J. Fluid Mech. (2004), *vol.* 509, *pp.* 63–81.
DOI: 10.1017/S0022112004008900 Printed in the United Kingdom

Oblique collisions and rebound of spheres from a wetted surface

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(Received 24 March 2003 and in revised form 26 January 2004)

Plastic and metal spheres were dropped from various heights onto a quartz disk covered with a thin layer of viscous oil and inclined at various angles with the horizontal. Rebound was observed only above a critical approach velocity, similar to that observed for head-on collisions when the disk is horizontal. The tangential component of the sphere's velocity is reduced only a small amount by the collision, owing to sliding lubrication/friction forces that also impart a small rotational velocity to the sphere. In contrast, the normal component of velocity is reduced substantially by viscous losses, and so the rebound angle of the sphere relative to the surface of the disk is smaller than the impact angle. The normal component of restitution and the rebound angle increase with the normal Stokes number based on the normal component of the impact velocity.

1. Introduction

Collisions between wetted particles and between particles and the walls of containers occur in industrial processes such as filtration, coagulation, sedimentation and slurry transport. To understand and model these collisions, the physical interactions occurring during contact should be known. The contact of two dry bodies has been studied extensively and is often described by the Hertzian contact law (Love 1927; Landau & Lifshitz 1959). This law assumes perfectly elastic collisions with zero energy losses, so that the coefficient of restitution (the ratio of rebound velocity to approach velocity) is unity. However, the coefficient of dry restitution (e_{dry}) is typically found to be less than unity owing to plastic deformation (Johnson 1985), elastic waves (Hunter 1957), vibrations (Hunter 1957; Reed 1985; Sondergaard, Chaney & Brennen 1990), viscoelasticity of the solids (Falcon *et al.* 1998; Ramirez *et al.* 1999) and adhesive forces (Dahneke 1971). In many applications, however, the particles and/or surfaces are wet. Examples include agglomeration, pollen capture on wet leaves, inertial capture on coated fibres, polishing, multiphase contactors, and sand blasting of wet walls. Even if solid losses are accounted for, any fluid present on the solid surfaces would have to be in extremely small quantities for the collisions to be modelled by dry contact forces. Otherwise, fluid forces and viscous losses should also be taken into consideration when predicting the post-collision behaviour.

Davis, Serayssol & Hinch (1986) studied the dynamic deformation of a solid sphere immersed in a viscous fluid and in head-on close approach to a plane solid surface (or another sphere). They developed an elastohydrodynamic theory that couples the

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solid mechanics with the lubrication forces of the fluid. This theory predicts whether a particle would stick or rebound, when impacting a wetted surface. When the Stokes number is less than a critical value, the particle sticks to the opposing surface, whereas the particle bounces when its Stokes number exceeds the critical value. The Stokes number represents the ratio of the inertia of the sphere to the viscous forces in the lubrication layer and is defined by

$$St = \frac{mV_o}{6\pi\mu a^2}, \tag{1.1}$$

where $m = 4\pi\rho_s a^3/3$ is the mass of the ball, ρ_s is its density, a is its radius, μ is the viscosity of the fluid, and V_o is the incident or approach velocity. According to Davis *et al.* (1986), the critical Stokes number for rebound decreases weakly with increasing values of an elasticity parameter:

$$\epsilon = \left(4\theta\mu V_o a^{3/2}\right)/x_o^{5/2}, \tag{1.2}$$

where x_o is the initial separation of the spheres and the surface, $\theta = (1-\nu_1^2)/\pi E_1 + (1-\nu_2^2)/\pi E_2$; E_i and ν_i are Young's modulus and Poisson's ratio, respectively, for the sphere ($i = 1$) and plane ($i = 2$). The elasticity parameter is a ratio of viscous forces that cause deformation and the stiffness of the solids to resist deformation, and it is generally small compared to unity. For $St > St_c$, where St_c is the critical Stokes number, Davis *et al.* (1986) gave numerical results for the deformation and rebound of the sphere.

Davis (1987) extended the elastohydrodynamic theory to account for the interparticle forces, surface roughness and the discrete molecular nature of the fluid. Lian, Adams & Thorton (1996) gave a simple closed-form solution that is in good agreement with the theory of Davis *et al.* (1986). Barnocky & Davis (1988) performed experiments wherein they dropped spheres onto a horizontal surface covered with a thin oil layer. They determined the critical drop height above which the sphere rebounded when dropped onto a wet surface. Their findings are in qualitative agreement with the theoretical considerations of Davis *et al.* (1986) and Davis (1987). Lundberg & Shen (1992) studied the collisions between a stationary ball and a moving roller in the presence of an oil drop. They found that the coefficient of restitution decreases with an increase in the oil viscosity due to the viscous forces exerted by the oil. Davis, Rager & Good (2002) measured values of the wet coefficient of restitution for near-normal collisions of spheres with a flat target covered with a thin layer of viscous fluid. Their results show that no rebound was observed for $St < St_c$, where St_c is the observed critical Stokes number for rebound. For $St > St_c$, the coefficient of restitution increases rapidly with St, before asymptotically approaching the value e_{dry}. Good agreement between the experiments and a simple analytical model was achieved.

Several studies of collisions with targets fully immersed in fluids have also been published. Joseph *et al.* (2001) performed head-on collisions of a particle attached to a pendulum with a wall immersed in a liquid. They observed that no rebound occurred below $St \approx 10$, and the coefficient of restitution asymptotically approached the value for dry collisions for $St > 100$. They also found that microscopic surface roughness affects the repeatability of experiments at low impact velocities. Gondret *et al.* (1999, 2002) observed the bouncing motion of balls dropped onto solid plates, with the ambient fluid being either liquid or gas. They also observed rebound only

Material	Diameter, $2a$ (m)	Density, ρ_s (kg m^{-3})	Young's modulus, E (kg m s^{-2})	Poisson's ratio, ν
Nylon	0.0127, 0.0064	1140	2.84×10^9	0.35
Stainless steel	0.0095, 0.0064	7972	2.00×10^{11}	0.28
Teflon	0.0127, 0.0064	2300	4.00×10^8	0.46

Table 1. Properties of the spheres used in the experiments.

above a critical Stokes number, as suggested by the theory of Davis *et al.* (1986). Joseph *et al.* (2001) and Gondret *et al.* (1999, 2002) made plots of e/e_{dry} versus St, where e is the wet coefficient of restitution, representing the ratio of rebound and impact velocities in the presence of liquid. The data have trends similar to those predicted by the theory.

Zhang *et al.* (1999) dropped spheres from varying heights onto a stationary sphere immersed in a liquid. They studied colinear collisions as well as collisions where the centres of the two spheres were staggered. They developed a mechanistic model to account for the various stages in the collisional process. The experimental results match well with the model predictions. They also showed that the viscous effect on the compression and rebound of contacting particles is significant for single collisions only when the fluid is highly viscous or the elastic modulus of the particle is small.

All of the above work in the area of collisions in the presence of fluids, except that of Zhang *et al.* (1999), has been for collisions in which the approach velocity is essentially normal to the target surface. However, in practice, most collisions are not head-on. While oblique impacts under dry conditions have been the subject of important work (e.g. Walton 1993; Foerster *et al.* 1994; Labous, Rosato & Dave 1997; Gorham & Kharaz 2000; Louge & Adams 2002), the effects of liquids on oblique collisions are relatively unstudied. An exception is the paper by Joseph & Hunt (2004), who studied glass and steel particles immersed in a viscous liquid and undergoing oblique collisions with a wall. It is important to know whether the theories and predictions that hold for normal collisions of particles with liquid present can be modified to describe oblique collisions. In particular, we are interested in whether the normal component of motion can be described by the prior work for head-on collisions and if the tangential component of motion is significantly affected by the collision. In the current work, experiments were performed similar to those reported in Davis *et al.* (2002), but with the target disk inclined at an angle to the horizontal. A comparison of the results for near-normal collisions and with an extension of the theory of Davis *et al.* (2002) is presented.

