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ON THE MULTIFRACTAL ANALYSIS OF THE COVERING NUMBER ON THE GALTON–WATSON TREE

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Abstract

We consider, for t in the boundary of a Galton–Watson tree ($\partial\mathsf{T}$), the covering number $\mathsf{N}_n(t)$ by the generation-n cylinder. For a suitable set I and sequence (s_n), we almost surely establish the Hausdorff dimension of the set $\{t \in \partial\mathsf{T}\colon \mathsf{N}_n(t) - nb \sim s_n\}$ for $b \in I$.

Keywords: Random covering; Hausdorff dimension; indexed martingale; Galton–Watson tree

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1. Introduction and main results

Let (N, X) be a random vector with independent components taking values in $\mathbb{N}^2$, where $\mathbb{N}$ denotes the set of nonnegative integers. Then consider $\{(N_u, X_u)\}_{u\in\bigcup_{n\geq 0}\mathbb{N}_+^n}$ to be a family of independent copies of the vector (N, X) indexed by the set of finite words over the alphabet $\mathbb{N}_+$, the set of positive integers ($n = 0$ corresponds to the empty sequence denoted by $\varnothing$). Let T be the Galton–Watson tree with defining elements $\{N_u\}$. We have $\varnothing \in \mathsf{T}$; if $u \in \mathsf{T}$ and $i \in \mathbb{N}_+$ then ui, the concatenation of u and i, belongs to T if and only if $1 \leq i \leq N_u$ and if $ui \in \mathsf{T}$ then $u \in \mathsf{T}$. Similarly, for each $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$, denote by $\mathsf{T}(u)$ the Galton–Watson tree rooted at u and defined by the $\{N_{uv}\}$, $v \in \bigcup_{n\geq 0}\mathbb{N}_+^n$.

We assume that $\mathbb{E}(N) > 1$, so that the Galton–Watson tree is supercritical. We also assume that the probability of extinction is equal to 0, so that $\mathbb{P}(N \geq 1) = 1$.

For each infinite word $t = t_1t_2 \cdots \in \mathbb{N}_+^{\mathbb{N}_+}$ and $n \geq 0$, we set $t_{|n} = t_1 \cdots t_n \in \mathbb{N}_+^n$ ($t_{|0} = \varnothing$). If $u \in \mathbb{N}_+^n$ for some $n \geq 0$ then n is the length of u and it is denoted by $|u|$. Then we denote by $[u]$ the set of infinite words $t \in \mathbb{N}_+^{\mathbb{N}_+}$ such that $t_{||u|} = u$.

The set $\mathbb{N}_+^{\mathbb{N}_+}$ is endowed with the standard ultrametric distance

$$d\colon (u, v) \mapsto \mathrm{e}^{-\sup\{|w|\colon u\in[w],\ v\in[w]\}},$$

with the convention that $\exp(-\infty) = 0$. The boundary of the Galton–Watson tree T is defined as the compact set

$$\partial\mathsf{T} = \bigcap_{n\geq 1}\bigcup_{u\in\mathsf{T}_n}[u],$$

where $\mathsf{T}_n = \mathsf{T} \cap \mathbb{N}_+^n$.

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We consider X_u as the covering number of the cylinder $[u]$, that is, the cylinder $[u]$ is cut off with probability $p_0 = \mathbb{P}(X = 0)$ and is covered m times with probability $p_m = \mathbb{P}(X = m)$, $m = 1, 2, \ldots$.

For $t \in \partial \mathsf{T}$, let

$$\mathsf{N}_n(t) = \sum_{k=1}^{n} X_{t_1 \cdots t_k}.$$

Since this quantity depends on $t_1 \cdots t_n$ only, we also denote by $\mathsf{N}_n(u)$ the constant value of $\mathsf{N}_n(\cdot)$ over $[u]$ whenever $u \in \mathsf{T}_n$. The quantity $\mathsf{N}_n(t)$ is called the covered number (or more precisely the n-covered number) of the point t by the generation-k cylinder, $k = 1, 2, \ldots, n$.

We also define the α-dimensional Hausdorff measure of a set E by

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \mathbb{N}} \operatorname{diam}(U_i)^\alpha \right\},$$

where the infimum is taken over all the countable coverings $(U_i)_{i \in \mathbb{N}}$ of E of diameters less than or equal to δ. Then the Hausdorff dimension of E is defined as

$$\dim E = \sup\{\alpha > 0 \colon \mathcal{H}^\alpha(E) = \infty\} = \inf\{\alpha > 0 \colon \mathcal{H}^\alpha(E) = 0\},$$

with the conventions that

$$\sup \varnothing = 0 \quad \text{and} \quad \inf \varnothing = \infty.$$

Moreover, if E is a Borel set and μ is a measure supported on E, then its lower Hausdorff dimension is defined as

$$\underline{\dim}(\mu) = \inf\{\dim F \colon F \text{ Borel}, \mu(F) > 0\},$$

and we have

$$\underline{\dim}(\mu) = \operatorname{ess\ inf}_\mu \liminf_{r \to 0^+} \frac{\log \mu(B(t, r))}{\log (r)},$$

where the first infimum is taken over all t and $B(t, r)$ stands for the closed ball of radius r centered at t [10].

Consider an individual infinite branch $t_1 \cdots t_n \cdots$ of $\partial \mathsf{T}$. When $\mathbb{E}(X)$ is defined, the strong law of large numbers yields $\lim_{n \to \infty} n^{-1} \mathsf{N}_n(t) = \mathbb{E}(X)$. It is also well known (see [11]) in the theory of the birth process that $\lim_{n \to \infty} \mathsf{N}_n(t) = +\infty$ almost surely (a.s.) for every $t \in \mathcal{D} = \{0, 1\}^{\mathbb{N}}$ if and only if

$$p_0 = \mathbb{P}(X = 0) < \tfrac{1}{2}.$$

Then, if this condition is satisfied, every point is infinitely covered a.s.

