wS. The mapping that $\hat{U}$ induces between two different leaves of $\hat{S}$ is thus essentially the same as the map that U induces between two leaves of wS, say $wS(x_1)$ and $wS(x_2)$. That mapping takes the conditional μ-measure on $wS(x_1)$ to a measure equivalent to the conditional μ-measure on $wS(x_2)$ and so the same will be true for the mapping that $\hat{U}$ induces between $wS(x_1) \cap D$ and $wS(x_2) \cap D$

Having accomplished this, the analysis of §5 proceeds as before to give at first the K-property and then the Bernoulli property. Notice that ergodicity is automatic since that trivially carries over to the cross-section map. The place where this argument fails for the general cross-section is in the construction of the foliations $\hat{U}$, $\hat{S}$. In general, leaves like $\hat{U}$ do not survive because the orbits pass near the boundary repeatedly and the points get out of synchronization with respect to f_τ even though the flow lines remain together.

We summarize this discussion as a theorem.

THEOREM 6.1. *If f_t is a flow with a hyperbolic block satisfying the assumptions of HBF (§2) with smooth measure μ, and D is a smooth disk transverse to the flow, then the return map $f_\tau : D \to D$ with the induced measure μ_D is either Bernoulli or Bernoulli times a periodic transformation.*

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Remark. The result is valid regardless of whether or not the Pinsker algebra of the flow is trivial since in any event the hyperbolic structure pushes down to the disk D.

It is also very important to go the other way, i.e. from the existence of a hyperbolic structure on the disk D under f_τ to ergodic properties of the flow f_τ itself. This kind of result is important to be able to apply our result to the many examples (from the Sinai billiards onwards) which were analyzed in detail in terms of the Poincaré return map on some naturally occurring cross-section. Here, instead of avoiding the boundary one must check to see that the differences between the successive return times do not accumulate. For this, it is clear that one needs some control on the modulus of continuity of the return time mapping. The situation is as follows. We have a manifold X, with a mapping $f : X \to X$ preserving a smooth μ with a hyperbolic block satisfying HBT. In addition, we have a return function $r : X \to \mathbb{R}^+$ that we assume is integrable $d\mu$ and satisfies a Hölder condition X with respect to a fixed metric ρ. We consider

$$X^r = \{(x, u) : 0 \leq u \leq r(x)\}$$

with the flow f_t that maps (x, u) to $(x, u + t)$ until $u + t = r(x)$ when it continues at $(f(x), 0)$. On X^r, the product measure of μ with Lebesgue measure on $\mathbb{R}$ is denoted by μ^r. The stable leaf through (x, u) is constructed by taking a $y \in S(x)$ and then looking for that u' so that the difference between the running sum $\sum_0^m r(f^j x)$ and $\sum_0^m r(f^j y)$ is accounted for by the difference between u and u'. We need to know, of course, that this difference converges and this is exactly what follows from the exponential decay of $\rho(f^j x, f^j y)$ and the Hölder continuity of the function r.

Having constructed a local foliation U^r and S^r in X^r from U and S we again have to check that we get a hyperbolic block for the flow. For the measurability one observes that the new local foliations through x in the base are small pieces of the old foliations moved up by a continuous function (the sum of the corrections referred to above). Thus they can be described by the old measurable coordinates together with another measurable function giving the required height. For the rest of the space these local foliations are simply translated along the flow lines. It remains to check the absolute continuity.

Once again this will follow from the fact that in the flow direction the invariant measure is simply Lebesgue measure. To be precise, recall that we coordinatized the hyperbolic block as $Y \times Z$, with $\{y\} \times Z$ representing stable leaves $S(y, z)$ and $Y \times \{z\}$ representing unstable leaves. The weakly stable leaves for the flow in X^r will be relatively open subsets of $\cup_t f_t S(y, z)$. The various unstable leaves of f_t will be organized in wU as translates via f_t of a single lifted leaf $U^r(x)$. The measure in the t direction is simply Lebesgue measure and since U maps the conditional μ-measures on leaves of S to equivalent measures a simple application of Fubini's theorem shows that U^r continues to do this for the conditional measures of μ^r along the leaves of wS^r. Applying Proposition 3.1 once again shows that the invariant measure μ^r is equivalent to product measure on the hyperbolic block formed by a bundle of U^r-leaves and wS^r-leaves. This establishes the following theorem.

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THEOREM 6.2. *If (X, μ, f) has a hyperbolic block satisfying HBT and $r : X \to \mathbb{R}^+$ is an integrable return time that is Hölder continuous, then (X^r, μ^r, f_t) has a hyperbolic block with the invariant measure μ^r equivalent to product measure and thus on the ergodic component that contains the block either f_t is Bernoulli or Bernoulli $\times$ rotation.*

7. *Examples*

The results of our discussions above apply to almost all of the examples in the literature where absolute continuity has been established for stable and unstable foliations. All of the assumptions that we made concerning the two foliations are usually given explicitly with the exception of the key hypothesis that we made concerning measurable coordinates. In what follows, we shall point out where these hypotheses can be found in some of the basic papers.

In [**P77**], Theorem 2.2.1 asserts the existence and gives the properties of the stable and unstable manifolds. At a good point x, where TS_x and TU_x denote the subspaces of TM_x associated with the exponents that are less than zero and greater than zero, respectively, one has a local mapping $\phi(x) : TS_x \to TU_x$ and exponentiating the graph of this mapping gives us the local stable mapping at x. This function $\phi(x)$ is ultimately constructed by using an implicit function theorem and it depends in a measurable way on x. It is straightforward to deduce from this the existence of the measurable coordinates in the sense of our definition. Thus our results will apply in any situation where [**P77**] is applicable.

In [**KSLP86**] some generalizations of the above are given to mappings with some singularities. For the basic construction of the stable and unstable manifolds at good points they refer back to the above theorem of Pesin. Thus the existence of measurable coordinates in their situation follows, essentially, from the corresponding result in [**P77**] that we have just discussed. They go on to give a careful proof of the absolute continuity of these foliations and then verify that their results apply to a large class of billiards. It follows that our theory applies to their case and in particular extends the K-property that they verify to the Bernoulli property.

In [**PS89**] yet another treatment of the invariant manifold theory is given and in their Theorem 3.8 and the discussion following it, one sees that measurable coordinates can be given for the stable and unstable manifolds. They also construct hyperbolic blocks and use them in a fashion similar to ours; however, their discussion stops short of the K-property. This can now be filled in via our techniques.

There have been several papers in the last few years in which a hyperbolic structure has been obtained and the K-property established. Some examples are: [**KSS91**], [**KSS92**], [**Si92**], and [**Sz92**]. Usually for the K-property they ultimately refer back to the monograph of [**KSLP86**] and there our results apply to deduce the additional Bernoulli property. A recent preprint of Chernov and Haskell entitled 'Nonuniformly hyperbolic K-systems are Bernoulli' covers some of the same ground as our paper with similar methods.

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