2. Materials and methods

Table 1 gives the properties of the materials of the spheres used in the experiments. Both metal and plastic spheres (Small Parts, Inc.) were used.

Scanning electron micrographs (SEMs) reveal no protrusions or bumps for the stainless steel spheres and show only small (1–5 μm) widely separated pits (Barnocky & Davis 1988). The roughness elements for the Teflon spheres appear to be flat-topped, with shadows that suggest a typical roughness of about 10 μm (Galvin, Zhao & Davis 2001). The root-mean-square roughness, describing the variation in the surface elevation with respect to a flat or mean (reference) surface, was found to be 2 μm for nylon spheres of 0.64 cm diameter and 0.023 μm for steel spheres of 0.64 cm diameter

(Joseph *et al.* 2001). Measurements by the method of Smart & Leighton (1989) gave hydrodynamic roughness of approximately 20 μm for nylon balls (Davis *et al.* 2002), 4 μm for steel balls (Davis *et al.* 2002), and 20 μm for Teflon balls (present work).

A quartz disk ($E = 7.26 \times 10^{10}\,\mathrm{kg\,m^{-1}\,s^{-2}}$, $\nu = 0.17$) of thickness 0.0127 m and diameter 0.0508 m was used as the target. The target disk was covered with an oil layer. Three silicone fluids (Brookfield Engineering Laboratories) of viscosities 0.975, 4.98 and 12.4 $\mathrm{kg\,m^{-1}\,s^{-1}}$ and densities 968, 972 and 973 $\mathrm{kg\,m^{-3}}$, respectively, at 25°C were used, with thicknesses of 80 and 150 μm. The oil was applied to the surface of the quartz disk with a brush. The difference in weight between the wet disk and the dry disk and the specific gravity of the oil were used to calculate the average oil-layer thickness. Once the desired weight of oil was applied to the disk, it was smoothed out with the brush, and reweighed and allowed to stand for about 5–10 min so that the surface evened out by gravity. The disk was kept horizontal and rotated about its axis between experiments, to prevent the thin oil layer from flowing to one side.

The quartz disk is held in a metal holder having a circular hole in the centre. The holder has provisions for attaching a pair of screws at one end. These screw lengths are adjusted to give the desired angle of inclination with the horizontal. A release plate with holes of three different diameters (0.0127 m, 0.0064 m and 0.0095 m) is mounted on a vertical metal rod with the help of an adjustable clamp. The rod has markings that show the vertical distance from the base. The release plate can be moved vertically as well as horizontally so as to ensure that it is at the desired height and exactly above the quartz disk. The sphere is held in the appropriate hole in the release plate by placing a finger below it. Quick downward motion of the finger releases the ball with minimum rotation. Strobotic photography was employed for capturing the pre- and post-impact images of the ball with the oil-laden target. An Olympus OM-1 35 mm camera and a General Radio Company Type 1531-A stroboscope were used. *Scion Image*, a software package developed by the Scion corporation and freely available on the net, was used for analysing the images. The approach and rebound velocities were determined from the images and corrected for acceleration/deceleration due to gravity to determine the respective velocities at the time of impact. The incident angle of impact relative to the disk surface when the balls are dropped vertically downward is given by ϕ_i. Values of $\phi_i \approx 30°$, 45° and 87° were studied. The experiments with $\phi_i \approx 87°$ represent near-normal collisions, with an angle of about 3° with the normal chosen so that the ball images before and after impact could be distinguished. The rebound angle ϕ_r was measured from the pictures. Knowing ϕ_i and ϕ_r, the normal and tangential components of the approach and rebound velocities were found, and the values of the coefficients of normal restitution (e_n) and tangential restitution (e_t) were calculated. Figure 1 shows a diagram of the apparatus and the velocities and angles discussed here.

The plastic spheres were also marked with stripes on their surfaces. These stripes were used to determine the rotation of the spheres. When the spheres were dropped as described, the rotation was typically found to be at most 2°–3° between successive images prior to impact, representing an approach angular velocity of less than 10 $\mathrm{rad\,s^{-1}}$. The post-collision rotation was also found for the nylon and Teflon spheres. Most of the spheres with a diameter of 0.0127 m have a rebound rotational velocity in the range 0–50 $\mathrm{rad\,s^{-1}}$. The spheres with a diameter of 0.0064 m have a broader range of 0–400 $\mathrm{rad\,s^{-1}}$. The larger spheres have a higher moment of inertia and therefore rotate less after being subjected to a torque during the collision. Figure 2 provides a sample strobotic picture for a nylon ball.

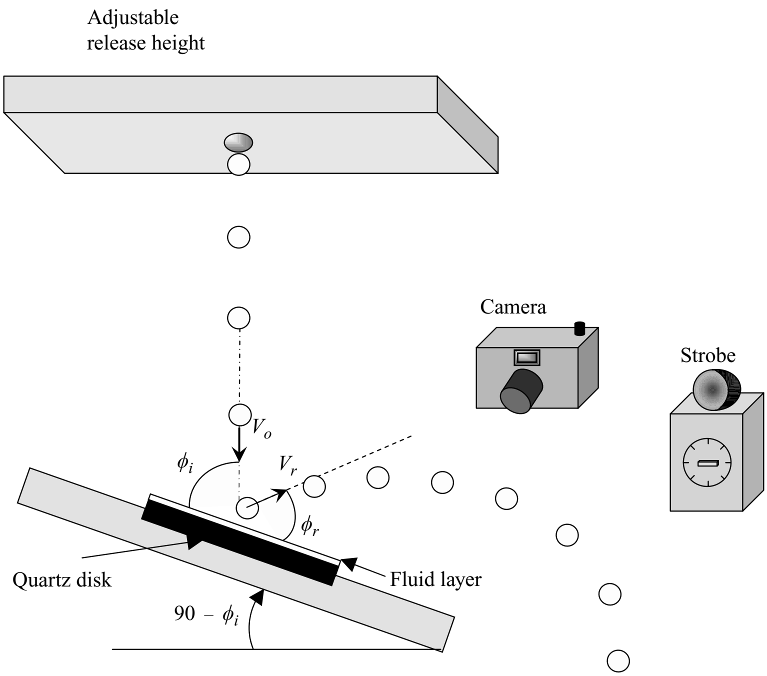

FIGURE 1. Diagram of the experimental apparatus and the dropped sphere.

FIGURE 2. Sample picture of a nylon ball with a 0.0127 m diameter dropped from a height of 0.53 m onto a quartz disk inclined at an angle of 45° to the horizontal and covered with a layer of oil of viscosity 0.975 kg m^{-1} s^{-1} and thickness of 80 μm, with the stroboscope running at 10 000 flashes per min.