For $b \in \mathbb{R}$, we consider the set

$$E_b = \left\{ t \in \partial \mathsf{T} \colon \lim_{n \to \infty} \frac{\mathsf{N}_n(t)}{n} = b \right\}.$$

These level sets can be described geometrically through their Hausdorff dimensions. They have been studied by many authors; see, e.g. [4], [7], [8], [12], and [17], and [2] and [3] for the general case. All these papers also deal with the multifractal analysis of associated Mandelbrot measures (see also [13], [16], and [18] for the study of the Mandelbrot measures dimension).

For the sake of simplicity, we will assume that the free energy of X defined as

$$\tau(q) = \log \mathbb{E}\bigg(\sum_{i=1}^{N} \mathrm{e}^{qX_i} \bigg)$$

is finite over $\mathbb{R}$. Let τ^* stand for the Legendre transform of the function τ, where, by convention, the Legendre transform of a mapping $f\colon \mathbb{R} \to \mathbb{R}$ is defined as the concave and upper semi-continuous function

$$f^*(b) := \inf_{q\in\mathbb{R}} (f(q) - qb).$$

We say that the multifractal formalism holds at $b \in \mathbb{R}$ if $\dim E_b = \tau^*(b)$. We will assume without loss of generality that X is not constant, so that the function τ is strictly convex.

The interior of subset A of $\mathbb{R}$ is denoted by $\mathrm{int}(A)$. In the following, we define the sets

$$J = \{q \in \mathbb{R}; \tau(q) - q\tau'(q) > 0\}, \qquad \Omega^1_\alpha = \mathrm{int}\bigg\{ q\colon \mathbb{E}\bigg(\bigg| \sum_{i=1}^{N} \mathrm{e}^{qX_i} \bigg|^\alpha \bigg) < \infty \bigg\},$$

$$\Omega^1 = \bigcup_{\alpha\in(1,2]} \Omega^1_\alpha, \qquad \mathcal{J} = J \cap \Omega^1, \quad \text{and} \quad I = \{\tau'(q); q \in \mathcal{J}\}.$$

Remark 1. Define the set $L = \{\alpha \in \mathbb{R},\ \tau^*(\alpha) \geq 0\}$. We can show that L is a convex, compact, and nonempty set (see [1, Proposition 3.1]). If we add the assumption that $J = \mathcal{J}$ (for example, if we suppose that, for all $q \in J$, there exists $\alpha \in (1, 2]$ such that $\mathbb{E}[|\sum_{i=1}^{N} \mathrm{e}^{qX_i}|^\alpha] < \infty$), then $I = \mathrm{int}(L)$ (see also [1, Proposition 3.1]). In particular, I is an interval.

Next, we define, for $b \in \mathbb{R}$ and any positive sequence $s = \{s_n\}$ such that $s_n = \mathrm{o}(n)$, the set

$$E_{b,s} = \{t \in \partial \mathsf{T}\colon \mathsf{N}_n(t) - nb \sim s_n \text{ as } n \to +\infty\},$$

where $\mathsf{N}_n(t) - nb \sim s_n$ means that $(\mathsf{N}_n(t) - nb)_n$ and $(s_n)_n$ are two equivalent sequences. We can obtain the Hausdorff dimension of the set E_b via, for example, the methods used in [2], [3], [14], and [15], but such methods do not give results on $\dim E_{b,s}$.

Let $(\eta_n)_{n\geq 1}$ be a positive sequence defined by $\eta_n = s_n - s_{n-1}$ for $n \geq 1$ and suppose that the following hypothesis holds.

Hypothesis 1. *Let $s_n = \mathrm{o}(n)$ and $\eta_n = \mathrm{o}(1)$. Then there exists (ε_n) such that*

$$\varepsilon_n \to 0, \qquad \sum_{n\geq 1} \exp\bigg(-\varepsilon \sum_{k=1}^{n} \varepsilon_k\, \eta_k^2 \bigg) < +\infty \quad \textit{for all } \varepsilon > 0.$$

For example, to satisfy Hypothesis 1, we can choose, for $n \geq 1$,

$$s_n = \sum_{k=1}^{n} \frac{1}{k^\alpha} \quad \text{and} \quad \varepsilon_n = n^{-\gamma}$$

such that $\alpha \in (0, \frac{1}{2})$ and $1 - 2\alpha - \gamma > 0$.

We are able now to state our main result.

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Theorem 1. *Let $s = (s_n)_{n \geq 1}$ be a positive sequence. Under Hypothesis 1, we have, a.s., for all $b \in I$,*

$$\dim E_{b,s} = \dim E_b = \tau^*(b).$$

A special case of this theorem was treated in [11], where the authors considered the space $\{0, 1\}^{\mathbb{N}}$ and constructed, for each $b = \tau'(q) \in I$, a Mandelbrot measure μ_q. Let us mention that our theorem gives a stronger result in the sense that, a.s., for all $b \in I$, we have the multifractal formalism. This requires a simultaneous building of an inhomogeneous Mandelbrot measure and the computation of their Hausdorff dimensions.

2. Proof of Theorem 1

Let s be a positive sequence such that $s_n = \mathrm{o}(n)$ and $\eta_n = \mathrm{o}(1)$.

2.1. Upper bounds for the Hausdorff dimension

Let us define, for $q \in \mathbb{R}$, the pressure-like function of q by

$$\widetilde{\tau}(q) = \limsup_{n \to +\infty} \frac{1}{n} \ln \Bigg(\sum_{u \in \mathsf{T}_n} \exp\,(q \mathsf{N}_n(u)) \Bigg).$$

Proposition 1. *With probability* 1, *for all $b \in \mathbb{R}$,*

$$\dim E_{b,s} \leq \dim E_b \leq \widetilde{\tau}^*(b) \leq \tau^*(b),$$

a negative dimension meaning that E_b is empty.

