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RESEARCH ARTICLE

The cocked hat: formal statements and proofs of the theorems

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Abstract
Navigators have been taught for centuries to estimate the location of their craft on a map from three lines of position, for redundancy. The three lines typically form a triangle, called a cocked hat. How is the location of the craft related to the triangle? For more than 80 years navigators have also been taught that, if each line of position is equally likely to pass to the right and to the left of the true location, then the likelihood that the craft is in the triangle is exactly 1/4. This is stated in numerous reputable sources, but was never stated or proved in a mathematically formal and rigorous fashion. In this paper we prove that the likelihood is indeed 1/4 if we assume that the lines of position always intersect pairwise. We also show that the result does not hold under weaker (and more reasonable) assumptions, and we prove a generalisation to n lines.

1. Introduction

Navigators used to plot on a map *lines of position* or *lines of bearing*, which are rays emanating from a landmark (e.g., a lighthouse or radio beacon) at a particular bearing (angle relative to north) that was estimated to be the direction from the landmark (which we also refer to as observation point) to the ship or plane. Two such rays usually intersect at a point, which the navigator would take as an estimate of the true position of the craft. Navigators were encouraged to plot three rays, to make position estimation more robust. The three rays normally created a triangle, called a *cocked hat* (Dear and Kemp, 2006), as shown in Figure 1. The properties of the cocked hat have been investigated thoroughly (His Majesty's Navigation School, 1938; Stansfield, 1947; Daniels, 1951; Anderson, 1952; Cotter, 1961; Williams, 1991; Cook, 1993; Stuart, 2019), to help navigators interpret it and make good navigation decisions. The aim of this paper is to analyse the conditions under which an elegant property of the cocked hat holds. That property was stated without a proof more than 80 years ago (His Majesty's Navigation School, 1938), proved informally (and essentially incorrectly) 70 years ago (Stansfield, 1947), and has been widely disseminated ever since (Daniels, 1951; The Open University, 1984; Williams, 1991; Cook, 1993; Denny, 2012), including in course material (The Open University, 1984) and in a popular science book (Denny, 2012). The mathematically incorrect explanation in the Open University video (The Open University; 1984), which is presented there as a proof, demonstrates that it was widely believed that this explanation is indeed a proof when in fact it is not.

The property that we are interested in is the probability of the cocked hat containing the true position being 1/4. Under what conditions is this statement true?

This claim first appeared in a 1938 navigation manual (His Majesty's Navigation School, 1938, p. 166), without a proof and with only informal conditions on the error angles at the three landmarks,

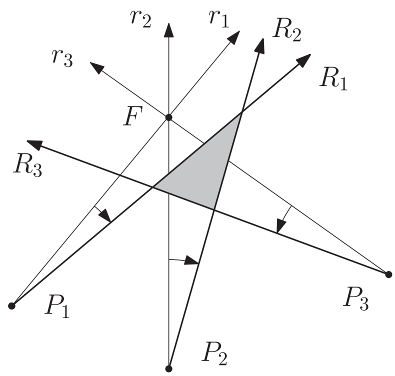

Figure 1. *Three observation points P_1, P_2, P_3, the target F, the three rays and the cocked hat (shaded).*

which we denote P_1, P_2 and P_3 (see Figure 1). The error angles ϵ_1, ϵ_2 and ϵ_3 are between the plotted rays, which we denote R_1, R_2 and R_3 and the rays r_i from P_i to the true position of the craft, which we denote by F (see Figure 1). The informal conditions are that the errors are independent (the manual does not use this term, but this is what it means) and fairly small, around 1 degree. A 1947 paper by Stansfield (Stansfield, 1947)[1] cites the claim, gives more formal conditions for it and sketches a proof. The conditions that Stansfield specified are remarkably weak: he claims that the result would hold if only two of the three errors have zero median. Stansfield writes that this assumption is equivalent to the following: 'for two of the stations the observed bearings are equally likely to pass to the right or the left of the true position'.

A 1951 paper by Daniels (1951) states Stansfield's result in a more modern statistical language, saying that the cocked hat is a 25% distribution-free confidence region; the term *distribution-free* means that the result is not dependent on a particular error distribution, say Gaussian, but only on a parameter of the distribution, here the zero median.[2] Daniels then considers the case of n landmarks and n rays starting from there. The lines of these rays split the plane into finitely connected components, some of them bounded, some of them not. Daniels claims without proof a particular formula, $2n/2^n$, for the probability that F belongs to the union of the unbounded components. The 25%-probability result was incorrectly extended again by Williams[3] in 1991. He claimed specific probabilities that the open regions around the cocked hat contain F, again with only an informal specification of the assumptions and with only a sketch of the proof. Williams's claims were shown to be false by Cook (1993), using specific error distributions, to which Williams answered with a witty (but scientifically wrong) rebuttal. Cook also repeated the claim that the probability of the cocked hat contains F is $1/4$.

