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STOCHASTIC COMPARISON ON ACTIVE REDUNDANCY ALLOCATION TO *K*-OUT-OF-*N* SYSTEMS WITH STATISTICALLY DEPENDENT COMPONENT AND REDUNDANCY LIFETIMES

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Abstract

In the literature on active redundancy allocation, the redundancy lifetimes are usually postulated to be independent of the component lifetimes for the sake of technical convenience. However, this unrealistic assumption leads to a risk of inaccurately evaluating system reliability, because it overlooks the statistical dependence of lifetimes due to common stresses. In this study, for k-out-of-n:F systems with component and redundancy lifetimes linked by the Archimedean copula, we show that (i) allocating more homogeneous redundancies to the less reliable components tends to produce a redundant system with stochastically larger lifetime, (ii) the reliability of the redundant system can be uniformly maximized through balancing the allocation of homogeneous redundancies in the context of homogeneous components, and (iii) allocating a more reliable matched redundancy to a less reliable component produces a more reliable system. These novel results on k-out-of-n:F systems in which component and redundancy lifetimes are statistically dependent are more applicable to the complicated engineering systems that arise in real practice. Some numerical examples are also presented to illustrate these findings.

Keywords: Archimedean copula; completely monotone; left tail permutation decreasing; likelihood ratio order; majorization; multivariate mixture; reversed hazard rate order; usual stochastic order

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1. Introduction

It is a common practice to allocate redundancies to a system at the component level, in order to enhance system reliability by reducing the chance of unexpected system failure, and this is of wide interest in reliability engineering and system security. In industrial engineering, two types of allocations are commonly practiced: (i) an *active* redundancy, also referred to as a *hot standby*, runs in parallel to system components and starts functioning at the same time as the system components are initiated; (ii) a *standby* redundancy, also referred

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to as a *cold standby*, is put in standby and starts functioning once some component fails. For pioneering discussions on redundancy allocation, the reader may refer to Boland *et al.* [6, 7], Shaked and Shanthikumar [40], Singh and Misra [43], and Singh and Singh [44], among others. For a review of recent advances in active redundancy allocation, we refer the reader to Li and Ding [25]. In the literature, some authors have studied the *general standby* model, which includes active redundancy and standby redundancy as special cases and hence is also referred to as *warm standby*. See, for example, She and Pecht [42], Amari *et al.* [1], and references therein.

In most of the research on redundancy allocation, component lifetimes and redundancy lifetimes are assumed to be independent. However, since components and active redundancies operate together and thus bear common stresses, their lifetimes are usually statistically dependent. Thus, evaluations of system reliability may be inaccurate under the assumption of independence. Recently, some authors have studied redundancy allocation in the context where system component lifetimes are mutually dependent but statistically independent of redundancy lifetimes. See, for example, You and Li [47] and You *et al.* [49]. In this study, we focus on active redundancy allocation to k-out-of-n:F systems with component and redundancy lifetimes being statistically dependent. Our results in this more general context enrich the research on active redundancy allocation and provide guidelines to practice engineering reliability.

In the theory of reliability, the coherent structure, taking into account monotonicity and system structure with respect to components, defines a very broad class of systems, including series, parallel, and k-out-of-n:F systems as special cases. See, for example, Barlow and Proschan [2] for a comprehensive discussion on coherent systems. A k-out-of-n:F system fails once k of its n components fail to operate properly. Apart from the well-known fact that the k-out-of-n:F structure includes series (i.e., $k = 1$) and parallel (i.e., $k = n$) systems as two typical cases, the lifetime of a k-out-of-n:F system takes the form of the kth smallest order statistic, and this facilitates the analytical study of system reliability. Thanks to the invention of the signature, a coherent system with exchangeable component lifetimes can be decomposed as a mixture of k-out-of-n:F systems according to system signature, and thus the k-out-of-n:F structure actually serves as the basic building block for coherent systems with homogeneous components. (For more on signatures, the reader may refer to Navarro *et al.* [35] and the monograph of Samaniego [38].) As a consequence, the k-out-of-n:F structure has received special attention in recent research on engineering reliability. In the literature, Boland *et al.* [6] were among the first to study the optimal allocation of multiple active redundancies to k-out-of-n:F systems by means of the majorization order. For example, for a k-out-of-n:F system with independent and stochastically ordered component lifetimes, the redundant system survival function was shown to be Schur-concave with respect to the allocation policy. Since the system hazard rate function plays a significant role in describing the way in which the system wears out, Misra *et al.* [30] further proved that the hazard rate function of a redundant series system is Schur-convex with respect to the allocation policy in the above context. Afterward, for a redundant k-out-of-n:F system with component and active redundancy lifetimes statistically independent and identically distributed (i.i.d.), Hu and Wang [19] showed that the redundant k-out-of-n:F system survival function is Schur-concave with respect to the allocation policy. In consideration of the fact that system components usually share common stresses and thus admit statistically dependent lifetimes, You and Li [47] examined the optimal allocation of active redundancies to a k-out-of-n:F system with stochastic arrangement increasing component lifetimes. In addition, You *et al.* [49] further studied the allocation of active redundancies to a

k-out-of-n:F system with component lifetimes being left tail permutation decreasing, which is an even weaker assumption than the stochastic arrangement increasing assumption. For more on related research in this line, the reader may refer to Shaked and Shanthikumar [40], Boland *et al.* [7], Singh and Misra [43], Singh and Singh [44], Li and Hu [26], Li and Ding [25], Belzunce *et al.* [3, 4], Zhao *et al.* [50], and Fang and Li [16, 17].

In most research on active redundancy allocation to k-out-of-n:F systems with statistically dependent component lifetimes, the redundancy lifetimes are usually assumed to be independent of component lifetimes. It is worth noting that such an assumption is rather restrictive in some real-world contexts. In many practical situations, system components and active redundancies operate in the same environment or share the same load, which means their lifetimes should be statistically dependent. This more general context is particularly apropos to complicated engineering systems in real practice. To the best of our knowledge, Belzunce *et al.* [3, 4] were the first to study the k-out-of-n:F system with component lifetimes statistically dependent on a single redundancy lifetime. Specifically, they established the usual stochastic order on the redundant k-out-of-n:F system lifetime under the assumption that component lifetimes conditioned by the redundancy lifetime are *stochastic arrangement increasing*. Subsequently, You and Li [48] rebuilt the usual stochastic order on the redundant k-out-of-n:F system lifetime in the context that, conditioned by the redundancy lifetime, component lifetimes are *left tail weakly stochastic arrangement increasing*. Recently, Torrado *et al.* [45] considered dependence among components and redundancies in their study of multi-level redundancy allocation for coherent systems formed by modules. In this vein, we compare redundant system lifetimes in the context in which component lifetimes and redundancy lifetimes are statistically dependent. For the sake of convenience, component lifetimes and redundancy lifetimes are assumed to be linked by an Archimedean copula, which is rather popular in statistics, operations management, actuarial and financial risks, etc.

This paper deals with k-out-of-n:F systems with component lifetimes and active redundancy lifetimes linked by an Archimedean copula. Our work is threefold. For multiple homogeneous redundancy lifetimes, we show that allocating more redundancies to the less reliable components tends to produce a stochastically larger system lifetime. For component lifetimes and redundancy lifetimes both homogeneous, we prove that the system reliability can be uniformly maximized by balancing the allocation of redundancies. In the context of matched redundancies, we find that system reliability can be improved by allocating a redundancy with a larger baseline reversed hazard rate to a component with smaller baseline reversed hazard rate.

The rest of the paper proceeds as follows. Section 2 reviews some basic concepts related to the main results and introduces a technical lemma which will be useful in developing the main result. In Sections 3 and 4, we present the main comparison results concerning the allocation of redundancies to k-out-of-n:F systems with components and redundancy lifetimes linked by an Archimedean copula. In Section 5, we develop the usual stochastic order on the lifetime of the system with matched redundancies under the multivariate mixture model, and then we present the result in the Archimedean copula setting as a direct consequence. Finally, Section 6 closes the study with some concluding remarks on further topics for research in this line.

Throughout the remaining sections, real vectors such as $\boldsymbol{x}=(x_1, \cdots, x_n)$ and $\boldsymbol{a}=(a_1, \cdots, a_n)$ and random vectors such as $\boldsymbol{X}=(X_1, \cdots, X_n)$ and $\boldsymbol{Y}=(Y_1, \cdots, Y_m)$ are all assumed to be nonnegative, and $I(A)$ denotes the *indicator* function on a set A, which takes the value 1 or 0 according to whether A occurs or not. For brevity, we denote the maximum by $\max(x, y)=x \vee y$ and the minimum by $\min(x, y)=x \wedge y$, and for sub-vectors we use the notation $\boldsymbol{x}_{\{1,2\}}=(x_3, \cdots, x_n)$. Also, all expected values are implicitly assumed to be finite whenever they appear. For convenience, the terms 'increasing' and 'decreasing' stand for 'nondecreasing' and 'nonincreasing', respectively.

2. Some preliminaries

Before proceeding to the main results, for ease of reference we review important notions such as majorization, stochastic orders, and copulas. We also introduce one technical lemma, which will be useful in developing our main results.

For a real vector $(a_1, \cdots, a_n)$, denote by $a_{[i]}$ the ith largest element of $a_1, \cdots, a_n$, for $i = 1, \cdots, n$.

Definition 2.1. A real vector $\boldsymbol{b} = (b_1, \cdots, b_n)$ is said to be *majorized* by another real vector $\boldsymbol{a} = (a_1, \cdots, a_n)$, and we write $\boldsymbol{b} \prec_{\mathrm{m}} \boldsymbol{a}$, if

$$\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i \qquad \text{and} \qquad \sum_{i=1}^{k} b_{[i]} \le \sum_{i=1}^{k} a_{[i]}, \quad \text{for } k = 1, \cdots, n-1.$$

Note that $\boldsymbol{b} \prec_{\mathrm{m}} \boldsymbol{a}$ implies that $\boldsymbol{a}$ is more dispersed than $\boldsymbol{b}$. A linear transformation M on $\mathbb{R}^n$ is called a *T-transform* if

$$M = \lambda J + (1 - \lambda) Q, \qquad \text{for some } \lambda \in [0, 1], \tag{2.1}$$

where J is the identity matrix, and Q is a permutation matrix that interchanges two coordinates. Majorization $\boldsymbol{b} \prec_{\mathrm{m}} \boldsymbol{a}$ is known to be realizable by using successive T-transforms.

Lemma 2.1. (Marshall *et al.* [29, Lemma 2.B.1].) *If $\boldsymbol{a} \prec_{\mathrm{m}} \boldsymbol{b}$, then $\boldsymbol{a}$ can be derived from $\boldsymbol{b}$ by successive applications of a finite number of T-transforms.*

A real function $g(\boldsymbol{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be *Schur-convex* (*Schur-concave*) if

$$g(\boldsymbol{b}) \le (\ge) g(\boldsymbol{a}), \qquad \text{for any } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n \text{ such that } \boldsymbol{b} \prec_{\mathrm{m}} \boldsymbol{a}.$$

Majorization is usually utilized to characterize various interesting inequalities associated with Schur-convex (Schur-concave) functions. In the sequel, we will employ majorization to compare the degree of balance of allocation policies. For a comprehensive exposition on majorization, we refer the reader to the monograph of Marshall *et al.* [29].

Definition 2.2. Consider a random variable X with cumulative distribution function (CDF) F, survival function (SF) $\bar{F} = 1 - F$, and probability density function (PDF) f, and another random variable Y with CDF G, SF $\bar{G} = 1 - G$, and PDF g. We make the following definitions:

(i) X is said to be smaller than Y in the sense of the *likelihood ratio* order (and we write $X \le_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing in x.

(ii) X is said to be smaller than Y in the sense of the *hazard rate* order (and we write $X \le_{\mathrm{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x.

(iii) X is said to be smaller than Y in the sense of the *reversed hazard rate* order (and we write $X \le_{\mathrm{rh}} Y$) if $G(x)/F(x)$ is increasing in x.

(iv) X is said to be smaller than Y in the sense of the *usual stochastic* order (and we write $X \le_{\mathrm{st}} Y$) if $\bar{F}(x) \le \bar{G}(x)$ for any x.

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Stochastic orders are popular in many areas of applied probability and statistics, including engineering reliability, operations management, quantitative risk, business and economics, etc. Standard references on stochastic orders with applications include Kaas *et al.* [21], Denuit *et al.* [13], Shaked and Shanthikumar [41], and Li and Li [23]. The reader may refer to these works for more detailed discussions.

For any $\{i, j\}$ such that $1 \le i < j \le n$, define the permutation

$$\tau_{ij}(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) = (x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n).$$

Definition 2.3. A multivariate real function $g(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ is said to be

(i) *arrangement increasing* (AI) with respect to $\{i, j\}$ such that $1 \le i < j \le n$ if

$$(x_i - x_j)[g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x}))] \le 0, \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n;$$

(ii) *left tail permutation decreasing* (LTPD) with respect to (i, j) such that $1 \le i < j \le n$ if

$$\int_{-\infty}^{t} \big[g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x}))\big]\mathrm{d}x_i \ge 0, \qquad \text{for any } t \le x_j \text{ and all } \boldsymbol{x} \in \mathbb{R}^n.$$

AI functions and LTPD functions are useful in many applied areas, such as econometrics, actuarial risk management and reliability theory, etc. For references on AI functions with applications, we refer the reader to Hollander *et al.* [18], Boland and Proschan [2], Boland *et al.* [6], and Li and You [27]. Absolutely continuous random vectors with an AI or LTPD PDF play a part in the development of our main results.

Definition 2.4. An absolutely continuous random vector $\boldsymbol{X}$ is said to be

(i) *stochastic arrangement increasing* (SAI) if the joint PDF $f(\boldsymbol{x})$ is AI with respect to any (i,j) such that $1 \le i < j \le n$;

(ii) *left tail permutation decreasing* (LTPD) if the joint PDF $f(\boldsymbol{x})$ is LTPD with respect to any (i,j) such that $1 \le i < j \le n$.

It should be remarked here that the SAI property implies the LTPD property. Both the SAI property and the LTPD property can be extended to generic random variables; in a general context, the latter is known as the *left tail weak stochastic arrangement increasing* (LWSAI) property. For more on the SAI and LTPD properties and their applications in quantitative risk and engineering reliability, we refer the reader to Cai and Wei [10, 11], Li and You [27], and references therein.

