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RIGIDITY FOR ELLIPTIC ISOMETRIC IMBEDDINGS

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Introduction

The main purpose of the present paper is to give the details of the results announced in the P. J. A. note [11], establishing some global theorems on rigidity for a certain class of isometric imbeddings.

We first introduce the notion of an elliptic imbedding: An imbedding f of a manifold M in a Euclidean space $\boldsymbol{R}^m$ is called elliptic if it is generic in a suitable sense and if the second fundamental form corresponding to any normal vector $\neq 0$ of the imbedding has at least two eigenvalues of the same sign. We then prove a rigidity theorem (Theorem 2.4) which may be roughly stated as follows: Let f_0 be an imbedding of M in $\boldsymbol{R}^m$. Assume that 1) f_0 is elliptic, 2) it is "infinitesimally rigid" and 3) M is compact. If two imbeddings f and f' of M in $\boldsymbol{R}^m$ lie both near to f_0 with respect to the C^3-topology and if they induce the same Riemannian metric on M, then there is a unique Euclidean transformation a of $\boldsymbol{R}^m$ such that $f' = af$. Moreover we apply Theorem 2.4 to the canonical isometric imbedding f_0 of a compact hermitian symmetric space $M = G/H$ in the Euclidean space $\boldsymbol{R}^m$, where $m = \dim G$. In fact it is shown that the imbedding f_0 satisfies the conditions 1), 2) and 3) stated above (Proposition 3.3 and Theorem 3.4). Thus we obtain a rigidity theorem (Theorem 3.5) for imbeddings around f_0, which turns out to be a partial generalization of the famous theorem of Cohn-Vossen.

In § 1 we first define an important differential operator $L = L^f$, which is associated with every imbedding f of M in $\boldsymbol{R}^m$ satisfying condition (C), where "condition (C)" is the very generality condition to impose on an elliptic imbedding. It is shown that the operator L is, in a suitable sense, equivalent to the differential operator Φ_{*f}, "the operator of infinitesimal isometric deformations" of f (Theorem 1.2). We then

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proceed to the definition of an elliptic imbedding and find that an imbedding $\boldsymbol{f}$ is elliptic if and only if the operator L is elliptic.

In § 2 we prove Theorem 2.4. The proof heavily depends on Theorem 2.3 which is the principle of upper semi-continuity concerning the operators L^f parametrized by the elliptic imbeddings $\boldsymbol{f}$ and which is analogous to Theorem 4 in Kodaira-Spencer [4]. Finally § 3 is devoted to the proof of Theorem 3.4, which needs some calculations on the Laplacian Δ of the hermitian symmetric space $M = G/H$.

Preliminary remark

Throughout the present paper we shall always assume the differentiability of class C^∞.

Let E be a differentiable vector bundle over a differentiable manifold M. E_p will denote the fibre of E over a point $p \in M$. $\Gamma(E)$ will denote the vector space of differentiable cross sections on M. *An inner product* $\langle\ ,\ \rangle$ in E is an assignment which assigns to every point $p \in M$ an inner product $\langle\ ,\ \rangle$ in the fibre E_p and which is differentiable in an appropriate sense.

§ 1. Elliptic imbeddings

1.1. The differential operator Φ_{*f}. Let M be a connected differentiable manifold. T denotes the tangent bundle of M and T^* its dual. S^2T^* denotes the vector bundle of symmetric tensors of type $\binom{0}{2}$ on M. For $\alpha, \beta \in T_p^*$, $\alpha \cdot \beta$ denotes the symmetric product of α and β, being an element of $S^2T_p^*$.

Let $\boldsymbol{R}^m$ be the space of m real variables and $x_1, \cdots, x_m$ the canonical coordinates of it. As usual $\boldsymbol{R}^m$ is an m-dimensional Euclidean space (flat Riemannian manifold) with respect to the Riemannian metric $ds^2 = \sum_i dx_i^2$. $\boldsymbol{R}^m$ is also considered as an m-dimensional Euclidean vector space with respect to the inner product $\langle\ ,\ \rangle$ defined as follows: $\langle \boldsymbol{a}, \boldsymbol{b}\rangle = \sum_i a_i b_i$, where $\boldsymbol{a} = (a_1, \cdots, a_m)$, $\boldsymbol{b} = (b_1, \cdots, b_m) \in \boldsymbol{R}^m$.

Let $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_m)$ be $\boldsymbol{R}^m$-valued 1-forms on M and let $\boldsymbol{f} = (f_1, \cdots, f_m)$ be an $\boldsymbol{R}^m$-valued function on M. $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle$ denotes the cross section of S^2T^* defined by $\langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle = \sum_i \alpha_i \beta_i$ or

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle(X, Y) = \tfrac{1}{2}(\langle \boldsymbol{\alpha}(X), \boldsymbol{\beta}(Y)\rangle + \langle \boldsymbol{\alpha}(Y), \boldsymbol{\beta}(X)\rangle) .$$

for all $X, Y \in T_p$ and $p \in M$. $\langle \boldsymbol{f}, \boldsymbol{\alpha}\rangle$ denotes the 1-form on M defined by

$\langle \boldsymbol{f}, \boldsymbol{\alpha}\rangle = \sum_i f_i \alpha_i$ or $\langle \boldsymbol{f}, \boldsymbol{\alpha}\rangle(X) = \langle \boldsymbol{f}, \boldsymbol{\alpha}(X)\rangle$. $d\boldsymbol{f}$ denotes the exterior derivative of $\boldsymbol{f}$: $d\boldsymbol{f} = (df_1, \cdots, df_m)$.

Let $\Gamma(M, m)$ be the vector space of differentiable maps of M to $\boldsymbol{R}^m$, which may be regarded as the vector space of $\boldsymbol{R}^m$-valued differentiable functions on M. Let $\mathfrak{E}(M, m)$ be the subset of $\Gamma(M, m)$ consisting of all the imbeddings of M in $\boldsymbol{R}^m$. We assume $\mathfrak{E}(M, m) \neq \phi$. For $\boldsymbol{f} \in \mathfrak{E}(M, m)$, we denote by $\Phi(\boldsymbol{f})$ the Riemannian metric $\boldsymbol{f}^* ds^2$ on M which is induced from ds^2 by the imbedding $\boldsymbol{f}$; We have

$$\Phi(\boldsymbol{f}) = \langle d\boldsymbol{f}, d\boldsymbol{f}\rangle .$$

Given a Riemannian metric ν on M, $\boldsymbol{f} \in \mathfrak{E}(M, m)$ is called *an isometric imbedding* of the Riemannian manifold (M, ν) in the Euclidean space $\boldsymbol{R}^m$ if $\Phi(\boldsymbol{f})$ coincides with the given ν.

The assignment $\boldsymbol{f} \to \Phi(\boldsymbol{f})$ gives a map Φ of the set $\mathfrak{E}(M, m)$ to the set $\mathfrak{R}(M)$ of Riemannian metrics on M. For $\boldsymbol{f} \in \mathfrak{E}(M, m)$, we define a differential operator Φ_{*f} of $\Gamma(M, m)$ to $\Gamma(S^2T^*)$, *the differential* of the map Φ at $\boldsymbol{f}$, by

$$\Phi_{*f}(\boldsymbol{u}) = 2\langle d\boldsymbol{f}, d\boldsymbol{u}\rangle$$

for all $\boldsymbol{u} \in \Gamma(M, m)$.

1.2. The second fundamental forms. Let $\boldsymbol{f}$ be an imbedding of M in $\boldsymbol{R}^m$. Let T' be the vector bundle on M which is induced from the tangent bundle $T(\boldsymbol{R}^m)$ of $\boldsymbol{R}^m$ by the imbedding $\boldsymbol{f}$. The Riemannian metric ds^2 on $\boldsymbol{R}^m$ induces an inner product in the vector bundle T' and the tangent bundle T of M may be identified with a subbundle of T'. This being said, *the normal bundle* N of $\boldsymbol{f}$ is the orthogonal complement of T in T' with respect to the inner product in T': $T' = T \oplus N$. T' being a trivial bundle in a canonical manner, every fibre T'_p of T' may be identified with the Euclidean vector space $\boldsymbol{R}^m$ and hence a cross section of T' may be regarded as an $\boldsymbol{R}^m$-valued function on M and vice versa.

Let ∇ be the covariant differentiation[1] (Riemannian connection) associated with the Riemannian metric $\nu = \Phi(\boldsymbol{f})$. Given an $\boldsymbol{R}^m$-valued tensor field $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_m)$ of type $\binom{0}{p}$ on M, we define the covariant derivative $\nabla\boldsymbol{\alpha}$ of $\boldsymbol{\alpha}$ in an obvious manner: $\nabla\boldsymbol{\alpha} = (\nabla\alpha_1, \cdots, \nabla\alpha_m)$. Note that $\nabla_X \boldsymbol{u} = d\boldsymbol{u}(X) = X\boldsymbol{u}$ for all $X \in T_p$ and $\boldsymbol{R}^m$-valued functions $\boldsymbol{u}$ on M.

The following proposition is known.

[1] As for the covariant differentiation ∇, we use the same notations as in [3].

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PROPOSITION 1.1. *For all* X, $Y \in T_p$, *the vectors* $\nabla_X \nabla_Y \boldsymbol{f}$ *are in the fibre* N_p *of the normal bundle* N.