Our aim in this paper is to show that the 25%-probability result is valid only for error distributions that guarantee that the three rays intersect at three distinct points and form a triangle.

We note that the use of the cocked hat in navigation is today obsolete, having been replaced by estimation of confidence regions, usually circles or ellipses, by computer algorithms.

2. Generalisations to rays that do not intersect

Two rays in the plane can intersect, but they can also fail to intersect. Lines of position plotted by navigators almost always intersected, because the error angles were small. Also, navigators were taught to choose landmarks so that no angle at the intersection is smaller than about 50 degrees – a small angle at the intersection implies ill-conditioning (high sensitivity of the intersection point to bearing errors).

[1] Stansfield developed the results published in the paper while serving in Operational Research Sections attached to the Royal Air Force Fighter Command and Coastal Command during World War II.

[2] Daniels was a statistician and served as the President of the Royal Statistical Society from 1974 to 1975. His paper incorrectly states that the Admiralty Navigation Manual proves the 25%-probability result. It does not; the first proof sketch appears in Stansfield's paper.

[3] Williams was a professional air navigator and served as President of the Royal Institute of Navigation from 1984–1987 (Charnley, 1993).

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Stansfield's formulation of the problem uses much more general assumptions on the errors, and no assumption about angles at the intersections. Stansfield, Daniels and the authors that followed only require that the three errors $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (-\pi, \pi]$ are random and independent, and that the median of their distributions is zero. We replace the zero-median assumption by a consistent but slightly more general condition, namely

$$\mathrm{Prob}(\epsilon_i < 0) = \mathrm{Prob}(\epsilon_i > 0) = \frac{1}{2} \tag{2.1}$$

for every $i \in [n]$, where $[n]$ is a shorthand for the set $\{1, 2, \ldots, n\}$. This means in particular that the target is *never on* R_i, which is a necessity because if $\mathrm{Prob}(\epsilon_i = 0) > 0$ were allowed, then $\mathrm{Prob}(F \in \Delta)$ could be close to one (e.g., if $\mathrm{Prob}(\epsilon_i = 0)$ is close to one), implying that the $1/4$ result does not hold in this case. We also consider the restriction of the errors to $[-\pi/2, \pi/2]$.

Under these weak assumptions on the error distribution, the three rays might fail to form a triangle (the cocked hat). How can we formally express the 25%-probability result when rays may fail to intersect? We propose four ways to express the result; the first three are fairly natural but are not sufficient for the $1/4$ result, even under the restriction $\epsilon_i \in [-\pi/2, \pi/2]$; the fourth is not particularly natural but is the only correct statement of the result.

Conjunction formulation. The probability that the three rays intersect at three points and that the triangle that they form contains F is $1/4$. In this formulation, we allow error distributions that could generate non-intersecting rays and we hope to prove that the probability that the rays intersect at fewer than three points or that the triangle does not contain F is exactly $3/4$. This is false.

Conditional probability formulation. The conditional probability that the triangle that the rays form contains F, conditioned on the rays forming a triangle, is $1/4$. In this formulation we again allow error distributions that generate non-intersecting rays, and we hope to prove that if the rays intersect at three points, then the probability that the triangle contains F is $1/4$. We do not care with what probability the rays fail to form a triangle. This again is false.

Lines formulation. We extend the rays r_i to infinite lines ℓ_i, which always form a triangle, and we hope to show that the triangle that they form contains F with probability $1/4$. Here we must restrict $\epsilon_i \in [-\pi/2, \pi/2]$, otherwise the same line could appear both on the left and on the right of F. Again, this claim is false.

Constrained distribution formulation. We assume that the distribution of errors is such that every pair of rays always intersects and we hope to show that the probability that the triangle contains F is $1/4$. We do not permit distributions under which two of the rays might fail to intersect. We show below that in this case $\mathrm{Prob}(F \in \Delta) = 1/4$.