Proof. It is clear, since $s_n = \mathrm{o}(n)$, that, a.s., for all $b \in \mathbb{R}$, we have $E_{b,s} \subset E_b$. Then, a.s.,

$$\dim E_{b,s} \leq \dim E_b.$$

In addition, we have

$$E_b = \bigcap_{\varepsilon > 0} \bigcup_{M \in \mathbb{N}^*} \bigcap_{n \geq M} \{t \in \partial \mathsf{T};\, |\mathsf{N}_n(t) - nb| \leq n\varepsilon\}.$$

Fix $\varepsilon > 0$. For $M \geq 1$, the set $E(M, \varepsilon, b) = \bigcap_{n \geq M} \{t \in \partial \mathsf{T};\, |\mathsf{N}_n(t) - nb| \leq n\varepsilon\}$ is covered by the union of those $[u]$ such that $u \in \mathsf{T}_n,\ n \geq M$, and $\mathsf{N}_n(u) - nb + n\varepsilon \geq 0$. Thus, for $\alpha \geq 0$, $n \geq M$, and $q > 0$,

$$\mathcal{H}^{\alpha}_{\mathrm{e}^{-n}}(E(M, \varepsilon, b)) \leq \sum_{u \in \mathsf{T}_n} \exp\,(-n\alpha)\ \exp\,(q \mathsf{N}_n(u) - nqb + nq\varepsilon).$$

Consequently, if $\zeta > 0$ and $\alpha > \widetilde{\tau}(q) + \zeta - qb + q\varepsilon$, by the definition of $\widetilde{\tau}(q)$, for large enough M, we have

$$\mathcal{H}^{\alpha}_{\mathrm{e}^{-n}}(E(M, \varepsilon, b)) \leq \exp\Big(-\frac{n\zeta}{2}\Big).$$

This yields $\mathcal{H}^{\alpha}(E(M, \varepsilon, b)) = 0$; hence, $\dim E(M, \varepsilon, b) \leq \alpha$. Since this holds for all $\zeta > 0$, we obtain $\dim E(M, \varepsilon, b) \leq \widetilde{\tau}(q) - qb + q\varepsilon$. It follows that

$$\dim E_b \leq \inf_{q > 0} \inf_{\varepsilon > 0} \sup_{M \in \mathbb{N}^*} \widetilde{\tau}(q) - qb + q\varepsilon.$$

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Similarly, if we take $q<0$, we obtain

$$\dim E_b \leq \inf_{q<0} \inf_{\varepsilon>0} \sup_{M\in\mathbb{N}^*} \widetilde{\tau}(q) - qb - q\varepsilon.$$

Then we have

$$\dim E_b \leq \widetilde{\tau}^*(b).$$

If $\widetilde{\tau}^*(b)<0$, we necessarily have $E_b=\varnothing$.

It remains to show that, with probability 1,

$$\widetilde{\tau}^*(b) \leq \tau^*(b) \quad \text{for all } b\in\mathbb{R}.$$

The functions $\widetilde{\tau}$ and τ are convex and thus continuous. We need only prove that the inequality $\widetilde{\tau}(q)\leq\tau(q)$ holds for each $q\in\mathbb{R}$ almost surely. Fix $q\in\mathbb{R}$. For $\alpha>\tau(q)$, we have

$$\begin{aligned}\mathbb{E}\bigg(\sum_{n\geq1}\exp(-n\alpha)\sum_{u\in\mathsf{T}_n}\exp(q\mathsf{N}_n(u))\bigg) &= \sum_{n\geq1}\exp(-n\alpha)\mathbb{E}\bigg(\sum_{i=1}^{N}\exp(qX_i)\bigg)^n \\ &= \sum_{n\geq1}\exp(n(\tau(q)-\alpha)).\end{aligned}$$

Consequently,

$$\sum_{n\geq1}\exp(-n\alpha)\sum_{u\in\mathsf{T}_n}\exp(q\mathsf{N}_n(u))<\infty, \quad \text{a.s.},$$

so that we have

$$\sum_{u\in\mathsf{T}_n}\exp(q\mathsf{N}_n(u)) = \mathrm{O}(\exp(n\alpha)) \quad \text{and} \quad \widetilde{\tau}(q)\leq\alpha.$$

Since $\alpha>\tau(q)$ is arbitrary, this completes the proof. □

2.2. Lower bounds for the Hausdorff dimension

2.2.1. *Construction of inhomogeneous Mandelbrot measures.* We define, for $(q,p)\in\mathcal{J}\times[1,\infty)$,

$$\varphi(p,q)=\exp(\tau(pq)-p\tau(q)).$$

We have the following result.

Lemma 1. *For all nontrivial compact sets $K\subset\mathcal{J}$, there exists a real number $1<p_K<2$ such that, for all $1<p\leq p_K$, we have*

$$\sup_{q\in K}\varphi(p_K,q)<1.$$

Proof. Let $q\in\mathcal{J}$. We have $\partial\varphi(1^+,q)/\partial p<0$. Therefore, there exists $p_q>1$ such that $\varphi(p_q,q)<1$. In a neighborhood V_q of q, we have

$$\varphi(p_q,q')<1 \quad \text{for all } q'\in V_q.$$

If K is a nontrivial compact of $\mathcal{J}$, it is covered by a finite number of such V_{q_i}.

Let $p_K=\inf_i p_{q_i}$. If $1<p\leq p_K$ and $\sup_{q\in K}\varphi(p,q)\geq1$, there exists $q\in K$ such that

$$\varphi(p,q)\geq1 \quad \text{and} \quad q\in V_{q_i} \quad \text{for some } i.$$

Let us recall that the mapping $p\mapsto\varphi(p,q)$ is log convex and that $\varphi(1,q)=1$. Since $1<p\leq p_{q_i}$, we have $\varphi(p,q)<1$, which is a contradiction. □

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Lemma 2. *For all compact sets $K \subset \mathcal{J}$, there exists $\widetilde{p}_K > 1$ such that*

$$\sup_{q\in K} \mathbb{E}\bigg(\bigg(\sum_{i=1}^{N} \mathrm{e}^{qX_i}\bigg)^{\widetilde{p}_K}\bigg) < \infty.$$

Proof. Since K is compact and the family of open sets $J \cap \Omega^1_\gamma$ increases to $\mathcal{J}$ as γ decreases to 1, there exists $\gamma \in (1, 2]$ such that $K \subset \Omega^1_\gamma$. Take $\widetilde{p}_K = \gamma$. The conclusion follows from the fact that the function $q \mapsto \mathbb{E}((\sum_{i=1}^{N} \mathrm{e}^{qX_i})^{\widetilde{p}_K})$ is convex over $\Omega^1_{\widetilde{p}_K}$ and thus continuous. □

