THE BULLETIN OF SYMBOLIC LOGIC
Volume 27, Number 4, December 2021

LEVEL THEORY, PART 1: AXIOMATIZING THE BARE IDEA OF A CUMULATIVE HIERARCHY OF SETS

TIM BUTTON

Abstract. The following bare-bones story introduces the idea of a cumulative hierarchy of pure sets: 'Sets are arranged in stages. Every set is found at some stage. At any stage S: for any sets found before S, we find a set whose members are exactly those sets. We find nothing else at S'. Surprisingly, this story already guarantees that the sets are arranged in well-ordered levels, and suffices for quasi-categoricity. I show this by presenting Level Theory, a simplification of set theories due to Scott, Montague, Derrick, and Potter.

> What we shall try to do here is
> to axiomatize the types in as
> simple a way as possible so that
> everyone can agree that the
> idea is natural.
>
> Scott (1974, p. 208)

The following bare-bones story introduces the idea of a cumulative hierarchy of pure sets:[1]

> **The Basic Story.** Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any sets found before **s**, we find a set whose members are exactly those sets. We find nothing else at **s**.

This story says nothing at all about the height of any hierarchy, and apparently says almost nothing about the order-type of the stages. It lays down nothing more than the *bare idea* of a pure cumulative hierarchy. Surprisingly, though, this bare idea already guarantees that the sets are arranged in well-ordered levels. Indeed, this bare idea is quasi-categorical. Otherwise put: the Basic Story pins down any cumulative hierarchy

2020 Mathematics Subject Classification. 03A05, 03E30.
Keywords. set theory, level theory, rank, ordinal.

[1]See, e.g., Shoenfield (1977, p. 323). I have modified Shoenfield's story in two ways. First: Shoenfield speaks of sets as 'formed' at stages; I avoid this way of speaking, to avoid begging the question against platonists. Second: Shoenfield speaks of forming 'collections consisting of sets' into sets; I simply speak plurally. Note that the Basic Story takes no stance on whether sets 'depend' upon their members in anything other than an heuristic sense (cf. Incurvati, 2012 and 2020, pp. 51–69).

1079-8986/21/2704-0006
DOI :10.1017/bsl.2021.13

completely, modulo that hierarchy's height, on which the Story takes no stance. The aim of this paper is to show all of this.

I begin by axiomatizing the Basic Story in the most obvious way possible, obtaining Stage Theory, ST. It is clear that any pure cumulative hierarchy satisfies ST. Unfortunately, ST has multiple primitives. To overcome this, I develop Level Theory, LT. Its only primitive is $\in$, but LT and ST say exactly the same things about sets (see Sections 1–4). As such, any cumulative hierarchy satisfies LT. Moreover, LT proves that the levels are well-ordered, and LT is quasi-categorical (see Sections 5 and 6).

My theory LT builds on work by Dana Scott, Richard Montague, John Derrick, and Michael Potter. I discuss their theories in Section 8, but I wish to be very clear at the outset: LT is significantly technically simpler than its predecessors, but it owes everything to them.

This paper is the first in a triptych. In Part 2, I explore potentialism, by considering a tensed variation of the Basic Story. In Part 3, I modify the Story again, to provide every set with a complement. Part 2 presupposes Part 1, but Parts 1 and 3 can be read in isolation.

§0. Preliminaries.

I use second-order logic throughout. Mostly, though, my use of second-order logic is just for convenience. Except when discussing quasi-categoricity (see Section 6), any second-order claim can be replaced with a first-order schema in the obvious way. In using second-order logic, I assume the Comprehension scheme, $\exists F \forall x (F(x) \leftrightarrow \phi)$, for any ϕ not containing 'F'.

For readability, I concatenate infix conjunctions, writing things like $a \subseteq r \in s \in t$ for $a \subseteq r \wedge r \in s \wedge s \in t$. I also use some simple abbreviations (where Ψ can be any predicate whose only free variable is x, and $\lhd$ can be any infix predicate):

$$(\forall x : \Psi)\phi := \forall x(\Psi(x) \rightarrow \phi), \qquad (\forall x \lhd y)\phi := \forall x(x \lhd y \rightarrow \phi),$$
$$(\exists x : \Psi)\phi := \exists x(\Psi(x) \wedge \phi), \qquad (\exists x \lhd y)\phi := \exists x(x \lhd y \wedge \phi).$$

When I announce a result or definition, I list in brackets the axioms I am assuming.

§1. Stage Theory.

The Basic Story, which introduces the bare idea of a cumulative hierarchy, mentions sets and stages. To begin, then, I will present a theory which quantifies distinctly over both sorts of entities. (It is a simple modification of Boolos's 1989 theory; see Sections 8.1 and 8.2.)

Stage Theory, ST, has two distinct sorts of variables, for *sets* (lower-case italic) and for **stages** (lower-case bold). It has three primitive predicates:

$\in$: a relation between sets; read '$a \in b$' as 'a is in b',
$<$: a relation between stages; read '$\mathbf{r} < \mathbf{s}$' as '$\mathbf{r}$ is before $\mathbf{s}$',
$\preceq$: a relation between a set and a stage; read '$a \preceq \mathbf{s}$' as 'a is found at $\mathbf{s}$'.

For brevity, I write $a \prec \mathbf{s}$ for $\exists\mathbf{r}(a \preceq \mathbf{r} < \mathbf{s})$, i.e., a is found before $\mathbf{s}$. Then ST has five axioms:[2]

$$
\begin{aligned}
\textbf{Extensionality}\quad & \forall a \forall b(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b),\\
\textbf{Order}\quad & \forall \mathbf{r}\forall \mathbf{s}\forall \mathbf{t}(\mathbf{r} < \mathbf{s} < \mathbf{t} \rightarrow \mathbf{r} < \mathbf{t}),\\
\textbf{Staging}\quad & \forall a \exists \mathbf{s}\ a \preceq \mathbf{s},\\
\textbf{Priority}\quad & \forall \mathbf{s}(\forall a \preceq \mathbf{s})(\forall x \in a) x \prec \mathbf{s},\\
\textbf{Specification}\quad & \forall F \forall \mathbf{s}((\forall x : F) x \prec \mathbf{s} \rightarrow (\exists a \preceq \mathbf{s})\forall x(F(x) \leftrightarrow x \in a)).
\end{aligned}
$$

The first two axioms make implicit assumptions explicit: whilst I did not mention Extensionality in the Basic Story of a cumulative hierarchy, I take it as analytic that sets are extensional;[3] similarly, Order records the analytic fact that 'before' is a transitive relation. The remaining three axioms can then be read off the Basic Story directly: Staging says that every set is found at some stage; Priority says that a set's members are found before it; and Specification says that, if we find every F before $\mathbf{s}$, then we find the set of Fs at $\mathbf{s}$. So all of ST's axioms are obviously true of the Basic Story. Otherwise put: any cumulative hierarchy obviously satisfies ST.[4]

This is ST's chief virtue. Its chief drawback is that it contains multiple primitives. To see why this is a defect, suppose that we were forced to axiomatize the bare idea of a cumulative hierarchy using something like ST's two-sorted logic. In that case, our grasp of the (cumulative iterative) notion of *set* would unavoidably depend upon a concept which we had not rendered set-theoretically, namely, *stage of a hierarchy*. And that would somewhat undercut the commonplace ambition, that our notion of *set* might serve as a certain kind of autonomous foundation for mathematics.

§2. Level Theory. To overcome this problem, I present Level Theory, LT. This theory's only primitive is $\in$, but it makes exactly the same claims about sets as ST does. I begin with a definition, due to Scott and Montague (see Section 8.3), which forms the linchpin of this paper:[5]

Definition 2.1 For any a, let a's *potentiation* be $\P a := \{x : \exists c(x \subseteq c \in a)\}$, if it exists.[6]

---

[2]Classical logic yields a 'cheap' proof of the existence of a stage and an empty set: by classical logic, there is some object, a; by Staging we have some $\mathbf{s}$ such that $a \preceq \mathbf{s}$; and with $F(x)$ given by $x \neq x$, Specification yields a set, $\emptyset$, such that $\forall x\ x \notin \emptyset$. Those who find such proofs *too* cheap can adopt a free logic and then add explicit existence axioms; I will retain classical logic.

[3]For brevity, I am considering hierarchies of pure sets; I revisit this in Appendixes A and B.

[4]Or, given footnote 2: any *non-null* hierarchy satisfies ST. I will not repeat this caveat.

[5]Montague et al. (unpublished, Definition 22.4, p. 161) and Scott (1974, p. 214). They used the '¶' symbol, but not the name 'potentiation'.

[6]By the notational conventions, $\P a = \{x : (\exists c \in a) x \subseteq c\} = \{x : \exists c(x \subseteq c \wedge c \in a)\}$. We do not initially assume that $\P a$ exists for every a; instead, we initially treat every expression of the form '$b = \P a$' as shorthand for '$\forall x(x \in b \leftrightarrow \exists c(x \subseteq c \in a))$', and must double-check whether $\P a$ exists. Ultimately, though, LT proves that $\P a$ exists for every a (Lemma 3.12.1).

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

The name *potentiation* emphasises the conceptual connection with powersets; note that $\P\{a\} = \wp a$.[7] The next two definitions employ this notion of potentiation (and thereby simplify definitions due to Derrick and Potter; see Section 8.4):[8]

Definition 2.2 Say that h is a *history*, written $Hist(h)$, iff $(\forall x \in h)x = \P(x \cap h)$. Say that s is a *level*, written $Lev(s)$, iff $(\exists h : Hist)s = \P h$.

The intuitive idea behind this definition is that a history is an initial sequence of levels, and that the levels go proxy for stages. It is not obvious that this will work as described; indeed, the next two sections are dedicated to establishing this fact. But, using the notion of a level, LT has just three axioms:[9]

Extensionality $\forall a \forall b(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$,
Separation $\forall F \forall a \exists b \forall x(x \in b \leftrightarrow (F(x) \wedge x \in a))$,
Stratification $\forall a(\exists s : Lev)a \subseteq s$.

§3. The well-ordering of the levels. In Section 4, I will show that LT makes exactly the same claims about sets as ST does. First, I must develop the elements of set theory within LT. To do so, I need some more definitions:

Definition 3.1 Say that a is *transitive* iff $(\forall x \in a)x \subseteq a$. Say that a is *potent* iff $\forall x(\exists c(x \subseteq c \in a) \rightarrow x \in a)$.

Transitivity is completely familiar. *Potency* is discussed in a few places, albeit with no standard name.[10] As my choice of name suggests, though, there is a tight link between the operation of potentiation (see Definition 2.1) and the property of potency:

Lemma 3.2 *If* $\P a$ *exists, then* $\P a$ *is potent*.

Lemma 3.3 (Extensionality). *a is potent iff* $a = \P a$.

Recall the conventions: Lemma 3.2 follows from the definitions alone, but Lemma 3.3 requires Extensionality. I leave the trivial proofs to the reader.

[7] NB: by design, LT does not prove that every set has a powerset; for that, we have LT + Endless (see Section 7).