We note that, from the navigator's perspective, the conditional probability is the most natural. You plot three rays. If they do not intersect at three points, you discard the measurements and try again, because you either picked bad observation points (e.g., two of them and your ship are almost collinear) or at least one of the bearings is way off. If they do intersect at three points, you want to know the (conditional) probability that the cocked hat contains F. From the statistician's perspective, any of the first three formulations makes sense. The fourth makes less statistical sense, because it is unusual to assume that independent error distributions satisfy some global structural constraint. In particular, it appears that Daniels may have believed that the lines formulation is correct, because he writes about geometrical lines in the plane, not about rays. He writes 'a particular set of n lines, no two of which are parallel, divides the plane in to $\frac{1}{2}(n^2 + n + 2)$ polygons'.

3. Counterexamples

We now show that the Conjunction, Conditional probability and Lines formulations are all false by giving counterexamples. Every example is a *two-ray distribution* that is concentrated on two rays R_i^+ and R_i^-: $\mathrm{Prob}(R_i = R_i^+) = \mathrm{Prob}(R_i = R_i^-) = \frac{1}{2}$. This is no coincidence, as we will see at the end of this section.

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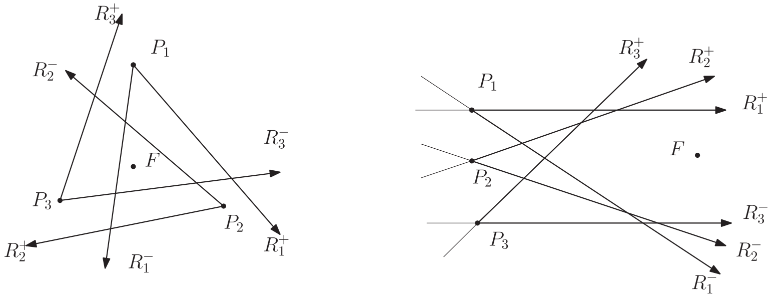

Figure 2. *Two counterexamples.*

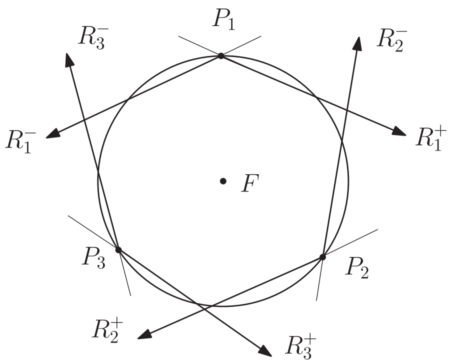

Figure 3. *The third counterexample.*

In the first example F is in the centroid of an equilateral triangle whose vertices are P_1, P_2 and P_3. Figure 2 (left) shows the two-ray error distributions. It is easy to see that R_i^+ intersects neither R_{i+1}^+ nor R_{i+1}^- (subscripts are meant modulo 3). Therefore, if R_i^+ is selected, then a cocked hat does not form. On the other hand, if R_1^-, R_2^- and R_3^- are selected, then they form a cocked hat that contains F. Therefore,

$$\text{Prob}(F \in \triangle \mid \text{ the rays form a cocked hat } \triangle) = 1,$$

$$\text{Prob}(\text{the rays form a cocked hat } \triangle \text{ and } F \in \triangle) = \frac{1}{8}.$$

This shows both that the Conjunction formulation is false and that the Conditional probability formulation is false. Note that all the error angles have magnitude less than $\pi/2$, so these formulations are false even with this restriction.

Figure 2 (right) shows another two-ray distribution. The error magnitudes are less than $\pi/2$, actually as small as you wish. The true position F lies outside all the triangles that the lines form, so the probability that the cocked hat (in the Lines formulation) contains F is zero. We can move F to the right by any amount and $F \notin \Delta$ will still hold. This example also shows that the conditional probability that a cocked hat formed by three rays contains F can also be zero.

The last example, given in Figure 3, shows that the probability that the triangle formed by the extension of the rays to lines contains F can be 1. We again note that the error angles are bounded in magnitude by $\pi/2$. In this example the three rays do not have three intersection points, so the cocked hat appears with probability zero. So this is another counterexample to the Conjunction formulation.

We close this section with a remark on two-ray distributions. The set of (Borel) probability distributions satisfying condition (2.1) is convex, and its extreme points are exactly the two-ray distributions, as one can easily check. Moreover, $\text{Prob}(F \in \Delta)$ is a linear function on the product of the distributions μ_1, μ_2, μ_3, where μ_i is the probability distribution of the ray R_i. Indeed, denoting by $I(E)$ the indicator