Proof. We have $\langle \nabla_Y \boldsymbol{f}, \nabla_Z \boldsymbol{f} \rangle = \nu(Y, Z)$ for all Y, $Z \in T_p$, whence

$$\langle \nabla_X \nabla_Y \boldsymbol{f}, \nabla_Z \boldsymbol{f} \rangle + \langle \nabla_Y \boldsymbol{f}, \nabla_X \nabla_Z \boldsymbol{f} \rangle = (\nabla_X \nu)(Y, Z) = 0$$

For all $X \in T_p$. Hence we have

$$\langle \nabla_Y \nabla_X \boldsymbol{f}, \nabla_X \boldsymbol{f} \rangle = 0 ,$$

$$\langle \nabla_X \nabla_X \boldsymbol{f}, \nabla_Y \boldsymbol{f} \rangle + \langle \nabla_X \boldsymbol{f}, \nabla_X \nabla_Y \boldsymbol{f} \rangle = 0$$

for all X, $Y \in T_p$. Since $\nabla_X \nabla_Y \boldsymbol{f} = \nabla_Y \nabla_X \boldsymbol{f}$ and since T_p is composed of the vectors $\nabla_Y \boldsymbol{f} (Y \in T_p)$, we have $\nabla_X \nabla_X \boldsymbol{f} \in N_p$ and hence $\nabla_X \nabla_Y \boldsymbol{f} \in N_p$, proving our assertion.

Let us now consider the following condition (C)[2] for the imbedding $\boldsymbol{f}$: At each $p \in M$, the fibre N_p of N is spanned by the vectors of the form $\nabla_X \nabla_Y \boldsymbol{f}$, where X, $Y \in T_p$.

For $\boldsymbol{a} \in N_p$, we define an element $\theta_{\boldsymbol{a}}$ of $S^2 T_p^*$ by

$$\theta_{\boldsymbol{a}}(X, Y) = \langle \boldsymbol{a}, \nabla_X \nabla_Y \boldsymbol{f} \rangle ,$$

which is usually called *the second fundamental form* of $\boldsymbol{f}$ corresponding to the normal vector $\boldsymbol{a}$. Then we see that $\boldsymbol{f}$ satisfies condition (C) if and only if the map θ: $N \ni \boldsymbol{a} \to \theta_{\boldsymbol{a}} \in S^2 T^*$ is injective.

1.3. The differential operator L. In what follows we assume that the imbedding $\boldsymbol{f}$ satisfies condition (C). By the above remark the image $\tilde{N}$ of N by θ forms a subbundle of $S^2 T^*$, which will be called *the bundle of second fundamental forms* of $\boldsymbol{f}$.

We define a differential operator D of $\Gamma(T^*)$ to $\Gamma(S^2 T^*)$ by

$$(D\varphi)(X, Y) = (\nabla_X \varphi)(Y) + (\nabla_Y \varphi)(X)$$

for all $\varphi \in \Gamma(T^*)$ and X, $Y \in T_p$, and denote by π the projection of $S^2 T^*$ onto the factor bundle $S^2 T^* / \tilde{N}$. Then the composition $L = \pi \circ D$ is a differential operator of $\Gamma(T^*)$ to $\Gamma(S^2 T^* / \tilde{N})$.

THEOREM 1.2. *Let* $\boldsymbol{f}$ *be an imbedding of* M *in* $\boldsymbol{R}^m$ *satisfying condition* (C) *and let* $\alpha \in \Gamma(S^2 T^*)$. *Then the solutions* $\boldsymbol{u}$ *of the equation*

[2] This is equivalent to the condition that, at each $p \in M$, the vector space $\boldsymbol{R}^m$ is spanned by the vectors of the form $(\partial f / \partial x_i)(p)$, $(\partial^2 f / \partial x_i \partial x_j)(p)$, where $x_1, \cdots, x_n$ is a coordinate system of M at p.

$\Phi_{*f}(\boldsymbol{u})=\alpha$ are in a one-to-one correspondence with the solutions φ of the equation $L\varphi=\pi\alpha$, and the correspondence $\boldsymbol{u}\to\varphi$ is given by the relation $\varphi=\langle\boldsymbol{u}, d\boldsymbol{f}\rangle$.

Proof. Let $\boldsymbol{u}\in\Gamma(M, m)$. Put $\varphi=\langle\boldsymbol{u}, d\boldsymbol{f}\rangle$ and denote by X the vector field on M which is dual to the 1-form φ with respect to the Riemannian metric ν. Put

$$\boldsymbol{v}=\boldsymbol{u}-X\boldsymbol{f}. \tag{1.1}$$

Then we have

$$\begin{aligned}\langle\boldsymbol{v}, d\boldsymbol{f}(Y)\rangle &= \langle\boldsymbol{u}, d\boldsymbol{f}(Y)\rangle-\langle d\boldsymbol{f}(X), d\boldsymbol{f}(Y)\rangle \\ &= \varphi(Y)-\nu(X, Y)=0\end{aligned}$$

for all $Y\in T_p$, showing that $\boldsymbol{v}$ may be regarded as a cross section of N. Furthermore from (1.1) we get

$$\nabla_Y\boldsymbol{u}=\nabla_Y\boldsymbol{v}+\nabla_Y\nabla_X\boldsymbol{f}+(\nabla_Y X)\boldsymbol{f} \tag{1.2}$$

for all $Y\in T_p$. Since $\langle\boldsymbol{v}, \nabla_Z\boldsymbol{f}\rangle=0$ for all $Z\in T_p$, we have $\langle\nabla_Y\boldsymbol{v}, \nabla_Z\boldsymbol{f}\rangle=-\langle\boldsymbol{v}, \nabla_Y\nabla_Z\boldsymbol{f}\rangle$ for all Y, $Z\in T_p$. Therefore it follows from (1.2) and Proposition 1.1 that

$$\begin{aligned}\langle\nabla_Y\boldsymbol{u}, \nabla_Z\boldsymbol{f}\rangle &= \langle(\nabla_Y X)\boldsymbol{f}, \nabla_Z\boldsymbol{f}\rangle+\langle\nabla_Y\boldsymbol{v}, \nabla_Z\boldsymbol{f}\rangle \\ &= \nu(\nabla_Y Y, Z)-\langle\boldsymbol{v}, \nabla_Y\nabla_Z\boldsymbol{f}\rangle\end{aligned}$$

and hence

$$\langle\nabla_Y\boldsymbol{u}, \nabla_Z\boldsymbol{f}\rangle=(\nabla_Y\varphi)(Z)-\theta_v(Y, Z),$$

where θ_v is the cross section of $\tilde{N}$ defined by $(\theta_v)_p=\theta(\boldsymbol{v}(p))$. Commuting Y and Z in the above equality, we also obtain

$$\langle\nabla_Z\boldsymbol{u}, \nabla_Y\boldsymbol{f}\rangle=(\nabla_Z\varphi)(Y)-\theta_v(Z, Y).$$

Therefore from these two equalities we get

$$2\langle d\boldsymbol{f}, d\boldsymbol{u}\rangle=D\varphi-2\theta_v. \tag{1.3}$$

Now suppose that $\boldsymbol{u}$ is a solution of the equation $\Phi_{*f}(\boldsymbol{u})=\alpha$. θ_v being a cross section of $\tilde{N}$, we know from (1.3) that φ is a solution of the equation $L\varphi=\pi\alpha$. Conversely let φ be a solution of the equation $L\varphi=\pi\alpha$. Since $D\varphi-\alpha$ is a cross section of $\tilde{N}$ and since $\theta: N\to\tilde{N}$ is an isomorphism, there is a unique cross section $\boldsymbol{v}$ of N such that $D\varphi-\alpha=2\theta_v$. If we put $\boldsymbol{u}=\boldsymbol{v}+X\boldsymbol{f}, X$ being the dual to φ, then we can see

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from the above argument that $\langle \boldsymbol{u}, df \rangle = \varphi$ and that $\boldsymbol{u}$ is a solution of the equation $\Phi_{*f}(\boldsymbol{u}) = \alpha$. We have thereby proved Theorem 1.2.

1.4. An elliptic imbedding. Let $\boldsymbol{f}$ be an imbedding of M in $\boldsymbol{R}^m$ satisfying condition (C). Let ξ be any covector at any point $p \in M$, i.e., $\xi \in T_p^*$. *The symbol* $\sigma(\xi)$ of the operator L at ξ is a linear map of T_p^* to $S^2T_p^*/\tilde{N}_p$ defined as follows: Take a function f on M such that $f(p) = 0$ and $df_p = \xi$ and, for any $\eta \in T_p^*$, take $\varphi \in \Gamma(T^*)$ such that $\varphi_p = \eta$. Then $\sigma(\xi)\eta = L(f\varphi)_p$. We have

$$D(f\varphi)(X, Y) = \nabla_X f \cdot \varphi(Y) + \nabla_Y f \cdot \varphi(X) + fD\varphi(X, Y)$$

for all X, $Y \in T_p$, whence $D(f\varphi)_p = 2\xi \cdot \eta$. Therefore we get

$$\sigma(\xi)\eta = 2\pi(\xi \cdot \eta) . \tag{1.4}$$

The operator L is called *elliptic* if the symbol $\sigma(\xi)$ is injective at any non-zero covector ξ. (For the theory of elliptic linear differential operators on manifolds, we refer to [10].) Furthermore we shall say that a subbundle of S^2T^* is *elliptic* if it contains no non-zero elements of the form $\xi \cdot \eta$, where ξ and η are covectors with the same origin, or equivalently if every symmetric form ($\neq 0$) in the subbundle has at least two eigenvalues of the same sign.