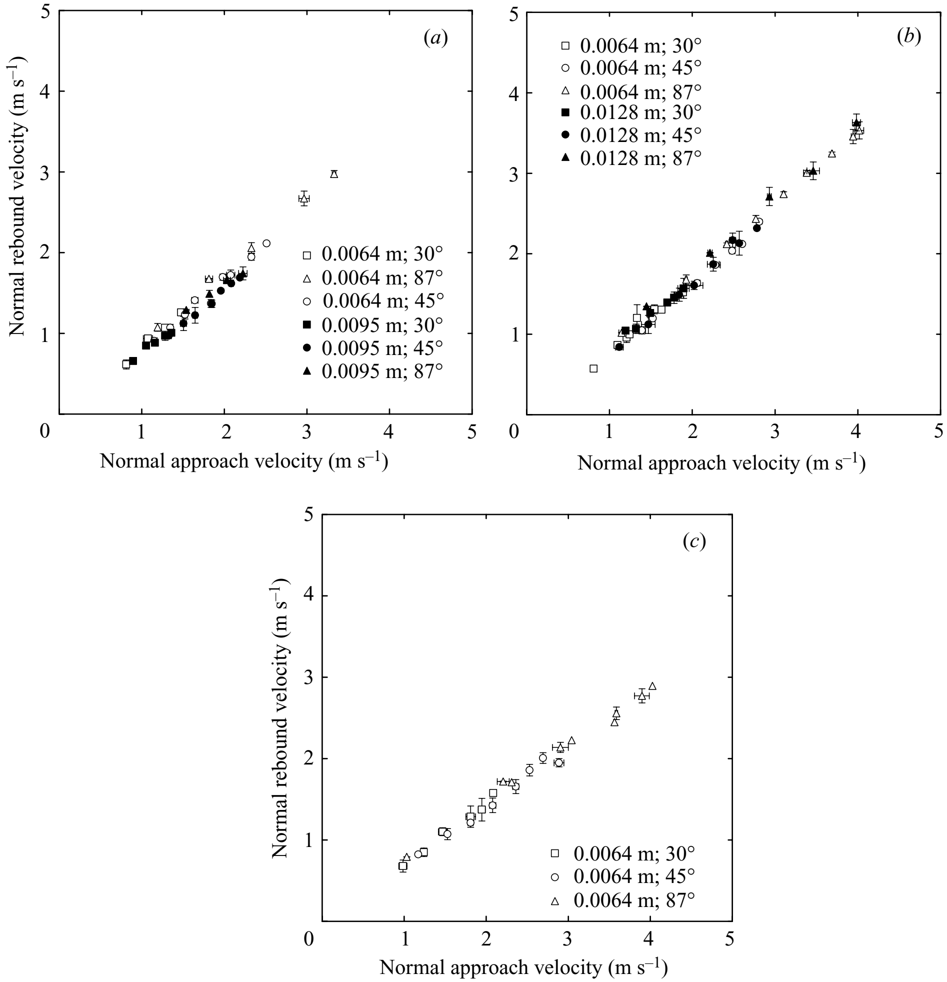

FIGURE 3. Normal approach velocity versus normal rebound velocity for impact with the dry quartz disk inclined at impact angles of 30°, 45° and 87°. (*a*) Stainless steel spheres of diameters 0.0064 m and 0.0095 m. (*b*) Nylon spheres of diameters 0.0064 m and 0.0127 m. (*c*) Teflon spheres of diameter 0.0064 m.

The critical Stokes number was determined experimentally by finding the critical height. The drop height of a sphere was reduced in discrete steps, and the maximum height at which the sphere rolled/slipped along the surface without any normal rebound was noted as the critical height. The critical height was found for each set of sphere diameter, material, oil viscosity and thickness, and angle of inclination of the disk. The approach velocity of the sphere, when dropped from the critical height, was determined and used in calculating St_c.

3. Results and discussion

Impact with a dry disk was examined first for each of the spheres. It has been reported previously that the dry coefficient of normal restitution may depend on the material properties, impact angle (Gorham & Kharaz 2000) and approach velocity (Labous *et al.* 1997; Gorham & Kharaz 2000). Figures 3(*a*), 3(*b*) and 3(*c*) provide plots

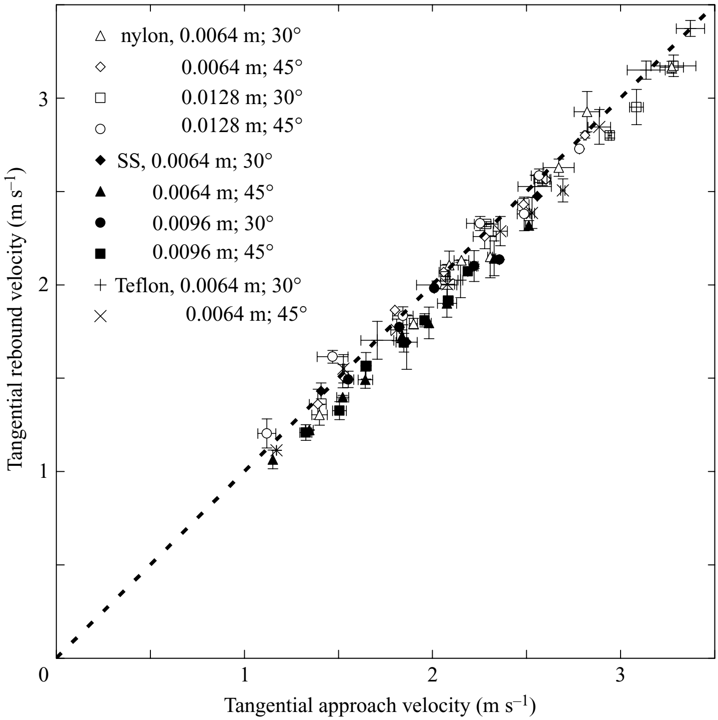

FIGURE 4. Tangential approach velocity versus tangential rebound velocity for nylon and steel spheres impacting the dry quartz disk at angles of 30° and 45°. The dotted line has slope unity.

of the normal rebound velocity versus the normal approach velocity for steel, nylon and Teflon spheres, respectively, for various impact angles under dry conditions. It can be seen that each data set lies almost on a straight line. Linear regression produced wide-ranged 95% confidence intervals for the intercept that contain zero. The intercept was then forced through zero to estimate the coefficient of normal restitution, e_{dry} (ratio of the normal component of the rebound velocity to the normal component of the approach velocity), from the slope of the line. The coefficients of dry restitution with $\pm$95% confidence intervals are $e_{dry} = 0.90 \pm 0.01$, 0.82 ± 0.02, 0.84 ± 0.04 for stainless steel balls of 0.0064 m diameter at $\phi_i \approx 87°$, 45°, 30°, respectively. Stainless steel balls of 0.0095 m diameter gave $e_{dry} = 0.81 \pm 0.02$, 0.76 ± 0.02, 0.75 ± 0.03 for $\phi_i \approx 87°$, 45°, 30°, respectively. Nylon balls of 0.0064 m diameter gave $e_{dry} = 0.88 \pm 0.01$, 0.81 ± 0.02, 0.81 ± 0.02 for $\phi_i \approx 87°$, 45°, 30°, respectively. Nylon balls of 0.0127 m diameter gave $e_{dry} = 0.90 \pm 0.02$, 0.82 ± 0.03, 0.81 ± 0.02 for $\phi_i \approx 87°$, 45°, 30°, respectively. Teflon balls of 0.0064 m diameter gave $e_{dry} = 0.72 \pm 0.02$, 0.70 ± 0.02, 0.72 ± 0.02 for $\phi_i \approx 87°$, 45°, 30°, respectively. The losses when using nylon and Teflon spheres are probably due to the combined effect of elastic waves, viscoelasticity and vibrations in the disk. Vibrations are thought to be the major cause of losses with the dense steel spheres. Gorham & Kharaz (2000) found that the dry coefficient of normal restitution did not change significantly with variation in the impact angle for elastic materials. However, we found that inclining the target disk increased the energy losses slightly for nylon and steel spheres, while those for the softer Teflon spheres remained almost constant.