[8] Potter (2004, p. 41).

[9] For ultra-economy, we can replace Separation+Stratification with $\forall F(\exists a \forall x(F(x) \leftrightarrow x \in a) \leftrightarrow (\exists s : Lev)(\forall x : F)x \in s)$. We can read this as: *a property determines a set iff its instances are all in some level* (cf. Button and Walsh, 2018, Definition 8.9). As in footnote 2, above, the use of classical logic offers a 'cheap' proof of the existence of $\emptyset$. Moreover, LT has a model whose *only* denizen is $\emptyset$.

[10] Potter (1990, p. 19) uses 'hereditary'; Doets (1999, p. 78) and Button and Walsh (2018, p. 193) use 'supertransitive'. Mathias (2001, p. 487) and Burgess (2004, p. 208) use 'supertransitive' for sets which are both transitive and potent; Lévy and Vaught (1961, p. 1046) use 'supercomplete' for such sets.

My aim now is to prove several results about levels, in the sense of Definition 2.2.[11] These results do not need LT's Stratification axiom, since any sets which were not subsets of levels would be irrelevant.[12]

Lemma 3.4 *Every level is potent and transitive.*

Proof. Fix a level, s, so $s = \P h$, for some history h. Potency follows by Lemma 3.2. For transitivity, fix $a \in s = \P h$; so $a \subseteq c \in h$ for some c, and $c = \P(c \cap h)$ as h is a history; so $a \subseteq \P(c \cap h) \subseteq \P h = s$. ⊣

Lemma 3.5 (Extensionality, Separation). *If every F is potent and something is F, then there is an $\in$-minimal F. Formally*: $\forall F((\exists x F(x) \wedge (\forall x : F)x \text{ is potent}) \rightarrow (\exists a : F)(\forall x : F)x \notin a)$.

Proof. Let F be as described and let u be an F. Using Separation twice, let:

$$c = \{x \in u : (\forall y : F)x \in y\} = \{x : (\forall y : F)x \in y\},$$
$$d = \{x \in c : x \notin x\}.$$

Clearly $d \notin c$, since otherwise $d \in d \leftrightarrow d \notin d$; so there is some a which is F with $d \notin a$. Now if x is F, then $d \subseteq c \subseteq x$, but $d \notin a$ and a is potent, so $x \notin a$. ⊣

Lemma 3.6 (Extensionality, Separation). *If some level is F, then there is an $\in$-minimal level which is F. Formally*: $\forall F((\exists s : Lev)F(s) \rightarrow (\exists s : Lev)(F(s) \wedge (\forall r : Lev)(F(r) \rightarrow r \notin s)))$.

Proof. All levels are potent, by Lemma 3.4; now use Lemma 3.5. ⊣

Lemma 3.7 (Extensionality, Separation). *Every member of a history is a level.*

Proof. For reductio, let h be a history with some non-level in it. Since $c = \P(c \cap h)$ for all $c \in h$, every member of h is potent by Lemma 3.2. Using Lemma 3.5, let a be an $\in$-minimal non-level in h. Now $a = \P(a \cap h)$, and $a \cap h = \{x \in a : x \in h\}$ exists by Separation. So, to obtain our desired contradiction, it suffices to show that $a \cap h$ is a history. Fix $b \in a \cap h$. So b is a level, by choice of a, and $b = \P(b \cap h)$ as $b \in h$. If $x \in b$, then $x \subseteq b$, since b is transitive by Lemma 3.4; so $x \in a$, since a is potent as above; hence, $b \subseteq a$. So $b = \P(b \cap h) = \P(b \cap (a \cap h))$. Generalising, $a \cap h$ is a history. ⊣

Lemma 3.8 (Extensionality, Separation). $s = \P\{r \in s : Lev(r)\}$, *for any level s.*

Proof. Let s be a level. If $a \subseteq r \in s$, then $a \in s$, as s is potent by Lemma 3.4. If $a \in s$, then as $s = \P h$ for some history h, we have $a \subseteq r \in h \subseteq \P h = s$ for some r, and r is a level by Lemma 3.7. ⊣

[11] The next few results simplify Potter (2004, pp. 41–6). Lemma 3.5 is inspired by Potter's Proposition 3.6.4; Lemma 3.7 by Potter's Proposition 3.4.1; Lemma 3.8 by Potter's Proposition 3.6.8; and Lemma 3.9 by Potter's Proposition 3.6.11.

[12] Cf. Scott (1974, 211n.1).

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

LEMMA 3.9 (Extensionality, Separation). *All levels are comparable.*[13] *Formally*: $(\forall s : Lev)(\forall t : Lev)(s \in t \vee s = t \vee t \in s)$.

PROOF. For reductio, suppose that some levels are incomparable. By Lemma 3.6, there is an $\in$-minimal level, s, which is incomparable with some level; and by Lemma 3.6 again, there is an $\in$-minimal level, t, which is incomparable with s. To complete the reductio, I will show that $s = t$.

To show that $s \subseteq t$, fix $a \in s$. So $a \subseteq r \in s$ for some level r, by Lemma 3.8. Now r is comparable with t, by choice of s. But if either $r = t$ or $t \in r$, then $t \in s$ as s is transitive, contradicting our assumption; so $r \in t$. Now $a \subseteq r \in t$, so that $a \in t$ as t is potent. Generalising, $s \subseteq t$.

Exactly similar reasoning, based on the choice of t, shows that $t \subseteq s$. So $t = s$. ⊣

Rolling Lemmas 3.6 and 3.9 together, we obtain the fundamental theorem of level theory:

THEOREM 3.10 (Extensionality, Separation). *The levels are well-ordered by membership.*

Combining this result with Stratification, we obtain a powerful tool, which intuitively allows us to consider the level at which a set is first found:

DEFINITION 3.11 (LT). Let ℓa be the $\in$-least level with a as a subset; i.e., $a \subseteq \ell a$ and $\neg(\exists s : Lev)a \subseteq s \in \ell a$.

LEMMA 3.12 (LT). *For all sets a, b, and all levels r, s:*

(1) *ℓa and $¶a$ both exist, and $¶a \subseteq \ell a$,*
(2) $a \notin \ell a$,
(3) *$r \subseteq s$ iff $s \notin r$,*
(4) $s = \ell s$,
(5) *if $b \subseteq a$, then $\ell b \subseteq \ell a$,*
(6) *if $b \in a$, then $\ell b \in \ell a$*
(7) $\ell a = ¶\{\ell x : x \in a\}$,
(8) *if every member of a is a level, then $¶a = \ell a$.*

PROOF. (1) ℓa exists by Stratification and Theorem 3.10. Now if $x \subseteq c \in a \subseteq \ell a$, then $x \in \ell a$ since ℓa is potent; so $¶a \subseteq \ell a$ exists by Separation.

(2) There is no level t with $a \subseteq t \in \ell a$, so $a \notin \ell a$ by Lemma 3.8.

(3) If $r \subseteq s$ then $s \notin r$ by the well-ordering of levels. Conversely, if $s \notin r$, then either $r \in s$ or $r = s$ by comparability; and $r \subseteq s$ either way, as s is transitive.

(4) By (2), $s \notin \ell s$. By (3), $\ell s \notin s$. So $s = \ell s$, by comparability.

(5) If $b \subseteq a$ then $b \subseteq \ell a$. So $\ell a \notin \ell b$, by definition of ℓb, so $\ell b \subseteq \ell a$ by (3).

(6) If $b \in a$ then $b \in \ell a$. By (2), $b \notin \ell b$; so $\ell a \nsubseteq \ell b$, and hence $\ell b \in \ell a$ by (3).

(7) Let $k = \{\ell x : x \in a\}$. If $c \in ¶k$ then $c \subseteq \ell x$ for some $x \in a$; now $\ell x \in \ell a$ by (6), so $c \in \ell a$. Conversely, if $c \in \ell a$ then $c \subseteq r \in \ell a$ for some

[13] Say that x is *comparable* with y iff $x \in y \vee x = y \vee y \in x$.

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

level r by Lemma 3.8; since $a \nsubseteq r$ by definition of ℓa, there is some $x \in a \setminus r$; now $\ell x \notin r$ as r is potent, so that $r \subseteq \ell x$ by (3) and hence $c \subseteq \ell x$; so $c \in \P k$.

(8) In this case, $a = \{\ell x : x \in a\}$ by (4), so $\ell a = \P a$ by (7). ⊣

§4. The set-theoretic equivalence of ST and LT. Having explained how to work within LT, I will now make good on my earlier promise, and show that LT and ST make exactly the same claims about sets. More precisely, I will prove the following:

THEOREM 4.1 ST $\vdash \phi$ *iff* LT $\vdash \phi$, *for any* LT-*sentence* ϕ.

To show that ST says no more about sets than LT does, I define a translation, $*$, from ST-formulas into LT-formulas. In effect, $*$ treats stages as levels, ordered by membership. Specifically, its non-trivial actions are as follows:[14]

$$(\mathbf{s} < \mathbf{t})^* := \mathbf{s} \in \mathbf{t}, \qquad (x \preceq \mathbf{s})^* := x \subseteq \mathbf{s}, \qquad (\forall \mathbf{s}\phi)^* := (\forall \mathbf{s} : Lev)(\phi^*).$$

After translation, we treat all first-order variables—whether bold or italic—as being of the same sort. Fairly trivially, for any LT-sentence ϕ, if ST $\vdash \phi$ then ST$^* \vdash \phi$. The left-to-right half of Theorem 4.1 now follows from this simple observation, together with the fact that $* : \text{ST} \longrightarrow \text{LT}$ is an interpretation:

LEMMA 4.2 (LT). ST* *holds.*

PROOF. Extensionality* is Extensionality. Staging* is Stratification. Order* holds by Lemma 3.4. Note that Lemma 3.8 allows us to simplify $(x \prec \mathbf{s})^*$, i.e., $(\exists \mathbf{r}(x \preceq \mathbf{r} < \mathbf{s}))^*$, to $(x \in \mathbf{s})$. Now Priority* holds trivially. And Specification* holds as if $(\forall x : F)x \in \mathbf{s}$, then $\{x : F(x)\} \subseteq \mathbf{s}$ by Separation.[15] ⊣

To obtain the right-to-left half of Theorem 4.1, I must first prove some quick results in ST:

LEMMA 4.3 (ST). *Separation holds.*

PROOF. By Staging, $a \preceq \mathbf{s}$ for some $\mathbf{s}$. By Priority, $(\forall x \in a)x \prec \mathbf{s}$. Now use Specification. ⊣

LEMMA 4.4 (ST). $\forall \mathbf{s} \forall a(a \preceq \mathbf{s} \leftrightarrow (\forall x \in a)x \prec \mathbf{s})$.