For a random vector $\boldsymbol{X} = (X_1, \cdots, X_n)$ with univariate marginal CDFs $F_1, \cdots, F_n$, if there exists a mapping $C : [0, 1]^n \mapsto [0, 1]$ such that the CDF of $\boldsymbol{X}$ may be represented as

$$F(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n))$$

for all $x_1, \cdots, x_n$, then $C(u_1, \cdots, u_n)$ is called the *copula* of $\boldsymbol{X}$.

A function φ on $(0, +\infty)$ is said to be *n-monotone* if $(-1)^k\varphi^{(k)}(t) \ge 0$ for any $k = 0, 1, \cdots, n$ and all $t \in (0, +\infty)$, where $\varphi^{(0)}(t) \equiv \varphi(t)$ and $\varphi^{(k)}(t)$ denotes the kth-order derivative for $k > 0$. The function φ is said to be *completely monotone* if $(-1)^k\varphi^{(k)}(t) \ge 0$ for any

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$t > 0$ and $k = 0, 1, \cdots$. Obviously, an n-monotone function $\varphi(t)$ is such that $(-1)^k\varphi^{(k)}(t)$ is always decreasing in t for $k = 1, \cdots, n-1$.

Definition 2.5. For an n-monotone function $\varphi : [0, +\infty) \mapsto (0, 1]$ with $\varphi(0) = 1$ and $\lim_{t\to\infty} \varphi(t) = 0$, the function

$$C(u_1, \cdots, u_n) = \varphi\big(\varphi^{-1}(u_1) + \cdots + \varphi^{-1}(u_n)\big)$$

is called an *Archimedean* copula, and φ is referred to as the *generator function*.

As for two random variables with reversed hazard rate order and the generator of an Archimedean copula, we introduce a technical lemma which will be useful in developing our main results.

Lemma 2.2. *For $X_i \sim F_i$ with PDF f_i, $i = 1, 2$, and a log-convex generator φ, if $X_1 \le_{\rm rh} X_2$, then*

$$f_1(x)\psi'(F_1(x)) \ge f_2(x)\psi'(F_2(x)) \qquad \textit{for all } x \ge 0,$$

where $\psi = \varphi^{-1}$ is the generalized inverse of φ.

Proof. Since φ is log-convex, $\frac{\varphi(x)}{\varphi'(x)}$ is decreasing. Because $\psi = \varphi^{-1}$ is decreasing, it follows that

$$x\psi'(x) = \frac{\varphi(\psi(x))}{\varphi'(\psi(x))}$$

is increasing. Also, $X_1 \le_{\rm rh} X_2$ implies $X_1 \le_{\rm st} X_2$, i.e., $F_1(x) \ge F_2(x)$ for all x. Thus, it holds that

$$F_1(x)\psi'(F_1(x)) \ge F_2(x)\psi'(F_2(x)), \qquad \text{for all } x.$$

In view of the fact that $\psi'(x) \le 0$ for all x, we have

$$0 \le \frac{F_1(x)\psi'(F_1(x))}{F_2(x)\psi'(F_2(x))} \le 1, \qquad \text{for all } x.$$

On the other hand, $X_1 \le_{\rm rh} X_2$ implies $\frac{f_1(x)}{F_1(x)} \le \frac{f_2(x)}{F_2(x)}$ for all x. Therefore, it holds that

$$\frac{F_1(x)\psi'(F_1(x))}{F_2(x)\psi'(F_2(x))} \cdot \frac{f_1(x)}{F_1(x)} \le \frac{f_2(x)}{F_2(x)},$$

which is equivalent to $f_1(x)\psi'(F_1(x)) \ge f_2(x)\psi'(F_2(x))$ for all x. □

A widely used tool in statistical practice, copulas are utilized to model statistical dependence among multiple random variables in many applied areas, such as biostatistics, econometrics, actuarial risk, etc. For comprehensive expositions on copula theory, one may refer to the monographs of Joe [20] and Nelsen [36]. In the past two decades, much attention has been paid to Archimedean copulas because of their mathematical tractability and the flexibility they allow in specifying the dependence structure of multivariate distributions. In the sequel, we will employ Archimedean copulas to model the dependence structure of the component and redundancy lifetimes.

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3. On systems with stochastically ordered component lifetimes

Consider a redundant k-out-of-n:F system with dependent component lifetimes $\boldsymbol{X} = (X_1, \cdots, X_n)$ and dependent active redundancy lifetimes $\boldsymbol{Y} = (Y_1, \cdots, Y_m)$. Let $\boldsymbol{r} = (r_1, \cdots, r_n)$ be the allocation policy, under which $r_i \geq 0$ redundancies are allocated to the ith component, $i = 1, \cdots, n$, and $r_1 + \cdots + r_n = m$, i.e.,

$$\boldsymbol{r} \in \mathcal{A}_{n,m} = \{(r_1, \cdots, r_n) : r_1 + \cdots + r_n = m,\ r_i \geq 0,\ i = 1, \cdots, n\}.$$

For $k = 1, \cdots, n$, denote by $(X_1, \cdots, X_n)_k$ the kth smallest order statistic based on $X_1, \cdots, X_n$. Then, with the allocation policy $\boldsymbol{r}$, the redundant k-out-of-n:F system attains the lifetime

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) = \big(X_1 \vee Y_1 \vee \cdots \vee Y_{r_1}, \cdots\cdots, X_n \vee Y_{r_1+\cdots+r_{n-1}+1} \vee \cdots \vee Y_m\big)_k. \tag{3.1}$$

Li and Ding [24, Theorem 1] proved for k-out-of-n:F systems that, under the framework of i.i.d. redundancy lifetimes and independent component lifetimes, more redundancies should be allocated to the components with stochastically smaller lifetimes: if $X_1, \cdots, X_n$ are independent, $Y_1, \cdots, Y_m$ are i.i.d., and $\boldsymbol{X}$ is independent of $\boldsymbol{Y}$, then, for any $\boldsymbol{r} \in \mathcal{A}_{n,m}$, $k = 1, \cdots, n$, and any (i,j) such that $1 \leq i < j \leq n$,

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \leq_{\text{st}} T_{k:n}\big(\boldsymbol{X}, \boldsymbol{Y}; \tau_{ij}(\boldsymbol{r})\big) \qquad \text{whenever } X_i \leq_{\text{st}} X_j \text{ and } r_i \leq r_j. \tag{3.2}$$

Subsequently, You and Li [47] presented a similar conclusion on the allocation of redundancies to k-out-of-n:F systems in the setting of SAI component lifetimes and i.i.d. redundancy lifetimes: if $(X_1, \cdots, X_n)$ is SAI, $Y_1, \cdots, Y_m$ are i.i.d., and $\boldsymbol{X}$ is independent of $\boldsymbol{Y}$, then, for any $\boldsymbol{r} \in \mathcal{A}_{n,m}$, $k = 1, \cdots, n$, and any (i,j) such that $1 \leq i < j \leq n$,

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \leq_{\text{st}} T_{k:n}\big(\boldsymbol{X}, \boldsymbol{Y}; \tau_{ij}(\boldsymbol{r})\big) \qquad \text{whenever } r_i \leq r_j. \tag{3.3}$$

You *et al.* [49, Theorem 4.3] further strengthened (3.3) from SAI component lifetimes to LTPD ones: if $\boldsymbol{X}$ is LTPD, $Y_1, \cdots, Y_m$ are i.i.d., and $\boldsymbol{X}$ is independent of $\boldsymbol{Y}$, then, for any $\boldsymbol{r} \in \mathcal{A}_{n,m}$, $k = 1, \cdots, n$, and any (i,j) such that $1 \leq i < j \leq n$,

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \leq_{\text{st}} T_{k:n}\big(\boldsymbol{X}, \boldsymbol{Y}; \tau_{ij}(\boldsymbol{r})\big) \qquad \text{whenever } r_i \leq r_j. \tag{3.4}$$

As per Proposition 4.1 of Cai and Wei [11], for $(X_1, \cdots, X_n)$ linked by an Archimedean copula with a completely monotone generator, $X_1 \leq_{\text{rh}} \cdots \leq_{\text{rh}} X_n$ implies that $(X_1, \cdots, X_n)$ is LWSAI. Thus, based on the result of (3.4), one can extend the result of (3.2) from independence of component lifetimes to an Archimedean copula, at the cost of upgrading the usual stochastic order of component lifetimes to the reversed hazard rate order.

It is worth mentioning that all of the above research work is developed under the assumption that system component lifetimes $\boldsymbol{X}$ and redundancy lifetimes $\boldsymbol{Y}$ are statistically independent of each other. Such an assumption is often impractical in real-world applications of reliability engineering. In this section we study the allocation of m redundancies to n components in the novel context in which the $m + n$ lifetimes involved are statistically dependent, by conducting stochastic comparison on the redundant system lifetime. Specifically, we compare the redundant k-out-of-n:F system with stochastically ordered component lifetimes and redundancy lifetimes linked by an Archimedean copula.

Now let us present our first main result on the redundant k-out-of-n:F system with component and redundancy lifetimes linked by an Archimedean copula. This result serves as

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an interesting generalization for both Li and Ding [24, Theorem 1] and You *et al.* [49, Theorem 4.3].

Theorem 3.1. *Suppose that the lifetimes* $(X_1, \cdots, X_n, Y_1, \cdots, Y_m)$ *are linked by one Archimedean copula with a log-convex and* $(n+1)$*-monotone generator, and* $Y_1, \cdots, Y_m$ *are identically distributed. Then, for* $k = 1, \cdots, n$*, any* $\boldsymbol{r} \in \mathcal{A}_{n,m}$*, and (i,j) such that* $1 \le i < j \le n$*,*

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \le_{\rm st} T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \tau_{ij}(\boldsymbol{r})) \tag{3.5}$$

whenever $X_i \le_{\rm rh} X_j$ *and* $r_i \le r_j$*.*

Proof. Without loss of generality, we consider $(i, j) = (1, 2)$ with $r_2 > r_1 \ge 0$. Define

$$\begin{aligned} Z_1 &= \max(Y_1, \cdots\cdots, Y_{r_1}),\\ Z &= \max(Y_{r_1+1}, \cdots, Y_{r_2}),\\ Z_2 &= \max(Y_{r_2+1}, \cdots\cdots, Y_{r_1+r_2}),\\ Z_i &= \max\big(Y_{r_1+\cdots+r_{i-1}+1}, \cdots\cdots, Y_{r_1+\cdots+r_{i-1}+r_i}\big), \qquad i = 3, \cdots, n. \end{aligned}$$

As per (3.1), we have

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \overset{st}{=} \big(X_1 \vee Z_1, X_2 \vee Z_2 \vee Z, X_3 \vee Z_3, \cdots\cdots, X_n \vee Z_n\big)_k \tag{3.6}$$

and

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \tau_{12}(\boldsymbol{r})) \overset{st}{=} \big(X_1 \vee Z_1 \vee Z, X_2 \vee Z_2, X_3 \vee Z_3, \cdots\cdots, X_n \vee Z_n\big)_k, \tag{3.7}$$

for any $k = 1, \cdots, n$, where $\overset{st}{=}$ means equality in distribution.

Let F_i and f_i be the CDF and PDF, respectively, of X_i, for $i = 1, \cdots, n$, and let G and g be the common CDF and PDF, respectively, of the Y_j, $j = 1, \cdots, m$. For the Archimedean copula with generator φ, let $\psi = \varphi^{-1}$, and define

$$\begin{aligned} \ell(z, \boldsymbol{x}) &= (r_2 - r_1)\psi(G(z)) + [r_1\psi(G(x_1)) + \psi(F_1(x_1))]\\ &\quad + [r_1\psi(G(x_2)) + \psi(F_2(x_2))] + \sum_{i=3}^{n} [r_i\psi(G(x_i)) + \psi(F_i(x_i))]. \end{aligned} \tag{3.8}$$

Then the random vector $\big(Z, X_1 \vee Z_1, \cdots, X_n \vee Z_n\big)$ has CDF

$$H(z, \boldsymbol{x}) = \mathbb{P}(Z \le z, X_1 \vee Z_1 \le x_1 \cdots, X_n \vee Z_n \le x_n) = \varphi\big(\ell(z, \boldsymbol{x})\big)$$

and hence PDF

$$\begin{aligned} &h(z, \boldsymbol{x})\\ &= \varphi^{(n+1)}(\ell(z, \boldsymbol{x}))(r_2 - r_1)g(z)\psi'(G(z))[r_1g(x_1)\psi'(G(x_1)) + f_1(x_1)\psi'(F_1(x_1))]\\ &\quad \cdot [r_1g(x_2)\psi'(G(x_2)) + f_2(x_2)\psi'(F_2(x_2))] \prod_{i=3}^{n} [r_ig(x_i)\psi'(G(x_i)) + f_i(x_i)\psi'(F_i(x_i))]. \end{aligned}$$

Note that Z has PDF

$$l(z) = (r_2 - r_1)g(z)\psi'(G(z))\varphi'((r_2 - r_1)\psi(G(z))).$$

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The random vector $(X_1 \vee Z_1, X_2 \vee Z_2, \cdots, X_n \vee Z_n \mid Z = z)$ has PDF

$$\begin{aligned}\rho(\boldsymbol{x}|z) &= \frac{h(z, \boldsymbol{x})}{l(z)} \\ &= \hbar(z, \boldsymbol{x}_{\{1,2\}})[r_1\psi'(G(x_1))g(x_1) + \psi'(F_1(x_1))f_1(x_1)] \\ &\quad \cdot \varphi^{(n+1)}(\ell(z, \boldsymbol{x}))[r_1\psi'(G(x_2))g(x_2) + \psi'(F_2(x_2))f_2(x_2)], \end{aligned} \tag{3.9}$$

where

$$\hbar(z, \boldsymbol{x}_{\{1,2\}}) = \frac{\prod_{i=3}^{n}\left[r_i\psi'(G(x_i))g(x_i) + \psi'(F_i(x_i))f_i(x_i)\right]}{\varphi'\big((r_2 - r_1)\psi(G(z))\big)}. \tag{3.10}$$