Now, we will construct the inhomogeneous Mandelbrot measure. For $q \in \mathcal{J}$ and $k \geq 1$, we define $\psi_k(q)$ as the unique t such that

$$\tau'(t) = \tau'(q) + \eta_k.$$

For $u \in \bigcup_{n\geq 0} \mathbb{N}^n_+$ and $q \in \mathcal{J}$, we define, for $1 \leq i \leq N_u$,

$$V(ui, q) = \frac{\exp(qX_{ui})}{\mathbb{E}(\sum_{i=1}^{N} \exp(qX_i))} = \exp(qX_{ui} - \tau(q)),$$

and, for all $n \geq 0$,

$$Y^s_n(q, u) = \sum_{v_1\cdots v_n\in \mathsf{T}_n(u)} \prod_{k=1}^{n} V(u \cdot v_1 \cdots v_k, \psi_{|u|+k}(q)).$$

When $u = \varnothing$, this quantity will be denoted by $Y^s_n(q)$ and, when $n = 0$, its value equals 1.

The sequence $(Y^s_n(q, u))_{n\geq 1}$ is a positive martingale with expectation 1, which converges a.s. and in the L^1-norm to a positive random variable $Y^s(q, u)$ (see [13], [5], or [6, Theorem 1]). However, our study will need the almost-sure simultaneous convergence of these martingales to positive limits.

Proposition 2. (i) *Let K be a compact subset of $\mathcal{J}$. There exists $p_K \in (1, 2]$ such that, for all $u \in \bigcup_{n\geq 0} \mathbb{N}^n_+$, the continuous functions $q \in K \mapsto Y^s_n(q, u)$ converge uniformly, a.s. and in the L_{p_K}-norm, to a limit $q \in K \mapsto Y^s(q, u)$. In particular, $\mathbb{E}(\sup_{q\in K} Y^s(q, u)^{p_K}) < \infty$. Moreover, $Y^s(\cdot, u)$ is positive a.s.*

In addition, for all $n \geq 0$, $\sigma(\{(X_{u1}, \ldots, X_{uN_u}),\ u \in \mathsf{T}_n\})$ and $\sigma(\{Y^s(\cdot, u),\ u \in \mathsf{T}_{n+1}\})$ are independent, and the random functions $Y^s(\cdot, u)$, $u \in \mathsf{T}_{n+1}$, are independent copies of $Y^s(\cdot) := Y^s(\cdot, \varnothing)$.

(ii) *With probability 1, for all $q \in \mathcal{J}$, the weights*

$$\mu^s_q([u]) = \bigg[\prod_{k=1}^{n} \exp(\psi_k(q)X_{u_1\ldots u_k}) - \tau(\psi_k(q)))\bigg] Y^s(q, u)$$

define a measure on $\partial\mathsf{T}$.

The measure μ^s_q will be used to approximate from below the Hausdorff dimension of the set $E_{b,s}$.

The proof of Proposition 2 needs the following result.

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Lemma 3. *For $q \in \mathcal{J}$, $u \in \mathsf{T}$, and $p \in (1, 2)$, there exists a constant C_p depending only on p such that, for $n \geq 1$,*

$$\mathbb{E}(|Y_n^s(q) - Y_{n-1}^s(q)|^p) \leq C_p \mathbb{E}\bigg(\bigg|\sum_{i=1}^{N} V(i, \psi_n(q))\bigg|^p\bigg) \prod_{k=1}^{n-1} \mathbb{E}\bigg(\sum_{i=1}^{N} |V(i, \psi_k(q))|^p\bigg).$$

Proof. The definition of the process Y_n immediately gives

$$Y_n^s(q) - Y_{n-1}^s(q) = \sum_{u \in \mathsf{T}_{n-1}} \prod_{k=1}^{n-1} V(u_{|k}, \psi_k(q)) \bigg(\sum_{i=1}^{N_u} V(ui, \psi_n(q)) - 1\bigg).$$

For each $n \geq 1$, let $\mathcal{F}_n = \sigma\{(N_u, V_{u1}, \ldots) : |u| \leq n-1\}$ and let $\mathcal{F}_0$ be the trivial sigma-field. For $u \in \mathsf{T}_{n-1}$, we set $B_u(q) = \sum_{i=1}^{N_u} V(ui, \psi_n(q))$. By construction, the random variables $(B_u(q) - 1)$, $u \in \mathsf{T}_{n-1}$, are centered, independent, identically distributed (i.i.d.), and independent of $\mathcal{F}_{n-1}$. Consequently, conditionally on $\mathcal{F}_{n-1}$, we can apply Lemma 6 in Appendix B to the family $\{(B_u(q) - 1) \prod_{k=1}^{n-1} V(u_{|k}, \psi_k(q))\}$. Noting that the $B_u(q)$, $u \in \mathsf{T}_{n-1}$, have the same distribution yields

$$\begin{aligned}\mathbb{E}(|Y_n^s(q) - Y_{n-1}^s(q)|^p) &= \mathbb{E}\big(\mathbb{E}(|Y_n^s(q) - Y_{n-1}^s(q)|^p \mid \mathcal{F}_{n-1})\big) \\ &\leq 2^{p-1}\mathbb{E}(|B(q) - 1|^p)\mathbb{E}\bigg(\sum_{u \in \mathsf{T}_{n-1}} \prod_{k=1}^{n-1} |V(u_{|k}, \psi_n(q))|^p\bigg),\end{aligned}$$

where $B(q)$ stands for any of the identically distributed variables $B_u(q)$.