PROOF. Left-to-right is Priority. For right-to-left, suppose $(\forall x \in a)x \prec \mathbf{s}$; then $\{x : x \in a\} = a \preceq \mathbf{s}$ by Extensionality and Specification. ⊣

I next introduce *slices*. These will turn out to be levels, in the sense of Definition 2.2. Here is the definition of a slice, and some elementary results concerning slices:

[14]So the other clauses are: $(\neg\phi)^* := \neg\phi^*$; $(\phi \wedge \psi)^* := (\phi^* \wedge \psi^*)$; $(\forall x\phi)^* := \forall x\phi^*$; $(\forall F\phi)^* := \forall F\phi^*$; and $\alpha^* := \alpha$ for all atomic formulas α which are not of the forms mentioned in the main text.

[15]Note that the $*$-translation of any ST-Comprehension instance is an LT-Comprehension instance.

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

Definition 4.5 For each $\mathbf{s}$, let $\check{\mathbf{s}} = \{x : x \prec \mathbf{s}\}$, if it exists. Say that a is a *slice* iff $a = \check{\mathbf{s}}$ for some $\mathbf{s}$.

Lemma 4.6 (ST). *For any* $\mathbf{s}$:

(1) $\check{\mathbf{s}}$ *exists,*
(2) $\forall \mathbf{r} \forall a(a \preceq \mathbf{r} \leq \mathbf{s} \rightarrow a \preceq \mathbf{s})$,
(3) $\forall a(a \subseteq \check{\mathbf{s}} \leftrightarrow a \preceq \mathbf{s})$,
(4) $\check{\mathbf{s}}$ *is transitive,*
(5) $\check{\mathbf{s}} = \P\{\check{\mathbf{r}} : \check{\mathbf{r}} \in \check{\mathbf{s}}\}$.

Proof. (1) By Specification and Extensionality.

(2) Let $a \preceq \mathbf{r} \leq \mathbf{s}$. Now $(\forall x \in a)x \prec \mathbf{r}$ by Priority, so $(\forall x \in a)x \prec \mathbf{s}$ by Order, and $a \preceq \mathbf{s}$ by Lemma 4.4.

(3) $a \subseteq \check{\mathbf{s}}$ iff $(\forall x \in a)x \in \check{\mathbf{s}}$ iff $(\forall x \in a)x \prec \mathbf{s}$ iff $a \preceq \mathbf{s}$ by Lemma 4.4.

(4) If $a \in \check{\mathbf{s}}$, then $a \preceq \mathbf{r} < \mathbf{s}$ for some $\mathbf{r}$; hence $a \preceq \mathbf{s}$ and $a \subseteq \check{\mathbf{s}}$ by (2) and (3).

(5) If $a \in \check{\mathbf{s}}$, then $a \preceq \mathbf{r} < \mathbf{s}$ for some $\mathbf{r}$; hence $a \subseteq \check{\mathbf{r}} \preceq \mathbf{r} < \mathbf{s}$ by (3), so $a \subseteq \check{\mathbf{r}} \in \check{\mathbf{s}}$. If $a \subseteq \check{\mathbf{r}} \in \check{\mathbf{s}}$, then $a \subseteq \check{\mathbf{r}} \preceq \mathbf{t} < \mathbf{s}$ for some $\mathbf{t}$; now $a \subseteq \check{\mathbf{r}} \subseteq \check{\mathbf{t}}$ by (3), so $a \preceq \mathbf{t}$ by (3), i.e., $a \in \check{\mathbf{s}}$. ⊣

It is now easy to show that $\in$ well-orders the slices: just transcribe the proofs of Lemmas 3.6 and 3.9 within ST, replacing 'levels' with 'slices', noting that ST proves Separation (see Lemma 4.3), and replacing appeal to Lemmas 3.4 and 3.8 with Lemmas 4.6.4 and 4.6.5. We can then go on to prove that the levels are the slices.

Lemma 4.7 (ST). *s is a level iff s is a slice.*

Proof. For induction on slices, suppose: $(\forall \check{\mathbf{q}} \in \check{\mathbf{s}})(\forall a \subseteq \check{\mathbf{q}})(a$ is a slice $\leftrightarrow$ $Lev(a))$. I will show that $(\forall a \subseteq \check{\mathbf{s}})(a$ is a slice $\leftrightarrow Lev(a))$. The result will follow by Staging and Lemma 4.6.3.

First, fix a level $r \subseteq \check{\mathbf{s}}$. Let $h = \{q \in r : Lev(q)\}$; so $r = \P h$ by Lemma 3.8. (Note that ST proves all of Lemmas 3.2–3.9, verbatim, since ST proves Separation.) Fix $a \in r$; so $a \in \check{\mathbf{s}}$, so $a \subseteq \check{\mathbf{q}} \in \check{\mathbf{s}}$ for some $\check{\mathbf{q}}$ by Lemma 4.6.5; hence, by the induction hypothesis, a is a slice iff a is a level. So $h = \{\check{\mathbf{q}} : \check{\mathbf{q}} \in r\}$. Noting that $h \subseteq \check{\mathbf{s}}$, let $\check{\mathbf{t}}$ be the $\in$-least slice such that $h \subseteq \check{\mathbf{t}}$. Since r is transitive and the slices are well-ordered, $h = \{\check{\mathbf{q}} : \check{\mathbf{q}} \in \check{\mathbf{t}}\}$. So $r = \P h = \check{\mathbf{t}}$ by Lemma 4.6.5, i.e., r is a slice.

Next, fix $\check{\mathbf{r}} \subseteq \check{\mathbf{s}}$. Let $h = \{\check{\mathbf{q}} : \check{\mathbf{q}} \in \check{\mathbf{r}}\}$; so $\check{\mathbf{r}} = \P h$ by Lemma 4.6.5; and $h = \{q \in \check{\mathbf{r}} : Lev(q)\}$ by the induction hypothesis. Fix $q \in h$; since $\check{\mathbf{r}}$ is transitive, $q \cap h = \{p \in q : Lev(p)\}$, so that $q = \P(q \cap h)$ by Lemma 3.8. So h is a history and $\check{\mathbf{r}} = \P h$ is a level. ⊣

This allows us to prove the last axiom of LT within ST:

Lemma 4.8 (ST). *Stratification holds.*

Proof. Fix a; by Staging, $a \preceq \mathbf{s}$ for some $\mathbf{s}$, i.e., $a \subseteq \check{\mathbf{s}}$ by Lemma 4.6.3, and $\check{\mathbf{s}}$ is a level by Lemma 4.7. ⊣

So ST $\vdash$ LT, completing the proof of Theorem 4.1.

§5. The inevitability of well-ordering. A simple argument now establishes that LT axiomatizes the bare idea of a cumulative hierarchy of sets:

(a) Any cumulative hierarchy of sets satisfies ST (see Section 1).
(b) LT is set-theoretically equivalent to ST (see Theorem 4.1).
(c) So: any cumulative hierarchy of sets satisfies LT (from (a) and (b)).

Otherwise put: LT is true of the Basic Story I told at the start of this paper, and which I repeat here for ease of reference:

> **The Basic Story.** Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any sets found before **s**, we find a set whose members are exactly those sets. We find nothing else at **s**.

In fact, (c) takes on an even deeper significance when we reflect on just *how* bare-bones this Basic Story is. The Story says that some stages are 'before' others, and we can safely assume that 'before' is a transitive relation on stages (hence ST's Order axiom).[16] But it is not obvious, for example, that it would be inconsistent to augment the Story by saying *for every stage there is an earlier stage*, or *between any two stages there is another stage*. This might prompt us to start entertaining cumulative hierarchies which are ordered like the integers, or the rationals, or more exotically still. A very simple argument, however, puts an abrupt end to such speculation:

(d) LT proves the well-ordering of the levels (see Theorem 3.10).
(e) So: any cumulative hierarchy of sets has well-ordered levels (from (c) and (d)).

Scott was the first to prove a well-ordering result from a similarly spartan starting point (see Section 8.3), and he put the point beautifully: 'This at first surprising result shows how little choice there is in setting up the type hierarchy'.[17] Scott's deep observation deserves to be much more widely known.

The connection between ST and LT also helps to demystify the definition of *level*. Working in ST, suppose that h is an initial sequence of slices; if $\check{\mathbf{s}} \in h$, then $\check{\mathbf{s}} \cap h$ is the set of all slices less than $\check{\mathbf{s}}$, so that $\check{\mathbf{s}} = \P(\check{\mathbf{s}} \cap h)$ by Lemma 4.6.5. These observations motivate Definition 2.2. We say that h is a history iff $(\forall x \in h)x = \P(x \cap h)$, in the hope that, so defined, a history will be an initial sequence of slices; if it is, then the next slice in the sequence is the potentiation of that history, by Lemma 4.6.5; and this is how we define levels.

[16] In similar spirit, Shoenfield (1977, p. 323) says: 'We should certainly expect *before* to be a partial ordering of the stages; and this is the only fact about this relation which we need for our axioms'. But Shoenfield obtains well-ordering by arguing for Foundation using a proof due to Scott (see Section 8.1) and then using Replacement to define the V_αs; LT, of course, does not include Replacement (see Section 7).

[17] Scott (1974, p. 210).

§6. The quasi-categoricity of LT. We just saw that every cumulative hierarchy of sets has well-ordered levels. In fact, we can push this point further. By design, LT says nothing about the height of any hierarchy. But, as I will show in this section, LT is quasi-categorical. Informally, we can spell out LT's quasi-categoricity as follows:

(f) Any two hierarchies satisfying LT are structurally identical for so far as they both run, but one may be taller than the other.

Since every cumulative hierarchy satisfies LT, we obtain:

(g) Any two cumulative hierarchies are structurally identical for so far as they both run, but one may be taller than the other (from (c) and (f)).

So, echoing Scott: when we set up a cumulative hierarchy, our only choice is how tall to make it.

It just remains to establish (f), i.e., to show that LT is quasi-categorical. In fact, there are at least two ways to explicate the informal idea of quasi-categoricity, and LT is quasi-categorical on both explications. (Note that both ways make essential use of second-order logic; this is the only section of the paper where my use of second-order logic is not merely for convenience.)

The first notion of quasi-categoricity is familiar from Zermelo. Working in some (set-theoretic) model theory, we define the V_αs as usual:

$$V_0 = \emptyset; \qquad V_{\alpha+1} = \wp V_\alpha; \qquad V_\alpha = \bigcup_{\beta \in \alpha} V_\beta \text{ when } \alpha \text{ is a limit.}$$

Each V_α then naturally yields a set-theoretic structure, $\mathcal{V}_\alpha$, whose domain is V_α, and which interprets '$\in$' as membership-restricted-to- V_α, i.e., $\{\langle x, y\rangle \in V_\alpha \times V_\alpha : x \in y\}$. We then have the following result, using full second-order logic: $\mathcal{M} \vDash$ ZF iff $\mathcal{M} \cong \mathcal{V}_\alpha$ for some strongly inaccessible α.[18] There is an analogous quasi-categoricity result for LT:[19]

THEOREM 6.1 (In full second-order logic). $\mathcal{M} \vDash$ LT *iff* $\mathcal{M} \cong \mathcal{V}_\alpha$ *for some ordinal* $\alpha > 0$.