By (1.4) we have the following

PROPOSITION 1.3. *The differential operator L associated with $\boldsymbol{f}$ is elliptic if and only if the bundle $\tilde{N}$ of second fundamental forms of $\boldsymbol{f}$ is elliptic.*

We shall say that an imbedding $\boldsymbol{f}$ of M in $\boldsymbol{R}^m$ is *elliptic* if it satisfies condition (C) and if the bundle $\tilde{N}$ is elliptic.

PROPOSITION 1.4. *Let $\boldsymbol{f}$ be an imbedding of M in $\boldsymbol{R}^m$. We assume that $\boldsymbol{f}$ is elliptic and that M is compact. Then the solution space of the equation $\Phi_{*f}(\boldsymbol{u}) = 0$ is finite dimensional.*

This follows immediately from Theorem 1.2 and a well known theorem on elliptic operators.

§2. A rigidity theorem for elliptic imbeddings

2.1. Imbeddings and Euclidean transformations. Let $O(m)$ be the orthogonal group of degree m. *A Euclidean transformation* of $\boldsymbol{R}^m$ is a transformation a of the form:

$$a\boldsymbol{x} = b\boldsymbol{x} + \boldsymbol{c} \qquad (\boldsymbol{x} \in \boldsymbol{R}^m)\,,$$

where $b \in O(m)$ and $\boldsymbol{c} \in \boldsymbol{R}^m$. The set $E(m)$ of Euclidean transformations forms a Lie group.

Let $\mathfrak{o}(m)$ be the Lie algebra of skew-symmetric matrices of degree m, being the Lie algebra of $O(m)$. *An infinitesimal Euclidean transformation* of $\boldsymbol{R}^m$ is a map A of $\boldsymbol{R}^m$ to itself of the form:

$$A\boldsymbol{x} = B\boldsymbol{x} + \boldsymbol{c} \qquad (\boldsymbol{x} \in \boldsymbol{R}^m)\,,$$

where $B \in \mathfrak{o}(m)$ and $\boldsymbol{c} \in \boldsymbol{R}^m$. The set $\mathfrak{e}(m)$ of infinitesimal Euclidean transformations forms a Lie algebra, being the Lie algebra of $E(m)$.

Let M and $\boldsymbol{R}^m$ be as in § 1. For $\boldsymbol{f} \in \mathfrak{E}(M, m)$ and $a \in E(m)$, we have $a\boldsymbol{f} \in \mathfrak{E}(M, m)$ and

$$\Phi(a\boldsymbol{f}) = \Phi(\boldsymbol{f})\,.$$

Thus the group $E(m)$ acts on $\mathfrak{E}(M, m)$ as a transformation group and the map Φ is an invariant. It is clear that $E(m)$ leaves invariant the subset $\mathfrak{E}_0$ of $\mathfrak{E}(M, m)$ composed of all the imbeddings satisfying condition (C) as well as the subset $\mathfrak{E}_1$ of $\mathfrak{E}(M, m)$ composed of all the elliptic imbeddings.

PROPOSITION 2.1. *The group $E(m)$ freely acts on the subset $\mathfrak{E}_0$ of $\mathfrak{E}(M, m)$.*

Proof. Suppose that $a\boldsymbol{f} = \boldsymbol{f}$, where $a \in E(m)$ and $\boldsymbol{f} \in \mathfrak{E}_0$. Then $b\boldsymbol{f} + \boldsymbol{c} = \boldsymbol{f}$, whence $b\nabla_X\boldsymbol{f} = \nabla_X\boldsymbol{f}$ and $b\nabla_X\nabla_Y\boldsymbol{f} = \nabla_X\nabla_Y\boldsymbol{f}$ for all X, $Y \in T_p$. Since the vectors $\nabla_X\boldsymbol{f}$, $\nabla_X\nabla_Y\boldsymbol{f}$ span the vector space $\boldsymbol{R}^m$, we have $b = e$, the identity, and $\boldsymbol{c} = 0$, proving our assertion.

PROPOSITION 2.2. *Let $\boldsymbol{f}$ be an imbedding of M in $\boldsymbol{R}^m$.*

(1) *For all $A \in \mathfrak{e}(m)$, $A\boldsymbol{f}$ is a solution of the equation $\Phi_{*f}(\boldsymbol{u})$ 0.*

(2) *If $\boldsymbol{f}$ satisfies condition (C), then the map $\mathfrak{e}(m) \ni A \to A\boldsymbol{f} \in \Gamma(M, m)$ is injective.*

This is clear from the invariance of Φ and Proposition 2.1.

Let $\rho(\boldsymbol{f})$ denote the dimension of the solution space of the equation $\Phi_{*f}(\boldsymbol{u}) = 0$, which is also the dimension of the solution space of the equation $L\varphi = 0$ by Theorem 1.2. Then Proposition 2.2 implies

$$\rho(\boldsymbol{f}) \geqq \dim E(m) = \tfrac{1}{2}m(m+1)\,,$$

provided that $\boldsymbol{f}$ satisfies condition (C).

2.2. Upper semi-continuity of the dimension $\rho(f)$. Let us introduce the C^r-topology in the set $\Gamma(M,m)$. (Let J^r be the vector bundle on M consisting of all the r-jets of local differentiable maps of M in $\boldsymbol{R}^m$. For $\boldsymbol{u} \in \Gamma(M,m)$, let $j_p^r\boldsymbol{u}$ denote the r-jet of $\boldsymbol{u}$ at $p \in M$. Then the assignment $p \to j_p^r\boldsymbol{u}$ gives a cross section $j^r\boldsymbol{u}$ of J^r, and the map $j^r: \Gamma(M,m) \ni \boldsymbol{u} \to j^r\boldsymbol{u} \in \Gamma(J^r)$ is injective. This being said, *the C^r-topology* in $\Gamma(M,m)$ is the topology in $\Gamma(M,m)$ which is induced from the compact-open topology in $\Gamma(J^r)$ by the injective map j^r.) We denote by $\mathfrak{E}(M,m)_{C^r}$ the subset $\mathfrak{E}(M,m)$ of $\Gamma(M,m)$ equipped with the C^r-topology.

We shall prove the following

THEOREM 2.3 (cf. [4], Theorem 4). *Let f_0 be an imbedding of M in $\boldsymbol{R}^m$. We assume that f_0 is elliptic and that M is compact. Then there is a neighborhood $U_1(f_0)$ of f_0 in $\mathfrak{E}(M,m)_{C^3}$ such that $\rho(f) \leqq \rho(f_0)$ for every $f \in U_1(f_0)$.*

In what follows, $\nabla^f, \tilde{N}^f, L^f$ with respectively mean the covariant differentiation ∇, the bundle $\tilde{N}$, the operator L which correspond to an imbedding $f \in \mathfrak{E}(M,m)$.

The Riemannian metric $\nu = \Phi(f)$ differentiably depends on the 1-jet j^1f of f, that is, the components g_{ij} of ν with respect to a coordinate system $x_1, \cdots, x_n$ of M are differentiable functions of j^1f: $\nu = \sum_{i,j} g_{ij}dx_idx_j$ and $g_{ij} = G_{ij}\circ j^1f$, where G_{ij} are differentiable functions defined on an open set of J^1.

(2.1) It follows that the covariant differentiation $\nabla = \nabla^f$ differentiably depends on the 2-jet j^2f of f, that is, the Riemann-Christoffel symbols Γ^i_{jk} associated with g_{ij} are differentiable functions of j^2f in an analogous sense to the above.

(2.2) Hence the derivative $\nabla\nabla f$ differentiably depends on j^2f, that is, the components of $\nabla\nabla f$ with respect to the coordinate system $x_1, \cdots, x_n$ are differentiable functions of j^2f.

We choose once for all a subbundle F of S^2T^* complementary to $\tilde{N}^{f_0}$. By (2.2) we know that there is a neighborhood $V(f_0)$ of f_0 in $\mathfrak{E}(M,m)_{C^2}$ such that every $f \in V(f_0)$ is elliptic. If we choose $V(f_0)$ sufficiently small, we also find that $S^2T^* = \tilde{N}^f \oplus F$ for every $f \in V(f_0)$. Hence the operator $L^f(f \in V(f_0))$ may be regarded as an operator of $\Gamma(T^*)$ to $\Gamma(F)$. From (2.1) we infer that the operator L^f differentiably depends on the 2-jet j^2f of f. Namely let $x_1, \cdots, x_n$ be a coordinate system of

M at any point p_0 and let $\alpha_1, \cdots, \alpha_l$ be a moving frame of F defined on a neighborhood of p_0. Then we have

$$L^f \varphi = \sum_\lambda \left(\sum_{i,j} a^\lambda_{ij} \frac{\partial u_i}{\partial x_j} + \sum_i b^\lambda_i u_i \right) \alpha_\lambda ,$$

where $\varphi \in \Gamma(T^*)$ and $\varphi = \sum_i u_i dx_i$, and the components $a^\lambda_{ij}, b^\lambda_i$ are differentiable functions of $j^2 \boldsymbol{f}$.

We are now able to prove Theorem 2.3 in a standard fashion by using some basic facts in the theory of elliptic operators. For completeness we shall accomplish the proof in Appendix.