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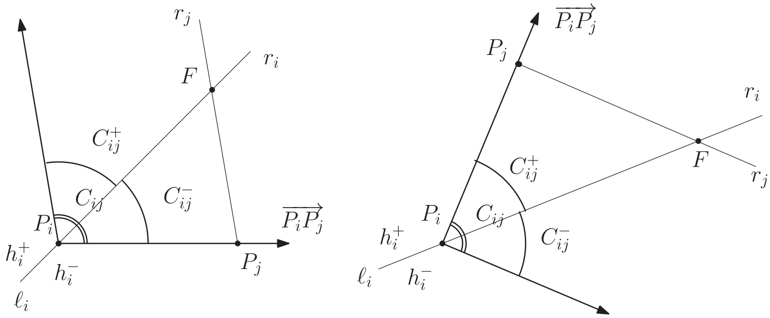

Figure 4. *Illustration for Lemma 4.1: the case $P_j \in h_i^-$ on the left, and the case $P_j \in h_i^+$ on the right.*

function of an event E, we have

$$\text{Prob}(F \in \Delta) = \int I(F \in \Delta)\, \mathrm{d}\mu_1\, \mathrm{d}\mu_2\, \mathrm{d}\mu_3, \tag{3.1}$$

a linear function of each μ_i, so if it takes the value $1/4$ on the two-ray distributions, then it takes the same value on all distributions satisfying (2.1). We will come back to such distributions again in Section 5.

4. Intersecting rays

We now start the analysis when rays must intersect in pairs. We assume throughout that the $n+1$ points $P_1, \ldots, P_n, F$ are in general position, so that no three are collinear and so that no other degeneracies arise.

We introduce some notation. We let $\overrightarrow{XY}$ denote the ray emanating from X in the direction of Y when X and Y are distinct points in the plane; here we assume that $X \notin \overrightarrow{XY}$. Thus $r_i = \overrightarrow{P_iF}$ is the ray starting at P_i in the direction of the target F, and ℓ_i is the line containing r_i. From each P_i out goes a random ray R_i making a (signed) angle $\varepsilon_i \in (-\pi, \pi)$ with r_i. Our basic assumption, besides (2.1), is that two random rays always intersect, that is, for distinct $i, j \in [n]$ we have

$$\text{Prob}(R_i \cap R_j = \emptyset) = 0. \tag{4.1}$$

So rays R_i and R_j intersect almost surely but their intersection point is not P_i or P_j because of our convention that $X \notin \overrightarrow{XY}$.

Further notation: h_i^- (respectively h_i^+) are the half-planes bounded by ℓ_i, with h_i^- consisting of points X such that the ray $\overrightarrow{P_iX}$ comes from a clockwise rotation from r_i with angle less than π, and h_i^+ is its complementary half-plane. When r and r' are two rays we denote by $\text{cone}(P_i, r, r')$ the cone whose apex is P_i and whose bounding rays are translated copies of r and r'. Such a cone always has angle less than π, because r and r' will never have opposite directions.

Define $C_{ij} = \text{cone}(P_i, r_j, \overrightarrow{P_iP_j})$ for distinct $i, j \in [n]$.

Lemma 4.1. *The cone C_{ij} contains r_i and* $\text{Prob}(R_i \subset C_{ij}) = 1$.

Proof. Assume first that $P_j \in h_i^-$. We define first the cones $C_{ij}^- = \text{cone}(P_i, r_i, \overrightarrow{P_iP_j})$ and $C_{ij}^+ = \text{cone}(P_i, r_i, r_j)$; see Figure 4. Note that the angle of C_{ij}^- (respectively C_{ij}^+) is equal to the angle at P_i (and at F) of the triangle with vertices P_i, P_j, F. Then $C_{ij} = C_{ij}^- \cup C_{ij}^+$ because the angle of this cone is the sum of the angles of C_{ij}^- and C_{ij}^+, so smaller than π. Then $r_i \subset C_{ij}$ indeed, as shown in Figure 4 (left).

Suppose now that $\varepsilon_i > 0$, which is the same as $R_i \subset h_i^+$. If R_i does not lie in C_{ij}^+, then $R_i \subset h_i^+ \setminus C_{ij}^+$. The last set is a convex cone, disjoint from h_j^-, as they are separated by the line ℓ_j. So no R_j with $\varepsilon_j < 0$ can intersect R_i, contradicting (4.1). So $R_i \subset C_{ij}^+$.