Thus, for any $x_2 \geq t \geq 0$, it holds that

$$\begin{aligned}\int_0^t \rho(\boldsymbol{x}|z)\mathrm{d}x_1 &= \hbar(z, \boldsymbol{x}_{\{1,2\}})[r_1\psi'(G(x_2))g(x_2) + \psi'(F_2(x_2))f_2(x_2)] \\ &\quad \cdot \int_0^t \varphi^{(n+1)}(\ell(z, \boldsymbol{x}))\mathrm{d}[r_1\psi(G(x_1)) + \psi(F_1(x_1))] \\ &= \hbar(z, \boldsymbol{x}_{\{1,2\}})[r_1\psi'(G(x_2))g(x_2) + \psi'(F_2(x_2))f_2(x_2)] \\ &\quad \cdot \left[\varphi^{(n)}(\ell(z, t, x_2, \cdots, x_n)) - \varphi^{(n)}(+\infty)\right], \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\int_0^t \rho(\tau_{12}(\boldsymbol{x})|z)\mathrm{d}x_1 &= \hbar(z, \boldsymbol{x}_{\{1,2\}})[r_1\psi'(G(x_2))g(x_2) + \psi'(F_1(x_2))f_1(x_2)] \\ &\quad \cdot \left[\varphi^{(n)}(\ell(z, x_2, t, \boldsymbol{x}_{\{1,2\}})) - \varphi^{(n)}(+\infty)\right]. \end{aligned} \tag{3.12}$$

As per Lemma 2.2, the fact that $X_1 \leq_{\mathrm{rh}} X_2$ along with the log-convex generator φ implies that

$$f_1(x)\psi'(F_1(x)) \geq f_2(x)\psi'(F_2(x)), \quad \text{for all } x \geq 0, \tag{3.13}$$

and hence

$$\begin{aligned} &- r_1 g(x_2)\psi'(G(x_2)) - f_2(x_2)\psi'(F_2(x_2)) \\ &\geq -r_1 g(x_2)\psi'(G(x_2)) - f_1(x_2)\psi'(F_1(x_2)) \\ &\geq 0, \qquad \text{for all } x_2 \geq 0. \end{aligned} \tag{3.14}$$

For $x_2 \geq t$, taking the integral over $[t, x_2]$ on both sides of (3.13), we have

$$\psi(F_1(x_2)) - \psi(F_1(t)) \geq \psi(F_2(x_2)) - \psi(F_2(t)),$$

and thus from (3.8) it follows that

$$\begin{aligned} &\ell(z, t, x_2, \boldsymbol{x}_{\{1,2\}}) - \ell(z, x_2, t, \boldsymbol{x}_{\{1,2\}}) \\ &= \psi(F_1(t)) + \psi(F_2(x_2)) - \psi(F_1(x_2)) - \psi(F_2(t)) \\ &\leq 0, \qquad \text{for any } x_2 \geq t. \end{aligned}$$

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For the $(n+1)$-monotone generator φ, $(-1)^n\varphi^{(n)}(x)$ is decreasing, and thus it holds that

$$\begin{aligned}&(-1)^n\big[\varphi^{(n)}(\ell(z, t, x_2, \boldsymbol{x}_{\{1,2\}}))-\varphi^{(n)}(+\infty)\big]\\ &\geq(-1)^n\big[\varphi^{(n)}(\ell(z, x_2, t, \boldsymbol{x}_{\{1,2\}}))-\varphi^{(n)}(+\infty)\big]\geq 0.\end{aligned}$$

Also, from (3.10) it follows that

$$\begin{aligned}&(-1)^{n-1}\hbar(z, \boldsymbol{x}_{\{1,2\}})\\ &=\frac{-1}{\varphi'((r_2-r_1)\psi(G(z)))}\prod_{i=3}^{n}[-r_1\psi'(G(x_i))g(x_i)-\psi'(F_i(x_i))f_i(x_i)]\\ &\geq 0, \qquad \text{for any } z \text{ and } \boldsymbol{x}_{\{1,2\}}.\end{aligned}$$

As a consequence, based on (3.11), (3.12), and (3.14), for any $t\leq x_2$ we have

$$\begin{aligned}\int_0^t\rho(\boldsymbol{x}|z)\mathrm{d}x_1&=(-1)^{n-1}\hbar(z, \boldsymbol{x}_{\{1,2\}})[-r_1\psi'(G(x_2))g(x_2)-\psi'(F_2(x_2))f_2(x_2)]\\ &\quad\cdot\big[(-1)^n\varphi^{(n)}(\ell(z, t, x_2, \boldsymbol{x}_{\{1,2\}}))-(-1)^n\varphi^{(n)}(+\infty)\big]\\ &\geq(-1)^{n-1}\hbar(z, \boldsymbol{x}_{\{1,2\}})[-r_1\psi'(G(x_2))g(x_2)-\psi'(F_1(x_2))f_1(x_2)]\\ &\quad\cdot\big[(-1)^n\varphi^{(n)}(\ell(z, x_2, t, \boldsymbol{x}_{\{1,2\}}))-(-1)^n\varphi^{(n)}(+\infty)\big]\\ &=\int_0^t\rho(\tau_{12}(\boldsymbol{x})|z)\mathrm{d}x_1.\end{aligned}$$

That is, $(X_1\vee Z_1, X_2\vee Z_2, \cdots, X_n\vee Z_n\mid Z=y)$ is LTPD with respect to (1, 2).

On the other hand, for any increasing function u, since, for $x_1\leq x_2$ and z, the difference

$$\begin{aligned}&u((x_1\vee z, x_2, \boldsymbol{x}_{\{1,2\}})_k)-u((x_1, x_2\vee z, \boldsymbol{x}_{\{1,2\}})_k)\\ &=\begin{cases}u((z, x_2, \boldsymbol{x}_{\{1,2\}})_k)-u((x_1, z, \boldsymbol{x}_{\{1,2\}})_k), & x_1\leq x_2\leq z,\\ u((z, x_2, \boldsymbol{x}_{\{1,2\}})_k)-u((x_1, x_2, \boldsymbol{x}_{\{1,2\}})_k), & x_1\leq z\leq x_2,\\ 0, & z\leq x_1\leq x_2,\end{cases}\end{aligned}$$

is nonnegative and decreasing in $x_1\in(-\infty, x_2]$, the function

$$[u((x_1\vee z, \boldsymbol{x}_{\{1,2\}})_k)-u((x_1, x_2\vee z, \boldsymbol{x}_{\{1,2\}})_k)]I(x_1\leq x_2)$$

is nonnegative and decreasing in x_1, irrespective of z. Then, as per Lemma 7.1(b) of Barlow and Proschan [2], we have, for any $x_2\geq 0$,

$$\int_0^{x_2}[u((x_1\vee z, \boldsymbol{x}_{\{1,2\}})_k)-u((x_1, x_2\vee z, \boldsymbol{x}_{\{1,2\}})_k)][\rho(\boldsymbol{x}|z)-\rho(\tau_{12}(\boldsymbol{x})|z)]\,\mathrm{d}x_1\geq 0.$$

As a result, it holds that

$$
\begin{aligned}
&\mathbb{E}[u((X_1 \vee Z_1 \vee z, X_2 \vee Z_2, \cdots, X_n \vee Z_n)_k \mid Z = z)] \\
&\quad - \mathbb{E}[u((X_1 \vee Z_1, X_2 \vee Z_2 \vee z, \cdots, X_n \vee Z_n)_k \mid Z = z)] \\
&= \underset{\mathbb{R}^n}{\int \cdots \int} [u((x_1 \vee z, \boldsymbol{x}_{\{1,2\}})_k) - u((x_1, x_2 \vee z, \boldsymbol{x}_{\{1,2\}})_k)]\rho(\boldsymbol{x}|z) \prod_{i=1}^{n} \mathrm{d}x_i \\
&= \underset{\mathbb{R}^{n-1}}{\int \cdots \int} \int_{x_1 \le x_2} [u((x_1 \vee z, \boldsymbol{x}_{\{1,2\}})_k) - u((x_1, x_2 \vee z, \boldsymbol{x}_{\{1,2\}})_k)]\rho(\boldsymbol{x}|z)\mathrm{d}x_1 \prod_{i=2}^{n} \mathrm{d}x_i \\
&\quad + \underset{\mathbb{R}^{n-1}}{\int \cdots \int} \int_{x_1 \ge x_2} [u((x_1 \vee z, \boldsymbol{x}_{\{1,2\}})_k) - u((x_1, x_2 \vee z, \boldsymbol{x}_{\{1,2\}})_k)]\rho(\boldsymbol{x}|z)\mathrm{d}x_1 \prod_{i=2}^{n} \mathrm{d}x_i \\
&= \underset{\mathbb{R}^{n-1}}{\int \cdots \int} \int_{x_1 \le x_2} [u((x_1 \vee z, \boldsymbol{x}_{\{1,2\}})_k) - u((x_1, x_2 \vee z, \boldsymbol{x}_{\{1,2\}})_k)]\rho(\boldsymbol{x}|z)\mathrm{d}x_1 \prod_{i=2}^{n} \mathrm{d}x_i \\
&\quad + \underset{\mathbb{R}^{n-1}}{\int \cdots \int} \int_{x_1 \le x_2} [u((x_2 \vee z, x_1, \boldsymbol{x}_{\{1,2\}})_k) - u((x_2, x_1 \vee z, \boldsymbol{x}_{\{1,2\}})_k)] \\
&\qquad\qquad \cdot \rho(\tau_{12}(\boldsymbol{x})|z)\mathrm{d}x_1 \prod_{i=2}^{n} \mathrm{d}x_i \\
&= \underset{\mathbb{R}^{n-1}}{\int \cdots \int} \int_0^{x_2} [u((x_1 \vee z, x_2, \boldsymbol{x}_{\{1,2\}})_k) - u((x_1, x_2 \vee z, \boldsymbol{x}_{\{1,2\}})_k)] \\
&\qquad\qquad \cdot [\rho(\boldsymbol{x}|z) - \rho(\tau_{12}(\boldsymbol{x})|z)]\mathrm{d}x_1 \prod_{i=2}^{n} \mathrm{d}x_i \\
&\ge 0, \qquad \text{for any } x_2 \ge 0.
\end{aligned}
$$

Therefore, by using double expectation we conclude that

$$
\begin{aligned}
&\mathbb{E}[u((X_1 \vee Z_1 \vee Z, X_2 \vee Z_2, \cdots, X_n \vee Z_n)_k)] \\
&\ge \mathbb{E}[u((X_1 \vee Z_1, X_2 \vee Z_2 \vee Z, \cdots, X_n \vee Z_n)_k)].
\end{aligned}
$$

Now, from (3.6) and (3.7) it follows immediately that

$$
\mathbb{E}\big[u\big(T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r})\big)\big] \ge \mathbb{E}\big[u\big(T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \tau_{12}(\boldsymbol{r}))\big)\big].
$$

Owing to the arbitrariness of the increasing u, this implies the desired (3.5). □

As is pointed out in You and Li [46], the log-convex generator of an Archimedean copula leads to positive dependence in the sense of the left tail decreasing in sequence. The reader may refer to Joe [20] and Colangelo *et al.* [12] for more details on this notion of positive dependence. Furthermore, as members of the Archimedean family, the independence copula, Clayton copula, Gumbel copula, and AMH copula (nonnegative parameter) all are known to have log-convex generator. According to the proof of Theorem 2.14 of Müller and Scarsini [33],

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the completely monotone generator of an Archimedean copula is always log-convex. Thus, we obtain the following corollary.

Corollary 3.1. *Suppose that the lifetimes* $(X_1, \cdots, X_n, Y_1, \cdots, Y_m)$ *are linked by one Archimedean copula with a completely monotone generator, and* $Y_1, \cdots, Y_m$ *are identically distributed. Then, for* $k = 1, \cdots, n$*, any* $\boldsymbol{r} \in \mathcal{A}_{n,m}$*, and (i,j) such that* $1 \le i < j \le n$*,*

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \le_{\rm st} T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \tau_{ij}(\boldsymbol{r}))$$

whenever $X_i \le_{\rm rh} X_j$ *and* $r_i \le r_j$*.*

In the context of a single active redundancy, i.e., $m = 1$, define

$$\mathbf{1}_i = (\underbrace{0, \cdots, 0}_{i-1}, 1, \underbrace{0, \cdots, 0}_{n-i}) \in \mathcal{A}_{n,1}, \qquad \text{for } i = 1, \cdots, n.$$

As a direct consequence of Theorem 3.1, we obtain the next corollary, which was originally developed by You and Li [48, Theorem 5].

Corollary 3.2. *Suppose that the component and redundancy lifetimes* $(X_1, \cdots, X_n, Y_1)$ *are linked by one Archimedean copula with a log-convex and* $(n+1)$*-monotone generator. If* $X_1 \le_{\rm rh} \cdots \le_{\rm rh} X_n$*, then*

$$T_{k:n}(\boldsymbol{X}, Y_1; \mathbf{1}_1) \ge_{\rm st} T_{k:n}(\boldsymbol{X}, Y_1; \mathbf{1}_2) \ge_{\rm st} \cdots\cdots \ge_{\rm st} T_{k:n}(\boldsymbol{X}, Y_1; \mathbf{1}_n), \tag{3.15}$$

for any $k = 1, \cdots, n$*.*

In fact, Belzunce *et al.* [4, Theorem 3.7] also derived the usual stochastic order of (3.15) under the assumption that, conditioned by the redundancy lifetime Y_1, the component lifetimes $(X_1, \cdots, X_n)$ are SAI. In according with Proposition 4.1 of Cai and Wei [11], $X_1 \le_{\rm rh} \cdots \le_{\rm rh} X_n$ implies that $(X_1, \cdots, X_n)$ is LWSAI if $(X_1, \cdots, X_n)$ is linked by the Archimedean copula associated with a completely monotone generator. Since a completely monotone generator is always log-convex, Corollary 3.2 successfully relaxes the assumption of SAI component lifetimes in Theorem 3.7 of Belzunce *et al.* [4] to the LWSAI assumption in the context of the Archimedean copula for the component and redundancy lifetimes.