This shows that any two hierarchies satisfying LT (read that phrase as 'any models of LT') are structurally identical (read that phrase as 'are isomorphic') for so far as they both run (read that phrase in the light of the well-ordering of the V_αs, established in the model theory). In short, LT is quasi-categorical, on a model-theoretic ('external') way of understanding quasi-categoricity.

[18] Zermelo (1930). For an accessible proof, see Button and Walsh (2018, §8.A).

[19] Button and Walsh (2018, §8.C) prove this for Potter's theory (see Section 8.4); the same proof works for LT. The same remark applies to the other results mentioned in this section. We could obtain external categoricity using only first-order logic, if we augmented LT with some axiom of the form 'there are exactly *n* levels'.

There is also, though, an object-language ('internal') way to understand quasi-categoricity.[20] Since this idea is less familiar, I will spend some time unpacking it.

In embracing Extensionality, LT assumes that everything is a pure set. There is a quick-and-dirty way to avoid this assumption. First, introduce a new predicate, *Pure*; intuitively, this should apply to the pure sets. Next, relativise LT to *Pure*, via the following formula:[21]

$$\begin{aligned}\mathrm{LT}(\mathit{Pure}, \varepsilon) := {} & (\forall a : \mathit{Pure})(\forall b : \mathit{Pure})(\forall x(x \mathrel{\varepsilon} a \leftrightarrow x \mathrel{\varepsilon} b) \to a = b) \wedge \\ & \forall F(\forall a : \mathit{Pure})(\exists b : \mathit{Pure})\forall x(x \mathrel{\varepsilon} b \leftrightarrow (F(x) \wedge x \mathrel{\varepsilon} a)) \wedge \\ & (\forall a : \mathit{Pure})(\exists s : \mathit{Lev})a \subseteq s \wedge \\ & \forall x \forall y(x \mathrel{\varepsilon} y \to (\mathit{Pure}(x) \wedge \mathit{Pure}(y))).\end{aligned}$$

The first three conjuncts tell us that the pure sets satisfy LT;[22] the last says that, when we use 'ε', we restrict our attention to membership facts between pure sets. Using this formula, I can now state the internal quasi-categoricity result (I have labelled the lines to facilitate its explanation):

THEOREM 6.2 *This is a deductive theorem of impredicative second-order logic*:

$$\begin{aligned}&(\mathrm{LT}(\mathit{Pure}_1, \varepsilon_1) \wedge \mathrm{LT}(\mathit{Pure}_2, \varepsilon_2)) \to \\ &\quad \exists R(\forall v \forall y(R(v, y) \to (\mathit{Pure}_1(v) \wedge \mathit{Pure}_2(y))) \wedge && (1)\\ &\quad\quad ((\forall v : \mathit{Pure}_1)\exists y R(v, y) \vee (\forall y : \mathit{Pure}_2)\exists v R(v, y)) \wedge && (2)\\ &\quad\quad \forall v \forall y \forall x \forall z((R(v, y) \wedge R(x, z)) \to (v \mathrel{\varepsilon_1} x \leftrightarrow y \mathrel{\varepsilon_2} z)) \wedge && (3)\\ &\quad\quad \forall v \forall y \forall z((R(v, y) \wedge R(v, z)) \to y = z) \wedge && (4)\\ &\quad\quad \forall y \forall v \forall x((R(v, y) \wedge R(x, y)) \to v = x) \wedge && (5)\\ &\quad\quad \forall v(\exists y R(v, y) \to (\forall x \subseteq_1 \ell_1 v)\exists z R(x, z)) \wedge && (6)\\ &\quad\quad \forall y(\exists v R(v, y) \to (\forall z \subseteq_2 \ell_2 y)\exists x R(x, z))). && (7)\end{aligned}$$

Intuitively, the point is this. Suppose two people are using their versions of LT, subscripted with '1' and '2' respectively. Then there is some second-order entity, a relation R, which takes us between their sets (1), exhausting the sets of one or the other person (2); which preserves membership (3); which is functional (4) and injective (5); and whose domain is an initial segment of one (6) or the other's (7) hierarchy. Otherwise put: LT is (internally) quasi-categorical.

---

[20]This has been brought out by Parsons (1990, 2008), McGee (1997), and Väänänen and Wang (2015). The remainder of this section presents specific elements of Button and Walsh (2018, Chapter 11).

[21]Here, '$\subseteq$' and '*Lev*' should be defined in terms of 'ε' rather than '$\in$'; similarly for 'ℓ' in Theorem 6.2. For now, we can treat '*Pure*' as a primitive; but see Definition B.1.

[22]With one insignificant caveat (see footnotes 2 and 4): whereas classical logic guarantees that any model of LT contains an empty set, LT(*Pure*, ε) allows that there may be no pure sets.

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

As a bonus, this internal *quasi*-categoricity result can be lifted into an internal *total*-categoricity result. To explain how, consider this abbreviation (where 'P' is a second-order function-variable):

$$\underset{\infty}{\exists} x\Phi(x) := \exists P(\forall x\Phi(P(x)) \wedge (\forall y : \Phi)\exists! x\ P(x) = y).$$

This formalizes the idea that there are as many Φs as there are objects *simpliciter*, i.e., that there is a bijection between the Φs and the universe. We can use this notation to state our internal total-categoricity result:

THEOREM 6.3 *This is a deductive theorem of impredicative second-order logic*:

$$\begin{aligned}(\mathrm{LT}(Pure_1, \varepsilon_1) \wedge \underset{\infty}{\exists} x\ Pure_1(x) \wedge \mathrm{LT}(Pure_2, \varepsilon_2) \wedge \underset{\infty}{\exists} x\ Pure_2(x)) \rightarrow \\ \exists R(\forall v \forall y(R(v, y) \rightarrow (Pure_1(v) \wedge Pure_2(y))) \wedge \\ (\forall v : Pure_1)\exists! y R(v, y) \wedge (\forall y : Pure_2)\exists! v R(v, y) \wedge \\ \forall v \forall y \forall x \forall z((R(v, y) \wedge R(x, z)) \rightarrow (v\ \varepsilon_1\ x \leftrightarrow y\ \varepsilon_2\ z))).\end{aligned}$$

Intuitively, if both LT-like hierarchies are as large as the universe, then there is a structure-preserving *bijection* between them. To see the significance of this result, note that it is common to claim that there are *absolutely infinitely many* pure sets. Whatever exactly this is meant to mean, it must surely entail that $\underset{\infty}{\exists} x\ Pure(x)$. So Theorem 6.3 tells us that absolutely infinite LT-like hierarchies are (internally) isomorphic.

§7. LT as a subtheory of ZF.

I have shown that any cumulative hierarchy satisfies LT, so that, in setting up a cumulative hierarchy, our only freedom of choice concerns its height. To make all of this more familiar, though, it is worth commenting on LT's relationship to ZF, the 'industry standard' set theory.

Unsurprisingly, ZF proves LT. In more detail: working in ZF, define the V_αs as usual; we can then show that the V_αs are the levels;[23] so Stratification holds as every set is a subset of some V_α.

Of course, ZF is much stronger than LT, since LT deliberately says nothing about the height of the cumulative hierarchy. If we want to set up a tall hierarchy, then three axioms naturally suggest themselves (where 'P' is a second-order function-variable in the statement of Unbounded):[24]

Endless	$(\forall s : Lev)(\exists t : Lev)s \in t,$
Infinity	$(\exists s : Lev)((\exists q : Lev)q \in s \wedge (\forall q : Lev)(q \in s \rightarrow (\exists r : Lev)q \in r \in s)),$
Unbounded	$\forall P \forall a(\exists s : Lev)(\forall x \in a)P(x) \in s.$

[23] *Proof sketch*. Working in ZF, fix α, and suppose for induction that $(\forall \beta < \alpha)(\forall x \subseteq V_\beta)(Lev(x) \leftrightarrow \exists \delta \quad x = V_\delta)$. Fix $V_\gamma \subseteq V_\alpha$; then $V_\gamma = \P\{V_\delta : V_\delta \in V_\gamma\} = \P\{s \in V_\gamma : Lev(s)\}$ by the induction hypothesis, which is a level by Lemma 3.8. Similarly, if $s \subseteq V_\alpha$ is a level, then $\exists \delta \quad s = V_\delta$.

[24] For Endless, cf. Montague (1965, p. 142), Scott (1974, p. 212), Potter (1990, pp. 20–1, and 2004, pp. 61–2). For Infinity, see Potter (2004, pp. 68–70) and Boolos's (1989, p. 8) axiom Inf, which I discuss in Section 8.2.

Endless says there is no last level. Infinity says that there is an infinite level, i.e., a level with no immediate predecessor. Unbounded states that the hierarchy of levels is so tall that no set can be mapped unboundedly into it. We now have some nice facts, whose proofs I leave to the reader:[25]

PROPOSITION 7.1

(1) LT *proves Separation, Union, and Foundation.*
(2) LT + *Endless proves Pairing and Powersets.*
(3) LT + *Endless* + *Infinity proves Zermelo's axiom of infinity.*[26]
(4) LT + *Endless* + ¬*Infinity is equivalent to* $\mathrm{ZF}_{\mathrm{fin}}$.[27]
(5) LT + *Infinity* + *Unbounded proves Endless.*
(6) LT + *Infinity* + *Unbounded is equivalent to* ZF.

Facts (1)–(3) show that LT + Endless + Infinity extends Zermelo's Z. This extension is strict, since Stratification is independent from Z.[28] Fact (6) then offers a neat way to conceive of ZF, as extending the theory which holds of any cumulative hierarchy, i.e., LT, with specific claims about the hierarchy's height.

§8. Conclusion, and LT's predecessors. The theory LT holds of every cumulative hierarchy. Since LT is also quasi-categorical, the only choice we have, in setting up a cumulative hierarchy, is over the hierarchy's height.

I will close this paper by discussing LT's predecessors, in roughly chronological order.

8.1. Scott. At a talk in 1957, Scott presented what seems to have been the first theory of stages. This was an axiomatic theory of *ranks*, in the sense of the V_αs. Writing '$a < b$' for 'a has lesser rank than b', Scott's suggested axioms were Extensionality and:[29]

$$\forall a \forall b (a < b \leftrightarrow (\exists x < b) x \not< a),$$
$$\forall F (\forall a ((\forall x < a) F(x) \rightarrow F(a)) \rightarrow \forall a F(a)),$$
$$\forall F \forall a \exists b \forall x (x \in b \leftrightarrow (F(x) \land x < a)).$$

This 1957 theory is clearly satisfied in any $\mathcal{V}_\alpha$ with $\alpha > 0$, when $\in$ and $<$ are given the obvious interpretations. However, it has some unintended models.

[25] Cf. Scott (1974, p. 212), Potter (1990, pp. 20–4, and 2004, pp. 47–9, 61–2).

[26] That is, $(\exists w \ni \emptyset)(\forall x \in w) x \cup \{x\} \in w$.

[27] The latter is the theory with all of ZF's axioms except that: (i) Zermelo's axiom of infinity is replaced with its negation; and (ii) it has a new axiom, $\forall a (\exists t \supseteq a)(t$ is transitive$)$.