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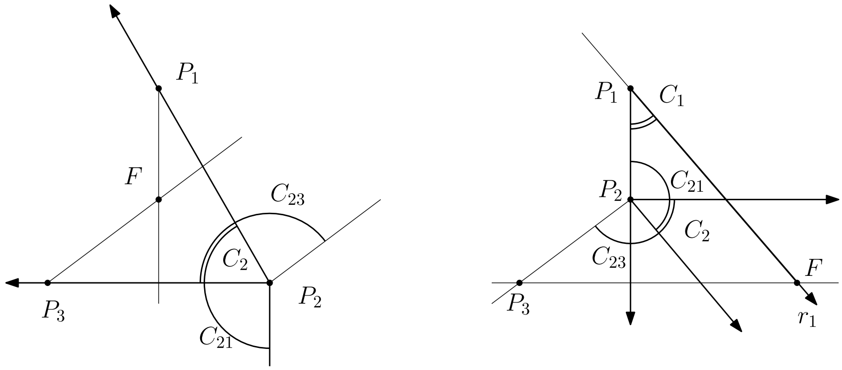

Figure 5. *The cone C_2 in Case 1 (left) and Case 2 (right).*

Let h denote the half-plane containing F and bounded by the line through P_i and P_j. Observe that by the previous argument $R_j \subset h$ because the complementary half-plane to h is disjoint from C_{ij}^+, so R_j can intersect $R_i \subset C_{ij}^+$ only if it lies in h.

Suppose next that $\varepsilon_i < 0$. We show that $R_i \subset C_{ij}^-$. If not, then $R_i \subset h_i^- \setminus C_{ij}^-$. The last set is a convex cone again, disjoint from h, so $R_i \cap R_j = \emptyset$ for all R_j with $\varepsilon_j < 0$, contradicting (4.1).

The argument for the case $P_j \in h_i^+$ is symmetric (see Figure 4 (right)), but otherwise identical and is therefore omitted. □

We remark here that Lemma 4.1 implies that the cone $\bigcap_{j \neq i} C_{ij}$ is convex (that is, its angle is smaller than π), it contains r_i, and $\text{Prob}(R_i \subset \bigcap_{j \neq i} C_{ij}) = 1$, of course only if $n \geq 2$. (For $n = 1$, condition (4.1) is void.) Define K_i as the smallest (with respect to inclusion) convex cone satisfying $\text{Prob}(R_i \subset K_i) = 1$. Note that $K_i \subset \bigcap_{j \neq i} C_{ij}$. For later reference we state the following corollary.

Corollary 4.1. *Under conditions* (2.1) *and* (4.1) *K_i is a convex cone, $r_i \subset K_i$ and* $\text{Prob}(R_i \subset K_i) = 1$ *for every $i \in [n]$.*

Theorem 4.1. *Under conditions* (2.1) *and* (4.1) *we have*

$$\text{Prob}(F \in \Delta) = \frac{1}{4}.$$

Proof. Set $T = \text{conv}\{P_1, P_2, P_3, F\}$, the convex hull of P_1, P_2, P_3 and F. We will have to consider three cases separately: when T is a triangle with F inside T (Case 1); when T is a triangle with F a vertex of T (Case 2); and when T is a quadrilateral (Case 3).

Case 1. Define $C_i = \text{cone}(P_i, \overrightarrow{P_i P_{i-1}}, \overrightarrow{P_i P_{i+1}})$ for $i = 1, 2, 3$, where the subscripts are taken mod 3; see Figure 5 (left).

We claim that $R_i \subset C_i$ for all i. By symmetry it suffices to show this for $i = 2$. By Lemma 4.1 $R_2 \subset C_{21} \cap C_{23}$. So it is enough to check that $C_2 = C_{21} \cap C_{23}$, and this is evident: the rays bounding C_2 are $\overrightarrow{P_2 P_1}$ (which bounds C_{21}) and $\overrightarrow{P_2 P_3}$ (which bounds C_{23}).

We can now finish the proof of the theorem in Case 1. There are eight sub-cases with equal probabilities that correspond to the signs of ϵ_1, ϵ_2 and ϵ_3, as shown in Figure 6. Only in two of them, namely when all ϵ_i have the same sign, do we have $F \in \triangle$, so the probability of this event is $1/4$.

Case 2. We assume (by symmetry) that P_2 is inside the triangle T. We define the cones $C_1 = \text{cone}(P_1, r_2, \overrightarrow{P_1 P_2})$, $C_2 = \text{cone}(P_2, r_1, r_3)$ and $C_3 = \text{cone}(P_3, r_2, \overrightarrow{P_3 P_2})$ and we claim that $R_i \subset C_i$ for all i. From Lemma 4.1 we have that $R_2 \subset C_{21} \cap C_{23}$. The bounding rays of C_2 are a translate of r_2 (bounding C_{21}) and a translate of r_3 (bounding C_{23}), so $C_2 = C_{21} \cap C_{23}$ (see Figure 5 (right)).