Also, it is worth remarking that the log-convex generator gives rise to LTPD component lifetimes in the setting of Theorem 3.1, and this finally leads to the usual stochastic order on the redundant system lifetime. As a consequence, in the setting where component lifetimes and homogeneous redundancy lifetimes are linked by such an Archimedean copula, more redundancies should be allocated to the components with stochastically smaller lifetimes. Naturally, one may wonder whether it is possible to upgrade the usual stochastic order on the system lifetime to the hazard rate order. In what follows we present two numerical examples related to the first main result.

Example 3.1 below reveals that a dependence structure outside of the Archimedean family of copulas may jeopardize the conclusion of Theorem 3.1 if the other assumptions are unchanged; however, there may also exist some non-Archimedean copula such that the usual stochastic order in Theorem 3.1 still holds.

Example 3.1. (*Extended Gumbel copula.*) Consider component and redundancy lifetimes X_1, X_2, Y, all having the standard exponential distribution, such that $X_1 \le_{\rm rh} X_2$ is trivially true.

Assume for the vector (X_1, X_2, Y) the following 3-dimensional extended Gumbel copula due to Embrechts *et al.* [15, Example 6.13]:

$$C(u_1, u_2, u_3) = \exp\left\{ -\left[(-\ln u_1)^{\theta_1} + \left((-\ln u_2)^{\theta_2} + (-\ln u_3)^{\theta_2}\right)^{\frac{\theta_1}{\theta_2}}\right]^{\frac{1}{\theta_1}}\right\}.$$

Because $\theta_1 > \theta_2$, this copula is not symmetric and hence is not an Archimedean copula. The vector of lifetimes (X_1, X_2, Y) admits the CDF

$$L(x_1, x_2, y) = \exp\left\{ -\left[(-\ln(1-e^{-x_1}))^{\theta_1} + \left((-\ln(1-e^{-x_2}))^{\theta_2} + (-\ln(1-e^{-y}))^{\theta_2}\right)^{\frac{\theta_1}{\theta_2}}\right]^{\frac{1}{\theta_1}}\right\},$$

where $x_1, x_2, y \geq 0$ and $\theta_1 > \theta_2 > 0$.

Corresponding to the allocation policy $\boldsymbol{r} = (0, 1)$, the system lifetimes are

$$T_{1:2}(\boldsymbol{X}, Y; \boldsymbol{r}) = X_1 \wedge (X_2 \vee Y), \qquad T_{1:2}(\boldsymbol{X}, Y; \tau_{12}(\boldsymbol{r})) = (X_1 \vee Y) \wedge X_2.$$

It is not difficult to check that $T_{1:2}(\boldsymbol{X}, Y; \boldsymbol{r})$ and $T_{1:2}(\boldsymbol{X}, Y; \tau_{12}(\boldsymbol{r}))$ respectively have CDFs

$$\mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \boldsymbol{r}) \leq t) = \mathbb{P}(X_1 \leq t) + \mathbb{P}(Y \leq t, X_2 \leq t) - \mathbb{P}(X_1 \leq t, X_2 \leq t, Y \leq t),$$

$$\mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \tau_{12}(\boldsymbol{r})) \leq t) = \mathbb{P}(X_2 \leq t) + \mathbb{P}(X_1 \leq t, Y \leq t) - \mathbb{P}(X_1 \leq t, X_2 \leq t, Y \leq t),$$

for all $t \geq 0$. Thus, it follows that

$$\begin{aligned}
&\mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \boldsymbol{r}) \leq t) - \mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \tau_{12}(\boldsymbol{r})) \leq t)\\
&= \mathbb{P}(X_2 \leq t, Y \leq t) - \mathbb{P}(X_1 \leq t, Y \leq t)\\
&= L(\infty, t, t) - L(t, \infty, t)\\
&= \exp\left\{-2^{1/\theta_2}[-\ln(1-e^{-t})]\right\} - \exp\left\{-2^{1/\theta_1}[-\ln(1-e^{-t})]\right\}\\
&= \left(1-e^{-t}\right)^{2^{1/\theta_2}} - \left(1-e^{-t}\right)^{2^{1/\theta_1}}\\
&< 0, \qquad \text{for all } t \geq 0 \text{ and } \theta_1 > \theta_2.
\end{aligned}$$

This invalidates the conclusion $T_{1:2}(\boldsymbol{X}, Y; \boldsymbol{r}) \leq_{\text{st}} T_{1:2}(\boldsymbol{X}, Y; \tau_{12}(\boldsymbol{r}))$ claimed by Theorem 3.1.

Next, Example 3.2 illustrates that it is infeasible to upgrade the usual stochastic order of (3.5) to the hazard rate order in the context of Theorem 3.1.

Example 3.2. (*Clayton copula.*) Assume X_1, X_2, Y have exponential distributions with hazard rates 2,1,3, respectively, and (X_1, X_2, Y) is linked by a Clayton copula. Then (X_1, X_2, Y) admits the CDF

$$L(x_1, x_2, y) = \left[(1-e^{-2x_1})^{-\alpha} + (1-e^{-x_2})^{-\alpha} + (1-e^{-3y})^{-\alpha} - 2\right]^{-1/\alpha},$$

where $x_1, x_2, y \geq 0$ and $\alpha \in [-1, \infty)\backslash\{1\}$.

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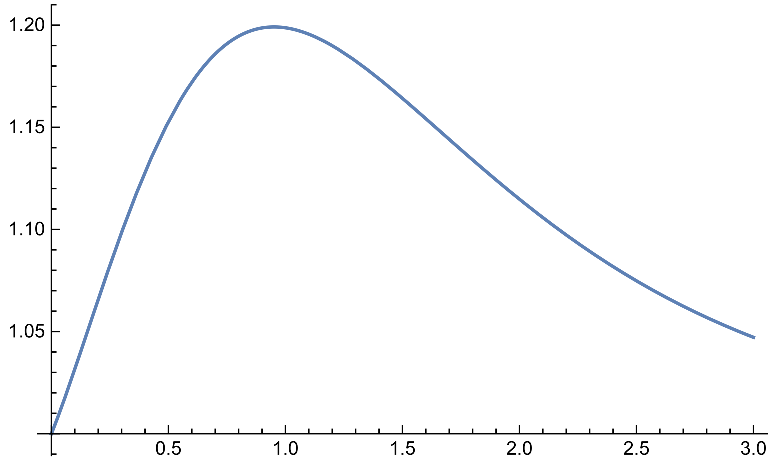

FIGURE 1. The curve of $\mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \tau_{12}(\boldsymbol{r})) > t)/\mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \boldsymbol{r}) > t)$ for $t \in (0, 3)$.

Let $\boldsymbol{r} = (0, 1)$; then it is easy to check that, for all $t \geq 0$,

$$\begin{aligned}
&\mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \tau_{12}(\boldsymbol{r})) > t)/\mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \boldsymbol{r}) > t)\\
&= \frac{\mathbb{P}(X_1 > t, X_2 > t) + \mathbb{P}(X_1 \leq t, X_2 > t, Y > t)}{\mathbb{P}(X_1 > t, X_2 > t) + \mathbb{P}(X_1 > t, X_2 \leq t, Y > t)}\\
&= 1 + \frac{\mathbb{P}(X_1 \leq t, X_2 > t, Y > t) - \mathbb{P}(X_1 > t, X_2 \leq t, Y > t)}{\mathbb{P}(X_1 > t, X_2 > t) + \mathbb{P}(X_1 > t, X_2 \leq t, Y > t)}\\
&= 1 + \frac{\mathbb{P}(X_2 > t, Y > t) - \mathbb{P}(X_1 > t, Y > t)}{\mathbb{P}(X_1 > t, X_2 > t) + \mathbb{P}(X_1 > t, X_2 \leq t, Y > t)}\\
&= 1 + \frac{L(t, \infty, \infty) - L(\infty, t, \infty) + L(\infty, t, t) - L(t, \infty, t)}{1 - L(t, \infty, \infty) - L(\infty, t, t) + L(t, t, t)}.
\end{aligned}$$

For $\alpha = 2$, as is clearly seen in Figure 1, the ratio

$$\mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \tau_{12}(\boldsymbol{r})) > t)/\mathbb{P}(T_{1:2}(\boldsymbol{X}, Y; \boldsymbol{r}) > t)$$

is not increasing with respect to $t \in (0, 3)$, and this fact directly negates the hazard rate order $T_{1:2}(\boldsymbol{X}, Y; \boldsymbol{r}) \leq_{\text{hr}} T_{1:2}(\boldsymbol{X}, Y; \tau_{12}(\boldsymbol{r}))$.

In industrial engineering, when taking care of extremely critical systems, reliability engineers sometimes seek the optimal way to allocate active redundancies in the sense of uniformly maximizing the resulted reliability function. In this context, an allocation policy with the corresponding system lifetime not stochastically lengthened is not feasible. Thus, an allocation policy $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*) \in \mathcal{A}_{n,m}$ is said to be stochastically optimal if

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}^*) \not\leq_{\text{st}} T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}), \qquad \text{for any } \boldsymbol{r} \in \mathcal{A}_{n,m}.$$

As a direct consequence of Theorem 3.1, we end up with a typical feature of the stochastically optimal allocation policy, which suggests that we should consider only the arrangement

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decreasing allocation policies; this helps narrow down the feasible set for the corresponding optimization problem.

Theorem 3.2. *Suppose the component and redundancy lifetimes* $(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ *are linked by one Archimedean copula with a log-convex and* $(n+1)$*-monotone generator. If* $X_1 \leq_{\rm rh} \cdots \leq_{\rm rh} X_n$ *and* $Y_1, \cdots, Y_m$ *are identically distributed, then the optimal allocation policy* $\boldsymbol{r}^* \in \mathcal{A}_{n,m}$ *is such that* $r_1^* \geq \cdots \geq r_n^*$*.*

4. On systems with homogeneous component lifetimes

In this section, we pay specific attention to the allocation of multiple redundancies to k-out-of-n:F systems with homogeneous and dependent component lifetimes. For the case of i.i.d. components and redundancies, Hu and Wang [19, Theorem 3.3] showed that the SF of the redundant k-out-of-n:F system is Schur-concave, i.e., if $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are i.i.d., then

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \leq_{\rm st} T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}), \qquad k = 1, \cdots, n,$$

for any $\boldsymbol{r}, \boldsymbol{s} \in \mathcal{A}_{n,m}$ such that $\boldsymbol{r} \prec_{\rm m} \boldsymbol{s}$. This confirms the intuition that for systems with symmetric structure, homogeneous components, and homogeneous redundancies, a more balanced allocation policy tends to produce a redundant system with a stochastically larger lifetime in the context of mutual independence of the component and redundancy lifetimes involved. Naturally, one conjectures that such an intuition is still true in the setting of symmetric statistical dependence among component and redundancy lifetimes.

Along these lines, we present the second main result on the redundant k-out-of-n:F system with component and redundancy lifetimes linked by an Archimedean copula, which partially verifies the above conjecture.

Theorem 4.1. *Suppose the component lifetimes* $(X_1, \ldots, X_n)$ *are identically distributed, the redundancy lifetimes* $(Y_1, \ldots, Y_m)$ *are identically distributed, and* $(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ *is linked by an Archimedean copula with* $(n+1)$*-monotone generator. Then*

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \leq_{\rm st} T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}), \qquad k = 1, \ldots, n, \tag{4.1}$$

for any $\boldsymbol{r}, \boldsymbol{s} \in \mathcal{A}_{n,m}$ *such that* $\boldsymbol{r} \prec_{\rm m} \boldsymbol{s}$*.*

Proof. For the transform M of (2.1), without loss of generality, let us say the permutation Q interchanges the ith and jth coordinates for $1 \leq i < j \leq n$. Then, for $\boldsymbol{x} = (x_1, \ldots, x_n)$,

$$M\boldsymbol{x}^T = (x_1, \ldots, x_{i-1}, \lambda x_i + (1-\lambda)x_j, x_{i+1}, \ldots, x_{j-1}, (1-\lambda)x_i + \lambda x_j, x_{j+1}, \ldots, x_n)^T,$$

and hence

$$x_i + x_j \equiv \lambda x_i + (1-\lambda)x_j + (1-\lambda)x_i + \lambda x_j.$$

Owing to Lemma 2.1 and the transitivity of the usual stochastic order, it suffices to show (4.1) only for $\boldsymbol{r}$ and $\boldsymbol{s}$ such that $(r_j, r_l) \prec_{\rm m} (s_j, s_l)$ for some $j < l$ and $r_i = s_i$ for $i \notin \{j, l\}$.