2.3. A rigidity theorem for elliptic imbeddings.

THEOREM 2.4. *Let $\boldsymbol{f}_0$ be an imbedding of M in $\boldsymbol{R}^m$. We assume that $\boldsymbol{f}_0$ is elliptic, $\rho(\boldsymbol{f}_0) = \frac{1}{2}m(m+1)$ and that M is compact. Then there is a neighborhood $U(\boldsymbol{f}_0)$ of $\boldsymbol{f}_0$ in $\mathfrak{S}(M, m)_{C^3}$ having the following property: If $\boldsymbol{f}, \boldsymbol{f}' \in U(\boldsymbol{f}_0)$ and if $\Phi(\boldsymbol{f}) = \Phi(\boldsymbol{f}')$, there is a unique Euclidean transformation a of $\boldsymbol{R}^m$ such that $\boldsymbol{f}' = a\boldsymbol{f}$.*

Proof. For $\boldsymbol{f}, \boldsymbol{f}' \in \mathfrak{S}(M, m)$, we put $\boldsymbol{u} = \boldsymbol{f}' - \boldsymbol{f}$ and $\boldsymbol{h} = \boldsymbol{f} + \frac{1}{2}\boldsymbol{u}$. Then we can find a neighborhood $U(\boldsymbol{f}_0)$ of $\boldsymbol{f}_0$ in $\mathfrak{S}(M, m)_{C^3}$ such that, for any $\boldsymbol{f}, \boldsymbol{f}' \in U(\boldsymbol{f}_0)$, $\boldsymbol{h}$ is in $U_1(\boldsymbol{f}_0)$, where $U_1(\boldsymbol{f}_0)$ is a neighborhood of $\boldsymbol{f}_0$ in $\mathfrak{S}(M, m)_{C^3}$ having the property in Theorem 2.3. Let $\boldsymbol{f}, \boldsymbol{f}' \in U(\boldsymbol{f}_0)$ be such that $\Phi(\boldsymbol{f}) = \Phi(\boldsymbol{f}')$. This clearly gives $\Phi_{*\boldsymbol{h}}(\boldsymbol{u}) = 2\langle d\boldsymbol{h}, d\boldsymbol{u}\rangle = 0$. Since $\boldsymbol{h} \in U_1(\boldsymbol{f}_0)$, we have

$$\tfrac{1}{2}m(m+1) \leqq \rho(\boldsymbol{h}) \leqq \rho(\boldsymbol{f}_0) = \tfrac{1}{2}m(m+1) ,$$

whence $\rho(\boldsymbol{h}) = \frac{1}{2}m(m+1)$. From these facts and Proposition 2.2 we see that there are a unique $B \in \mathfrak{o}(m)$ and a unique $\boldsymbol{c} \in \boldsymbol{R}^m$ such that

$$\boldsymbol{u} = B\boldsymbol{h} + \boldsymbol{c} = B(\boldsymbol{f} + \tfrac{1}{2}\boldsymbol{u}) + \boldsymbol{c} .$$

We have $\det(1 - \frac{1}{2}B) \neq 0$, because B is a skew-symmetrix matrix. Therefore it follows that

$$\boldsymbol{u} = (1 - \tfrac{1}{2}B)^{-1}B\boldsymbol{f} + (1 - \tfrac{1}{2}B)^{-1}\boldsymbol{c}$$

and hence

$$\boldsymbol{f}' = (1 + (1 - \tfrac{1}{2}B)^{-1}B)\boldsymbol{f} + (1 - \tfrac{1}{2}B)^{-1}\boldsymbol{c} .$$

An easy calculation shows that $1 + (1 - \frac{1}{2}B)^{-1}B$ is an orthogonal matrix,

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proving the existence part of Theorem 2.4. The uniqueness part follows from Proposition 2.1.

2.4. Remarks on the operator L. Let f be an imbedding of M in $\boldsymbol{R}^m$ satisfying condition (C). If we put $n = \dim M$, then we have $m = n + \dim N$ and $\dim N = \dim \tilde{N} \leqq \dim S^2T^* = \frac{1}{2}n(n+1)$, where N (resp. $\tilde{N}$) is the normal bundle (resp. the bundle of second fundamental forms) of f. Hence the dimensions n and m must satisfy the inequalities:

$$n \leqq m \leqq \tfrac{1}{2}n(n+3)\,.$$

We shall now explain (without proof) how the properties of the operator L associated with f depend on the dimensions n and m and how the operator is connected with the imbedding problem for Riemannian manifolds.

a) The operator L is over-determined when $n \leqq m < \frac{1}{2}n(n+1)$.

b) PROPOSITION 2.5. *In order that the operator L is elliptic, it is necessary that*[3] $n \leqq m < \frac{1}{2}n(n+1)$ *or* $n = 2$ *and* $m = 3$.

c) The fibre $\tilde{N}_p$ of $\tilde{N}$ at $p \in M$, being a subspace of $S^2T_p^*$, may be identified with a vector space of symmetric endomorphisms of the Euclidean vector space T_p (The inner product in T_p is defined by the Riemannian metric $\nu = \Phi(f)$.). This being said, we define a subspace $\mathfrak{g}_p$ of $\mathrm{Hom}(T_p, T_p)$ by $\mathfrak{g}_p = \mathfrak{o}(T_p) + \tilde{N}_p$, where $\mathfrak{o}(T_p)$ is the vector space of skew-symmetric endomorphisms of T_p. We note that the vector bundle $\mathfrak{g} = \bigcup_p \mathfrak{g}_p$ may be regarded as the symbol[4] of the equation $L\varphi = 0$. Let $\mathfrak{g}_p^{(i)}$ be the i-th prolongation[5] of the subspace $\mathfrak{g}_p$ of $\mathrm{Hom}(T_p, T_p)$. Then we have $\dim \mathfrak{g}_p^{(1)} = n \dim \tilde{N}$.

We say that the imbedding f is involutive if $\mathfrak{g}_p$ is involutive[6] at every $p \in M$. In order that f is involutive, it is necessary that $\frac{1}{2}n(n+1) \leqq m \leqq \frac{1}{2}n(n+3)$.

PROPOSITION 2.6. *The imbedding f is involutive if and only if the equation $L\varphi = 0$ is involutive.*

THEOREM 2.7. *If f is involutive, then the equation $\Phi_{*f}(\boldsymbol{u}) = \alpha$ has a local solution for any given local cross section α of S^2T^*, where everything should be considered in the real analytic category.*

[3] It is easy to show that $n \leqq m < \frac{1}{2}n(n+1)$ or n is even and $m = \frac{1}{2}n(n+1)$. The second case occurs only when $n = 2$, which follows immediately from [1], Theorem 1.

[4] See [8].

[5] and [6] See [9].

The É. Cartan's result [2] indicates that every Riemannian manifold M of dimension n can be locally isometrically imbedded in the Euclidean space of dimension $\frac{1}{2}n(n+1)$ by an involutive imbedding, where as above everything should be considered in the real analytic category.

d) The case where $m = \frac{1}{2}n(n+3)$. Since $\tilde{N} = S^2T^*$, the operator L is reduced to 0 and hence the equation $\Phi_{*f}(\boldsymbol{u}) = \alpha$ has a global solution for any given $\alpha \in \Gamma(S^2T^*)$, which is a basic fact in the theory of Nash [6]. An example of such an imbedding is the canonical isometric imbedding of the real projective space $P^n(\boldsymbol{R})$ in the space of symmetric matrices of degree $n+1$ with vanishing trace (cf. 3.1, Example.)

e) The case where $n = 2$ and $m = 3$. The imbedding $\boldsymbol{f}$ is always involutive, and it is elliptic if and only if the second fundamental form corresponding to any normal vector ($\neq 0$) is definite. Consider the case where $M = S^2$, the unit sphere in $\boldsymbol{R}^3$. Then $\boldsymbol{f}$ is elliptic if and only if the image $\boldsymbol{f}(S^2)$ of S^2 by $\boldsymbol{f}$ is an ovaloid in $\boldsymbol{R}^3$, and when $\boldsymbol{f}$ is elliptic, we know the following facts: 1^0. $\rho(\boldsymbol{f}) = \frac{1}{2}m(m+1) = 6$, 2^0. the equation $\Phi_{*f}(\boldsymbol{u}) = \alpha$ has a global solution for any given $\alpha \in \Gamma(S^2T^*)$. These facts play an important role in the solution [7] of the Weyl problem.

f) The case where $n \geqq 3$ and $m = n+1$. Consider the case where the rank of the second fundamental form corresponding to any normal vector ($\neq 0$) is at least 3. Then the imbedding $\boldsymbol{f}$ is elliptic, and we can prove the following facts: 1^0. The equation $L\varphi = 0$ is of finite type or more precisely, $\dim \mathfrak{g}_p = \frac{1}{2}n(n-1)+1$, $\dim \mathfrak{g}_p^{(1)} = n$ and $\mathfrak{g}_p^{(2)} = 0$, 2^0. the equation is formally integrable[7], 3^0. $\rho(\boldsymbol{f}) = \frac{1}{2}m(m+1)$ without compactness assumption.

§ 3. Rigidity for some classes of elliptic imbeddings

3.1. The canonical isometric imbedding of a compact hermitian symmetric space. Let M be a global hermitian symmetric space. (For the theory of hermitian symmetric spaces, we refer to [3].) Let I and ν be the almost complex structure and the Riemannian metric respectively on M. Let G be the largest connected group of automorphisms (holomorphic isometries) of M. The group G transitively acts on M and hence the space M may be expressed as the homogeneous space G/H, where H is the isotropy group of G at a fixed point 0 of M. In the following we assume that G is compact and semi-simple.

[7] See [8].

Let $\mathfrak{g}$ be the Lie algebra of G and B its Killing form. Note that B is negative definite. Let $\mathfrak{h}$ be the Lie algebra of H and $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form B. We have $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Let π be the projection of G onto M. Then the linear map $\mathfrak{m} \ni X \to \pi_* X_e \in T_0 = T(M)_0$ is an isomorphism, by which we shall identify the two vector spaces $\mathfrak{m}$ and T_0. This being said, we make the second assumption that the Riemannian metric ν is induced from the Killing form B, that is,

$$\nu(X, Y) = -B(X, Y)$$

for all $X, Y \in T_0$.