The cases $i = 1$ and 3 are symmetric and very simple. We only consider $i = 1$. Again, by Lemma 4.1 $R_1 \subset C_{12}$ and then $C_1 = C_{12}$ implying $R_1 \subset C_1$.

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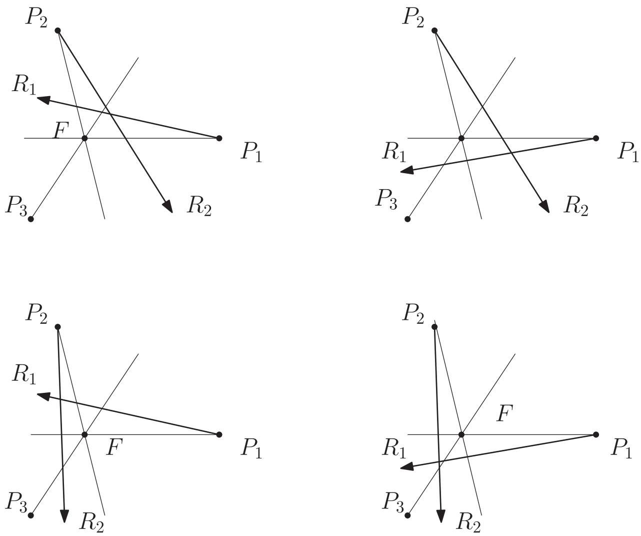

Figure 6. *Illustration for the proof of Theorem 4.1.*

Again, there are eight subcases, corresponding to the eight possible sign patterns of $\varepsilon_1, \varepsilon_2, \varepsilon_3$. It is easy to see that $F \in \Delta$ in exactly two of them.

Case 3. We assume again by symmetry that the segment P_2F is a diagonal of the quadrilateral T. Define cones $C_1 = \text{cone}(P_1, r_2, \overrightarrow{P_1P_3})$, $C_2 = \text{cone}(P_2, r_1, r_3)$ and $C_3 = \text{cone}(P_3, r_2, \overrightarrow{P_3P_2})$. We claim again that $R_i \subset C_i$ for all i. The proof is similar to the previous ones using Lemma 4.1 and is omitted here. Again, $F \in \Delta$ in exactly two out of the eight cases. □

5. Daniels' statement

We assume now that there are $n \geq 3$ observation points $P_1, \ldots, P_n$ plus the target point F and that these $n + 1$ points are in general position. A random ray R_i starts at each P_i satisfying conditions (2.1) and (4.1). The lines of the rays R_i split the plane into connected components. Let U denote the union of the $2n$ unbounded components. Here comes Daniels' statement.

Theorem 5.1. *Under conditions* (2.1) *and* (4.1) *we have*

$$\text{Prob}(F \in U) = \frac{2n}{2^n}.$$

The case $n = 2$ is trivial and not interesting. The case $n = 3$ is just Theorem 4.1. We note that condition (4.1) is a necessity, even for $n = 3$, as the counterexamples in Section 3 show.

We are going to prove this theorem under the assumption that each R_i is a two-ray distribution, that is, $\text{Prob}(R_i = R_i^+) = \text{Prob}(R_i = R_i^-) = 1/2$ and explain, after the proof, how this special case implies the theorem. We also assume that the $2n$ rays R_i^+, R_i^-, together with the points $P_1, \ldots, P_n, F$ are in general position. This is not a serious restriction because the general case of two-ray distributions follows from this by a routine limiting argument.

Proof. To simplify the exposition, we set $D_i = \text{cone}(P_i, R_i^+, R_i^-)$, which is equivalent to $D_i = \text{conv}(R_i^+ \cup R_i^-)$. Lemma 4.1 implies that $r_i \subset D_i$ for every $i \in [n]$. Let S be a circle centred at F such that $S \subset D_i$ for every $i \in [n]$. Observe that for distinct $i, j \in [n]$, the intersection $D_i \cap D_j$ is a convex quadrilateral containing S and of course F; see Figure 7 (left). This follows from condition (4.1): both R_i^+ and R_i^- intersect both R_j^+ and R_j^- and the four intersection points are the vertices of $D_i \cap D_j$, which is then a convex quadrilateral.