Figure 4 provides plots of the tangential component of the rebound velocity versus the tangential component of the approach velocity for dry collisions. It can be seen

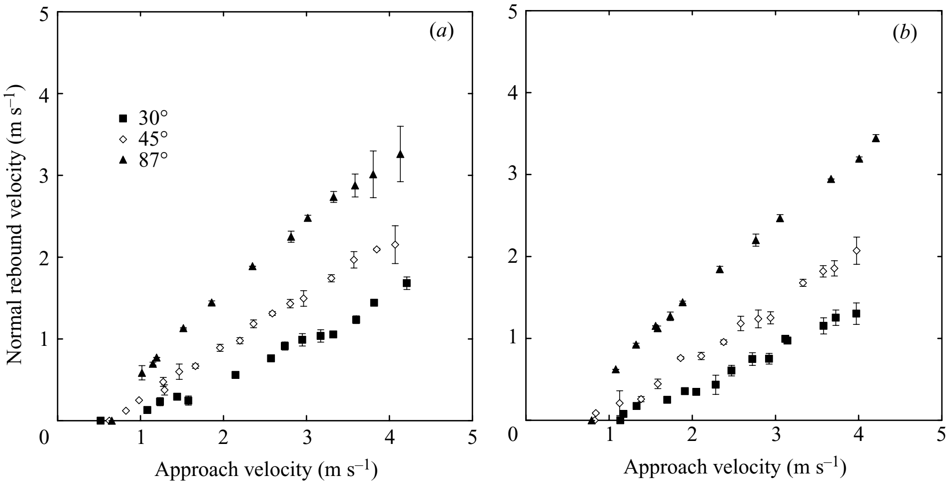

FIGURE 5. Normal rebound velocity versus approach velocity for stainless steel spheres of 0.0064 m diameter impacting the quartz disk covered with an oil layer of thickness 80 μm and viscosity (*a*) 0.975 kg m^{-1} s^{-1} and (*b*) 12.4 kg m^{-1} s^{-1}.

that all the data lie near the line with slope unity. Thus, there is not a significant reduction in velocity in the tangential direction, which implies that Coulombic friction for the materials investigated does not have a significant effect. Foerster *et al.* (1994) previously found that the tangential restitution coefficient did not show a statistical dependence on the tangential approach velocity.

Figures 5(*a*) and 5(*b*) provide plots of the normal rebound velocity versus the approach velocity for steel spheres impacting a quartz disk covered with a thin oil layer and inclined at three different angles. In all cases, there is a critical approach velocity, below which the sphere sticks to the wetted target and does not rebound. Above the critical approach velocity, it is observed that, as the incident angle increases for the same approach velocity, the normal rebound velocity increases. This behaviour is expected because, as the incident angle is increased toward 90°, the normal component of velocity of the approaching sphere increases. A greater normal approach velocity leads to a greater rebound in the direction normal to the disk. Also, the critical approach velocity for rebound is lower for smaller viscosity owing to less viscous dissipation.

Figures 6(*a*) and 6(*b*) provide plots of the wet coefficient of normal restitution versus the normal Stokes number, $St \sin\phi_i$, which is a measure of the ratio of the inertia of the sphere normal to the wetted surface to the viscous forces exerted by the fluid layer, for stainless steel spheres colliding with a wetted quartz disk at various angles, two different fluid viscosities, and fixed fluid-layer thickness. The trends are the same as that observed in near-normal collisions (Joseph *et al.* 2001; Davis *et al.* 2002; Gondret *et al.* 2002), i.e. e_n is zero for $St \leqslant St_c$, increases for $St > St_c$ and approaches an asymptotic value for $St \gg St_c$. We note that, for lower impact angles, the values of e_n do not even out to form a plateau for the range of normal St investigated. Because of experimental limitations, the maximum height from which the ball was dropped onto the target is 1.6 m, corresponding to an approach velocity of 5.6 m s^{-1}. For high inclinations, this drop height is not sufficient to give a normal approach velocity high enough to achieve the asymptotic value of e_n. The critical Stokes number for rebound is smaller for the more viscous oil, as predicted by

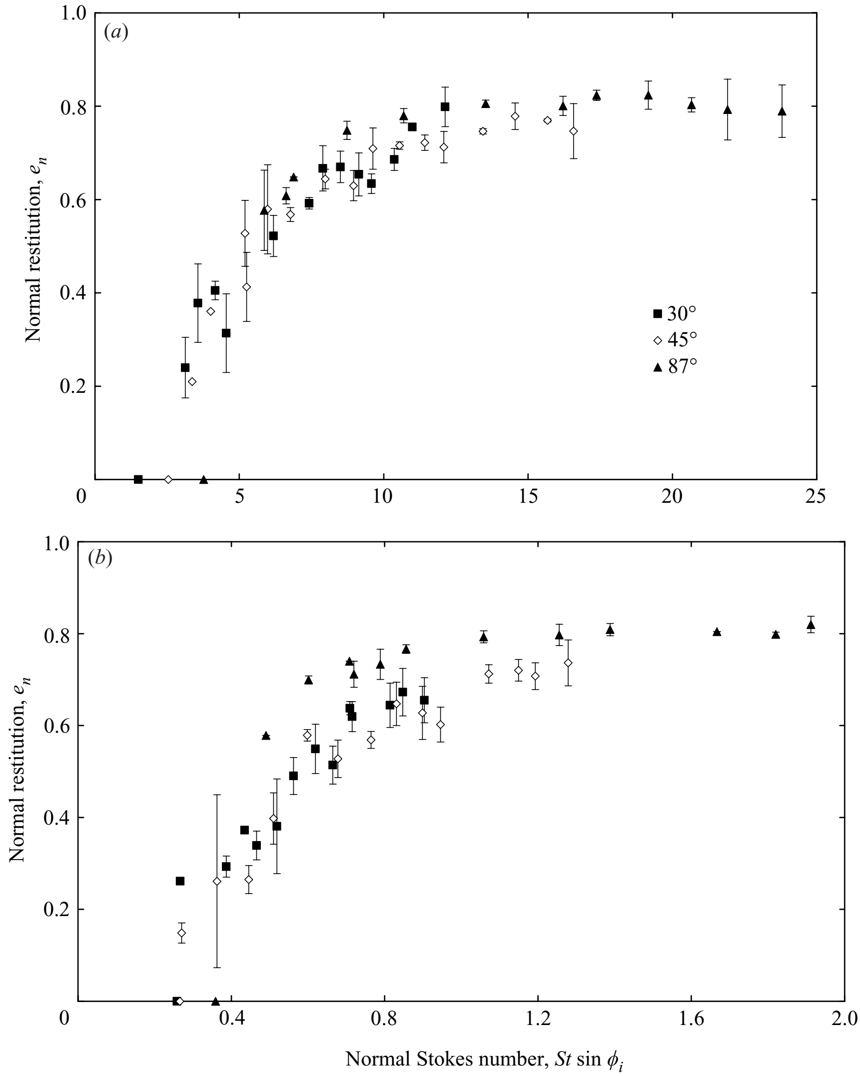

FIGURE 6. Coefficient of normal restitution versus the normal Stokes number for stainless steel spheres of 0.0064 m diameter impacting the quartz disk covered with an oil layer of thickness 80 μm and viscosity (*a*) 0.975 $kg\,m^{-1}\,s^{-1}$ and (*b*) 12.4 $kg\,m^{-1}\,s^{-1}$.