It is well known that there is a unique element Z_0 in the centre of $\mathfrak{h}$ such that $IX = [Z_0, X]$ for all $X \in T_0$ and such that H is the centralizer of Z_0 in G, i.e., $H = \{a \in G \mid \mathrm{ad}\, a Z_0 = Z_0\}$. (See [3], Theorem 9.6.) By the second property of Z_0, we see that the map $G \ni a \to \mathrm{ad}\, a Z_0 \in \mathfrak{g}$ induces a map $\boldsymbol{f}$ of M to $\mathfrak{g}$. Furthermore we see that $\boldsymbol{f}$ is an imbedding and that it is equivariant, i.e.,

$$\boldsymbol{f}(ap) = \mathrm{ad}\, a \boldsymbol{f}(p)$$

for all $a \in G$ and $p \in M$.

Let us now define an inner product $\langle\, ,\rangle$ in $\mathfrak{g}$ as follows:

$$\langle X, Y \rangle = -B(X, Y)$$

for all $X, Y \in \mathfrak{g}$. Thus if we put $m = \dim \mathfrak{g}$, $\mathfrak{g}$ may be regarded as the m-dimensional Euclidean space $\boldsymbol{R}^m$ with respect to a fixed orthonormal base $X_1, \cdots, X_m$ of $\mathfrak{g}$. Note that $\mathrm{ad}\, a: \mathfrak{g} \to \mathfrak{g}$ is an orthogonal transformation for all $a \in G$. Let ∇ be the covariant differentiation associated with the Riemannian metric ν and R its curvature.

PROPOSITION 3.1. (1) *$\boldsymbol{f}$ is an isometric imbedding of the hermitian symmetric space M in the Euclidean space $\mathfrak{g} = \boldsymbol{R}^m$* ([5]).

(2) *$\boldsymbol{f}$ satisfies condition (C).*

(3) $\nabla_Z \nabla_X \nabla_Y \boldsymbol{f} = (R(IX, Y)IZ)\boldsymbol{f}$

for all $X, Y, Z \in T_p$ and $p \in M$.

First we shall prove the following.

LEMMA 3.2. *For all $X, Y \in T_0 = \mathfrak{m}$, we have:*

(1) $Xf = -IX$.
(2) $\nabla_X \nabla_Y f = [IX, Y] = -[X, IY]$.

Proof. If we put $u(t) = \pi(\exp tX)$, we have

$$f(u(t)) = \mathrm{ad}(\exp tX)Z_0 = \sum_k \frac{t^k}{k!}(\mathrm{ad}\, X)^k Z_0 .$$

We have $(du/dt)(0) = \pi_* X = X$ and hence

$$Xf = \mathrm{ad}\, X Z_0 = [X, Z_0] = -IX ,$$

proving (1). Let $\tilde{X}$ be the vector field on M induced by the 1-parameter group of transformations: $\boldsymbol{R} \times M \ni (t, p) \to (\exp tX)p \in M$. Then $(d^2 f \circ u / dt^2)(0) = X\tilde{X}f$ and hence

$$X\tilde{X}f = (\mathrm{ad}\, X)^2 Z_0 = [X, [X, Z_0]] = [IX, X] .$$

On the other hand

$$X\tilde{X}f = \nabla_X \nabla_X f + (\nabla_X \tilde{X})f .$$

$u(t)$ is an integral curve of $\tilde{X}$ and at the same time a geodesic, whence $\nabla_X \tilde{X} = 0$. Therefore $\nabla_X \nabla_X f = [IX, X]$. Since $\nabla_X \nabla_Y f = \nabla_Y \nabla_X f$ and since $[IX, Y] + [X, IY] = [Z_0, [X, Y]] = 0$, it follows that $\nabla_X \nabla_Y f = [IX, Y]$, proving (2).

Proof of Proposition 3.1. By Lemma 3.2, (1) we have $\langle Xf, Yf \rangle = -B(IX, IY) = -B(X, Y) = \nu(X, Y)$ for all $X, Y \in T_0$. Since f is equivariant, it follows that $\langle Xf, Yf \rangle = \nu(X, Y)$ for all $X, Y \in T_p$ and $p \in M$, proving (1). We have $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$. Hence (2) follows from Lemma 3.2, (2) and the equivariance of f. By Lemma 3.2 and the equivariance of f, we have

$$\nabla_X \nabla_Y f = [\nabla_{IX} f, \nabla_Y f]$$

for all $X, Y \in T_p$ and $p \in M$. Therefore by using $\nabla I = 0$, we have, for all $Z \in T_p$,

$$\begin{aligned} \nabla_Z \nabla_X \nabla_Y f &= [\nabla_Z \nabla_{IX} f, \nabla_Y f] + [\nabla_{IX} f, \nabla_Z \nabla_Y f] \\ &= [[\nabla_{IZ} f, \nabla_{IX} f], \nabla_Y f] + [\nabla_{IX} f, [\nabla_{IZ} f, \nabla_Y f]] \\ &= [\nabla_{IZ} f, [\nabla_{IX} f, \nabla_Y f]] . \end{aligned}$$

Considering the case where $p = 0$ and using Lemma 3.2, (1), we have

$$\nabla_Z \nabla_X \nabla_Y f = -[Z, [X, IY]]$$

for all $X, Y, Z \in T_0$. Since

$$R(IX, Y)IZ = -[[IX, Y], IZ],$$

it follows that

$$\nabla_Z \nabla_X \nabla_Y f = -[[IX, Y], Z] = (R(IX, Y)IZ)f .$$

Now (3) follows from this equality and the equivariance of f.

We denote by E the subbundle of S^2T^* which consists of all the hermitian forms ($\in S^2T^*$) with respect to the almost complex structure I. As is easily observed, E is elliptic.

PROPOSITION 3.3. *The bundle $\tilde{N}$ of second fundamental forms of f is a subbundle of the bundle E of hermitian forms. In particular the imbedding f is elliptic.*

Proof. By Lemma 3.2, (2) and the equivariance of f, we have

$$\nabla_X \nabla_Y f = \nabla_{IX} \nabla_{IY} f$$

for all $X, Y \in T_p$. Proposition 3.3 is immediate from this fact.

EXAMPLE. Let us consider the case where the hermitian symmetric space $M = G/H$ is the n-dimensional complex projective space $P^n(\boldsymbol{C})$. We have

$$G = U(n+1)/C \quad \text{and} \quad H = U(1) \times U(n)/C ,$$

where C is the centre of $U(n+1)$ and the isotropy group H is considered at the point $0 = (1, 0, \cdots, 0)$. Hence $\mathfrak{g} \cong \mathfrak{su}(n+1)$ and $\dim \mathfrak{g} = n^2 + 2n$. Let $z_0, \cdots, z_n$ be the homogeneous coordinates of $P^n(\boldsymbol{C})$. Then the imbedding $f = f^{(n)}$ is given by

$$f^{(n)}((z_0, \cdots, z_n)) = \sqrt{-1}\left(\frac{\delta_{ij}}{n+1} - \frac{z_i \bar{z}_j}{|z|^2}\right) \quad (\in \mathfrak{su}(n+1)) ,$$

where $|z|^2 = \sum_k |z_k|^2$. Finally we note that the two bundles $\tilde{N}$ and E just coincide.

3.2. A rigidity theorem associated with the isometric imbedding $f: M \to \mathfrak{g}$. In the next paragraph we shall prove the following

THEOREM 3.4. *Let f be the isometric imbedding of the compact* *hermitian symmetric space M in the Euclidean space $\mathfrak{g} = \boldsymbol{R}^m$ defined in* 3.1. *Then we have $\rho(\boldsymbol{f}) = \frac{1}{2}m(m+1)$.*

By Proposition 3.3, Theorems 2.4 and 3.4 we have the following

THEOREM 3.5. *Let $\boldsymbol{f}$ be as in Theorem* 3.4. *Then there is a neighborhood $U(\boldsymbol{f})$ of $\boldsymbol{f}$ in $\mathfrak{E}(M,m)_{C^3}$ having the following property: If $\boldsymbol{f}',\boldsymbol{f}'' \in U(\boldsymbol{f})$ and if $\Phi(\boldsymbol{f}') = \Phi(\boldsymbol{f}'')$, there is a unique Euclidean transformation a of $\mathfrak{g}$ such that $\boldsymbol{f}'' = a\boldsymbol{f}'$.*

Remark 1. Consider the isometric imbedding $\boldsymbol{f}^{(1)}$ of $P^1(\boldsymbol{C})$ in $\mathfrak{su}(2) = \boldsymbol{R}^3$. (See 3.1, Example.) $P^1(\boldsymbol{C})$ is isomorphic with the sphere $S^2(\sqrt{2})$ of radius $\sqrt{2}$ in $\boldsymbol{R}^3$ as Riemannian manifolds. Therefore Theorem 3.5 is a partial generalization of the theorem of Cohn-Vossen.