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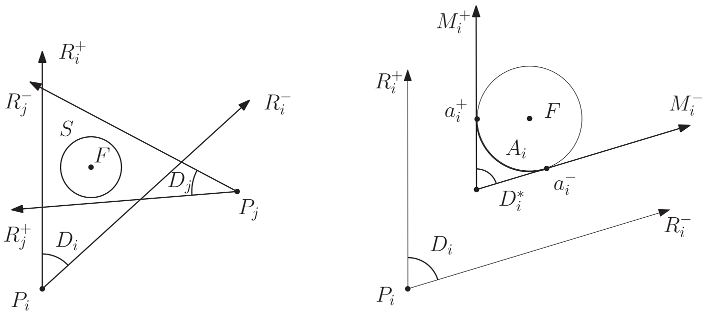

Figure 7. *The intersection $D_i \cap D_j$ and the translated cone D_i^*.*

Let L_i^+ (respectively L_i^-) denote the line of the ray R_i^+ (and R_i^-). For a selection $\delta_1, \ldots, \delta_n \in \{1, -1\}$ of signs, the lines $L_1^{\delta_1}, \ldots, L_n^{\delta_n}$ split the plane into finitely many connected components. We are going to show that, out of the 2^n possible selections, there are exactly $2n$ for which F lies in an unbounded component.

We reduce this statement to another one about arcs on the unit circle. First comes a simpler reduction. Translate each cone D_i into a new (and actually unique) position D_i^* so that its rays touch the circle S (see Figure 7 (right)). Let Q_i^+, M_i^+ (respectively Q_i^-, M_i^-) be the translated copies of R_i^+, L_i^+ (and R_i^-, L_i^-). Note that $D_i^* \cap D_j^*$ is again a convex quadrilateral.

We *claim* next that, for a fixed selection $\delta_1, \ldots, \delta_n$ of signs, F lies in an unbounded component for the lines $L_1^{\delta_1}, \ldots, L_n^{\delta_n}$ if and only if it lies in the corresponding unbounded component for the lines $M_1^{\delta_1}, \ldots, M_n^{\delta_n}$. This is simple. The point F lies in an unbounded component for the lines $L_i^{\delta_i}$ if and only if there is a half-line R starting at F and disjoint from each $L_i^{\delta_i}$, which happens if and only if R is disjoint from the lines $M_i^{\delta_i}$ as well.

Assume now that S is the unit circle. Let a_i^+ (respectively a_i^-) be the points where M_i^+ (and M_i^-) touch S, and let A_i be the shorter arc on S between a_i^+ and a_i^-; see Figure 7 (right). It is clear that a_i^+ and a_i^- are not opposite points on S, so A_i is well defined. These arcs completely determine D_i^*. They satisfy the following conditions:

(i) each A_i is shorter than π, and
(ii) $A_i \cup A_j$ is an arc in S longer than π for all $i, j \in [n]$, $i \neq j$.

The latter condition follows from the fact that $D_i^* \cap D_j^*$ is a convex quadrilateral. □

We call a selection $\delta_1, \ldots, \delta_n$ *special* if it gives an unbounded component containing F. We *claim* that a selection is special if and only if the points $a_1^{\delta_1}, \ldots, a_n^{\delta_n}$ lie on an arc of S shorter than π. This is also simple. If there is such an arc, call it I and let Q be the centre point of the complementary arc $S \setminus I$. The ray $\overrightarrow{FQ}$ avoids every line $M_i^{\delta_i}$. If there is no such arc, then the connected component containing F (and S) is bounded, as one can check easily. Therefore it suffices to prove the following lemma.

Lemma 5.1. *Under the above conditions, there are exactly $2n$ special selections.*

Proof. For a special selection $\delta = (\delta_1, \ldots, \delta_n)$, let $I(\delta)$ denote the shortest arc on S containing every $a_i^{\delta_i}$, $i \in [n]$. Thus $I(\delta)$ is the shorter arc between points $a_i^{\delta_i}$ and $a_j^{\delta_j}$ for some distinct $i, j \in [n]$, and they are the *endpoints* of $I(\delta)$.

Claim 5.1. *Each a_i^+ (and a_i^-) is the endpoint of $I(\delta)$ for exactly two special selections δ.*

It suffices to prove this claim, since it implies Lemma 5.1 and then Theorem 5.1 as well.

Proof. It is enough to work with a_1^+. Using the notation on Figure 8, we assume that a_1^- is from the half-circle S^+ so $A_1 \subset S^+$. □

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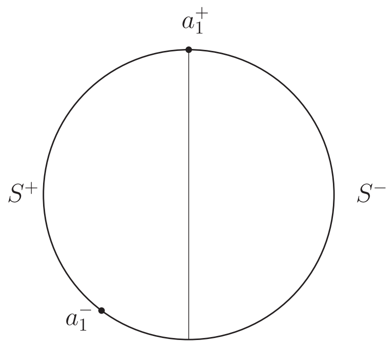

Figure 8. *The definition of S^+ and S^-.*

Define $X = \{a_1^+, a_1^-, \ldots, a_n^+, a_n^-\}$ and $Y = X \setminus \{a_1^+, a_1^-\}$. Observe first that S^+ cannot contain any A_i, $i > 1$, as otherwise $A_1, A_i \subset S^+$, contradicting condition (ii). Moreover, S^- cannot contain two arcs A_i, A_j with distinct $i, j > 1$ because of condition (ii) again. It follows then that $|S^+ \cap Y| = n-1$ or $n-2$.