Using the branching property and the independence of the random vectors $(N_u, X_{u1}, \ldots)$ used in the constructions yields

$$\begin{aligned}&\mathbb{E}\bigg(\sum_{u \in \mathsf{T}_{n-1}} \prod_{k=1}^{n-1} |V(u_{|k}, \psi_k(q))|^p\bigg) \\ &\quad = \mathbb{E}\bigg[\mathbb{E}\bigg(\sum_{u \in \mathsf{T}_{n-2}} \prod_{k=1}^{n-2} |V(u_{|k}, \psi_k(q))|^p\bigg)\bigg(\sum_{i=1}^{N_u} |V(ui, \psi_{n-1}(q))|^p\bigg) \bigg| \mathcal{F}_{n-2}\bigg)\bigg] \\ &\quad = \mathbb{E}\bigg(\sum_{i=1}^{N} |V(i, \psi_{n-1}(q))|^p\bigg)\mathbb{E}\bigg(\sum_{u \in \mathsf{T}_{n-2}} \prod_{k=1}^{n-2} |V(u_{|k}, \psi_k(q))|^p\bigg).\end{aligned}$$

Then a recursion using the branching property and the independence of the random vectors $(N_u, X_{u1}, \ldots)$ yields

$$\mathbb{E}\bigg(\sum_{u \in \mathsf{T}_{n-1}} \prod_{k=1}^{n-1} |V(u_{|k}, \psi_k(q))|^p\bigg) = \prod_{k=1}^{n-1} \mathbb{E}\bigg(\sum_{i=1}^{N} |V(i, \psi_k(q))|^p\bigg).$$

Using the inequality

$$|x + y|^r \leq 2^{r-1}(|x|^r + |y|^r), \qquad r > 1,$$

we obtain

$$\mathbb{E}\bigg(\bigg|\sum_{i=1}^{N_u} V(ui, \psi_n(q)) - 1\bigg|^p\bigg) \leq 2^{p-1}\mathbb{E}\bigg(\bigg|\sum_{i=1}^{N_u} V(ui, \psi_n(q))\bigg|^p + 1\bigg).$$

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Since

$$1=\left(\mathbb{E}\left(\sum_{1=1}^{N_u} V(ui,\,\psi_n(q))\right)\right)^p \leq \mathbb{E}\left|\sum_{i=1}^{N_u} V(ui,\,\psi_n(q))\right|^p,$$

then it follows from Lemma 6 in Appendix B that

$$\mathbb{E}\left(\left|\sum_{i=1}^{N_u} V(ui,\,\psi_n(q))-1\right|^p\right)\leq 2^p\mathbb{E}\left(\left|\sum_{i=1}^{N_u} V(ui,\,\psi_n(q))\right|^p\right)=2^p\mathbb{E}\left(\left|\sum_{i=1}^{N} V(i,\,\psi_n(q))\right|^p\right).$$

Finally, we have

$$\mathbb{E}(|Y_n^s(q)-Y_{n-1}^s(q)|^p)\leq 2^p\mathbb{E}\left(\left|\sum_{i=1}^{N} V(i,\,\psi_n(q))\right|^p\right)\prod_{k=1}^{n-1}\mathbb{E}\left(\sum_{i=1}^{N}|V(i,\,\psi_k(q))|^p\right). \qquad \square$$

Proof of Proposition 2(i). Recall that the uniform convergence result uses an argument developed in [6]. Fix a compact $K\subset\mathcal{J}$. Since $\eta_k=o(1)$, we can fix, without loss of generality, a compact neighborhood $K'\subset\mathcal{J}$ of K and suppose that

$$\psi_k(q)\in K' \quad \text{for all } q\in K \text{ and all } k\geq 1.$$

Fix a compact neighborhood K'' of K'. By Lemma 2, we can find $\widetilde{p}_{K''}>1$ such that

$$\sup_{q\in K''}\mathbb{E}\left(\left(\sum_{i=1}^{N}\mathrm{e}^{qX_i}\right)^{\tilde{p}_{K''}}\right)<\infty.$$

By Lemma 1, we can fix $1<p_K\leq\min(2,\tilde{p}_{K''})$ such that $\sup_{q\in K'}\phi(p_K,q)<1$. Then, for each $q\in K'$, there exists a neighborhood $V_q\subset\mathbb{C}$ of q whose projection to $\mathbb{R}$ is contained in K'' and such that, for all $u\in\mathsf{T}$ and $z\in V_q$, the random variables

$$V(u,z)=\frac{\exp(zX_u)}{\mathbb{E}(\sum_{i=1}^{N}\exp(zX_i))} \quad\text{and}\quad \Gamma(z)=\frac{\mathbb{E}(\sum_{i=1}^{N}X_i\exp(zX_i))}{\mathbb{E}(\sum_{i=1}^{N}\exp(zX_i))}$$

are well defined. For $z\in V_q$ and $k\geq 1$, we define $\psi_k(z)$ as the unique t such that

$$\Gamma(t)=\Gamma(z)+\eta_k.$$

Moreover, we have

$$\sup_{z\in V_q}\phi(p_K,z)<1, \quad\text{where}\quad \phi(p_K,z)=\frac{\mathbb{E}(\sum_{i=1}^{N}|\mathrm{e}^{zX_i}|^{p_K})}{|\mathbb{E}(\sum_{i=1}^{N}\mathrm{e}^{zX_i})|^{p_K}}.$$

By extracting a finite covering of K' from $\bigcup_{q\in K'}V_q$, we find a neighborhood $V\subset\mathbb{C}$ of K' such that

$$\sup_{z\in V}\phi(p_K,z)<1 \text{ and } \psi_k(z) \text{ is defined for all } z\in V.$$

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Since the projection of V to $\mathbb{R}$ is included in K'' and the mapping $z \mapsto \mathbb{E}(\sum_{i=1}^N e^{zX_i})$ is continuous and does not vanish on V, by considering a smaller neighborhood of K' included in V if necessary, we can assume that

$$C_V = \sup_{z\in V} \mathbb{E}\biggl(\biggl|\sum_{i=1}^N e^{zX_i}\biggr|^{p_K}\biggr)\biggl|\mathbb{E}\biggl(\sum_{i=1}^N e^{zX_i}\biggr)\biggr|^{-p_K} < \infty.$$

Now, for $u \in \mathsf{T}$, we define the analytic extension to V of $Y_n^s(q, u)$ given by

$$\begin{aligned} Y_n^s(z, u) &= \sum_{v\in \mathsf{T}_n(u)} \prod_{k=1}^n V(u\cdot v_1 \cdots v_k, \psi_{|u|+k}(z)) \\ &= \biggl[\prod_{k=1}^n \mathbb{E}\biggl(\sum_{i=1}^N e^{\psi_k(z)X_i}\biggr)\biggr]^{-1} \sum_{v\in \mathsf{T}_n(u)} \prod_{k=1}^n e^{\psi_{|u|+k}(z)X(uv_{|k})}. \end{aligned}$$