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For the sake of convenience, we set $j=1$ and $l=2$. Since the Archimedean copula is symmetric, and $\boldsymbol{X}$ and $\boldsymbol{Y}$ are each identically distributed, it holds that

$$\begin{aligned}
&T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \\
&\overset{st}{=} \left(X_1 \vee Y_1 \vee \cdots \vee Y_{r_1}, X_2 \vee Y_{r_1+1} \vee \cdots \vee Y_{r_1+r_2}, \cdots, X_n \vee Y_{\sum_{i=1}^{n-1} r_i+1} \vee \cdots \vee Y_m\right)_k \\
&\overset{st}{=} \left(X_2 \vee Y_1 \vee \cdots \vee Y_{r_1}, X_1 \vee Y_{r_1+1} \vee \cdots \vee Y_{r_1+r_2}, \cdots, X_n \vee Y_{\sum_{i=1}^{n-1} r_i+1} \vee \cdots \vee Y_m\right)_k \\
&\overset{st}{=} \left(X_1 \vee Y_{r_1+1} \vee \cdots \vee Y_{r_1+r_2}, X_2 \vee Y_1 \vee \cdots \vee Y_{r_1}, \cdots, X_n \vee Y_{\sum_{i=1}^{n-1} r_i+1} \vee \cdots \vee Y_m\right)_k \\
&\overset{st}{=} T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \tau_{12}(\boldsymbol{r})).
\end{aligned}$$

Thus, we focus only on (4.1) with $s_1 < r_1 \le r_2 \le s_2$, $s_1 + s_2 = r_1 + r_2$, and $r_i = s_i$, $i = 3, \ldots, n$. Define

$$\begin{aligned}
Z_1 &= \max(Y_1, \ldots, Y_{s_1}), \\
Z &= \max(Y_{s_1+1}, \ldots, Y_{r_1}), \\
Z_l &= \max\left(Y_{r_1+\cdots+r_{l-1}+1}, \ldots, Y_{r_1+\cdots+r_{l-1}+r_l}\right), \qquad l = 2, \ldots, n.
\end{aligned}$$

Then we have

$$\begin{aligned}
&T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \\
&\overset{st}{=} \left(X_1 \vee Y_1 \vee \cdots \vee Y_{r_1}, \cdots, X_n \vee Y_{r_1+\cdots+r_{n-1}+1} \vee \cdots \vee Y_m\right)_k \\
&= \left(X_1 \vee Z_1 \vee Z, X_2 \vee Z_2, X_3 \vee Z_3, \ldots, X_n \vee Z_n\right)_k,
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \\
&\overset{st}{=} (X_1 \vee Y_1 \vee \cdots \vee Y_{s_1}, \cdots, X_n \vee Y_{r_1+\cdots+r_{n-1}+1} \vee \cdots \vee Y_m)_k \\
&= (X_1 \vee Z_1, X_2 \vee Z_2 \vee Z, X_3 \vee Z_3, \ldots, X_n \vee Z_n)_k.
\end{aligned} \tag{4.3}$$

Let φ be the generator of the Archimedean copula of $(\boldsymbol{X}, \boldsymbol{Y})$, let F and f respectively be the common univariate marginal CDF and PDF of $\boldsymbol{X}$, and let G and g respectively be the common univariate marginal CDF and PDF of $\boldsymbol{Y}$. Define

$$\begin{aligned}
\ell(z, t, x_2, \boldsymbol{x}_{\{1,2\}}) = (r_1 - s_1)\psi(G(z)) + \psi(F(t)) + s_1\psi(G(t)) \\
+ \sum_{i=2}^{n} \left[\psi(F(x_i)) + r_i\psi(G(x_i))\right],
\end{aligned} \tag{4.4}$$

and

$$\hbar(z, \boldsymbol{x}_{\{1,2\}}) = \frac{1}{\varphi'((r_1 - s_1)\psi(G(z)))} \prod_{i=3}^{n} \left[\psi'(F(x_i))f(x_i) + r_i\psi'(G(x_i))g(x_i)\right]. \tag{4.5}$$

Then $\left(Z, X_1 \vee Z_1, X_2 \vee Z_2, \ldots, X_n \vee Z_n\right)$ attains the CDF

$$H(z, \boldsymbol{x}) = \varphi(\ell(z, \boldsymbol{x}))$$

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and hence the PDF

$$\begin{aligned} &h(z, \boldsymbol{x}) \\ &= \varphi^{(n+1)}(\ell(z, \boldsymbol{x}))(r_1 - s_1)\psi'(G(z))g(z)\big[\psi'(F(x_1))f(x_1) + s_1\psi'(G(x_1))g(x_1)\big] \\ &\cdot \prod_{i=2}^{n} \big[\psi'(F(x_i))f(x_i) + r_i\psi'(G(x_i))g(x_i)\big]. \end{aligned}$$

Also, since Z has PDF

$$l(z) = (r_1 - s_1)g(z)\psi'(G(z))\varphi'((r_1 - s_1)\psi(G(z))),$$

the vector $\big(X_1 \vee Z_1, X_2 \vee Z_2, \cdots, X_n \vee Z_n \,|\, Z = z\big)$ attains the PDF

$$\begin{aligned} \rho(\boldsymbol{x}|z) &= \frac{h(z, \boldsymbol{x})}{l(z)} \\ &= \big[\psi'(F(x_1))f(x_1) + s_1\psi'(G(x_1))g(x_1)\big]\big[\psi'(F(x_2))f(x_2) + r_2\psi'(G(x_2))g(x_2)\big] \\ &\quad \cdot \varphi^{(n+1)}(\ell(z, \boldsymbol{x}))\hbar(z, \boldsymbol{x}_{\{1,2\}}). \end{aligned} \tag{4.6}$$

Consequently, for any $x_2 \geq t \geq 0$,

$$\begin{aligned} \int_0^t \rho(\boldsymbol{x}|z)\mathrm{d}x_1 &= \big[\psi'(F(x_2))f(x_2) + r_2\psi'(G(x_2))g(x_2)\big]\hbar(z, \boldsymbol{x}_{\{1,2\}}) \\ &\quad \cdot \int_0^t \varphi^{(n+1)}(\ell(z, \boldsymbol{x}))\mathrm{d}[\psi(F(x_1)) + s_1\psi(G(x_1))g(x_1)\big)] \\ &= \big(\psi'(F(x_2))f(x_2) + r_2\psi'(G(x_2))g(x_2)\big)\hbar(z, \boldsymbol{x}_{\{1,2\}}) \\ &\quad \cdot \big[\varphi^{(n)}(\ell(z, t, x_2, \boldsymbol{x}_{\{1,2\}})) - \varphi^{(n)}(+\infty)\big] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \int_0^t \rho(\tau_{12}(\boldsymbol{x})|z)\mathrm{d}x_1 &= \big[\psi'(F(x_2))f(x_2) + s_1\psi'(G(x_2))g(x_2)\big]\hbar(z, \boldsymbol{x}_{\{1,2\}}) \\ &\quad \cdot \big[\varphi^{(n)}(\ell(z, x_2, t, \boldsymbol{x}_{\{1,2\}})) - \varphi^{(n)}(+\infty)\big]. \end{aligned} \tag{4.8}$$

Since G is increasing and ψ is decreasing, it holds that

$$(r_2 - s_1)\psi(G(x_2)) \leq (r_2 - s_1)\psi(G(t)), \qquad \text{for } r_2 \geq s_1 \text{ and } x_2 \geq t,$$

and then by (4.4) we have

$$\ell(z, t, x_2, \boldsymbol{x}_{\{1,2\}}) - \ell(z, x_2, t, \boldsymbol{x}_{\{1,2\}}) = (r_2 - s_1)\psi(G(x_2)) - (r_2 - s_1)\psi(G(t)) \leq 0.$$

As a result, the decreasing property of $(-1)^n\varphi^{(n)}(x)$ implies that

$$\begin{aligned} &(-1)^n\varphi^{(n)}(\ell(z, t, x_2, \boldsymbol{x}_{\{1,2\}})) - (-1)^n\varphi^{(n)}(+\infty) \\ &\geq (-1)^n\varphi^{(n)}(\ell(z, x_2, t, \boldsymbol{x}_{\{1,2\}})) - (-1)^n\varphi^{(n)}(+\infty) \\ &\geq 0, \qquad \text{for } x_2 \geq t. \end{aligned}$$

Also, $(-1)^{n-1}\hbar(z, \boldsymbol{x}_{\{1,2\}}) \geq 0$ and $-\psi'(x) \geq 0$ imply that

$$\begin{aligned}
&\big[-\psi'(F(x_2))f(x_2) - r_2\psi'(G(x_2))g(x_2)\big](-1)^{n-1}\hbar\big(z, \boldsymbol{x}_{\{1,2\}}\big) \\
\geq &\big[-\psi'(F(x_2))f(x_2) - s_1\psi'(G(x_2))g(x_2)\big](-1)^{n-1}\hbar\big(z, \boldsymbol{x}_{\{1,2\}}\big) \\
\geq &\,0, \qquad \text{for } r_2 \geq s_1.
\end{aligned}$$

Therefore, from (4.7) and (4.8) it follows immediately that, for any $t \leq x_2$,

$$\begin{aligned}
\int_0^t \rho(\boldsymbol{x}|z)\mathrm{d}x_1 &= \big[-\psi'(F(x_2))f(x_2) - r_2\psi'(G(x_2))g(x_2)\big](-1)^{n-1}\hbar(z, \boldsymbol{x}_{\{1,2\}}) \\
&\quad \cdot \big[(-1)^n\varphi^{(n)}(\ell(z, t, x_2, \boldsymbol{x}_{\{1,2\}})) - (-1)^n\varphi^{(n)}(+\infty)\big] \\
&\geq \big[-\psi'(F(x_2))f(x_2) - s_1\psi'(G(x_2))g(x_2)\big](-1)^{n-1}\hbar(z, \boldsymbol{x}_{\{1,2\}}) \\
&\quad \cdot \big[(-1)^n\varphi^{(n)}(\ell(z, x_2, t, \boldsymbol{x}_{\{1,2\}})) - (-1)^n\varphi^{(n)}(+\infty)\big] \\
&= \int_0^t \rho(\tau_{12}(\boldsymbol{x})|z)\mathrm{d}x_1.
\end{aligned}$$

That is, $(X_1 \vee Z_1, X_2 \vee Z_2, \cdots, X_n \vee Z_n \mid Z = z)$ is LTPD with respect to (1, 2).

Now, completely similarly to the preceding proof of Theorem 3.1, we can show that

$$\begin{aligned}
&\mathbb{E}\big[u((X_1 \vee Z_1 \vee Z, X_2 \vee Z_2, \cdots, X_n \vee Z_n)_k)\big] \\
\geq &\,\mathbb{E}\big[u((X_1 \vee Z_1, X_2 \vee Z_2 \vee Z, \cdots, X_n \vee Z_n)_k)\big].
\end{aligned}$$

In light of (4.2) and (4.3), we reach

$$\mathbb{E}\big[u\big(T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r})\big)\big] \geq \mathbb{E}\big[u\big(T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s})\big)\big].$$

Owing to the arbitrariness of u, this implies the usual stochastic order of (4.1). □

In accordance with Theorem 4.1, by balancing the allocation of active redundancies, one can stochastically maximize the lifetime of the redundant k-out-of-n:F system with identically distributed component lifetimes and identically distributed redundancy lifetimes having some Archimedean copula. Corresponding to the independence copula, the generator $\varphi(x) = e^{-x}$ is completely monotone, and then Theorem 4.1 successfully generalizes Theorem 3.3 of Hu and Wang [19] by equipping component and redundancy lifetimes with the Archimedean copula with an $(n+1)$-monotone generator. Also, it is worth remarking that the generator of the Archimedean copula is required to be log-convex in the setting of Theorem 3.1. By contrast, here this assumption is no longer required, because of the homogeneity of the system components, and thus the component and redundancy lifetimes can be either positive or negative dependent.

In what follows, using one numerical example, we point out that the usual stochastic order on the redundant system lifetime cannot be upgraded to the hazard rate order in the context of Theorem 4.1.

Example 4.1. (*Clayton copula.*) Assume that $(X_1, X_2, Y_1, Y_2, Y_3, Y_4)$ is linked by the Clayton copula with parameter α, that X_1, X_2 are of common univariate Pareto distribution with

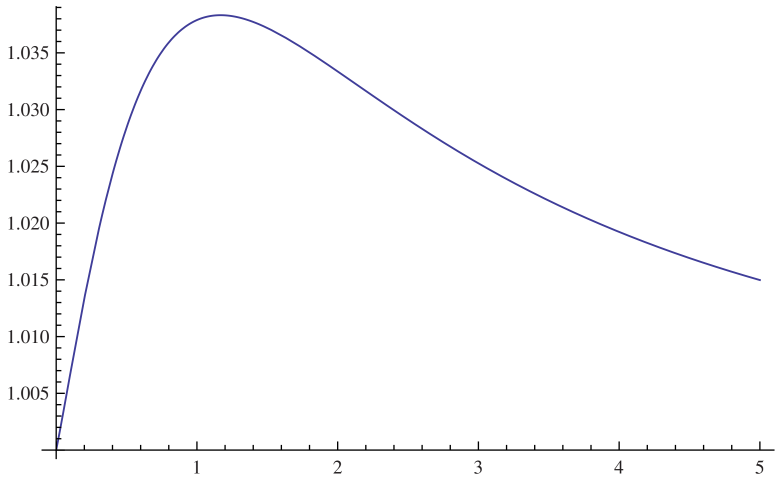

FIGURE 2. The curve of $\mathbb{P}(T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) > t)/\mathbb{P}(T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) > t)$ for $t \in (0, 5)$.

parameter β, and that Y_1, Y_2, Y_3, Y_4 are of common univariate Pareto distribution with parameter γ. Then $(X_1, X_2, Y_1, Y_2, Y_3, Y_4)$ attains the CDF

$$L(x_1, x_2, y_1, y_2, y_3, y_4) = \left[\left(1 - \frac{1}{(1+x_1)^\beta}\right)^{-\alpha} + \left(1 - \frac{1}{(1+x_2)^\beta}\right)^{-\alpha} + \sum_{i=1}^{4}\left(1 - \frac{1}{(1+y_i)^\gamma}\right)^{-\alpha} - 5\right]^{-\frac{1}{\alpha}},$$

where $x_i > 0$, $y_j > 0$, $i = 1, 2$, $j = 1, 2, 3, 4$, and $\alpha, \beta, \gamma > 0$.

Set $\boldsymbol{r} = (2, 2)$ and $\boldsymbol{s} = (1, 3)$. Obviously, $\boldsymbol{r} \prec_{\mathrm{m}} \boldsymbol{s}$,

$$T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) = (X_1 \vee Y_1 \vee Y_2) \wedge (X_2 \vee Y_3 \vee Y_4),$$

and

$$T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) = (X_1 \vee Y_1) \wedge (X_2 \vee Y_2 \vee Y_3 \vee Y_4).$$

It is routine to check that, for all $t \geq 0$,

$$1 - \mathbb{P}(T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) > t) = L(t, \infty, t, t, \infty, \infty) + L(\infty, t, \infty, \infty, t, t) - L(t, t, t, t, t, t)$$

and

$$1 - \mathbb{P}(T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) > t) = L(\infty, t, \infty, t, t, t) + L(t, \infty, t, \infty, \infty, \infty) - L(t, t, t, t, t, t).$$

For $(\alpha, \beta, \gamma) = (2, 2, 3)$, the ratio

$$\mathbb{P}(T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) > t)/\mathbb{P}(T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) > t)$$

is seen to be non-monotone with respect to $t \in (0, 5)$ in Figure 2, and this gives rise to $T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \not\leq_{\mathrm{hr}} T_{1:2}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r})$ although $\boldsymbol{r} \prec_{\mathrm{m}} \boldsymbol{s}$.