Case 1: when $|S^+ \cap Y| = n-1$. Then $|S^- \cap Y| = n-1$ as well and S^+ contains exactly one element from each pair $\{a_i^+, a_i^-\}$, $i > 1$, and then so does S^-. This gives exactly two special selections δ and ε with $I(\delta) \subset S^+$ and $I(\varepsilon) \subset S^-$, with a_1^+ an endpoint of both.

Case 2: when $|S^+ \cap Y| = n-2$. Then S^+ contains no $I(\delta)$ with δ special, $|S^- \cap Y| = n$ and so $A_i \subset S^-$ for a unique $i > 1$. This again gives two special selections δ and ε where a_1^+ is the endpoint of $I(\delta)$ and $I(\varepsilon)$. In fact δ and ε coincide except at position i: $\delta_j = \varepsilon_j$ for all $j \in [n]$ but $j = i$ and $\delta_1 = \varepsilon_1 = 1$. □

We explain now how the case of two-ray distributions implies Theorem 5.1, or rather give a sketch of this and leave the technical details to the interested reader. Assume each ray R_i follows a generic distribution μ_i for all $i \in [n]$ still satisfying conditions (2.1) and (4.1). Note that, by Corollary 4.1, $\text{Prob}(R_i \subset K_i) = 1$. Using this, one can check that every μ_i can be approximated with high precision by a convex combination of two-ray distributions, each having $R_i^+, R_i^- \subset K_i$. One has to show as well that this approximation can be chosen so that $R_i^{\delta_i} \cap R_j^{\delta_j} \neq \emptyset$ for distinct $i, j \in [n]$ and for every choice of signs δ_i, δ_j. As in (3.1), $\text{Prob}(F \in U)$ is a linear function of the underlying distributions μ_i, and this linear function equals $2n/2^n$ on the product of two-ray distributions. Therefore, this linear function equals $2n/2^n$ on any convex combination of products of two-ray distributions and consequently $\text{Prob}(F \in U)$ must be equal to $2n/2^n$ on the product of the μ_i.

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References

Anderson E. W. (1952). The treatment of navigational errors. *Journal of Navigation*, **5**(2), 103–124. doi:10.1017/S0373463300044519

Charnley J. (1993). J. E. D. Williams. *Journal of Navigation*, **46**(2), 290–292. An obituary. doi:10.1017/S0373463300011668

Cook I. (1993). Random cocked hats. *Journal of Navigation*, **46**(1), 132–135. A response by J. E. D. Williams appears on pages 135–137. doi:10.1017/S0373463300011383

Cotter C. H. (1961). The cocked hat. *Journal of Navigation*, **14**(2), 223–230. doi:10.1017/S0373463300039564

Daniels H. E. (1951). The theory of position finding. *Journal of the Royal Statistical Society, Series B (Methodological)*, **13**(2), 186–207. https://www.jstor.org/stable/2984060

Dear I. C. B. and Kemp P. (2006). *The Oxford Companion to Ships and the Sea*. 2nd ed. Oxford, UK: Oxford University Press.

Denny M. (2012). *The Science of Navigation: From Dead Reckoning to GPS*. Baltimore, Maryland: The Johns Hopkins University Press.

His Majesty's Navigation School (1938). *Admiralty Navigation Manual*. Vol. **3**. London, UK: His Magesty's Stationary Office.

Stansfield R. G. (1947). Statistical theory of D.F. fixing. *Journal of the Institution of Electrical Engineers Part IIIA: Radiocommunication*, **94**(5), 762–770. doi:10.1049/ji-3a-2.1947.0096

Stuart R. G. (2019). Probabilities in a Gaussian cocked hat. *Journal of Navigation*, **72**(6), 1496–1512. doi:10.1017/S0373463319000110

The Open University (1984). *M245 Probability and statistic: Item 15, Findings One's Bearings*. A video that is part of course materials. UK. https://www.open.ac.uk/library/digital-archive/video:FOUM190W

Williams J. E. D. (1991). The cocked hat. *Journal of Navigation*, **44**(2), 269–271. doi:10.1017/S0373463300009991

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