We also denote $Y_n^s(z, \varnothing)$ by $Y_n^s(z)$. The same lines as in the proof of Lemma 3 show that

$$\mathbb{E}(|Y_n^s(z) - Y_{n-1}^s(z)|^{p_K}) \le C_{p_K}\mathbb{E}\biggl(\biggl|\sum_{i=1}^N V(i, \psi_n(z))\biggr|^{p_K}\biggr) \prod_{k=1}^{n-1} \mathbb{E}\biggl(\sum_{i=1}^N |V(i, \psi_k(z))|^{p_K}\biggr).$$

Note that $\mathbb{E}(\sum_{i=1}^N |V(i, z)|^{p_K}) = \phi(p_K, \psi_k(z))$. Then

$$\begin{aligned} \mathbb{E}(|Y_n^s(z) - Y_{n-1}^s(z)|^{p_K}) &\le C_{p_K}\mathbb{E}\biggl(\biggl|\sum_{i=1}^N V(i, \psi_n(z))\biggr|^{p_K}\biggr) \prod_{k=1}^{n-1} \phi(p_K, \psi_k(z)). \\ &\le C_{p_K} C_V \prod_{k=1}^{n-1} \sup_{z\in V} \phi(p_K, z), \end{aligned}$$

where we have used the fact that $\psi_k(z) \in V$ for all $k \ge 1$.

With probability 1, the functions $z \in V \mapsto Y_n^s(z)$, $n \ge 0$, are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V$. Theorem 2 gives

$$\sup_{z\in D(z_0,\rho)} |Y_n^s(z) - Y_{n-1}^s(z)| \le 2\int_{[0,1]} |Y_n^s(\zeta(t)) - Y_{n-1}^s(\zeta(t))| \, dt,$$

where, for $t \in [0, 1]$, $\zeta(t) = z_0 + 2\rho e^{i2\pi t}$.

Furthermore, Jensen's inequality and Fubini's theorem give

$$\begin{aligned} &\mathbb{E}\Bigl(\sup_{z\in D(z_0,\rho)} |Y_n^s(z) - Y_{n-1}^s(z)|^{p_K}\Bigr) \\ &\quad\le \mathbb{E}\biggl(\biggl(2\int_{[0,1]} |Y_n^s(\zeta(t)) - Y_{n-1}^s(\zeta(t))| \, dt\biggr)^{p_K}\biggr) \\ &\quad\le 2^{p_K}\mathbb{E}\biggl(\int_{[0,1]} |Y_n^s(\zeta(t)) - Y_{n-1}^s(\zeta(t))|^{p_K} \, dt\biggr) \\ &\quad\le 2^{p_K}\int_{[0,1]} \mathbb{E}|Y_n^s(\zeta(t)) - Y_{n-1}^s(\zeta(t))|^{p_K} \, dt \\ &\quad\le 2^{p_K} C_V C_{p_K} \prod_{k=1}^{n-1} \sup_{z\in V} \phi(p_K, z). \end{aligned}$$

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Since $\sup_{z\in V}\phi(p_K, z) < 1$, it follows that

$$\sum_{n\geq 1}\Big\| \sup_{z\in D(z_0,\rho)} |Y_n^s(z) - Y_{n-1}^s(z)|\Big\|_{p_K} < \infty.$$

This implies that $z\mapsto Y_n^s(z)$ converge uniformly a.s. and in the L^{p_K}-norm over the compact $D(z_0, \rho)$ to a limit $z\mapsto Y^s(z)$. This also implies that

$$\Big\| \sup_{z\in D(z_0,\rho)} Y^s(z)\Big\|_{p_K} < \infty.$$

Since K can be covered by finitely many such discs $D(z_0, \rho)$, we get the uniform convergence, a.s. and in the L^{p_K}-norm, of the sequence $(q\in K\mapsto Y_n^s(q))_{n\geq 1}$ to $q\in K\mapsto Y^s(q)$. Moreover, since $\mathcal{J}$ can be covered by a countable union of such compact K, we get the simultaneous convergence for all $q\in\mathcal{J}$. The same holds simultaneously for all the functions $q\in\mathcal{J}\mapsto Y_n^s(q, u)$, $u\in\bigcup_{n\geq 0}\mathbb{N}_+^n$, because $\bigcup_{n\geq 0}\mathbb{N}_+^n$ is countable.

To complete the proof of Proposition 2(i), we must show that, with probability 1, $q\in K\mapsto Y^s(q)$ does not vanish. Without loss of generality, we can suppose that $K=[0, 1]$. If I is a dyadic closed subcube of $[0, 1]$, we denote by E_I the event {there exists $q\in I$: $Y^s(q)=0$}. Let I_0 and I_1 stand for the two dyadic intervals of I in the next generation. The event E_I being a tail event of probability 0 or 1. If we suppose that $\mathbb{P}(E_I)=1$ then there exists $j\in\{0, 1\}$ such that $\mathbb{P}(E_{I_j})=1$. Suppose now that $\mathbb{P}(E_K)=1$. The previous remark allows us to construct a decreasing sequence $(I(n))_{n\geq 0}$ of dyadic subscubes of K such that $\mathbb{P}(E_{I(n)})=1$. Let q_0 be the unique element of $\bigcap_{n\geq 0} I(n)$. Since $q\mapsto Y^s(q)$ is continuous, we have $\mathbb{P}(Y^s(q_0)=0)=1$, which contradicts the fact that $(Y_n^s(q_0))_{n\geq 1}$ converge to $Y^s(q_0)$ in L^1. □

2.2.2. *Proof of Theorem 1*. The proof of Theorem 1 can be deduced from the two following propositions. Their proof are developed in the next subsections.

Proposition 3. *Suppose that Hypothesis 1 holds. Then, with probability* 1*, for all* $q\in\mathcal{J}$,

$$\mathsf{N}_n(t) - nb \sim s_n \quad \textit{for } \mu_q^s\textit{-almost every } t\in\partial\mathsf{T},$$

where $b=\tau'(q)$.