[28] Potter (2004, p. 293ff) makes a similar point. The independence is immediate from the fact that there are models of (even second-order) Z which fail to satisfy $\forall a (\exists c \supseteq a)(c$ is transitive$)$; see Drake (1974, p. 111). For detailed discussions of Z's weaknesses, as either a first- or second-order theory, see Mathias (2001). (As mentioned in the introduction, although I have formulated LT as a second-order theory, it has a natural first-orderization. Read uniformly as either first-order or second-order theories, and closing under provability, the point is: Z $\subsetneq$ LT + Endless + Infinity $\subsetneq$ ZF.)

[29] Scott (1960); I have tweaked the presentation slightly.

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

Example 8.1 Let the domain have two sets : $\emptyset$ and a Quine atom $a = \{a\}$. Let $a < \emptyset$. This is a model of the 1957 theory, since $<$ is trivially a well-order, and since the only sets given by the third axiom are $\emptyset$ and $\{a\} = a$.

Example 8.2 Let the domain have four sets : $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$. Permute the usual rank relation, so that $\{\emptyset\} < \emptyset < \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$, with $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ incomparable.

At a talk in 1967, Scott provided a vastly improved theory of stages. I will present the 1967 theory in a slightly simplified form, starting with a definition given later by Potter (see Section 8.4):

Definition 8.3 For each set a, let $\text{Acc}a = \{x : (\exists c \in a)(x \in c \lor x \subseteq c)\}$, if it exists.

Scott's 1967 theory treats the notion of *level* as a primitive, which applies to certain sets. Temporarily using bold-face letters to range over these *levels*, the 1967 theory comprises just Extensionality, Separation, and two further axioms:[30]

Accumulation $\forall \mathbf{s}\ \mathbf{s} = \text{Acc}\{\mathbf{r} : \mathbf{r} \in \mathbf{s}\}$,
Restriction $\forall a \exists \mathbf{s}\ a \subseteq \mathbf{s}$.

Scott's 1967 theory (unlike his 1957 theory) does not explicitly state that the levels are well-ordered; instead, the 1967 theory *proves* the well-ordering of the levels (cf. Section 5).[31] We have Scott to thank for a truly remarkable bit of mathematics-cum-conceptual-analysis.

Scott's 1967 theory obviously inspires ST: compare his Restriction axiom with my Staging (and Stratification), and his Accumulation axiom with my Lemma 4.6.5 (and Lemma 3.8). Moreover, Scott's 1967 theory and ST make exactly the same claims about sets (cf. Theorem 4.1). But I used ST in Section 1, rather than Scott's 1967 theory, since ST is easier to motivate. In particular, Scott simply instructs us to write '$\mathbf{s} \in \mathbf{t}$' for '$\mathbf{s}$ is before $\mathbf{t}$', and his justification of Accumulation amounts to stipulating that 'a given level is *nothing more than* the accumulation of all the members and subsets of all the *earlier* levels'.[32] Both claims are very natural, and they are true in LT; but it is not immediately obvious that they are true of the Basic Story I told in the introduction. (In fairness to Scott, he does not start with that story, but with a related justificatory tale.)

8.2. Boolos and Shoenfield. The second source of inspiration for ST is Boolos. He first presented a theory of stages in 1971, which included explicit axioms stating that the stages are well-ordered;[33] this theory has several

[30]Scott (1974, pp. 208–9). Scott allowed urelements, which I am ignoring for ease of presentation (though see Appendix A).

[31]Scott's (1974, pp. 211–2) proof uses the idea of a *grounded* set, introduced by Montague (1955).

[32]Both quotes from Scott (1974, p. 209); his emphasis; variables adjusted to match surrounding text.

[33]See Boolos's (1971, pp. 223–4) I–IV and Induction Axioms.

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

similarities with Shoenfield's 1967 theory.[34] Boolos then presented a better theory of stages in 1989, explicitly drawing from Scott's 1967 theory to prove (rather than assume) a principle of induction for stages.[35] My theory ST tweaks Boolos's 1989 theory in three ways.

First. Boolos has qualms about how to justify Extensionality;[36] I have no such qualms.

Second. Boolos aims to vindicate the traditional Zermelian axioms of Foundation, Union, Pairing, Separation, Powersets, and Infinity. To secure these last two axioms, his 1989 theory contains:

Inf $\exists \mathbf{t}(\exists \mathbf{r}\, \mathbf{r} < \mathbf{t} \wedge (\forall \mathbf{r} < \mathbf{t})\exists \mathbf{s}(\mathbf{r} < \mathbf{s} < \mathbf{t}))$,
Net $\forall \mathbf{r}\forall \mathbf{s}\exists \mathbf{t}(\mathbf{r} < \mathbf{t} \wedge \mathbf{s} < \mathbf{t})$.

Boolos's Inf guarantees there is a stage with infinitely many predecessors, and his Net guarantees that there is no last stage. Since ST is deliberately silent on the height of any cumulative hierarchy, it has no similar axioms. However, if I had wanted to augment ST with the claim that there is no last stage, I would have offered $\forall \mathbf{s}\exists \mathbf{t}\ \mathbf{s} < \mathbf{t}$ (cf. Endless, from Section 7). Boolos's Net says more than this; it guarantees that stages are directed. Boolos's proof of Pairing relies upon this directedness,[37] but I cannot see why Boolos felt independently entitled to adopt Net rather than the weaker principle.

Third. The remainder of Boolos's 1989 theory comprises Order, Staging, and these two axioms:[38]

When $\forall \mathbf{s}\forall a(a \preceq \mathbf{s} \leftrightarrow (\forall x \in a)x \prec \mathbf{s})$,
Spec $\forall F\forall \mathbf{s}((\forall x : F)x \prec \mathbf{s} \rightarrow \exists a\forall x(F(x) \leftrightarrow x \in a))$.

In the presence of Extensionality, the axioms When+Spec are equivalent to ST's Priority+Specification; but we need Extensionality to prove the right-to-left direction of When from Priority+Specification (see Lemma 4.4). Moreover, given Boolos's qualms about Extensionality, he cannot provide an intuitive justification for the right-to-left direction of When. If $(\forall x \in a)x \prec \mathbf{s}$, then there should certainly be some $b \preceq \mathbf{s}$ such that $\forall x(x \in b \leftrightarrow x \in a)$; but only Extensionality can justify the assertion that $b = a$. Crucially for Boolos's aims, though, Powersets can fail if we replace When+Spec with Priority+Specification in Boolos's theory: without Extensionality or the right-to-left direction of When, we might keep finding new empty sets at every stage in the hierarchy; there will then be no stage by which every subset of a set has been found, and hence no stage at which any powerset can be found.

[34] Shoenfield (1967, pp. 238–40).

[35] Scott (1974, pp. 211–2) and Boolos (1989, pp. 11–12); Boolos cites Shoenfield's (1977, p. 327) presentation of Scott.

[36] Boolos (1989, pp. 10–11).

[37] Boolos's (1989, p. 19) proof is as follows. Fix a and b; by Staging, there are $\mathbf{r}$ and $\mathbf{s}$ with $a \preceq \mathbf{r}$ and $b \preceq \mathbf{s}$. By Net, there is some $\mathbf{t}$ after both $\mathbf{r}$ and $\mathbf{s}$. So by Spec there is a set whose members are exactly a and b.

[38] Boolos (1989, p. 8) formulates Spec as a first-order scheme, but considers the second-order axiom on the next page.

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

8.3. Scott and Montague. I now want to return to Scott's 1967 theory. As mentioned in Section 8.1, this theory initially takes the notion of *level* as primitive. However, Scott notes that the primitive can be eliminated, by proving within the 1967 theory that s is a level iff $\P s \subseteq s \wedge (\forall a \in s)(\exists h \in s)(\forall k \subseteq h)(\P k \in s \wedge (\P k \in h \vee a \subseteq \P k))$. Scott developed this ideologically-spartan theory in joint work with Montague; they described their theory as 'rank free', so I will call it RF.[39] It has just three axioms: Extensionality, Separation, and

Hierarchy $\forall a \exists h (\forall k \subseteq h)(\exists s = \P k)(s \in h \vee a \subseteq s)$.

The point of calling it 'rank free' was to highlight that RF takes no stance on the number of ranks in the hierarchy. More precisely, we have the external quasi-categoricity result that $\mathcal{M} \vDash \text{RF}$ iff $\mathcal{M} \cong \mathcal{V}_\alpha$ for some $\alpha > 0$ (assuming full second-order logic; cf. Theorem 6.1). To establish this, Montague and Scott first say that h is a *hierarchy* iff $(\forall k \subseteq h)(h \subseteq \P k \vee \P k = \bigcap(h \setminus \P k))$. They then let $\mathrm{R}a := \bigcap\{\P h : h \text{ is a hierarchy} \wedge a \subseteq \P h\}$ for each a, and show that $\mathrm{R}a$ serves the role of a's 'rank' (cf. LT's notion of ℓa, as laid down in Definition 3.11).

Unfortunately, as Scott himself put it, the deductions from these axioms and definitions 'are quite lengthy'.[40] This led Scott to dismiss the significance of RF, writing: 'there seems to be no technical or conceptual advantage in reducing the number of primitive notions to the minimum'.[41]

Still, these lengthy deductions were intended to form a section of a monograph on axiomatic set theory. A complete manuscript of this monograph exists, Montague et al. (unpublished), containing very minor markups, handwritten notes to the printers, and an accompanying list of 'Things to be Done' which amounts to nothing more than writing an Introduction and dealing with the mundane logistics of publication. Everything, in short, was almost ready to print.

Sadly, it was never printed. This was a serious loss. As explained in Section 1, there are good philosophical reasons for 'reducing the number of primitive notions to the minimum'. Moreover, whilst Montague's and Scott's *deductions* were 'quite lengthy', the *axioms* of RF are quite elegant. The lengthiness of the deductions from RF is down to the awkwardness of the definitions of *hierarchy* and $\mathrm{R}a$. If Montague and Scott had been aware of the definition of *history* and *level*, as given in Definition 2.2, they could have given some much briefer deductions. Indeed, these definitions make it easy to prove that RF and LT are equivalent. One direction of this equivalence is easy:

Proposition 8.4 (LT). RF *holds.*

[39]Montague (1965, p. 139), Montague et al. (unpublished, pp. 161–2), and Scott (1974, p. 214).

[40]Scott (1974, p. 214). Indeed, it occupies 13 dense sides of Montague et al. (unpublished, pp. 161–74).

[41]Scott (1974, p. 214).

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

Proof. It suffices to prove Hierarchy. Fix a, let $h = \{s \in \ell a : Lev(s)\}$ and fix $k \subseteq h$. Now $\P k = \ell k$ by Lemma 3.12.8; so if $\P k = \ell k \notin h$, then $\ell k \notin \ell a$, so $a \subseteq \ell a \subseteq \ell k = \P k$ by Lemma 3.12.3. ⊣

For the other direction of the equivalence, I must first prove some quick facts in RF:

Lemma 8.5 (RF). *For all a:*

(1) *if $\P a$ exists, then $\P a \notin a$,*
(2) *$\P a$ exists,*
(3) *if every member of a is a level, then $\P a$ is a level.*

Proof. (1) If $\P a \in a$, then $(\forall c \subseteq \P a) c \in \P a$. But this is impossible: by Separation, let $d = \{x \in \P a : x \notin x\}$; then $d \notin \P a$.