Davis *et al.* (1986), owing to a larger elasticity parameter. More important, the data for various impact angles in figure 6 appear to nearly collapse on the same curve, except that the restitution for wet head-on collisions is slightly greater than for oblique collisions, as was also observed for dry collisions of the steel balls. This trend implies that the normal restitution of a stainless steel sphere, normalized by the dry value, depends primarily on the normal component of the approach velocity of the sphere and is nearly independent of the impact angle. It further suggests that the normal and tangential components of velocity are essentially decoupled. Joseph & Hunt (2004) also found that the normal coefficient of restitution in oblique collisions is essentially independent of the tangential component of velocity. Thus, the theories that predict

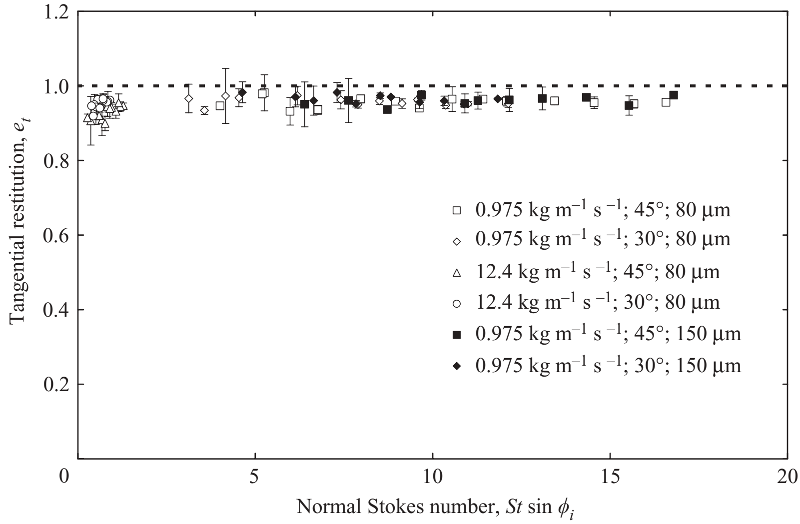

FIGURE 7. Coefficient of tangential restitution versus the normal Stokes number for stainless steel spheres of 0.0064 m diameter impacting a 80 and 150 μm thick oil layer of 0.975 or 12.4 kg m^{-1} s^{-1} viscosity.

the coefficient of restitution for head-on wet collisions can be extended to predict the coefficient of normal restitution for oblique wet collisions.

In figure 7, the coefficient of tangential restitution given by e_t (ratio of the tangential component of the rebound velocity to the tangential component of the approach velocity) is plotted versus the normal Stokes number for stainless steel spheres colliding with a wetted quartz disk. The values of e_t are close to unity, except for the most viscous oil at small Stokes numbers. Thus, the sliding lubrication forces exerted by the oil layer do not significantly reduce the tangential component of the sphere velocity. Note that e_t is an apparent coefficient of tangential restitution, based on the velocity of the centre-of-mass of the ball, in contrast to the common definition for dry collisions based on the velocity at the point of contact (Labous *et al.* 1997); the two definitions are the same only when rotation is negligible.

In figure 8, the rebound angle is plotted versus the normal Stokes number for stainless steel spheres colliding with the wetted quartz disk. The rebound angle is given by the following relationship:

$$\phi_r = \tan^{-1}\left(\frac{V_{rn}}{V_{rt}}\right) = \tan^{-1}\left(\frac{e_n V_{on}}{e_t V_{ot}}\right) = \tan^{-1}\left(\frac{e_n \tan\phi_i}{e_t}\right), \tag{3.1}$$

where the subscripts n and t refer to the normal and tangential components, respectively, of the respective velocities. Since $e_n < e_t$, the rebound angle is expected to be smaller than the impact angle, as seen in figure 8.

In figure 9, the coefficient of normal restitution resulting from the collisions of stainless steel spheres with a quartz disk with two different oil-layer thicknesses is compared. As expected, the coefficient of normal restitution for the thicker oil layer is lower. A thicker oil layer leads to more viscous dissipation, thereby reducing the normal restitution and increasing the critical Stokes number, as seen in Davis *et al.* (2002) for head-on collisions.

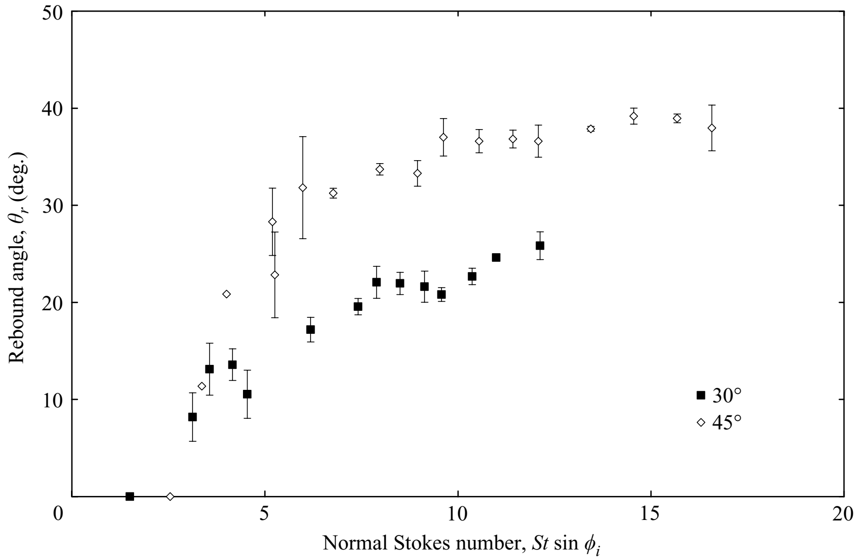

FIGURE 8. Rebound angle versus the normal Stokes number for stainless steel balls of 0.0064 m diameter impacting an 80 μm thick oil layer of 0.975 or 12.4 kg m^{-1} s^{-1} viscosity.

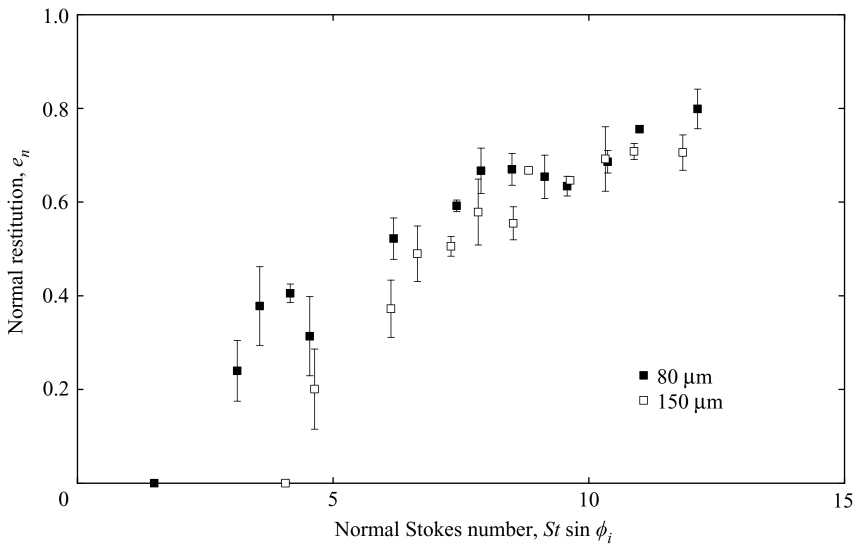

FIGURE 9. Coefficient of normal restitution versus the normal Stokes number for stainless steel spheres of 0.0064 m diameter impacting an oil layer of viscosity 0.975 kg m^{-1} s^{-1} at an impact angle of 30° for two oil-layer thicknesses.