Remark 2. It is well known that the sphere $S^2(\sqrt{2})$ in $\boldsymbol{R}^3$ or the isometric imbedding $\boldsymbol{f}^{(1)}$ is locally deformable. (For example, see [2].) Therefore the isometric imbedding $\boldsymbol{f}$ of the hermitian symmetric space M in $\mathfrak{g}$ is locally deformable, provided that $P^1(\boldsymbol{C})$ appears in the de Rham decomposition of M. We shall see in the proof of Theorem 3.4 that the equation $L\varphi = 0$ associated with the isometric imbedding $\boldsymbol{f}^{(n)}$ of $P^n(\boldsymbol{C})$ in $\mathfrak{su}(n+1)$ is of infinite type, which suggests that $\boldsymbol{f}^{(n)}$ is locally deformable. However the problem to examine it seems to be rather complicated and difficult.

3.3. Proof of Theorem 3.4. The proof is devided into several steps.

I. M is a Kahlerian manifold: $\nabla I = 0$ and M is also a symmetric space: $\nabla R = 0$. Since the Riemannian metric ν is induced from the Killing form B of $\mathfrak{g}$, we know that M is an Einstein space ([3], Proposition 9.7):

$$\sum_i R(X, e_i)e_i = -\frac{1}{2}\sum_i R(Ie_i, e_i)IX = \frac{1}{2}X\,.$$

(In the following X, Y, Z will denote any vectors at any point $p \in M$ and $e_1, \cdots, e_{2n}$ any orthogonal frame at the point p, where $n = \dim_C M$.)

For $\alpha, \beta \in \Gamma(\otimes^p T^*)$ we define a function $\langle\alpha, \beta\rangle$ by

$$\langle\alpha,\beta\rangle = \frac{1}{p\,!}\sum_{i_1,\cdots,i_p}\alpha(e_{i_1},\cdots,e_{i_p})\beta(e_{i_1},\cdots,e_{i_p})$$

and put

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle dv ,$$

the inner product of α and β, where dv is the volume element of the oriented Riemannian manifold M.

Let δ be the adjoint operator of the exterior differentiation d of $\Gamma(\Lambda^p T^*)$ to $\Gamma(\Lambda^{p+1} T^*)$ with respect to the inner products $(\, ,)$. Then the operator $\Delta = \delta d + d\delta$ is the so called Laplacian. In terms of ∇, we have:

$$(d\varphi)(X_1, \cdots, X_{p+1}) = \sum_i (-1)^{i-1} (\nabla_{X_i}\varphi)(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1}) ,$$

$$(\delta\varphi)(X_1, \cdots, X_p) = -\sum_i (\nabla_{e_i}\varphi)(e_i, X_1, \cdots, X_p) .$$

We have

$$\Delta f = -\sum_i \nabla_{e_i}\nabla_{e_i} f$$

for a function f and

$$\Delta\varphi = -\sum_i \nabla_{e_i}\nabla_{e_i}\varphi + \tfrac{1}{2}\varphi$$

for a 1-form φ.

II. Let E' be the subbundle of S^2T^* which consists of all the anti-hermitian forms ($\in S^2T^*$) with respect to I: $S^2T^* = E \oplus E'$. ($\alpha \in S^2T^*$ is anti-hermitian if $\alpha(IX, IY) = -\alpha(X, Y)$.) We define a differential operator L_0 of $\Gamma(T^*)$ to $\Gamma(E')$ by

$$(L_0\varphi)(X, Y) = \tfrac{1}{2}((D\varphi)(X, Y) - (D\varphi)(IX, IY))$$

for all $\varphi \in \Gamma(T^*)$, i.e., $L_0\varphi$ is the anti-hermitian part of $D\varphi$. $\tilde{N}$ is a subbundle of E by Proposition 3.3 and hence a solution of the equation $L\varphi = 0$ is necessarily a solution of the equation $L_0\varphi = 0$.

Remark. Let φ be a local 1-form on M and X the vector field dual to φ with respect to ν. The fact that φ is a solution of the equation $L_0\varphi = 0$ means that $D\varphi = L_X\nu$, L_X being the Lie derivation, is hermitian. Hence if X is an analytic vector field, then φ is a solution of the equation $L_0\varphi = 0$. It follows that the equation $L_0\varphi = 0$ is of infinite type. In particular consider the case where $M = P^n(C)$. Then $\tilde{N} = E$ and hence L may be identified with L_0. Consequently the equation $L\varphi = 0$ is of infinite type.

Let L_0^* be the adjoint operator of L_0 with respect to the inner products $(\, ,)$. In terms of ∇, L_0^* may be expressed as follows:

$$(L_0^*\alpha)(X) = -\sum_i (\nabla_{e_i}\alpha)(e_i, X)$$

for all $\alpha \in \Gamma(E')$. Indeed we have, for all $\varphi \in \Gamma(T^*)$,

$$\begin{aligned}\langle L_0\varphi, \alpha\rangle &= \sum_{i,j} (\nabla_{e_i}\varphi)(e_j)\alpha(e_i, e_j)\\ &= \sum_i (\nabla_{e_i}\beta)(e_i) - \sum_{i,j} \varphi(e_j)(\nabla_{e_i}\alpha)(e_i, e_j)\,,\end{aligned}$$

where β is the 1-form defined by $\beta(X) = \sum_j \varphi(e_j)\alpha(X, e_j)$. Since the integral of $\delta\beta\cdot dv$ over M is 0, we obtain the desired formula.

For $\varphi \in \Gamma(T^*)$ we define $\varphi I \in \Gamma(T^*)$ by $(\varphi I)(X) = \varphi(IX)$.

LEMMA 3.6.

$$2L_0^*L_0\varphi = \varDelta\varphi + d\delta\varphi - (d\delta(\varphi I))I - 2\varphi\,.$$

Proof. We have:

$$\sum_i (\nabla_{e_i}D\varphi)(e_i, X) = \sum_i (\nabla_{e_i}\nabla_{e_i}\varphi)(X) + \sum_i (\nabla_{e_i}\nabla_X\varphi)(e_i)\,,$$

$$\begin{aligned}\sum_i (\nabla_{e_i}\nabla_X\varphi)(e_i) &= \sum_i (\nabla_X\nabla_{e_i}\varphi)(e_i) - \sum_i \varphi(R(e_i, X)e_i)\\ &= (-d\delta\varphi + \tfrac{1}{2}\varphi)(X)\,.\end{aligned}$$

Hence

$$\sum_i (\nabla_{e_i}D\varphi)(e_i, X) = (-\varDelta\varphi - d\delta\varphi + \varphi)(X)\,. \tag{3.1}$$

We have:

$$\sum_i (\nabla_{e_i}D\varphi)(Ie_i, IX) = \sum_i (\nabla_{e_i}\nabla_{Ie_i}\varphi)(IX) + \sum_i (\nabla_{e_i}\nabla_{IX}\varphi)(Ie_i)\,,$$

$$\begin{aligned}\sum_i (\nabla_{e_i}\nabla_{Ie_i}\varphi)(IX) &= -\sum_i (\nabla_{Ie_i}\nabla_{e_i}\varphi)(IX)\\ &= -\sum_i (\nabla_{e_i}\nabla_{Ie_i}\varphi)(IX) + \sum_i \varphi(R(Ie_i, e_i)IX)\,,\end{aligned}$$

$$\sum_i (\nabla_{e_i}\nabla_{IX}\varphi)(Ie_i) = (-(d\delta(\varphi I))I - \tfrac{1}{2}\varphi)(X)\,.$$

Hence $\sum_i (\nabla_{e_i}\nabla_{Ie_i}\varphi)(IX) = -\frac{1}{2}\varphi(X)$ and

$$\sum_i (\nabla_{e_i}D\varphi)(Ie_i, IX) = (-(d\delta(\varphi I))I - \varphi)(X)\,. \tag{3.2}$$

Lemma 3.6 is immediate from (3.1) and (3.2).

III. By Lemma 3.6, every solution of the equation $L_0\varphi = 0$ is a solution of the equation:

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$$\varDelta\varphi = -d\delta\varphi + (d\delta(\varphi I))I + 2\varphi \tag{3.3}$$

and the converse is also true. Let us now consider the following equation:

$$\varDelta f = f\,, \tag{3.4}$$

f being functions.

LEMMA 3.7. (1) *If f is a solution of equation* (3.4), *then df is a solution of equation* (3.3).

(2) *Every solution φ of equation* (3.3) *can be uniquely written as*

$$\varphi = df + \varphi_1\,,$$

where f is a solution of equation (3.4) *and φ_1 is a solution of equation* (3.3) *combined with the equation $\delta\varphi = 0$.*

Proof. This lemma is easy from the following facts: $d\varDelta = \varDelta d$, $\delta\varDelta = \varDelta\delta$ and $\delta((df)I) = 0$ for any function f.

A 1-form φ on M is called a Killing form if the vector field X dual to φ with respect to ν is a Killing vector field, i.e., $L_X\nu = D\varphi = 0$.

LEMMA 3.8. *The solutions f of equation* (3.4) *are in a one-to-one correspondence with the Killing forms φ and the correspondence $f \to \varphi$ is given by the relation $\varphi = (df)I$. In particular the solutions of equation* (3.4) *form a vector space of dimension $m(= \dim \mathfrak{g})$.*

This fact is well known. (For example, see [12], Chapter IV.) The next lemma is also known.

LEMMA 3.9. *If we put $\boldsymbol{f} = (f_1, \cdots, f_m)$, i.e., $\boldsymbol{f} = \sum_i f_i X_i$, then the functions $f_1, \cdots, f_m$ form a base of the solution space of equation* (3.4).