(2) Fix a, and let h witness Hierarchy. Let $k = h$, so that $\P h$ exists and $\P h \in h \vee a \subseteq \P h$, i.e., $a \subseteq \P h$ by (1). Since $\P h$ is potent by Lemma 3.2, $\P a \subseteq \P h$ exists by Separation on $\P h$.

(3) Using Separation and (2), let $h = \{s \in \P a : Lev(s)\}$. I will first prove that $\P h = \P a$, and then that h is a history, so that $\P h = \P a$ is a level.

To see that $\P a = \P h$: since $h \subseteq \P a$, we have $\P h \subseteq \P\P a = \P a$ by Lemmas 3.2 and 3.3; and if $x \in \P a$ then $x \subseteq r \in a$ for some level r, so $r \in h$, and hence $x \in \P h$.

To see that h is a history, fix $s \in h$; it suffices to show that $s = \P(s \cap h)$. Since s is a level, $\P(s \cap h) \subseteq \P s = s$ by Lemmas 3.3 and 3.4. To see that $s \subseteq \P(s \cap h)$, fix $x \in s$; now $x \subseteq r \in s$ for some level r by Lemma 3.8; and $r \subseteq s \in \P a$ by Lemma 3.4, so $r \in \P a$ by Lemma 3.2 and hence $r \in h$; so $x \subseteq r \in (s \cap h)$, i.e., $x \in \P(s \cap h)$. ⊣

Proposition 8.6 (RF). LT *holds.*

Proof. It suffices to prove Stratification. Fix a, and let h witness Hierarchy, i.e., $(\forall k \subseteq h)(\P k \in h \vee a \subseteq \P k)$. Let $k = \{s \in h : Lev(s)\}$. By Lemma 8.5.3, $\P k$ is a level. Now if $\P k \in h$, then $\P k \in k$, contradicting Lemma 8.5.1; so $a \subseteq \P k$. ⊣

This last proof helps to explain the intuitive idea behind RF's axiom Hierarchy.[42] Roughly, the h guaranteed to exist by Hierarchy has this property: for any initial sequence of levels $k \subseteq h$, the next level after all of them is $\P k$; and if a is not a subset of $\P k$, then $\P k$ is in h; and *hence* (but here I invoke a transfinite induction) the members of h are all the levels up to the first level including a. In short, the fundamental idea behind RF is quite elegant.

8.4. Derrick and Potter. As mentioned in Section 2, my definition of *level* is inspired by Derrick and Potter,[43] but I have simplified it. Here is a little more detail about that simplification. In his 1990 book, Potter explicitly built on Scott's 1967 theory and also on Derrick's unpublished lecture notes.[44]

[42] Cf. Montague et al. (unpublished: 162).
[43] See especially Potter (1990, pp. 16–20 and 2004, pp. 41–7).
[44] Potter (1993, pp. 183–4, 1990, p. 22, and 2004, p. vii, 54).

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

Now, Scott's Accumulation axiom (see Section 8.1) formalizes the claim that 'a given level is *nothing more than* the accumulation of all the members and subsets of all the *earlier* levels'.[45] This suggests the use of the Acc-operator, and so Potter offers Definition 8.3.[46] Potter then supplies the definition of *history* and *level* given in Definition 2.2, but using Acc rather than ¶. So, Potter stipulates that h is a history iff $(\forall x \in h)x = \text{Acc}(x \cap h)$, and that s is a level iff $s = \text{Acc}h$ for some history h. Potter then proves that, so defined, the levels are well-ordered. And his own theory of levels is, in effect, just LT, with this slightly different explicit definition of '*Lev*'.[47] But the use of ¶, rather than Acc, simplifies things significantly, as illustrated by the brevity of Section 3.

Appendix A. Adding urelements In this paper, I restricted my attention to pure sets.[48] This was only for ease of exposition; in this appendix and the next, I will remove this simplifying assumption.

To accommodate urelements, we must tweak the Basic Story. The easiest way to do this (which I revisit in Appendix B) is to assume that the urelements are 'always' available to be collected into sets:

> **The Urelemental Story.** Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any things, each of which is either a set found before **s** or an urelement, we find a set whose members are exactly those things. We find nothing else at **s**.

To formalize this Story, we need a new primitive predicate, enabling us to distinguish sets from urelements: we take *Set* as primitive, and define $Ur(x) := \neg Set(x)$. Stage Theory with Urelements, STU, now has six axioms:[49]

$$
\begin{array}{rl}
\textbf{Empty-U} & (\forall u : Ur)\forall x \;\; x \notin u, \\
\textbf{Ext-U} & (\forall a : Set)(\forall b : Set)(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b), \\
\textbf{Order} & \forall \mathbf{r}\forall \mathbf{s}\forall \mathbf{t}(\mathbf{r} < \mathbf{s} < \mathbf{t} \rightarrow \mathbf{r} < \mathbf{t}), \\
\textbf{Staging-U} & (\forall a : Set)\exists \mathbf{s} \;\; a \preceq \mathbf{s}, \\
\textbf{Priority-U} & \forall \mathbf{s}(\forall a : Set)(a \preceq \mathbf{s} \rightarrow (\forall x \in a)(Ur(x) \vee x \prec \mathbf{s})), \\
\textbf{Spec-U} & \forall F \forall \mathbf{s}((\forall x : F)(Ur(x) \vee x \prec \mathbf{s}) \rightarrow \\
& \qquad (\exists a : Set)(a \preceq \mathbf{s} \wedge \forall x(F(x) \leftrightarrow x \in a))).
\end{array}
$$

[45] Scott (1974, p. 209).

[46] Potter (1990, p. 16 and 2004, p. 41, 50).

[47] There are three other small differences: (1) Potter allows urelements; (2) he provides a first-order theory; and (3) he offers a slightly more restricted version of Separation, whose second-order formulation is $\forall F(\forall s : Lev)\exists b \forall x(x \in b \leftrightarrow (F(x) \wedge x \in s))$, but this trivially entails the unrestricted version of Separation given (Potter's version of) Stratification.

[48] Montague (1965, p. 139), Scott (1974, p. 214), and Potter (1990, 2004) accommodate urelements from the outset.

[49] As in footnote 2: STU gives us a stage **s** 'for free', so that $\{x : Ur(x)\}$ exists by Spec-U.

Empty-U says that no urelement has any members; the other axioms relativise ST to sets. As in Section 1, any cumulative hierarchy obviously satisfies STU, on the assumption that the urelements are all 'always' available to be arbitrarily collected into sets.

We obtain Level Theory with Urelements, LTU, by tweaking LT's key definitions. Specifically, I offer the following re-definition:[50]

Definition A.1 (For Appendix A only). Say that a is potent iff $\forall x((Ur(x) \vee (\exists c : Set)x \subseteq c \in a) \rightarrow x \in a)$. Let $\P a := \{x : Ur(x) \vee (\exists c : Set)x \subseteq c \in a\}$, if it exists. Say that $Hist(h)$ iff $(\forall x \in h)x = \P(x \cap h)$. Say that $Lev(s)$ iff $(\exists h : Hist)s = \P h$.

The axioms of LTU are then Empty-U, Ext-U, Stratification (with '*Lev*' as redefined), and:[51]

Sep-U $\forall F \forall a (\exists b : Set)\forall x(x \in b \leftrightarrow (F(x) \wedge x \in a))$.

The proofs of Sections 3 and 4 go now through with trivial changes. Specifically, the (redefined) levels are well-ordered, and STU and LTU make exactly the same demands on sets and urelements.

The (quasi-)categoricity results of Section 6 also carry over to LTU. Let $\mathcal{A}$ and $\mathcal{B}$ be models of LTU in full second-order logic, and suppose there is a bijection between their respective collections of urelements, $Ur^{\mathcal{A}}$ and $Ur^{\mathcal{B}}$. This bijection can be lifted to a quasi-isomorphism: $\mathcal{A}$ and $\mathcal{B}$ are isomorphic 'so far as they go', but the levels of one may outrun the other. This external result can also be 'internalised', yielding results analogous to Theorems 6.2 and 6.3.

Note that LTU, like LT before it, takes no stance on the height of the hierarchy. In particular, it has no version of Replacement. In this regard, LTU differs sharply from ZF(C)U, which is something like the 'industry standard' for iterative set theory with urelements. It is particularly noteworthy that LTU allows that the set of urelements may be larger than any pure set.[52] (For a trivial example, suppose there are exactly three urelements and exactly two levels; for a less trivial example, suppose there are exactly $\beth_{\omega+1}$ urelements but only an $\omega + \omega$ sequence of levels.)

Appendix B. Adding absolutely infinitely many urelements The Urelemental Story accommodates urelements in a humdrum way. However, there has been recent interest in a less humdrum approach, according to which there are *absolutely infinitely many* urelements. Here is a brisk,

[50]The first level is therefore $\{x : Ur(x)\}$. This follows Montague (1965, p. 139), Potter (1990, p. 16, and 2004, p. 41). By contrast, Scott's (1974, p. 214 first level is $\emptyset$, and the urelements are members of every subsequent level.

[51]Together, Stratification and Sep-U deliver the existence of $\{x : Ur(x)\}$; see the previous two footnotes.

[52]LTU could therefore be used in place of, e.g., Menzel's ZFCU$'$ (2014, pp. 67–71), which is designed to accommodate the claim that the set of urelements is not equinumerous with any pure set.

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

three-premise argument in favour of that approach, inspired by Christopher Menzel:[53]

(a) There are absolutely infinitely many levels in the cumulative hierarchy.
(b) There are at least as many ordinals as there are levels in the cumulative hierarchy.
(c) Ordinals are not really sets; they are urelements.

Each premise is not implausible,[54] and they jointly entail that there are absolutely infinitely many urelements. In this appendix, I will explore that idea (without endorsing it).

B.1. Preliminary motivations and observations. There is an immediate technical issue: in this kind of cumulative setting, no set has absolutely infinitely many members.[55] This follows from a simple version of Cantor's Theorem. For reductio, suppose that some set, a, has absolutely infinitely many members. As discussed in Section 6, this entails that $\underset{\infty}{\exists} x\ x \in a$, i.e., there is a map, P, such that $\forall x\ P(x) \in a$ and $(\forall y \in a)\exists! x\ P(x) = y$. By P's injectivity and Separation,[56] there is some $d = \{x \in a : x \notin P^{-1}(x)\}$. Since $P(d) \in a$, contradiction follows familiarly.

So: if there are absolutely infinitely many urelements, then there is no set of all urelements.[57] But the existence of such a set is a trivial consequence of Spec-U, as laid down in Appendix A. So, those who think that there are absolutely infinitely many urelements must reject Spec-U. Furthermore, since Spec-U follows from the third sentence of the Urelemental Story of Appendix A, they must change their story.