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A system $T(\boldsymbol{X})$ with component lifetimes $\boldsymbol{X} = (X_1, \ldots, X_n)$ is said to be a *mixed* system if the reliability function can be represented as

$$\mathbb{P}(T(\boldsymbol{X}) > t) = \sum_{k=1}^{n} p_k \mathbb{P}(X_{k,n} > t),$$

for all $t \geq 0$ and some $p_k \in [0, 1]$, $k = 1, \cdots, n$, such that $p_1 + \cdots + p_n = 1$. For example, the lifetime of a coherent system with exchangeable component lifetimes can be represented as a finite mixture of k-out-of-n:F system lifetimes with $(p_1, \cdots, p_n)$ being the system signature (see Navarro *et al.* [35]), and thus it is a mixed system. For such systems and generalized mixed systems we refer the reader to Navarro [34] and some of the references therein.

Corresponding to the allocation policy $\boldsymbol{r}$, denote by $T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r})$ the lifetime of a redundant mixed system with component lifetimes $\boldsymbol{X}$ and redundancy lifetimes $\boldsymbol{Y}$, i.e.,

$$\mathbb{P}(T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) > t) = \sum_{k=1}^{n} p_k \mathbb{P}\big(T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) > t\big), \qquad \text{for all } t \geq 0.$$

Since the usual stochastic order is closed under taking the mixture (see Shaked and Shanthikumar [41, Theorem 1.A.3.(d)]), the ordering result of Theorem 4.1 can be extended from the k-out-of-n structure to the mixed structure. Here we present such an extension, omitting the technical proof.

Theorem 4.2. *Suppose for a mixed system that the component lifetimes $X_1, \cdots, X_n$ are identically distributed, the redundancy lifetimes $Y_1, \cdots, Y_m$ are identically distributed, and the vector $(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ are linked by one Archimedean copula with an $(n+1)$-monotone generator. Then*

$$T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \leq_{\rm st} T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}),$$

for any $\boldsymbol{r}, \boldsymbol{s} \in \mathcal{A}_{n,m}$ such that $\boldsymbol{r} \prec_{\rm m} \boldsymbol{s}$.

To close this section, we employ one example to illustrate Theorem 4.3 in the setting of a coherent system without exchangeable component lifetimes.

Example 4.2. (*Nonexchangeable component lifetimes.*) For component lifetimes $\boldsymbol{X} = (X_1, X_2, X_3)$ and redundancy lifetimes $\boldsymbol{Y} = (Y_1, Y_2)$, let us consider the system with structure

$$T(\boldsymbol{X}) = \max\big\{X_1, \min\{X_2, X_3\}\big\}.$$

Let the allocation policy $\boldsymbol{s}$ assign Y_1, Y_2 to X_1, X_2 respectively, and let the allocation policy $\boldsymbol{r}$ assign Y_1, Y_2 to $X_1, X_2 \wedge X_3$ respectively. Then it is routine to derive the following equations:

$$\begin{aligned}
&\mathbb{P}\big(T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \leq t\big) \\
&= \mathbb{P}\big(X_1 \vee Y_1 \leq t, (X_2 \vee Y_2) \wedge X_3) \leq t\big) \\
&= \mathbb{P}\big(X_1 \vee Y_1 \leq t, X_3 \leq t\big) + \mathbb{P}\big(X_1 \vee Y_1 \leq t, (X_2 \vee Y_2) \leq t\big) \\
&\quad - \mathbb{P}\big(X_1 \vee Y_1 \leq t, X_3 \leq t, (X_2 \vee Y_2) \leq t\big) \\
&= \mathbb{P}(X_1 \leq t, Y_1 \leq t, X_3 \leq t) + \mathbb{P}(X_1 \leq t, Y_1 \leq t, Y_2 \leq t, X_2 \leq t) \\
&\quad - \mathbb{P}(X_1 \leq t, X_2 \leq t, X_3 \leq t, Y_1 \leq t, Y_2 \leq t), \quad \text{for all } t \geq 0,
\end{aligned}$$

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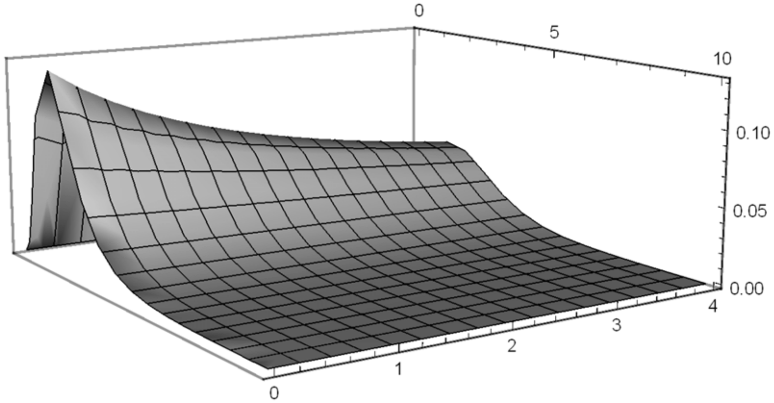

FIGURE 3. The difference $\mathbb{P}(T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \le t) - \mathbb{P}(T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \le t)$: $\alpha \in (0, 4)$, $t \in (0, 10)$.

and

$$\begin{aligned}
&\mathbb{P}\big(T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \le t\big) \\
&= \mathbb{P}\big(X_1 \vee Y_1 \le t, (X_2 \wedge X_3) \vee Y_2 \le t\big) \\
&= \mathbb{P}\big(X_1 \le t, Y_1 \le t, Y_2 \le t, (X_2 \wedge X_3) \le t\big) \\
&= \mathbb{P}(X_1 \le t, X_2 \le t, Y_1 \le t, Y_2 \le t) + \mathbb{P}(X_1 \le t, Y_1 \le t, X_3 \le t, Y_2 \le t) \\
&\quad - \mathbb{P}(X_1 \le t, X_2 \le t, X_3 \le t, Y_1 \le t, Y_2 \le t), \quad \text{for all } t \ge 0.
\end{aligned}$$

Evidently, it holds that

$$\begin{aligned}
&\mathbb{P}\big(T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \le t\big) - \mathbb{P}\big(T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \le t\big) \\
&= \mathbb{P}(X_1 \le t, Y_1 \le t, X_3 \le t) - \mathbb{P}(X_1 \le t, Y_1 \le t, X_3 \le t, Y_2 \le t) \\
&\ge 0, \qquad \text{for all } t \ge 0.
\end{aligned}$$

Assume that $X_1 \sim \mathcal{E}(2)$, $X_i \sim \mathcal{E}(1)$, $Y_i \sim \mathcal{E}(1)$, $i = 1, 2$, and the vector of lifetimes $(X_1, X_2, X_3, Y_1, Y_2)$ is coupled by the Clayton copula with parameter α. Based on the above two equations, one can obtain closed-form expressions for the two CDFs.

As is seen in Figure 3, the nonnegative difference

$$\mathbb{P}\big(T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \le t\big) - \mathbb{P}\big(T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r}) \le t\big)$$

confirms that $T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \le_{\text{st}} T(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r})$. In addition, as a parallel system with two component lifetimes X_1 and $X_2 \wedge X_3$, all active allocation policies for (Y_1, Y_2) produce the same redundant systems.

5. On systems with matched active redundancies

Consider component lifetimes $\boldsymbol{X} = (X_1, \cdots, X_n)$ and matched redundancy lifetimes $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$. Define the redundant k-out-of-n:F system lifetime as

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}) = \big(X_1 \vee Y_1, \cdots\cdots, X_n \vee Y_n\big)_k, \qquad k = 1, \cdots, n.$$

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In this section we consider multivariate mixture models; i.e., $(\boldsymbol{X}, \boldsymbol{Y})$ admits the CDF

$$F(\boldsymbol{x}, \boldsymbol{y}) = \underset{\mathbb{R}^{2n}}{\int \cdots \int} \prod_{i=1}^{n} F_i(x_i, \boldsymbol{\theta}) G_i(y_i, \boldsymbol{\theta}) \mathrm{d}F_{\boldsymbol{\Theta}}(\boldsymbol{\theta}), \tag{5.1}$$

where $F_{\boldsymbol{\Theta}}$ is the distribution function of a $2n$-dimensional random vector $\boldsymbol{\Theta} = (\Theta_1, \cdots, \Theta_{2n})$, while $F_i(x_i, \boldsymbol{\theta})$ and $G_i(y_i, \boldsymbol{\theta})$ are univariate CDFs. For more on stochastic comparison of multivariate mixture models, we refer the reader to Belzunce *et al.* [5] and references therein.

For $i = 1, \cdots, n$, let $X_i(\boldsymbol{\theta}) = [X_i | \boldsymbol{\Theta} = \boldsymbol{\theta}]$ and $Y_i(\boldsymbol{\theta}) = [Y_i | \boldsymbol{\Theta} = \boldsymbol{\theta}]$ be absolutely continuous random variables with CDFs $F_i(x, \boldsymbol{\theta})$ and $G_i(y, \boldsymbol{\theta})$ and PDFs $f_i(x, \boldsymbol{\theta})$ and $g_i(y, \boldsymbol{\theta})$, respectively. In the present result, we investigate conditions on the CDFs $F_i(x, \boldsymbol{\theta})$ and $G_i(y, \boldsymbol{\theta})$, $i = 1, \cdots, n$, such that the system lifetimes $T_{k:n}(\boldsymbol{X}, \boldsymbol{Y})$ and $T_{k:n}(\boldsymbol{X}, \tau_{ij}(\boldsymbol{Y}))$ are of usual stochastic order.

Theorem 5.1. *Consider the multivariate mixture model of (5.1) for component and matched redundancy lifetimes $(\boldsymbol{X}, \boldsymbol{Y})$. For (i,j) such that $1 \le i < j \le n$,*

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}) \le_{\mathrm{st}} T_{k:n}(\boldsymbol{X}, \tau_{ij}(\boldsymbol{Y})), \qquad k = 1, \cdots, n, \tag{5.2}$$

whenever $X_i(\boldsymbol{\theta}) \le_{\mathrm{rh}} X_j(\boldsymbol{\theta})$ and $Y_i(\boldsymbol{\theta}) \le_{\mathrm{lr}} Y_j(\boldsymbol{\theta})$.

Proof. Taking the derivative on both sides of (5.1), we get the PDF of $(\boldsymbol{X}, \boldsymbol{Y})$ as

$$f(\boldsymbol{x}, \boldsymbol{y}) = \underset{\mathbb{R}^{2n}}{\int \cdots \int} \prod_{i=1}^{n} f_i(x_i, \boldsymbol{\theta}) g_i(y_i, \boldsymbol{\theta}) \mathrm{d}F_{\boldsymbol{\Theta}}(\boldsymbol{\theta}). \tag{5.3}$$

Without loss of generality, let us set $i = 1$ and $j = 2$. Define $\boldsymbol{x} \vee \boldsymbol{y} = (x_1 \vee y_1, \cdots, x_n \vee y_n)$ for $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (x_1, \cdots, y_n)$. Then

$$(\boldsymbol{x} \vee \boldsymbol{y})_k = (\tau_{12}(\boldsymbol{x}), \tau_{12}(\boldsymbol{y}))_k, \qquad (\tau_{12}(\boldsymbol{x}), \boldsymbol{y})_k = (\boldsymbol{x}, \tau_{12}(\boldsymbol{y}))_k, \qquad k = 1, \cdots, n.$$

It is easy to verify that for $x_2, y_2 \ge 0$ and any increasing function u,

$$\begin{aligned}
&\mathbb{E}[u(T_{k:n}(\boldsymbol{X}, \tau_{12}(\boldsymbol{Y})))] - \mathbb{E}[u(T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}))] \\
&= \underset{\mathbb{R}^{2n-2}}{\int \cdots \int} \underset{x_1 \le x_2}{\iint} \big[u((\boldsymbol{x}, \tau_{12}(\boldsymbol{y}))_k) - u((\boldsymbol{x}, \boldsymbol{y})_k)\big]\big[f(\boldsymbol{x}, \boldsymbol{y}) - f(\tau_{12}(\boldsymbol{x}), \boldsymbol{y})\big] \prod_{i=1}^{n} \mathrm{d}x_i \mathrm{d}y_i \\
&= \underset{\mathbb{R}^{2n-4}}{\int \cdots \int} \underset{y_1 \le y_2}{\iint} \underset{x_1 \le x_2}{\iint} \big[u((\boldsymbol{x}, \tau_{12}(\boldsymbol{y}))_k) - u((\boldsymbol{x}, \boldsymbol{y})_k)\big]\big[f(\boldsymbol{x}, \boldsymbol{y}) - f(\tau_{12}(\boldsymbol{x}), \boldsymbol{y})\big] \prod_{i=1}^{n} \mathrm{d}x_i \mathrm{d}y_i \\
&\quad + \underset{\mathbb{R}^{2n-4}}{\int \cdots \int} \underset{y_1 \le y_2}{\iint} \underset{x_1 \le x_2}{\iint} \big[u((\boldsymbol{x}, \boldsymbol{y})_k) - u((\boldsymbol{x}, \tau_{12}(\boldsymbol{y}))_k)\big]\big[f(\boldsymbol{x}, \tau_{12}(\boldsymbol{y})) \\
&\qquad\qquad - f(\tau_{12}(\boldsymbol{x}), \tau_{12}(\boldsymbol{y}))\big] \prod_{i=1}^{n} \mathrm{d}x_i \mathrm{d}y_i
\end{aligned}$$

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$$= \int\cdots\int_{\mathbb{R}^{2n-4}} \iint_{y_1\le y_2} \iint_{x_1\le x_2} \big[u((\boldsymbol{x},\boldsymbol{y})_k) - u((\boldsymbol{x},\tau_{12}(\boldsymbol{y}))_k)\big]\cdot\big[f(\boldsymbol{x},\boldsymbol{y}) - f(\tau_{12}(\boldsymbol{x}),\boldsymbol{y})$$

$$- f(\boldsymbol{x},\tau_{12}(\boldsymbol{y})) + f(\tau_{12}(\boldsymbol{x}),\tau_{12}(\boldsymbol{y}))\big]\prod_{i=1}^{n}\mathrm{d}x_i\mathrm{d}y_i$$

$$= \int\cdots\int_{\mathbb{R}^{2n-2}} \int_0^{y_2}\int_0^{x_2} \big[u((\boldsymbol{x},\tau_{12}(\boldsymbol{y}))_k) - u((\boldsymbol{x},\boldsymbol{y})_k)\big]\cdot\big[f(\boldsymbol{x},\boldsymbol{y}) - f(\tau_{12}(\boldsymbol{x}),\boldsymbol{y})$$

$$- f(\boldsymbol{x},\tau_{12}(\boldsymbol{y})) + f(\tau_{12}(\boldsymbol{x}),\tau_{12}(\boldsymbol{y}))\big]\mathrm{d}x_1\mathrm{d}y_1\prod_{i=2}^{n}\mathrm{d}x_i\mathrm{d}y_i. \tag{5.4}$$