In figures 10(*a*) and 10(*b*), plots of the coefficient of normal restitution versus the normal Stokes number are given for Teflon and nylon spheres, respectively, colliding with the wetted quartz disk at various angles. The normal restitution for these plastic spheres shows more data scatter than for the steel spheres (see figure 6), and the collapse of the data on a single curve is not as good. The plastic spheres have much larger surface roughness than do the steel spheres, and the roughness elements may

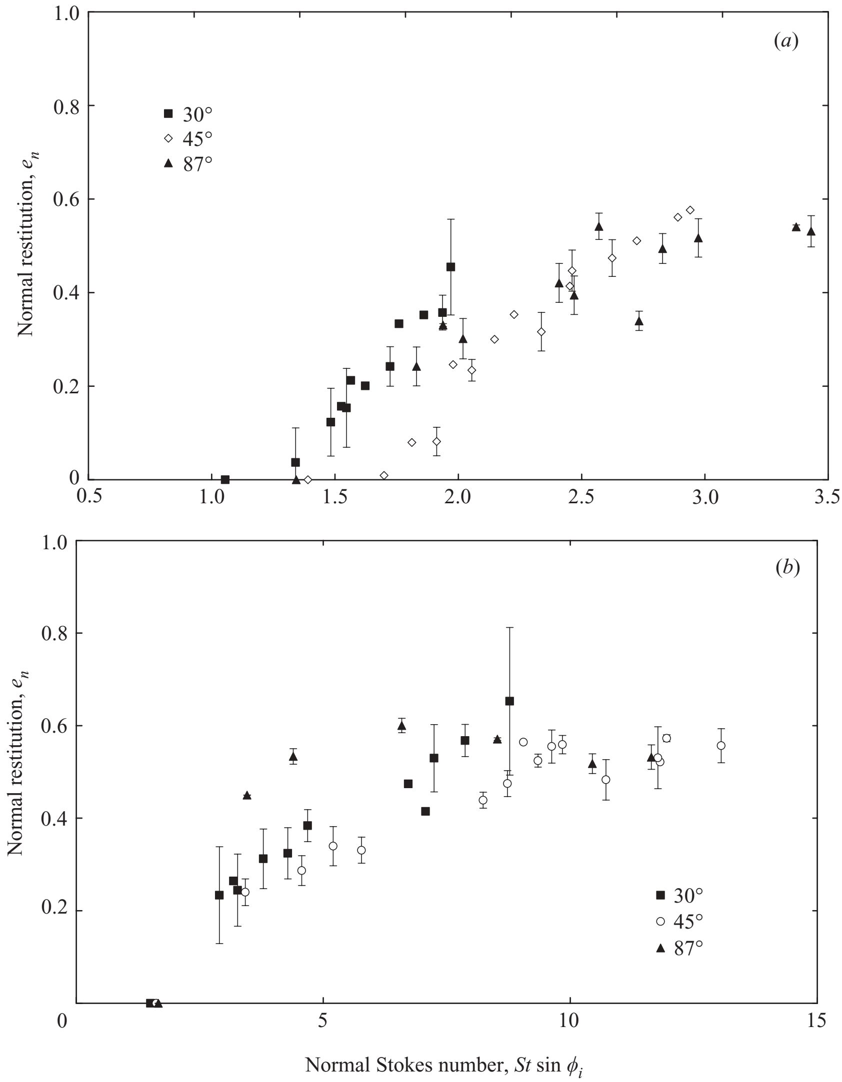

FIGURE 10. Coefficient of normal restitution versus the normal Stokes number for impact with a quartz disk covered with an oil layer of thickness 80 μm and viscosity 0.975 kg m^{-1} s^{-1} using (*a*) nylon spheres of 0.0127 m diameter and (*b*) Teflon spheres of 0.0127 m diameter.

affect the collision and rebound processes. The wet coefficient of tangential restitution was found to be nearly unity for the plastic spheres, as was also seen for the steel spheres.

The change in rotational velocity ($\Delta\omega$, positive for clockwise rotation) as a result of the collision is shown in figure 11 for Teflon spheres of 0.64 cm diameter colliding with the dry surface and when it is wetted with a thin layer of the least viscous oil. While there is considerable scatter in the data for both dry and wet collisions, which again might be attributed to surface non-uniformities, a key result is that less rotation occurs for the wetted surface. This result indicates that the liquid layer serves as a lubricant to reduce the tangential friction forces during the collision.

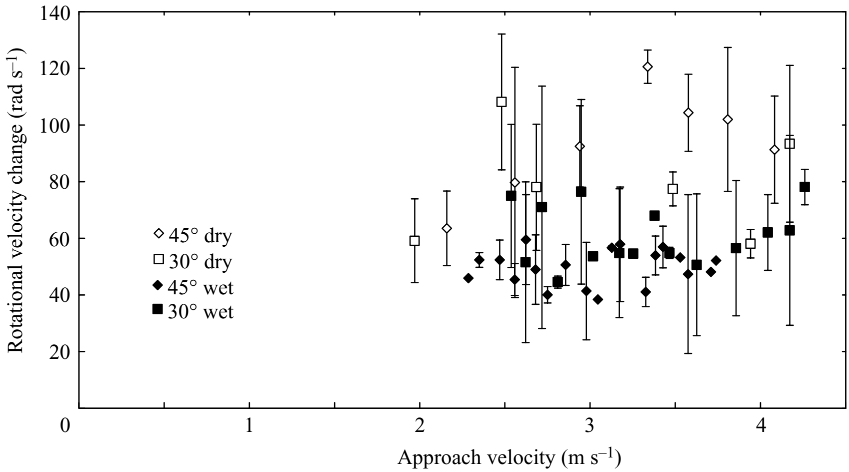

FIGURE 11. Rotational velocity change versus impact velocity for Teflon spheres of 0.0064 m diameter impacting the dry quartz disk and the same disk covered with an oil layer of thickness 80 μm and viscosity 0.975 kg m^{-1} s^{-1}.

4. Comparisons with theory

4.1. *Normal component of velocity*

Davis *et al.* (2002) described an approximate theory for head-on elastohydrodynamic collisions and rebound. It is based on lubrication theory for the viscous forces slowing an undeformed sphere as it approaches the wetted target surface, coupled with Hertzian elasticity theory to estimate the deformation and rebound. Interfacial and wetting forces due to the moving contact line as the sphere penetrates the oil layer are shown to be negligible (Barnocky & Davis 1988). The theory predicts the following rebound behaviour:

$$e_{wet} = e_{dry}(1 - St_c/St), \quad St > St_c, \tag{4.1a}$$

$$e_{wet} = 0, \quad St \leqslant St_c, \tag{4.1b}$$

where e_{wet} is the coefficient of restitution for a head-on collision with a wet surface and e_{dry} is the corresponding value for dry collisions and accounts for solid losses. The critical Stokes number for rebound can be determined experimentally or predicted by the theory:

$$St_c = \ln(x_o/x_r). \tag{4.2}$$

Here, x_o is the initial separation of the sphere and the opposing surface and is estimated as $x_o = 2\delta/3$ to account for sufficient penetration of the sphere into the liquid layer for lubrication forces to become important (Barnocky & Davis 1988), where δ is the fluid-layer thickness. The separation at which significant deformation occurs is x_r, and scaling arguments from elastohydrodynamic theory yield (Davis *et al.* 2002)

$$x_r \approx \left(3\pi\theta\mu a^{3/2} V_o/\sqrt{2}\right)^{2/5}. \tag{4.3}$$

Then, using (1.2) and (4.2),

$$St_c = 0.4\ln(1/\epsilon) - 0.2. \tag{4.4}$$

However, if the sphere or opposing surface has microscopic surface roughness elements with heights greater than x_r, then the surfaces may contact on the tips