Many alternative stories are possible, but the simplest approach is simply to bolt a Limitation of Size principle onto the Urelemental Story, insisting that the Basic Story remains correct of the pure sets, whilst denying that any set is absolutely infinite. This leads to the following:[58]

---

[53] Menzel (1986, p. 41ff); cf. Rumfitt (2015, pp. 271–5). Menzel (2014, p. 57) also offers a second (very different) argument to the same conclusion.

[54] Claim (a) can be motivated by a principle of plenitude concerning sets. Claim (b) can be motivated by combining the fact that the levels of any (pure) cumulative hierarchy are well-ordered (see Section 5) with the idea that any system of well-ordered objects exemplifies an ordinal (provided that the objects are all members of some set). Claim (c) can be motivated by a kind of platonistic structuralism, according to which ordinals are indeed *objects*, but not *sets*, since sets have structure which is not purely order-theoretic. For the record, I do not subscribe to this kind of platonistic structuralism.

[55] *Pace* Menzel (1986, pp. 44–51, 2014, pp. 71–9). Note that my argument does not involve Powersets (which Menzel ultimately rejects). Menzel escapes formal inconsistency, whilst retaining (a first-order version of) Separation, only because his set-theoretic object language has no way to pick out a suitable map, P, which witnesses the absolute infinity of his set $\{x : Ur(x)\}$.

[56] I take it that rejecting Separation is not an option in this setting; though see Level Theory, Part 3 for an approach which rejects Separation.

[57] Uzquiano (2015, pp. 330–1) also suggests the use of a set theory with urelements but no set of urelements, though for somewhat different reasons.

[58] Cf. Uzquiano (2015, p. 331).

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

The $\underset{\infty}{\text{U}}$relemental Story. Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any things—provided both that (i) there are not absolutely infinitely many of them, and that (ii) each of them is either a set found before **s** or an urelement—we find a set whose members are exactly those things. We find nothing else at **s**. (NB: since the Basic Story is correct of the pure sets, we do not find absolutely infinitely many pure sets before **s**.)

In the remainder of this appendix, I will briefly sketch (equivalent) stage-theoretic and level-theoretic formalizations of this Story. For readability, I leave all proofs to the reader, with hints in footnotes.

B.2. The stage-theoretic approach. To axiomatize the $\underset{\infty}{\text{U}}$relemental Story, we need a predicate, '*Pure*', to pick out the pure sets (cf. Section 6). Since we have assumed that the Basic Story holds of the pure sets, we can define '*Pure*' explicitly:

DEFINITION B.1 Say that a is pure, $Pure(a)$, iff both $Set(a)$ and there is some transitive $c \supseteq a$ whose members are all sets.

To axiomatize the $\underset{\infty}{\text{U}}$relemental Story, we also need a way to formalize 'there are absolutely infinitely many Φs'. There are familiar concerns about the possibility of formalizing this idea.[59] Nonetheless, if there are absolutely infinitely many Φs, then certainly $\underset{\infty}{\exists} x\Phi(x)$ (cf. Section 6). Conversely, if $\underset{\infty}{\exists} x\Phi(x)$, then no property can have *more* instances than Φ. So, '$\underset{\infty}{\exists} x\Phi(x)$' will serve as our proxy for 'there are absolutely infinitely many Φs'.[60]

I can now lay down the theory ST$\underset{\infty}{\text{U}}$. Its axioms are Empty-U, Ext-U, Order, Staging-U, Priority-U, and the following:

$$\begin{aligned}
&\textbf{Spec-}\underset{\infty}{\textbf{U}} && \forall F \forall \mathbf{s}((\neg \underset{\infty}{\exists} x F(x) \wedge (\forall x : F)(Ur(x) \vee x \prec \mathbf{s})) \rightarrow \\
& && \qquad (\exists a : Set)(a \preceq \mathbf{s} \wedge \forall x(F(x) \leftrightarrow x \in a))),\\
&\textbf{LoS-}\underset{\infty}{\textbf{U}} && (\forall a : Set)\neg \underset{\infty}{\exists} x\, x \in a,\\
&\textbf{Pure-}\underset{\infty}{\textbf{U}} && \forall F \forall \mathbf{s}((\forall x : F)(Pure(x) \wedge x \prec \mathbf{s}) \rightarrow \neg \underset{\infty}{\exists} x F(x)),\\
&\textbf{Many-}\underset{\infty}{\textbf{U}} && \underset{\infty}{\exists} x\, Ur(x).
\end{aligned}$$

In brief: Spec-$\underset{\infty}{\text{U}}$ restricts Spec-U to capture conditions (i) and (ii) of the $\underset{\infty}{\text{U}}$relemental Story; LoS-$\underset{\infty}{\text{U}}$ enshrines Limitation of Size, which follows from condition (i) plus the fact that 'we find nothing else' at any stage; Pure-$\underset{\infty}{\text{U}}$ formalizes the parenthetical 'NB' of the Story; and Many-$\underset{\infty}{\text{U}}$ formalizes the claim that there are absolutely infinitely many urelements.

B.3. The level-theoretic approach. ST$\underset{\infty}{\text{U}}$ is a multi-sorted, stage-theoretic, formalization of the $\underset{\infty}{\text{U}}$relemental Story. That Story can instead be given a single-sorted formalization, LT$\underset{\infty}{\text{U}}$. To do this, I start by tweaking LT's key definitions:

[59] See, e.g., McGee (1992, p. 279).

[60] Very little of what I say depends upon this particular choice of proxy. In particular, I rely upon its logical properties only when claiming that both ST$\underset{\infty}{\text{U}}$ and LT$\underset{\infty}{\text{U}}$ prove Sep-U, and in my remarks on the quasi-categoricity of LT$\underset{\infty}{\text{U}}$.

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

Definition B.2 (For Appendix B only). Say that a is potent iff $(\forall x : Set)(\exists c(x \subseteq c \in a) \rightarrow x \in a)$. Let $\P a = \{x : Set(x) \wedge \exists c(x \subseteq c \in a)\}$, if it exists. Say that $Hist(h)$ iff $(\forall x \in h)x = \P(x \cap h)$. Say that $Lev(s)$ iff $(\exists h : Hist)s = \P h$.

Using these redefinitions, we can prove analogues of Lemmas 3.4–3.9 from Section 3. Specifically, given Ext-U and Sep-U, we can prove that the levels (so defined) are potent, transitive, pure,[61] and well-ordered by $\in$.

I can now lay down $\mathrm{LT}\underset{\infty}{\mathrm{U}}$. It uses a primitive one-place function symbol, $\mathbf{L}$, where '$\mathbf{L}a$' should be read as *a's level-index*. (I discuss the use of this primitive in Section B.4, below.) Then $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ has six axioms: Empty-U, Ext-U, LoS-$\underset{\infty}{\mathrm{U}}$, Many-$\underset{\infty}{\mathrm{U}}$, and two axioms governing $\mathbf{L}$:

Leveller $(\forall a : Set)((\exists s : Lev)\mathbf{L}a = s \wedge (\forall x : Set)(x \in a \rightarrow \mathbf{L}x \in \mathbf{L}a) \wedge (\forall s : Lev)(s \in \mathbf{L}a \rightarrow (\exists x : Set)(x \in a \wedge s \subseteq \mathbf{L}x)))$,

Consolidation $\forall F((\neg \underset{\infty}{\exists} x F(x) \wedge \exists a(\forall x : F)(Ur(x) \vee \mathbf{L}x \in a)) \rightarrow (\exists b : Set)\forall x(F(x) \leftrightarrow x \in b))$.

To understand these axioms, note that $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ guarantees that the (pure) levels are well-ordered by membership.[62] Now, Leveller ensures that the levels index the sets; intuitively, a's level-index is the least level greater than the level-index of every set in a. Consolidation then allows us to find all the impure sets we would want to find 'at' any given level. Finally, note that $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ proves a pure-analogue of Stratification:[63]

Lemma B.3 ($\mathrm{LT}\underset{\infty}{\mathrm{U}}$). *If a is pure, then $a \subseteq \mathbf{L}a$.*

Consequently, $\mathrm{LT}\underset{\infty}{\mathrm{U}}$'s pure sets can be thought of as satisfying LT. Indeed, if we define '$x \,\varepsilon\, y$' as '$Pure(x) \wedge Pure(y) \wedge x \in y$', then $\mathrm{LT}\underset{\infty}{\mathrm{U}} \vdash \mathrm{LT}(Pure, \varepsilon)$, as defined in Section 6. It follows that $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ is externally and internally (quasi-)categorical: any two hierarchies satisfying $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ have quasi-categorical pure sets; moreover, if there is a bijection between the hierarchies' urelemental bases, their impure sets are quasi-categorical. (However, $\mathrm{LT}\underset{\infty}{\mathrm{U}}$'s analogue of Theorem 6.1 is more restricted: if $\mathcal{M}$ is a standard, set-sized, model of $\mathrm{LT}\underset{\infty}{\mathrm{U}}$, then $|Ur^{\mathcal{M}}|$ is regular.)[64]

In fact, $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ and $\mathrm{ST}\underset{\infty}{\mathrm{U}}$ are provably equivalent, concerning sets and urelements. To prove that $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ interprets $\mathrm{ST}\underset{\infty}{\mathrm{U}}$, tweak the $*$-translation

[61] Since they are transitive, they witness their own purity.

[62] To see this, note $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ proves Sep-U, and combine this with the remarks after Definition B.2.

[63] Use induction on levels, together with the second conjunct of Leveller.

[64] Assuming Choice. *Proof.* Let $\kappa = |Ur^{\mathcal{M}}|$. By Consolidation, every smaller-than-κ subset of $Ur^{\mathcal{M}}$ is in $Set^{\mathcal{M}}$. So κ is infinite, by Many-$\underset{\infty}{\mathrm{U}}$. For each $\lambda < \kappa$, there are κ^{λ} subsets of $Ur^{\mathcal{M}}$ with cardinality λ, so that $\kappa^{\lambda} \leq |Set^{\mathcal{M}}|$. So if $\mathrm{cf}(\kappa) < \kappa$, then by König's Theorem $\kappa < \kappa^{\mathrm{cf}(\kappa)} \leq |Set^{\mathcal{M}}|$, contradicting Many-$\underset{\infty}{\mathrm{U}}$; hence $\mathrm{cf}(\kappa) = \kappa$. (Thanks to Gabriel Uzquiano for suggesting I consider how $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ interacts with regular cardinals.)

https://doi.org/10.1017/bsl.2021.13 Published online by Cambridge University Press

of Section 4, so that $(x \preceq \mathbf{s})^* := \mathbf{L}x \subseteq \mathbf{s}$.[65] It is then easy to show that LT$\underset{\infty}{\mathrm{U}}$ $\vdash$ ST$\underset{\infty}{\mathrm{U}}^*$ (cf. Lemma 4.2).