Proposition 4. *With probability* 1*, for all* $q\in\mathcal{J}$ *and* μ_q^s*-almost every* $t\in\partial\mathsf{T}$,

$$\lim_{n\to\infty}\frac{\log Y^s(q, t_{|n})}{n} = 0.$$

From Proposition 3, it follows, with probability 1, for all $q\in\mathcal{J}$ and $\mu_q^s(E_{b,s})=1$, that $\lim_{n\to+\infty}\mathsf{N}_n(t)/n=b$, $b=\tau'(q)$. In addition, with probability 1, for all $q\in\mathcal{J}$ and μ_q^s-almost every $t\in E_{b,s}$, from Propositions 3 and 4, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{\log(\mu_q^s[t_{|n}])}{\log(\operatorname{diam}([t_{|n}]))} &= \lim_{n\to\infty} -\frac{1}{n}\log\left(\prod_{k=1}^n \exp(\psi_k(q)X_{t_1\ldots t_k} - \tau(\psi_k(q)))Y^s(q, t_{|n})\right)\\
&= \lim_{n\to\infty} -\frac{1}{n}\sum_{k=1}^n \psi_k(q)X_{t_1\ldots t_k} + \frac{1}{n}\sum_{k=1}^n \tau(\psi_k(q)) - \frac{\log Y^s(q, t_{|n})}{n}\\
&= \lim_{n\to\infty} -\frac{1}{n}\sum_{k=1}^n \psi_k(q)X_{t_1\ldots t_k} + \frac{1}{n}\sum_{k=1}^n \tau(\psi_k(q)).
\end{aligned}$$

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Since $\eta_k = \circ(1)$ and then $\psi_k(q) \to q$, we obtain

$$\lim_{n\to\infty} \frac{\log(\mu_q^s[t_{|n}])}{\log(\operatorname{diam}([t_{|n}]))} = -q\tau'(q) + \tau(q) = \tau^*(\tau'(q)).$$

We deduce the result from the mass distribution principle (Theorem 3) and Proposition 1.

2.3. Proof of Proposition 3

Let K be a compact subset of $\mathcal{J}$. For $b = \tau'(q)$, $q \in \mathcal{J}$, $n \geq 1$, $\varepsilon > 0$, and $s = (s_n)_{n\geq 1}$, we set

$$E^1_{b,s,n,\varepsilon} = \left\{ t \in \partial\mathsf{T} \colon \sum_{k=1}^n X_{t_1\cdots t_k}(t) - b - \eta_k \geq \varepsilon \sum_{k=1}^n \eta_k \right\},$$
$$E^{-1}_{b,n,s,\varepsilon} = \left\{ t \in \partial\mathsf{T} \colon \sum_{k=1}^n X_{t_1\cdots t_k}(t) - b - \eta_k \leq -\varepsilon \sum_{k=1}^n \eta_k \right\}.$$

Suppose that we have shown that, for $\lambda \in \{-1, 1\}$, we have

$$\mathbb{E}\left(\sup_{q\in K} \sum_{n\geq 1} \mu_q^s(E^\lambda_{b,n,s,\varepsilon}) \right) < \infty. \tag{2.1}$$

Then, with probability 1, for all $q \in \mathcal{J}$, $\lambda \in \{-1, 1\}$, and $\varepsilon \in \mathbb{Q}_+^*$,

$$\sum_{n\geq 1} \mu_q^s(E^\lambda_{b,n,s,\varepsilon}) < \infty.$$

Consequently, by the Borel–Cantelli lemma, for μ_q^s-almost every t, we have

$$\sum_{k=1}^n X_{t_1\cdots t_k}(t) - b - \eta_k = o\left(\sum_{k=1}^n \eta_k \right),$$

so $\mathsf{N}_n(t) - nb \sim s_n$, which yields the desired result.

Let us prove (2.1) when $\lambda = 1$ (the case $\lambda = -1$ is similar). Let $\theta = (\theta_n)$ be a positive sequence and $q \in K$. Then

$$\sup_{q\in K} \mu_q^s(E^1_{b,n,s,\varepsilon}) \leq \sup_{q\in K} \sum_{u\in\mathsf{T}_n} \mu_q^s([u]) \mathbf{1}_{\{E^1_{b,n,s,\varepsilon}\}}(t_u),$$

where t_u is any point in $[u]$. Denote t_u simply by t. Then

$$\begin{aligned}
&\sup_{q\in K} \mu_q^s(E^1_{b,n,s,\varepsilon}) \\
&\quad \leq \sup_{q\in K} \sum_{u\in\mathsf{T}_n} \mu_q^s[u] \prod_{k=1}^n \exp\left(\theta_k X_{t_1\cdots t_k} - \theta_k b - \theta_k \eta_k (1+\varepsilon)\right) \\
&\quad \leq \sup_{q\in K} \sum_{u\in\mathsf{T}_n} \prod_{k=1}^n \exp\left((\psi_k(q) + \theta_k) X_{t_1\cdots t_k} - \tau(\psi_k(q)) - \theta_k b - \theta_k \eta_k (1+\varepsilon)\right) Y^s(q, u).
\end{aligned}$$

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For $q \in K$, $\theta = (\theta_n)$, and $n \geq 1$, set

$$H_n^s(q,\theta) = \sum_{u \in \mathsf{T}_n} \prod_{k=1}^n \exp\left((\psi_k(q) + \theta_k)X_{t_1 \cdots t_k} - \tau(\psi_k(q)) - \theta_k b - \theta_k \eta_k(1+\varepsilon)\right) M^s(u),$$

where

$$M^s(u) = \sup_{q \in K} Y^s(u,q).$$

Recall from the proof of Proposition 2 that there exists a neighborhood $V_K \subset \mathbb{C}$ of K such that

$$\Gamma(z) = \frac{\mathbb{E}\left(\sum_{i=1}^N X_i \exp(zX_i)\right)}{\mathbb{E}\left(\sum_{i=1}^N \exp(zX_i)\right)} \quad \text{and} \quad \psi_k(z) \text{ for } k \geq 1$$

are well defined for $z \in V_K$.