By (5.3), we have, for $x_2 \ge 0$,

$$\int_0^{x_2} f(\boldsymbol{x},\boldsymbol{y})\mathrm{d}x_1 = \int\cdots\int_{\mathbb{R}^{2n}}\int_0^{x_2}\prod_{i=1}^{n} f_i(x_i,\boldsymbol{\theta})g_i(y_i,\boldsymbol{\theta})\mathrm{d}x_1\mathrm{d}F_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$$

$$= \int\cdots\int_{\mathbb{R}^{2n}} F_1(x_2,\boldsymbol{\theta})\prod_{i=2}^{n} f_i(x_i,\boldsymbol{\theta})\prod_{i=1}^{n} g_i(y_i,\boldsymbol{\theta})\mathrm{d}F_{\boldsymbol{\Theta}}(\boldsymbol{\theta}),$$

$$\int_0^{x_2} f(\tau_{12}(\boldsymbol{x}),\boldsymbol{y})\mathrm{d}x_1 = \int\cdots\int_{\mathbb{R}^{2n}} F_2(x_2,\boldsymbol{\theta})f_1(x_2,\boldsymbol{\theta})\prod_{i=2}^{n} f_i(x_i,\boldsymbol{\theta})\prod_{i=1}^{n} g_i(y_i,\boldsymbol{\theta})\mathrm{d}F_{\boldsymbol{\Theta}}(\boldsymbol{\theta}),$$

$$\int_0^{x_2} f(\boldsymbol{x},\tau_{12}(\boldsymbol{y}))\mathrm{d}x_1$$

$$= \int\cdots\int_{\mathbb{R}^{2n}} F_1(x_2,\boldsymbol{\theta})g_1(y_2,\boldsymbol{\theta})g_2(y_1,\boldsymbol{\theta})\prod_{i=2}^{n} f_i(x_i,\boldsymbol{\theta})\prod_{i=3}^{n} g_i(y_i,\boldsymbol{\theta})\mathrm{d}F_{\boldsymbol{\Theta}}(\boldsymbol{\theta}),$$

$$\int_0^{x_2} f(\tau_{12}(\boldsymbol{x}),\tau_{12}(\boldsymbol{y}))\mathrm{d}x_1$$

$$= \int\cdots\int_{\mathbb{R}^{2n}} F_2(x_2,\boldsymbol{\theta})f_1(x_2,\boldsymbol{\theta})g_1(y_2,\boldsymbol{\theta})g_2(y_1,\boldsymbol{\theta})\prod_{i=3}^{n} f_i(x_i,\boldsymbol{\theta})\prod_{i=3}^{n} g_i(y_i,\boldsymbol{\theta})\mathrm{d}F_{\boldsymbol{\Theta}}(\boldsymbol{\theta}).$$

Thus, it holds that

$$\int_0^{x_2} [f(\boldsymbol{x},\boldsymbol{y}) - f(\tau_{12}(\boldsymbol{x}),\boldsymbol{y}) - f(\boldsymbol{x},\tau_{12}(\boldsymbol{y})) + f(\tau_{12}(\boldsymbol{x}),\tau_{12}(\boldsymbol{y}))]\mathrm{d}x_1$$

$$= \int_{\mathbb{R}^{2n}} \left[\frac{f_2(x_2,\boldsymbol{\theta})}{F_2(x_2,\boldsymbol{\theta})} - \frac{f_1(x_2,\boldsymbol{\theta})}{F_1(x_2,\boldsymbol{\theta})}\right]\left[\frac{g_2(y_2,\boldsymbol{\theta})}{g_1(y_2,\boldsymbol{\theta})} - \frac{g_2(y_1,\boldsymbol{\theta})}{g_1(y_1,\boldsymbol{\theta})}\right]$$

$$\cdot F_2(x_2,\boldsymbol{\theta})F_1(x_2,\boldsymbol{\theta})g_1(y_1,\boldsymbol{\theta})g_1(y_2,\boldsymbol{\theta})\prod_{i=3}^{n} f_i(x_i,\boldsymbol{\theta})g_i(y_i,\boldsymbol{\theta})\mathrm{d}F_{\boldsymbol{\Theta}}(\boldsymbol{\theta}). \tag{5.5}$$

Since $X_1(\boldsymbol{\theta}) \le_{\mathrm{rh}} X_2(\boldsymbol{\theta})$ implies

$$\frac{f_2(x_2,\boldsymbol{\theta})}{F_2(x_2,\boldsymbol{\theta})} \ge \frac{f_1(x_2,\boldsymbol{\theta})}{F_1(x_2,\boldsymbol{\theta})},$$

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and $Y_1(\boldsymbol{\theta}) \leq_{\rm lr} Y_2(\boldsymbol{\theta})$ implies

$$\frac{g_2(y_2, \boldsymbol{\theta})}{g_1(y_2, \boldsymbol{\theta})} \geq \frac{g_2(y_1, \boldsymbol{\theta})}{g_1(y_1, \boldsymbol{\theta})}, \qquad \text{for } y_1 \leq y_2,$$

from (5.5) it follows that

$$\int_0^{x_2} [f(\boldsymbol{x}, \boldsymbol{y}) - f(\tau_{12}(\boldsymbol{x}), \boldsymbol{y}) - f(\boldsymbol{x}, \tau_{12}(\boldsymbol{y})) + f(\tau_{12}(\boldsymbol{x}), \tau_{12}(\boldsymbol{y}))]\mathrm{d}x_1 \geq 0.$$

On the other hand, according to the proof of Theorem 3.4 of You *et al.* [49], we have that for $y_1 \leq y_2$, the function

$$u((\boldsymbol{x}, \tau_{12}(\boldsymbol{y}))_k) - u((\boldsymbol{x}, \boldsymbol{y})_k)$$

is nonnegative and decreasing with respect to $x_1 \in (0, x_2)$. Thus, for $y_1 \leq y_2$, the function

$$\big[u((\boldsymbol{x}, \tau_{12}(\boldsymbol{y}))_k) - u((\boldsymbol{x}, \boldsymbol{y})_k)\big] I(x_1 \leq x_2)$$

is nonnegative and decreasing with respect to $x_1 \in (0, x_2)$. Thus, according to Barlow and Proschan [2, Lemma 7.1(b)], we have, for any $x_2 \geq 0$,

$$\int_0^{x_2} [u((\boldsymbol{x}, \tau_{12}(\boldsymbol{y}))_k) - u((\boldsymbol{x}, \boldsymbol{y})_k)][f(\boldsymbol{x}, \boldsymbol{y}) - f(\tau_{12}(\boldsymbol{x}), \boldsymbol{y}) - f(\boldsymbol{x}, \tau_{12}(\boldsymbol{y})) + f(\tau_{12}(\boldsymbol{x}), \tau_{12}(\boldsymbol{y}))]\mathrm{d}x_1 \geq 0.$$

Consequently, it follows from (5.4) that for any increasing u,

$$\begin{aligned}
&\mathbb{E}[u(T_{k:n}(\boldsymbol{X}, \tau_{12}(\boldsymbol{Y})))] - \mathbb{E}[u(T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}))] \\
&= \underset{\mathbb{R}^{2n-2}}{\int \cdots \int} \int_0^{y_2} \int_0^{x_2} [u((\boldsymbol{x}, \tau_{12}(\boldsymbol{y}))_k) - u((\boldsymbol{x}, \boldsymbol{y})_k)] \cdot [f(\boldsymbol{x}, \boldsymbol{y}) - f(\tau_{12}(\boldsymbol{x}), \boldsymbol{y}) \\
&\qquad - f(\boldsymbol{x}, \tau_{12}(\boldsymbol{y})) + f(\tau_{12}(\boldsymbol{x}), \tau_{12}(\boldsymbol{y}))]\mathrm{d}x_1\mathrm{d}y_1 \prod_{i=2}^n \mathrm{d}x_i\mathrm{d}y_i \\
&\geq 0.
\end{aligned}$$

This implies $T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}) \leq_{\rm st} T_{k:n}(\boldsymbol{X}, \tau_{ij}(\boldsymbol{Y}))$, exactly as desired in (5.2). □

According to Theorem 5.1, it is better to allocate a redundancy with larger baseline reversed hazard rate to a component with smaller baseline reversed hazard rate.

Now let us move our focus to the case where $(\boldsymbol{X}, \boldsymbol{Y})$ are linked by an Archimedean copula with completely monotone generator φ, i.e., $(\boldsymbol{X}, \boldsymbol{Y})$ has CDF

$$F(\boldsymbol{x}, \boldsymbol{y}) = \mathbb{E}\left[\prod_{i=1}^n H_i^{\Theta}(x_i)K_i^{\Theta}(y_i)\right] = \int_0^{\infty} \prod_{i=1}^n H_i^{\theta}(x_i)K_i^{\theta}(y_i)\mathrm{d}F_{\Theta}(\theta), \tag{5.6}$$

where H_i and K_i are some baseline CDFs, for $i = 1, \cdots, n$, and $F_{\Theta}(\theta)$ is the CDF of the random frailty Θ with *Laplace transform* $\mathcal{L}_{\Theta}$. Since X_i and Y_i have marginal CDFs

$$F_i(x) = \int_0^{\infty} H_i^{\theta}(x)\mathrm{d}F_{\Theta}(\theta) = \int_0^{\infty} \exp\{\theta \ln H_i(x)\}\mathrm{d}F_{\Theta}(\theta) = \mathcal{L}_{\Theta}\big(-\ln H_i(x)\big)$$

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and

$$G_i(y) = \int_0^\infty K_i^\theta(y) \mathrm{d}F_\Theta(\theta) = \int_0^\infty \exp\{\theta \ln K_i(y)\} \mathrm{d}F_\Theta(\theta) = \mathcal{L}_\Theta\big(-\ln K_i(y)\big),$$

respectively, for $i = 1, 2, \cdots, n$, from (5.6) it follows that

$$F(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{L}_\Theta\left(\sum_{i=1}^n \mathcal{L}_\Theta^{-1}\big(F_i(x_i)\big) + \sum_{i=1}^n \mathcal{L}_\Theta^{-1}\big(G_i(y_i)\big)\right).$$

Thus, the generator $\varphi(x) = \mathcal{L}_\Theta(x)$. For more detailed discussion on Archimedean copulas with completely monotone generator, one may refer to Marshall and Olkin [28], Denuit *et al.* [13], Mulero *et al.* [32], and Pellerey and Zalzadeh [37].

Corresponding to the baseline distributions, let $X_i^* \sim H_i$ and $Y_i^* \sim K_i$ be absolutely continuous random variables with PDFs $h_i(x)$ and $k_i(x)$, and the reversed hazard rates $\lambda_{X_i^*}(x)$ and $\lambda_{Y_i^*}(x)$, respectively, $i = 1, 2, \cdots, n$. As a direct consequence of Theorem 5.1, we obtain the following sufficient conditions on the baseline distribution functions H_i and K_i for $i = 1, \cdots, n$ to ensure that the system lifetimes $T_{k:n}(\boldsymbol{X}, \boldsymbol{Y})$ and $T_{k:n}(\boldsymbol{X}, \tau_{ij}(\boldsymbol{Y}))$ are of the usual stochastic order.

Corollary 5.1. *Suppose that the lifetimes* $(\boldsymbol{X}, \boldsymbol{Y})$ *of system components and matched redundancies are linked by an Archimedean copula with completely monotone generator. Then, for (i,j) such that* $1 \le i < j \le n$,

$$T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}) \le_{\text{st}} T_{k:n}(\boldsymbol{X}, \tau_{ij}(\boldsymbol{Y})), \qquad k = 1, \cdots, n, \tag{5.7}$$

whenever $X_i^* \le_{\text{rh}} X_j^*$ *and* $Y_i^* \le_{\text{lr}} Y_j^*$.

Note that the likelihood ratio order implies the reversed hazard rate order. In what follows, Example 5.1 illustrates that the likelihood ratio order of Corollary 5.1 may be relaxed to the reversed hazard rate order in some specific contexts.

Example 5.1. Consider the random vector (X_1, X_2, Y_1, Y_2) equipped with one Archimedean copula with generator φ and marginal distribution functions

$$F_1(x) = G_1(x) = \varphi(-\ln H_1(x)), \qquad F_2(x) = G_2(x) = \varphi(-\ln K_2(x)),$$

where the baseline CDFs are

$$H_1(x) = K_1(x) = 1 - e^{-x}, \qquad H_2(x) = K_2(x) = 1 - \frac{1}{1 + x^2}, \qquad \text{for all } x \ge 0.$$

As is shown in Figure 4, the likelihood ratio

$$\frac{k_2(x)}{k_1(x)} = \frac{2xe^x}{(1 + x^2)^2}$$

is not increasing on $(0, +\infty)$, and this negates $Y_1^* \le_{\text{lr}} Y_2^*$. However, in accordance with You *et al.* [49, Example 5.1], both $X_1^* \le_{\text{rh}} X_2^*$ and $Y_1^* \le_{\text{rh}} Y_2^*$ are valid.

We evaluate the two redundant system survival functions $\mathbb{P}\big(\min(X_1 \vee Y_1, X_2 \vee Y_2) > x\big)$ and $\mathbb{P}\big(\min(X_1 \vee Y_2, X_2 \vee Y_1) > x\big)$, respectively, and then plot their difference in Figures 5(a) and 5(b). It can be observed that

$$\mathbb{P}(\min(X_1 \vee Y_2, X_2 \vee Y_1) > x) \ge \mathbb{P}(\min(X_1 \vee Y_1, X_2 \vee Y_2) > x)$$

for all $x \ge 0$, and this confirms $\min(X_1 \vee Y_1, X_2 \vee Y_2) \le_{\text{st}} \min(X_1 \vee Y_2, X_2 \vee Y_1)$.