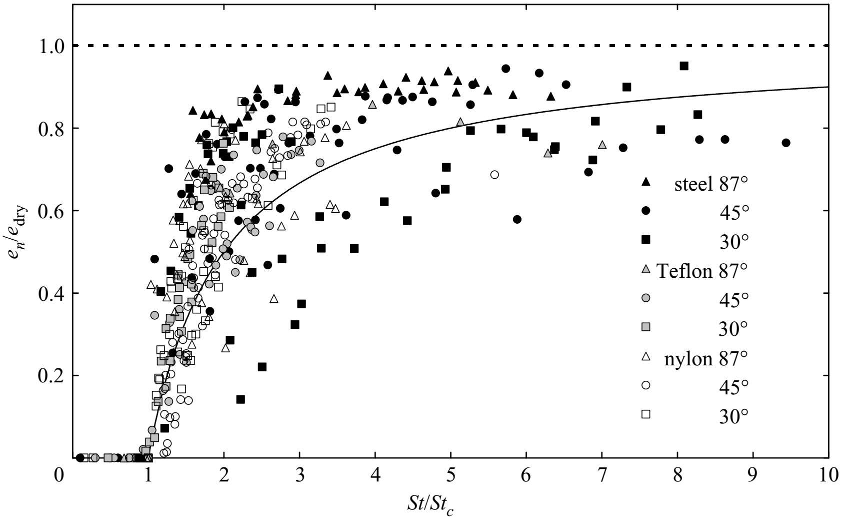

FIGURE 12. Normalized coefficient of normal restitution versus normalized Stokes number for stainless steel, Teflon and nylon balls of various sizes impacting a quartz disk covered with an oil layer of thickness 80 μm and viscosity 0.975, 4.98 and 12.4 kg m^{-1} s^{-1} for various impact angles. The solid line is the theory from (4.1*a*).

of the roughness elements and rebound more easily (owing to reduced lubrication losses). For the conditions of the experiments using oil of 0.975 kg m s^{-1} viscosity, typical values of x_r from (4.3) are 1–2 μm, 7–8 μm, and 10–12 μm for 0.0064 m steel, nylon and Teflon spheres, respectively. The hydrodynamic roughness measured by the method of Smart & Leighton (1989) is comparable to, or larger than, the respective x_r values. Thus, surface roughness elements are likely to influence the values of St_c and contribute to the data scatter.

For the oblique collisions considered in the present work, it is proposed, as a first approximation, that the normal and tangential components of the collisions are decoupled. Then, the above theory for head-on collisions may be applied to describe the normal component of oblique collisions. Equations (4.1)–(4.4) still hold, but V_o, ϵ, St and St_c should be replaced by the normal components $V_o \sin\phi_i$, $\epsilon \sin\phi_i$, $St \sin\phi_i$ and $St_c \sin\phi_i$, respectively.

In figure 12, the coefficient of normal restitution normalized by e_{dry} is plotted versus the ratio St/St_c. The experimental values of St_c are used for this figure, rather than the predicted values from (4.4), owing to the presence of surface roughness and the fact that (4.4) is restricted to conditions of $\epsilon \ll 1$ and $St \leqslant 1$ not met in some of the experiments. Both theory and experiment show that the normal coefficient of restitution is zero for $St < St_c$ and then increases rapidly for $St > St_c$ before approaching e_{dry} asymptotically for $St \gg St_c$. However, the experimental data appear to increase, on average, more rapidly with increasing St than predicted by (4.1). A possible explanation is that the fluid losses at low and moderate values of the Stokes number reduce the impulses exerted on the plate and balls and, therefore, reduce the solid losses. In addition, there is considerable scatter in the data, as was observed previously for head-on collisions (Joseph *et al.* 2001; Davis *et al.* 2002). The scatter may be due, at least in part, to variable roughness on the surface of the spheres and non-uniformities in the oil-layer thickness.

4.2. *Tangential component of velocity*

The net force exerted in the tangential direction owing to the squeezing flow from the normal motion of the sphere is zero by symmetry. The sliding motion of the sphere during an oblique collision, however, leads to a tangential viscous stress of the order

$$f = \mu V_o \cos\phi_i / x, \tag{4.5}$$

where $x < \delta$ is a characteristic sphere–plane separation. The fluid viscosity, μ, is assumed in this approximate model to be constant, whereas Barnocky & Davis (1989) and Joseph & Hunt (2004) considered the effects of increased viscosity (due to the large lubrication pressure) on the normal and tangential motion, respectively. During the penetration stage of the motion, as the sphere approaches the plane with little deformation, the characteristic area over which the sliding lubrication force acts is

$$A_p = 2\pi a x. \tag{4.6}$$

The characteristic time of action of the sliding lubrication force during penetration is approximately

$$t_p \approx 2\delta/(3V_o \sin\phi_i), \tag{4.7}$$

representing the time required for the sphere to penetrate a characteristic distance $2\delta/3$ (Barnocky & Davis 1988) at the characteristic normal velocity $V_o \sin\phi_i$.

In the deformation stage of motion, the time for the sphere to deform and reverse its direction of normal motion is given by the Hertzian contact theory (Goldsmith 1960):

$$t_d = 3.2\left[(\pi\theta)^2 a^5 \rho_s^2 / \left(V_o \sin\phi_i\right)\right]^{1/5}, \tag{4.8}$$

where ρ_s is the density of the solid sphere. The area of contact under the action of a given normal force, F_n, is (Landau & Lifshitz 1959)

$$A_d = \pi F_n^{2/3} (0.75\pi\theta a)^{2/3}. \tag{4.9}$$

The normal force acting on the sphere is the viscous lubrication force resisting the approach of the sphere, which following the asymptotic solution of Davis *et al.* (1986) is

$$F_n = 6\pi\mu a^2 V_o \sin\phi_i / x_r. \tag{4.10}$$

The viscous stress acting on the contact area is still given by (4.5), with $x \approx x_r$ from (4.3).

The lubrication force during rebound is negligible, owing to cavitation (Barnocky & Davis 1988). Thus, the change in the tangential component of momentum of the sphere is equal to the sum of the tangential impulses during penetration and rebound, yielding

$$1 - e_t \approx \frac{f_p A_p t_p + f_d A_d t_d}{m V_o \cos\phi_i} \approx \frac{0.22\delta}{a\, St \sin\phi_i}\left(1 + 9.5(\epsilon St \sin^2\phi_i)^{2/5} (\sin\phi_i)^{2/3}\right). \tag{4.11}$$

Since $\delta \ll a$, the tangential coefficient of restitution is very close to unity, except at very small values of $St \sin\phi_i$, as seen in figure 7 for metal spheres and figure 13 for plastic spheres. The data follow the trends of the theory, but show a slightly lower tangential restitution. There may have been small frictional losses due to solid–solid contacts on the roughness elements, not included in the above analysis of viscous effects. Indeed, Joseph & Hunt (2004) found larger effective friction coefficients for their rough glass spheres than for their smooth steel spheres, and they attributed the difference to solid–solid contacts for the rough spheres.