Proof. Let $e_1, \cdots, e_{2n}$ be an orthonormal base of $\mathfrak{m} = T_0$. Then we have

$$-\sum_i [[Ie_i, e_i], X] = \sum_i R(Ie_i, e_i)X = IX = [Z_0, X]$$

for all $X \in T_0$, whence $\sum_i [Ie_i, e_i] = -Z_0$. Therefore by Lemma 3.2, (2), we have

$$\sum_i \nabla_{e_i}\nabla_{e_i}\boldsymbol{f} = \sum_i [Ie_i, e_i] = -Z_0 = -\boldsymbol{f}(0)\,.$$

This together with the equivariance of $\boldsymbol{f}$ yields $\Delta\boldsymbol{f} = \boldsymbol{f}$. Hence $f_1, \cdots, f_m$ are solutions of equation (3.4). Lemma 3.9 now follows from Lemma 3.8.

LEMMA 3.10. *If f is a solution of equation* (3.4), *then df is a solution of the equation* $L\varphi = 0$.

Proof. By Lemma 3.9, it is sufficient to prove this for $f = f_i$. We have $Ddf_i = 2\nabla\nabla f_i$, and $\nabla\nabla f_i = \langle X_i, \nabla\nabla\boldsymbol{f}\rangle$ is a cross section of $\tilde{N}$. Hence we obtain $Ldf_i = 0$.

Remark. It can be proved that the dimension of the solution space of the equation $L_0\varphi = 0$ is equal to $l + 2m$, where l is the dimension of the solution space of the equation $\Delta\varphi = 2\varphi$, φ being 1-forms. Consequently in the case where $M = P^n(\boldsymbol{C})$, our problem is reduced to the problem to find the dimension l.

IV. Let $\boldsymbol{u}$ be a solution of the equation $\Phi_{*f}(\boldsymbol{u}) = 0$. By Theorem 1.2 there corresponds to $\boldsymbol{u}$ a solution φ of the equation $L\varphi = 0$. Since $\varphi = \langle \boldsymbol{u}, d\boldsymbol{f}\rangle$, we have

$$\langle \boldsymbol{u}, \nabla_Y \boldsymbol{f}\rangle = \varphi(Y)$$

and hence

$$\langle \nabla_X \boldsymbol{u}, \nabla_Y \boldsymbol{f}\rangle + \langle \boldsymbol{u}, \nabla_X\nabla_Y \boldsymbol{f}\rangle = (\nabla_X\varphi)(Y)\,.$$

We have $\Phi_{*f}(\boldsymbol{u}) = 2\langle d\boldsymbol{f}, d\boldsymbol{u}\rangle = 0$, meaning that $\langle \nabla_X \boldsymbol{u}, \nabla_Y \boldsymbol{f}\rangle$ is skew-symmetric with respect to X and Y. Furthermore $\langle \boldsymbol{u}, \nabla_X\nabla_Y \boldsymbol{f}\rangle$ is symmetric with respect to X and Y. Therefore we obtain:

$$\langle \boldsymbol{u}, \nabla_X\nabla_Y \boldsymbol{f}\rangle = \tfrac{1}{2}(D\varphi)(X, Y)\,, \tag{3.5}$$

$$\langle \nabla_X \boldsymbol{u}, \nabla_Y \boldsymbol{f}\rangle = \tfrac{1}{2}(d\varphi)(X, Y)\,. \tag{3.6}$$

LEMMA 3.11. (1) $\langle \Delta\boldsymbol{u} - \boldsymbol{u}, \nabla_X \boldsymbol{f}\rangle = -(d\delta\varphi)(X)\,.$

(2) $\langle \Delta\boldsymbol{u} - \boldsymbol{u}, \nabla_X\nabla_Y \boldsymbol{f}\rangle = (\tfrac{1}{2}D\Delta\varphi - D\varphi)(X, Y) + \sum_i (L_0\varphi)(R(e_i, Y)X, e_i)\,.$

Proof. By (3.5) and (3.6) we have

$$\sum_i \langle \nabla_{e_i}\boldsymbol{u}, \nabla_{e_i}\nabla_X \boldsymbol{f}\rangle + \sum_i \langle \boldsymbol{u}, \nabla_{e_i}\nabla_{e_i}\nabla_X \boldsymbol{f}\rangle = \tfrac{1}{2}\sum_i (\nabla_{e_i}D\varphi)(e_i, X)\,,$$

$$\sum_i \langle \nabla_{e_i}\nabla_{e_i}\boldsymbol{u}, \nabla_X \boldsymbol{f}\rangle + \sum_i \langle \nabla_{e_i}\boldsymbol{u}, \nabla_{e_i}\nabla_X \boldsymbol{f}\rangle = \tfrac{1}{2}\sum_i (\nabla_{e_i}d\varphi)(e_i, X)\,.$$

By Proposition 3.1, (3) we have

$$\sum_i \langle \boldsymbol{u}, \nabla_{e_i}\nabla_{e_i}\nabla_X \boldsymbol{f}\rangle = \sum_i \langle \boldsymbol{u}, (R(Ie_i, X)Ie_i)\boldsymbol{f}\rangle$$
$$-\tfrac{1}{2}\langle \boldsymbol{u}, X\boldsymbol{f}\rangle = -\tfrac{1}{2}\varphi(X)\,.$$

Hence we obtain

$$\begin{aligned}\langle \Delta\boldsymbol{u} - \boldsymbol{u}, \nabla_X \boldsymbol{f}\rangle &= \tfrac{1}{2}(-\Delta\varphi - d\delta\varphi + \varphi)(X) \\ &\quad + \tfrac{1}{2}(\delta d\varphi)(X) - \tfrac{1}{2}\varphi(X) \\ &= -(d\delta\varphi)(X)\,,\end{aligned}$$

proving (1).

By (3.5) and Proposition 3.1, (3) we have

$$\langle \nabla_Z \boldsymbol{u}, \nabla_X\nabla_Y \boldsymbol{f}\rangle + \varphi(R(IX, Y)IZ) = \tfrac{1}{2}(\nabla_Z D\varphi)(X, Y)\,.$$

It follows that

$$\begin{aligned}&\sum_i \langle \nabla_{e_i}\nabla_{e_i}\boldsymbol{u}, \nabla_X\nabla_Y \boldsymbol{f}\rangle + \sum_i \langle \nabla_{e_i}\boldsymbol{u}, \nabla_{e_i}\nabla_X\nabla_Y \boldsymbol{f}\rangle \\ &\qquad + \sum_i (\nabla_{e_i}\varphi)(R(IX, Y)Ie_i) = \tfrac{1}{2}\sum_i (\nabla_{e_i}\nabla_{e_i} D\varphi)(X, Y)\,.\end{aligned}$$

Therefore from (3.5), (3.6) and Proposition 3.1, (3) we obtain

$$\begin{aligned}&\langle \Delta\boldsymbol{u} - \boldsymbol{u}, \nabla_X\nabla_Y \boldsymbol{f}\rangle \\ &\quad = -\tfrac{1}{2}\sum_i (\nabla_{e_i}\nabla_{e_i} D\varphi)(X, Y) + \tfrac{1}{2}\sum_i (d\varphi)(e_i, R(IX, Y)Ie_i) \\ &\qquad + \sum_i (\nabla_{e_i}\varphi)(R(IX, Y)Ie_i) - \tfrac{1}{2}(D\varphi)(X, Y)\,.\end{aligned}$$

We have

$$\begin{aligned}&\sum_i (\nabla_{e_i}\nabla_{e_i} D\varphi)(X, Y) \\ &\quad = D(\sum_i \nabla_{e_i}\nabla_{e_i}\varphi)(X, Y) + \tfrac{1}{2}(D\varphi)(X, Y) + 2\sum_i (D\varphi)(R(X, e_i)Y, e_i) \\ &\quad = -(D\Delta\varphi)(X, Y) + (D\varphi)(X, Y) + 2\sum_i (D\varphi)(R(X, e_i)Y, e_i)\end{aligned}$$

Thus we have proved the equality:

$$\begin{aligned}&\langle \Delta\boldsymbol{u} - \boldsymbol{u}, \nabla_X\nabla_Y \boldsymbol{f}\rangle \\ &\quad = \tfrac{1}{2}(D\Delta\varphi)(X, Y) - (D\varphi)(X, Y) + \tfrac{1}{2}\sum_i (d\varphi)(e_i, R(IX, Y)Ie_i) \\ &\qquad + \sum_i (\nabla_{e_i}\varphi)(R(IX, Y)Ie_i) - \sum_i (D\varphi)(R(X, e_i)Y, e_i)\,.\end{aligned}$$

Since $A = R(IX, Y)I$ is symmetric with respect to ν, we have

$$\sum_i (d\varphi)(e_i, R(IX, Y)Ie_i) = 0\,,$$

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$$\sum_i (\nabla_{e_i}\varphi)(R(IX, Y)Ie_i) - \sum_i (D\varphi)(R(X, e_i)Y, e_i)$$
$$= \frac{1}{2}\sum_i (D\varphi)(e_i, R(IX, Y)Ie_i) - \sum_i (D\varphi)(R(X, e_i)Y, e_i)\,.$$

By Lemma 3.12 below the right hand side of the last equality is equal to $-\sum_i (L_0\varphi)(R(Y, e_i)X, e_i)$. We have thereby proved (2).

LEMMA 3.12. *For $\theta \in \Gamma(S^2T^*)$ we have the equality:*

$$\sum_i \theta(R(IX, Y)Ie_i, e_i) - 2\sum_i \theta(R(X, e_i)Y, e_i)$$
$$= 2\sum_i \theta'(R(e_i, Y)X, e_i)\,,$$

where θ' is the anti-hermitian part of θ.

This is a lemma in general Kählerian manifolds generalizing Proposition 4.5 in [3] and can be proved just in the same way as that proposition.