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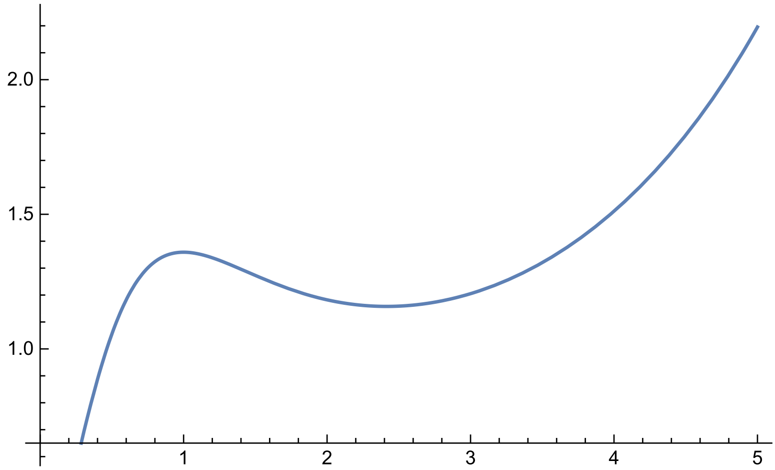
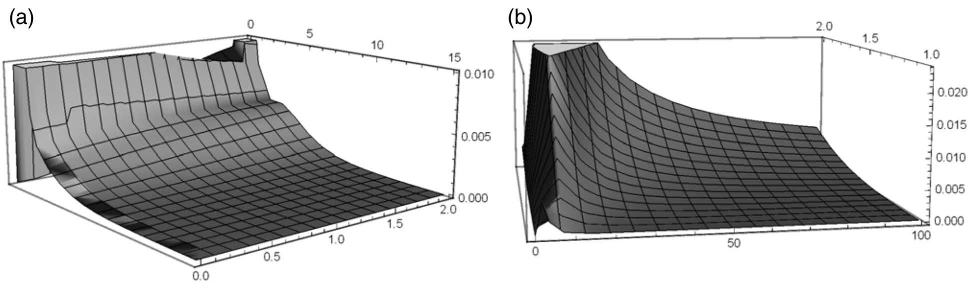


FIGURE 4. The curve of the ratio $k_2(x)/k_1(x)$ for $x \in (0, 5)$.



Clayton copula with $\alpha \in (0, 2)$ and $x \in (0, 15)$ Gumbel copula with $\theta \in (1, 2)$ and $x \in (0, 100)$

FIGURE 5. The difference $\mathbb{P}(\min(X_1 \vee Y_2, X_2 \vee Y_1) > x) - \mathbb{P}(\min(X_1 \vee Y_1, X_2 \vee Y_2) > x)$.

Motivated by Example 5.1, the next result asserts that the hazard rate orders in Corollary 5.1 can be relaxed to the usual stochastic orders for the series system with two components.

Corollary 5.2. *Assume for* (X_1, X_2, Y_1, Y_2) *an Archimedean copula with completely monotone generator. Then* $X_1^* \leq_{\rm st} X_2^*$ *and* $Y_1^* \leq_{\rm st} Y_2^*$ *imply that*

$$\min(X_1 \vee Y_1, X_2 \vee Y_2) \leq_{\rm st} \min(X_1 \vee Y_2, X_2 \vee Y_1).$$

Proof. Note that $\min(X_1 \vee Y_1, X_2 \vee Y_2)$ and $\min(X_1 \vee Y_2, X_2 \vee Y_1)$ respectively have SFs

$$\bar{F}(t) = 1 - \mathbb{P}(X_1 \leq t, Y_1 \leq t) - \mathbb{P}(X_2 \leq t, Y_2 \leq t) + \mathbb{P}(X_1 \leq t, Y_1 \leq t, X_2 \leq t, Y_2 \leq t)$$

and

$$\bar{G}(t) = 1 - \mathbb{P}(X_1 \leq t, Y_2 \leq t) - \mathbb{P}(X_2 \leq t, Y_1 \leq t) + \mathbb{P}(X_1 \leq t, Y_1 \leq t, X_2 \leq t, Y_2 \leq t),$$

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for all $t \geq 0$. From (5.6) it follows that

$$\begin{aligned}\bar{F}(t) - \bar{G}(t) &= \int_0^\infty H_1^\theta(t) K_2^\theta(t)\, \mathrm{d}F_\Theta(\theta) + \int_0^\infty H_2^\theta(t) K_1^\theta(t)\, \mathrm{d}F_\Theta(\theta) \\ &\quad - \int_0^\infty H_1^\theta(t) K_1^\theta(t)\, \mathrm{d}F_\Theta(\theta) - \int_0^\infty H_2^\theta(t) K_2^\theta(t)\, \mathrm{d}F_\Theta(\theta) \\ &= \int_0^\infty \big(H_1^\theta(t) - H_2^\theta(t)\big)\big(K_2^\theta(t) - K_1^\theta(t)\big)\, \mathrm{d}F_\Theta(\theta).\end{aligned}$$

By the assumption that $X_1^* \leq_{\text{st}} X_2^*$ and $Y_1^* \leq_{\text{st}} Y_2^*$, we have

$$H_1^\theta(t) \geq H_2^\theta(t) \qquad \text{and} \qquad K_1^\theta(t) \geq K_2^\theta(t),$$

for all $t \geq 0$ and $\theta > 0$. This implies that $\bar{F}(t) \leq \bar{G}(t)$ for all $t \geq 0$, yielding the desired result. □

Recall that the random variable Y_j^* is said to age faster than Y_i^* in terms of the reversed hazard rate if $\lambda_{Y_j^*}(t)/\lambda_{Y_i^*}(t)$ is increasing in $t \in (0, +\infty)$. For more on relative aging, the reader may refer to Sengupta and Deshpande [39], Misra and Francis [31], Li and Li [22], and references therein. At the end of this section, we present a numerical example suggesting that, in the conclusion of Theorem 5.1, the usual stochastic order cannot be further upgraded to either the hazard rate order or the reversed hazard rate order.

Example 5.2. (*Gumbel copula.*) Assume that Θ has Laplace transform $L_\Theta(t) = e^{-t^{1/\gamma}}$ for $\gamma \geq 1$, and that the baseline random variables X_1^*, X_2^*, X_3^*, Y_1^*, Y_2^*, and Y_3^* have exponential distributions with parameters α_1, α_2, α_3, β_1, β_2, and β_3, respectively. As per (5.6), $(\boldsymbol{X}, \boldsymbol{Y})$ has CDF

$$L(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{L}_\Theta\left(-\sum_{i=1}^3 \ln\left(1 - e^{-\alpha_i x_i}\right) - \sum_{i=1}^3 \ln\left(1 - e^{-\beta_i y_i}\right)\right), \tag{5.8}$$

where $x_i, y_i \geq 0$, $i = 1, 2, 3$. Let $\gamma = 2$, $\alpha_1 = \beta_1 = 1.5$, $\alpha_2 = \beta_2 = 1$, $\alpha_3 = \beta_3 = 0.6$. It is easy to check that $X_1^* \leq_{\text{rh}} X_2^* \leq_{\text{rh}} X_3^*$ and $Y_1^* \leq_{\text{rh}} Y_2^* \leq_{\text{rh}} Y_3^*$.

Since $e^{1.5x} - 3e^{0.5x}$ is increasing in $x \geq 0$, it holds that $e^{1.5x} - 3e^{0.5x} + 2 \geq 0$ for $x \geq 0$. Hence, by taking the derivative on the ratio of the reversed hazard rates, we get

$$\left(\frac{\lambda_{Y_2^*}(x)}{\lambda_{Y_1^*}(x)}\right)' = \frac{2}{3}\left(\frac{e^{1.5x} - 1}{e^x - 1}\right)' = \frac{e^x}{3} \frac{e^{1.5x} - 3e^{0.5x} + 2}{(e^x - 1)^2} \geq 0.$$

This implies that $\lambda_{Y_2^*}(x)/\lambda_{Y_1^*}(x)$ is increasing in $x > 0$. Similarly, $\lambda_{Y_3^*}(x)/\lambda_{Y_2^*}(x)$ is increasing in $x > 0$. Thus, from Shaked and Shanthikumar [41, Theorem 1.C.4.(b)], it follows that $Y_3^* \geq_{\text{lr}} Y_2^* \geq_{\text{lr}} Y_1^*$.

Based on (5.8), we derive CDFs

$$\begin{aligned}&\mathbb{P}(T_{1:3}(\boldsymbol{X} \vee \boldsymbol{Y}) \leq t) \\ &= \mathbb{P}(\min\{X_1 \vee Y_1, X_2 \vee Y_2, X_3 \vee Y_3\} \leq t) \\ &= \sum_{i=1}^3 \mathbb{P}(X_i \vee Y_i \leq t) - \sum_{1 \leq i \neq j \leq 3} \mathbb{P}(X_i \vee Y_i \leq t, X_j \vee Y_j \leq t) + \mathbb{P}(X_i \vee Y_i \leq t, i = 1, 2, 3) \\ &= L(t, \infty, \infty, t, \infty, \infty) + L(\infty, t, \infty, \infty, t, \infty) + L(\infty, \infty, t, \infty, \infty, t) + L(t, t, t, t, t, t) \\ &\quad - L(t, t, \infty, t, t, \infty) - L(t, \infty, t, t, \infty, t) - L(\infty, t, t, \infty, t, t), \qquad \text{for } t \geq 0,\end{aligned}$$

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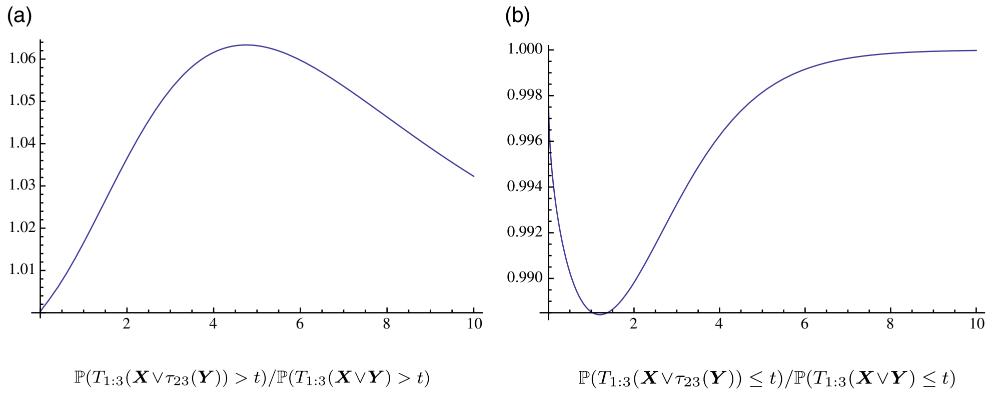

FIGURE 6. The ratios of the SFs and distribution functions with $t \in (0, 10)$.

and

$$\begin{aligned} &\mathbb{P}(T_{1:3}(\boldsymbol{X} \vee \tau_{23}(\boldsymbol{Y})) \leq t) \\ &= L(t, \infty, \infty, t, \infty, \infty) + L(\infty, t, \infty, \infty, \infty, t) + L(\infty, \infty, t, \infty, t, \infty) + L(t, t, t, t, t, t) \\ &\quad - L(t, t, \infty, t, \infty, t) - L(t, \infty, t, t, t, \infty) - L(\infty, t, t, \infty, t, t), \qquad \text{for } t \geq 0. \end{aligned}$$

In Figures 6(a) and 6(b), the ratios

$$\mathbb{P}(T_{1:3}(\boldsymbol{X} \vee \tau_{23}(\boldsymbol{Y})) > t)/\mathbb{P}(T_{1:3}(\boldsymbol{X} \vee \boldsymbol{Y}) > t)$$

and

$$\mathbb{P}(T_{1:3}(\boldsymbol{X} \vee \tau_{23}(\boldsymbol{Y})) \leq t)/\mathbb{P}(T_{1:3}(\boldsymbol{X} \vee \boldsymbol{Y}) \leq t)$$

are both observed to be non-monotone in $t \in (0, 10)$. As a consequence, neither $T_{1:3}(\boldsymbol{X} \vee \boldsymbol{Y}) \leq_{\text{hr}} T_{1:3}(\boldsymbol{X} \vee \tau_{23}(\boldsymbol{Y}))$ nor $T_{1:3}(\boldsymbol{X} \vee \boldsymbol{Y}) \leq_{\text{rh}} T_{1:3}(\boldsymbol{X} \vee \tau_{23}(\boldsymbol{Y}))$ is true.

6. Concluding remarks

This study deals with redundant k-out-of-n:F systems in the context in which stochastically ordered component lifetimes and multiple redundancy lifetimes are linked by an Archimedean copula. We show that (i) allocating more redundancies to weaker components yields a stochastically larger redundant system lifetime, and (ii) balancing the allocation of active redundancies tends to uniformly maximize the redundant system reliability in the context of homogeneous component lifetimes and homogeneous redundancy lifetimes. The novelty of this paper is to provide a comparison on redundant k-out-of-n:F systems with statistically dependent component and redundancy lifetimes; these assumptions are more applicable to complicated real-world engineering systems than the usual assumptions of independence.

The present results essentially extend some relevant work in the recent literature. However, further research is necessary along these lines, for example, (i) to explore similar comparison results in the setting in which component and redundancy lifetimes are linked by a generic copula function, and (ii) to check whether the usual stochastic order on system lifetimes still holds if the reversed hazard rate order on component lifetimes is replaced by the hazard rate order in Theorem 3.1. Note that for i.i.d. component and redundancy lifetimes, Ding and Li

[14] further strengthened the usual stochastic order of Hu and Wang [19, Theorem 3.3] to the hazard rate order $T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{s}) \leq_{\rm hr} T_{k:n}(\boldsymbol{X}, \boldsymbol{Y}; \boldsymbol{r})$ whenever $\boldsymbol{r} \leq_{\rm m} \boldsymbol{s}$. Although our attempt to prove it was unsuccessful, we conjecture that such a generalization can also be achieved for identically distributed component and redundancy lifetimes linked by an Archimedean copula.

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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