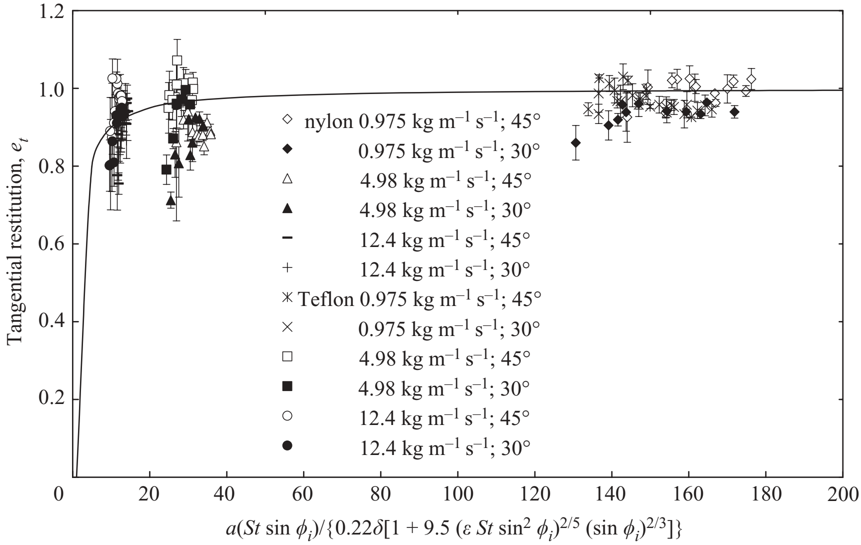

FIGURE 13. Coefficient of tangential restitution versus $a(St \sin \phi_i)/\{0.22\delta[1 + 9.5(\epsilon St \sin^2 \phi_i)^{2/5}(\sin \phi_i)^{2/3}]\}$ for nylon and Teflon spheres of diameter 0.0064 m impacting a quartz plate covered with an oil layer of 80 µm thickness and various viscosities. The solid line is the theory given by (4.11).

4.3. *Rotational velocity*

The tangential force also exerts a torque on the sphere, causing rotation:

$$\Delta\omega \approx \frac{(f_p A_p t_p + f_d A_d t_d)a}{I}, \tag{4.12}$$

where $\Delta\omega$ is the change in rotational velocity (positive for clockwise rotation) and $I = 2ma^2/5$ is the moment of inertia of the sphere. Then, using (4.5)–(4.10) and (1.2), a dimensionless rotational velocity imparted by the tangential forces during the collision is approximated by

$$\psi = \frac{a\Delta\omega}{V_o \cos\phi_i} \approx \frac{0.56\delta}{a\, St \sin\phi_i}\left(1 + 9.5(\epsilon St \sin^2\phi_i)^{2/5}(\sin\phi_i)^{2/3}\right), \tag{4.13}$$

where ψ is the dimensionless change in rotational velocity due to lubrication forces. This result indicates that the imparted rotation increases with increasing oil viscosity and layer thickness and with decreasing ball size and density, as was observed.

Figure 14 is a plot of the dimensionless change in rotational velocity for nylon and Teflon spheres (steel spheres are not included, as stripes on their surface are difficult to see) versus the reciprocal of the expression given by (4.13). The figure shows that the theory given by (4.13) provides a good estimate of the change in angular velocity due to impact with a wetted surface, despite its approximate nature. Both theory and experiment show that the dimensionless rotational velocity imparted during the collision is small at large values of the Stokes number, which follows from the fact that only a small loss in tangential restitution is converted to rotational energy, but it then increases when the Stokes number is decreased for smaller impact velocities, lighter spheres, and more viscous oil layers. At large Stokes numbers, the observed rotational velocities exceed the predictions, which again may indicate the role of solid–solid contacts adding a frictional force and torque. In this range,

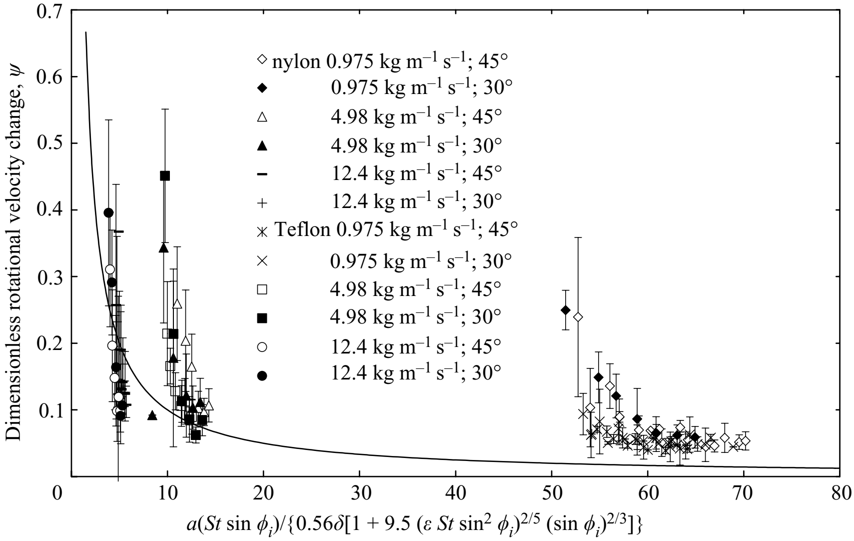

FIGURE 14. Non-dimensional change in angular velocity, ψ, versus $a(St\sin\phi_i)/\{0.56\delta[1+9.5\,(\epsilon St\sin^2\phi_i)^{2/5}(\sin\phi_i)^{2/3}]\}$ for impact with a quartz disk covered with an oil layer of thickness 80 μm and viscosity 0.975, 4.98 and 12.4 kg m^{-1} s^{-1} using nylon and Teflon spheres of 0.0064 m diameter. The solid line is the theoretical curve for ψ from (4.13).

the dimensionless rotation rates for nylon generally exceed those for Teflon, which may be due to the greater penetration (smaller x_r) of the stiffer nylon spheres, leading to more solid–solid contact.

The second term on the right-hand sides of (4.11) and (4.13), which corresponds to the impulse imparted to the sphere during the deformation stage, becomes more important for softer material (higher ϵ) and collisions with higher inertia (larger St). For the experiments in figures 13 and 14, the second term is in the range 0.8–2.2 times the first term for nylon, whereas it is 1.8–5.7 times the first term for Teflon (which is softer and denser), with the larger values in these ranges occurring on the right-hand side of the plots where St and ϵ are largest.

5. Conclusions

Elastohydrodynamic rebound experiments were performed by dropping steel, nylon and Teflon balls onto a quartz surface covered with a thin oil layer and inclined at various angles. As in head-on collisions, no rebound was observed unless the approach velocity exceeded a critical value, as the kinetic energy of the sphere is otherwise dissipated by viscous losses. The critical approach velocity for rebound increases with increasing deviation of the impact angle from a normal collision. The normal component of the rebound velocity for impact velocities greater than the critical value increases with a normal Stokes number: $St\sin\phi_i = (2\,a\rho_s V_o\sin\phi_i)/(9\mu)$ where ρ_s is the density of the sphere, a is its radius, V_o is the approach velocity, ϕ_i is the impact angle relative to the surface tangent, and μ is the fluid viscosity. Thus, previous findings for head-on collisions may be applied to the normal component of oblique collisions, provided that the normal component of the approach velocity as well as the normal component of the rebound velocity are used in the analysis. On the other hand, the tangential component of velocity for oblique collisions was observed to be reduced by only a small fraction for both dry and wet collisions, under the conditions investigated. As a result, the rebound angle relative to the impact surface is reduced owing to the reduced normal component of the rebound velocity. A theory was developed to account for the small effects of the fluid layer on the tangential and rotational motion of the sphere. This theory shows that losses in tangential restitution and the dimensionless rotational velocity due to oblique impact with a fluid layer are small compared to unity, except at small normal Stokes numbers.

This work is expected to have practical applications in processes such as fluidization, agglomeration, wet granular flow, and the collection of particles from gas streams by impact with wet filter elements. Of particular interest in these processes are predicting the conditions for which particles stick or rebound on colliding with a wet surface, and predicting the rebound velocity and angle for collisions where rebound does occur.

The authors would like to thank Matthew P. Hyman for helping with the experiments. This work was supported by NASA grant NCC3-796.

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