To show that ST$\underset{\infty}{\mathrm{U}}$ interprets LT$\underset{\infty}{\mathrm{U}}$, first note that ST$\underset{\infty}{\mathrm{U}}$ proves Sep-U and the converse of Priority-U (cf. Lemmas 4.3 and 4.4). Then tweak Definition 4.5 (cf. Definition B.2):

Definition B.4 (For Appendix B only). Let $\check{\mathbf{s}} := \{x \prec \mathbf{s} : \mathit{Pure}(x)\}$. Say that a is a *slice* iff $a = \check{\mathbf{s}}$ for some $\mathbf{s}$.

It follows that the slices are the levels (in the senses of Definitions B.2 and B.4; cf. Lemma 4.7).[66] We can then interpret LT$\underset{\infty}{\mathrm{U}}$'s unique primitive, $\mathbf{L}$, via ρ, defined as follows:[67]

Definition B.5. For each set a, let $\rho a := \bigcap\{\check{\mathbf{s}} : a \preceq \mathbf{s} \wedge \neg\exists \mathbf{r}(a \preceq \mathbf{r} < \mathbf{s})\}$.

Theorem B.6 ST$\underset{\infty}{\mathrm{U}}$ $\vdash \phi^\rho$ *iff* LT$\underset{\infty}{\mathrm{U}}$ $\vdash \phi$, *for any* LT$\underset{\infty}{\mathrm{U}}$*-sentence* ϕ, *where* ϕ^ρ *is the formula which results from* ϕ *by replacing each instance of* $\mathbf{L}$ *with* ρ.

The upshot is that no information about sets or urelements is lost or gained in moving from ST$\underset{\infty}{\mathrm{U}}$ to LT$\underset{\infty}{\mathrm{U}}$. Since any hierarchy which is described by the $\underset{\infty}{\mathrm{U}}$relemental Story satisfies ST$\underset{\infty}{\mathrm{U}}$, it also satisfies LT$\underset{\infty}{\mathrm{U}}$. And LT$\underset{\infty}{\mathrm{U}}$ is quasi-categorical. Our work on the $\underset{\infty}{\mathrm{U}}$relemental Story is complete.

B.4. Eliminating primitives. Or rather: almost complete. Given the discussion of Section 1, we may want to eliminate LT$\underset{\infty}{\mathrm{U}}$'s unique primitive, $\mathbf{L}$. This is easily done within second-order logic: just conjoin Leveller and Consolidation, and bind $\mathbf{L}$ with a (second-order) existential quantifier. But if we are willing to make some further assumptions, then we can eliminate $\mathbf{L}$ using certain *first-order* functions.[68]

Roughly, a *ranking-function*: (1) has a transitive domain (setting aside urelements); and (2) behaves like $\mathbf{L}$ where defined. More formally:

Definition B.7. Say that a function f is a *ranking-function* iff, for all $a \in \mathrm{dom}(f)$, both:

(1) $\mathit{Set}(a)$ and $(\forall x : \mathit{Set})(x \in a \to x \in \mathrm{dom}(f))$; and
(2) $\mathit{Lev}(f(a))$ and $(\forall x : \mathit{Set})(x \in a \to f(x) \in f(a))$ and $(\forall s : \mathit{Lev})(s \in f(a) \to (\exists x \in a) s \subseteq f(x))$.

[65] Stipulate that $(\mathit{Set}(x))^* := \mathit{Set}(x)$.

[66] For the analogue of Lemma 4.6: Ext-U, Pure-$\underset{\infty}{\mathrm{U}}$, and Spec-$\underset{\infty}{\mathrm{U}}$ guarantee that $\check{\mathbf{s}}$ exists for each stage $\mathbf{s}$; in clauses (2) and (3), the quantifier '$\forall a$' becomes '$(\forall a : \mathit{Pure})$'; and note that each slice witnesses its own purity.

[67] Since ST$\underset{\infty}{\mathrm{U}}$ does not prove that all stages are comparable (cf. the discussion of Boolos's Net from Section 8.2), it takes several steps to vindicate Definition B.5. First: show that stages obey $<$-induction. Second: show that if $a \preceq \mathbf{s}$ and $\neg\exists\mathbf{r}(a \preceq r < \mathbf{s})$ and $a \preceq \mathbf{t}$ and $\neg\exists\mathbf{r}(a \preceq r < \mathbf{t})$ then $\check{\mathbf{s}} = \check{\mathbf{t}}$; it follows that ρa is a slice. Third: combine this with the fact that the slices are levels, to show that ρ behaves like $\mathbf{L}$.

[68] Lévy and Vaught (1961, p. 1047) and Uzquiano (1999, p. 299) present a somewhat similar method for defining the rank of a set (via functions on ordinals). Here I treat first-order functions as sets of ordered pairs in the normal way, and $x \in \mathrm{dom}(f)$ abbreviates $\exists y \langle x, y\rangle \in f$. Of course, a fully first-order version of LT$\underset{\infty}{\mathrm{U}}$ would need to define '$\underset{\infty}{\exists} xF(x)$' differently (cf. footnote 60).

Say that $Ranks(f, a)$ iff f is a ranking-function with $a \in \text{dom}(f)$.

It is easy to show that ranking-functions agree wherever they are defined, i.e.:

LEMMA B.8 (Ext-U, Sep-U). *If $Ranks(f, a)$ and $Ranks(g, a)$, then $f(a) = g(a)$.*

We can now replace Leveller, in $\text{LT}\underset{\infty}{\text{U}}$, with $(\forall a : Set)\exists f\, Ranks(f, a)$. Note that this claim is *independent* of $\text{LT}\underset{\infty}{\text{U}}$: it guarantees that every set is a member of some set, and so guarantees that the hierarchy has no final stage (cf. Endless from Section 7). Still, this allows us to define $\mathbf{L}a := \bigcap\{f(a) : Ranks(f, a)\}$. We can use this definition in Consolidation, and prove Leveller via Definition B.7.

Acknowledgments. Thanks to Neil Barton, Sharon Berry, Luca Incurvati, Juliette Kennedy, Øystein Linnebo, Michael Potter, Chris Scambler, Will Stafford, James Studd, Rob Trueman, Gabriel Uzquiano, Sean Walsh, and anonymous referees for this journal.

REFERENCES

G. BOOLOS, *The iterative conception of set*. ***The Journal of Philosophy***, vol. 68 (1971), no. 8, pp. 215–231.

———, *Iteration again*. ***Philosophical Topics***, vol. 17 (1989), no. 2, pp. 5–21.

J.P. BURGESS, *E pluribus unum: Plural logic and set theory*. ***Philosophia Mathematica***, vol. 12 (2004), pp. 193–221.

T. BUTTON and S. WALSH, ***Philosophy and Model Theory***, Oxford University Press, Oxford, 2018.

K. DOETS, *Relatives of the Russell paradox*. ***Mathematical Logic Quarterly***, vol. 45 (1999), no. 1, pp. 73–83.

F.R. DRAKE, ***Set Theory: An Introduction to Large Cardinals***, North-Holland, London, 1974.

L. INCURVATI, *How to be a minimalist about sets*. ***Philosophical Studies***, vol. 159 (2012), no. 1, pp. 69–87.

———, ***Conceptions of Set and the Foundations of Mathematics***, Cambridge University Press, Cambridge, 2020.

A. LÉVY and R. VAUGHT, *Principles of partial reflection in the set theories of Zermelo and Ackermann*. ***Pacific Journal of Mathematics***, vol. 11 (1961), no. 3, pp. 1045–1062.

A.R.D. MATHIAS, *Slim models of Zermelo set theory*. ***The Journal of Symbolic Logic***, vol. 66 (2001), no. 2, pp. 487–496.

V. MCGEE, *Two problems with Tarski's theory of consequence*. ***Proceedings of the Aristotelian Society***, vol. 92 (1992), pp. 273–292.

———, *How we learn mathematical language*. ***Philosophical Review***, vol. 106 (1997), no. 1, pp. 35–68.

C. MENZEL, *On the iterative explanation of the paradoxes*. ***Philosophical Studies***, vol. 49 (1986), no. 1, pp. 37–61.

———, *Wide sets, ZFCU, and the iterative conception*. ***The Journal of Philosophy***, vol. 111 (2014), no. 2, pp. 57–83.

R. MONTAGUE, *On the paradox of grounded classes*. ***The Journal of Symbolic Logic***, vol. 20 (1955), no. 2, p. 140.

———, *Set theory and higher-order logic*, ***Formal Systems and Recursive Functions: Proceedings of the Eighth Logic Colloquium, July 1963*** (J. Crossley and M. Dummett, editors), North-Holland, Amsterdam, 1965, pp. 131–148.

R. Montague, D. Scott, and A. Tarski, ***An axiomatic approach to set theory***, Archive copy from the Bancroft Library (BANC MSS 84/69 c, carton 4, folder 29–30), unpublished manuscript, n.d.

C. Parsons, *The uniqueness of the natural numbers*. ***Iyyun***, vol. 39 (1990), no. 1, pp. 13–44.

———, ***Mathematical Thought and Its Objects***, Harvard University Press, Cambridge, MA, 2008.

M. Potter, ***Sets: An Introduction***, Oxford University Press, Oxford, 1990.

———, *Iterative set theory*. ***Philosophical Quarterly***, vol. 43 (1993), no. 171, pp. 178–193.

———, ***Set Theory and Its Philosophy***, Oxford University Press, Oxford, 2004.

I. Rumfitt, ***The Boundary Stones of Thought: An Essay in the Philosophy of Logic***, Oxford University Press, Oxford, 2015.

D. Scott, *The notion of rank in set-theory*, ***Summaries of Talks Presented at the Summer Institute for Symbolic Logic, Cornell University, 1957***, Institute for Defence Analysis, Princeton, NJ, 1960, pp. 267–269.

———, *Axiomatizing set theory*, ***Axiomatic Set Theory II*** (T. Jech, editor), Proceedings of Symposia in Pure Mathematics, vol. 13, American Mathematical Society, Providence, RI, 1974, pp. 207–214.

J.R. Shoenfield, ***Mathematical Logic***, Addison-Wesley, London, UK, 1967.

———, *Axioms of set theory*, ***Handbook of Mathematical Logic*** (J. Barwise, editor), North-Holland, London, 1977, pp. 321–344.

G. Uzquiano, *Models of second-order Zermelo set theory*, this Journal, vol. 5 (1999), no. 3, pp. 289–302.

———, *A neglected resolution of Russell's paradox of propositions*. ***The Review of Symbolic Logic***, vol. 8 (2015), pp. 328–344.

J. Väänänen and T. Wang, *Internal categoricity in arithmetic and set theory*. ***Notre Dame Journal of Formal Logic***, vol. 56 (2015), no. 1, pp. 121–134.

E. Zermelo, *Über Grenzzahlen und Mengenbereiche*: *Neue Untersuchungen über die Grundlagen der Mengenlehre*. ***Fundamenta Mathematicæ***, vol. 16 (1930), pp. 29–47.

DEPARTMENT OF PHILOSOPHY
UNIVERSITY COLLEGE LONDON
GOWER STREET, LONDON, WC1E 6BT, UK
E-mail: tim.button@ucl.ac.uk
URL: http://www.nottub.com