V. We know that a solusion of the equation $L\varphi = 0$ is also a solution of the equation $L_0\varphi = 0$. Hence by Lemmas 3.7 and 3.10 every solution φ of the equation $L\varphi = 0$ can be uniquely written as

$$\varphi = df + \varphi_1\,,$$

where f is a solution of equation (3.4) and φ_1 is a solution of the equations

$$L\varphi = \delta\varphi = 0\,. \tag{3.7}$$

Therefore by Theorem 1.2, Proposition 2.2 and Lemma 3.8 we arrive at the following conclusion: In order to prove $\rho(\boldsymbol{f}) = \frac{1}{2}m(m+1)$, it is sufficient to show that, for any solution φ of equations (3.7), there is $A \in \mathfrak{o}(m)$ such that $\varphi = \langle A\boldsymbol{f}, d\boldsymbol{f}\rangle$.

This being said, let φ be a solution of equations (3.7) and $\boldsymbol{u}$ the corresponding solution of the equation $\Phi_{*\boldsymbol{f}}(\boldsymbol{u}) = 0$. (The correspondence is given by Theorem 1.2.) Then our task is to show that there is $A \in \mathfrak{o}(m)$ such that $\boldsymbol{u} = A\boldsymbol{f}$.

LEMMA 3.13. *If we put $\boldsymbol{u} = (u_1, \cdots, u_m)$ then the functions $u_1, \cdots, u_m$ are solutions of equation* (3.4).

Proof. By Lemma 3.11, (1) and the fact that $\delta\varphi = 0$, we obtain

$$\langle \Delta\boldsymbol{u} - \boldsymbol{u}, \nabla_X\boldsymbol{f}\rangle = 0\,. \tag{3.8}$$

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Since φ satisfies equation (3.3), we have

$$\Delta\varphi = (d\delta(\varphi I))I + 2\varphi . \tag{3.9}$$

Since $\Delta(\varphi I) = (\Delta\varphi)I$, it follows that $\Delta(\varphi I) = -d\delta(\varphi I) + 2\varphi I$ and hence that $f = \delta(\varphi I)$ is a solution of equation (3.4). Therefore $(df)I$ is a Killing form by Lemma 3.8 and hence from (3.9) we get $D\Delta\varphi = 2D\varphi$. Consequently by Lemma 3.11, (2) we obtain

$$\langle \Delta\boldsymbol{u} - \boldsymbol{u}, \nabla_X\nabla_Y\boldsymbol{f}\rangle = 0 . \tag{3.10}$$

Since, at each $p \in M$, the vectors $\nabla_X\boldsymbol{f}, \nabla_X\nabla_Y\boldsymbol{f}(X, Y \in T_p)$ span the vector space $\mathfrak{g}$, it follows from (3.8) and (3.10) that $\Delta\boldsymbol{u} = \boldsymbol{u}$, proving Lemma 3.13.

By Lemmas 3.9 and 3.13 we see that u_i are linear combinations of $f_1, \cdots, f_m$. Therefore there is a matrix A of degree m such that $\boldsymbol{u} = A\boldsymbol{f}$.

LEMMA 3.14. *A is a skew-symmetric matrix i.e., $A \in \mathfrak{o}(m)$.*

Proof. If we put $\boldsymbol{u}' = -{}^tA\boldsymbol{f}$, then we have

$$\begin{aligned}\langle d\boldsymbol{f}, d\boldsymbol{u}'\rangle &= -\langle d\boldsymbol{f}, {}^tAd\boldsymbol{f}\rangle \\ &= -\langle Ad\boldsymbol{f}, d\boldsymbol{f}\rangle \\ &= -\langle d\boldsymbol{u}, d\boldsymbol{f}\rangle = 0 ,\end{aligned}$$

meaning that $\boldsymbol{u}'$ is a solution of the equation $\Phi_{*f}(\boldsymbol{u}) = 0$. We now show that the solution $\boldsymbol{u}'$ just corresponds to the given solution φ of the equation $L\varphi = 0$. Indeed by (3.5) and Lemma 3.9 we have $\langle \boldsymbol{u}, \boldsymbol{f}\rangle = \langle \boldsymbol{u}, \Delta\boldsymbol{f}\rangle = \delta\varphi = 0$. Hence we have

$$\begin{aligned}\langle \boldsymbol{u}', \nabla_X\boldsymbol{f}\rangle &= -\langle {}^tA\boldsymbol{f}, \nabla_X\boldsymbol{f}\rangle = -\langle \boldsymbol{f}, \nabla_X\boldsymbol{u}\rangle \\ &= -\nabla_X\langle \boldsymbol{f}, \boldsymbol{u}\rangle + \langle \nabla_X\boldsymbol{f}, \boldsymbol{u}\rangle \\ &= \varphi(X) ,\end{aligned}$$

i.e., $\langle \boldsymbol{u}', d\boldsymbol{f}\rangle = \varphi$, proving our assertion. Therefore we have $\boldsymbol{u}' = \boldsymbol{u}$ and hence $(A + {}^tA)\boldsymbol{f} = 0$, from which follows that $A + {}^tA = 0$ (cf. Proof of Proposition 2.1).

We have thus completed proof of Theorem 3.4.

APPENDIX

In this appendix we shall accomplish the proof of Theorem 2.3, as we promised.

We first introduce inner products $\langle\,,\rangle$ in the vector bundles T^* and F. We define an inner product $(\,,)$ in $\Gamma(T^*)$ by

$$(\varphi, \varphi') = \int_M \langle \varphi, \varphi' \rangle dv$$

for all $\varphi, \varphi' \in \Gamma(T^*)$, where dv is the volume density associated with some Riemannian metric on M, and define a norm $\|\ \|$ in $\Gamma(T^*)$ by $\|\varphi\|^2 = (\varphi, \varphi)$. In the same way we define an inner product $(\,,)$ and a norm $\|\ \|$ in $\Gamma(F)$. We also introduce a Sobolev norm $\|\ \|_1$ in $\Gamma(T^*)$.

For $f \in V(f_0)$, let L^{f*} be the adjoint operator of L^f with respect to the inner products $(\,,)$. Since L^f differentiably depends on $j^2 f$, it follows that the operator $\square^f = L^{f*}L^f$ differentiably depends on the 3-jet $j^3 f$ of f in an analogous sense. We also note that $\square^f$ is strongly elliptic, because L^f is elliptic.

LEMMA A. *There is a neighborhood $V_1(f_0)$ of f_0 in $\mathfrak{E}(M, m)_{C^2}$ such that*

$$\|\varphi\|_1^2 \leqq C(\|L^f \varphi\|^2 + \|\varphi\|^2)$$

for all $\varphi \in \Gamma(T^*)$ and $f \in V_1(f_0)$, where C is a positive constant independent of φ and f.

Proof. Since $\square^{f_0}$ is strongly elliptic, we have the Gårding inequality:

$$\|\varphi\|_1^2 \leqq C(\|L^{f_0} \varphi\|^2 + \|\varphi\|^2)$$

for all $\varphi \in \Gamma(T^*)$, where C is a positive constant independent of φ. Since L^f differentiably depends on $j^2 f$, we have: For any $\varepsilon > 0$, there is a neighborhood $V^\varepsilon(f_0)$ of f_0 in $\mathfrak{E}(M, m)_{C^2}$ such that

$$\|L^f \varphi - L^{f_0} \varphi\| \leqq \varepsilon \|\varphi\|_1$$

for all $\varphi \in \Gamma(T^*)$ and $f \in V^\varepsilon(f_0)$. Lemma A follows easily from these facts.

We are now in position to prove Theorem 2.3. Suppose that Theorem 2.3 is not true. Then there is a sequence f_i $(i = 1, 2, \cdots)$ of elements in $\mathfrak{E}(M, m)$ such that $f_i \to f_0$ in $\mathfrak{E}(M, m)_{C^3}$ and $\rho(f_i) > \rho(f_0)$. Hence, for each i, we can find $k = \rho(f_0) + 1$ elements $\varphi_1^{(i)}, \cdots, \varphi_k^{(i)}$ in $\Gamma(T^*)$ such that $L^{f_i} \varphi_\lambda^{(i)} = 0$ and $(\varphi_\lambda^{(i)}, \varphi_\mu^{(i)}) = \delta_{\lambda\mu}$. By Lemma A we have $\|\varphi_\lambda^{(i)}\|_1^2 \leqq C$. Therefore by the Rellich lemma, we may assume that, for each λ, the sequence $\varphi_\lambda^{(i)}$ $(i = 1, 2, \cdots)$ converges to an element φ_λ in the completion of $\Gamma(T^*)$

with respect to the norm $\| \ \|$. For every $\psi \in \Gamma(T^*)$, we have $(\varphi_\lambda^{(i)}, \Box^{f_i}\psi) = (\Box^{f_i}\varphi_\lambda^{(i)}, \psi) = 0$. Since $\Box^f$ differentiably depends on $j^3 f$, $\|\Box^{f_i}\psi - \Box^{f_0}\psi\| \to 0$. It follows that $(\varphi_\lambda, \Box^{f_0}\psi) = 0$. Therefore by the hypoellipticity of $\Box^{f_0}$, $\varphi_\lambda \in \Gamma(T^*)$ and $L^{f_0}\varphi_\lambda = 0$. Since $(\varphi_\lambda, \varphi_\mu) = \delta_{\lambda\mu}$, we have $\rho(f_0) \geqq k = \rho(f_0) + 1$, which is a contradiction. We have thus completed proof of Theorem 2.3.

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