For $\varepsilon > 0$, $z \in V_K$, and $n \geq 1$, we define

$$H_n^s(z,\theta) = \sum_{u \in \mathsf{T}_n} \prod_{k=1}^n \exp\left((\psi_k(z) + \theta_k)X_{u_{|k}} - \theta_k \Gamma(z) - \theta_k \eta_k(1+\varepsilon)\right) \times \mathbb{E}\left(\sum_{i=1}^N \exp(\psi_k(z)X_i)\right)^{-1} M^s(u).$$

Proposition 5. *There exists a neighborhood $V \subset V_K$ of K, a positive constant $\mathcal{C}_K$, and a positive sequence θ such that, for all $z \in V_K$ and all $n \in \mathbb{N}^*$,*

$$\mathbb{E}(|H_n^s(z,\theta)|) \leq \mathcal{C}_K \exp\left(-\frac{\varepsilon}{4}\sum_{k=1}^n \varepsilon_k \eta_k^2\right),$$

where the sequence $(\varepsilon_n)_n$ is the sequence used in Hypothesis 1.

Lemma 4. *There exists a positive sequence $\theta = (\theta_n)$ and a positive constant $\mathcal{C}_K$ such that, for all $q \in K$, we have*

$$\mathbb{E}\big(H_n^s(q,\theta)\big) \leq \mathcal{C}_K \exp\left(-\frac{\varepsilon}{2}\sum_{k=1}^n \varepsilon_k \eta_k^2\right).$$

Proof. Let $\theta = (\theta_n)$ be a positive sequence. Clearly we have

$$\begin{aligned}
&\mathbb{E}(H_n^s(q,\theta)) \\
&\quad = \prod_{k=1}^n \mathbb{E}\left(\sum_{i=1}^N \exp\big((\psi_k(q)+\theta_k)X_i\big)\exp\big(-\tau(\psi_k(q)) - \theta_k b - \theta_k\eta_k(1+\varepsilon)\big)\right)\mathbb{E}(M^s(u)), \\
&\quad \leq \mathcal{C}'_K \prod_{k=1}^n \exp\big(\tau(\psi_k(q)+\theta_k) - \tau(\psi_k(q)) - \theta_k b - \theta_k\eta_k(1+\varepsilon)\big),
\end{aligned}$$

where, by Proposition 2, $\mathcal{C}'_K = \mathbb{E}(M^s(u)) = \mathbb{E}(M^s(\varnothing)) < \infty$ for all $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$.

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Since $\eta_k = \mathrm{o}(1)$, we can fix a compact neighborhood K' of K and suppose that, for all $k \geq 1$ and all $q \in K$, we have $\psi_k(q) \in K'$. For $q \in K$ and $k \geq 1$, writing the Taylor expansion of the function $g\colon \theta \mapsto \widetilde{\tau}(\psi_k(q) + \theta)$ at 0 up to the second order, we obtain

$$g(\theta) = g(0) + \theta g'(0) + \theta^2 \int_0^1 (1-t) g''(t\theta)\, \mathrm{d}t,$$

with $g''(t\theta) \leq m_K = \sup_{t \in [0,1]} \sup_{q \in K'} g''(t\theta)$. It follows that, for all $k \geq 1$

$$\tau(\psi_k(q) + \theta_k) - \tau((\psi_k(q)) - \theta_k \tau'((\psi_k(q)) \leq \theta_k^2 m_K.$$

Recall that $\tau'(\psi_k(q)) = \tau'(q) + \eta_k$. Then

$$\begin{aligned}\mathbb{E}\big(H_n^s(q,\theta)\big) &\leq \mathcal{C}'_K \prod_{k=1}^n \exp\big(\tau(\psi_k(q) + \theta_k) - \tau(\psi_k(q)) - \theta_k b - \theta_k \eta_k (1+\varepsilon)\big) \\ &\leq \mathcal{C}'_K \prod_{k=1}^n \exp\big(-\theta_k \eta_k \varepsilon + \theta_k^2 m_K\big).\end{aligned}$$

Choose the sequence θ such that $\theta_k = \varepsilon_k \eta_k$. Then

$$\mathbb{E}(H_n^s(q,\theta)) \leq \mathcal{C}'_K \prod_{k=1}^n \exp\big(-\varepsilon_k \eta_k^2 (\varepsilon - \varepsilon_k m_K)\big).$$

Since $\varepsilon_k \to 0$ then, for large enough k, we have $\varepsilon - \varepsilon_k m_K > \varepsilon/2$. Then there exists a constant $\mathcal{C}_K$ such that

$$\mathbb{E}(H_n^s(q,\theta)) \leq \mathcal{C}_K \exp\left(-\frac{\varepsilon}{2} \sum_{k=1}^n \varepsilon_k \eta_k^2\right). \qquad \square$$

Proof of Proposition 5. Since $\mathbb{E}(|H_n^s(q,\theta)|) \leq \mathcal{C}_K \exp\big(-(\varepsilon/2)\sum_{k=1}^n \varepsilon_k \eta_k^2\big)$ for $q \in K$, there exists a neighborhood $V_q \subset V_K$ of q such that, for all $z \in V_q$, we have $\mathbb{E}(|H_n^s(z,\theta)|) \leq \mathcal{C}_K \exp\big(-(\varepsilon/4)\sum_{k=1}^n \varepsilon_k \eta_k^2\big)$. By extracting a finite covering of K from $\bigcup_{q \in K} V_q$, we find a neighborhood $V \subset V_K$ of K such that

$$\mathbb{E}(|H_n^s(z,\theta)|) \leq \mathcal{C}_K \exp\left(-\frac{\varepsilon}{4} \sum_{k=1}^n \varepsilon_k \eta_k^2\right). \qquad \square$$

With probability 1, the functions $z \in V \longmapsto H_n^s(z,\theta)$ are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V,\ \rho > 0$, such that $D(z_0, 2\rho) \subset V$. Theorem 2 gives

$$\sup_{z \in D(z_0,\rho)} |H_n^s(z,\theta)| \leq 2 \int_{[0,1]} |H_n(\zeta(t),\theta)|\, \mathrm{d}t,$$

where, for $t \in [0, 1]$,

$$\zeta(t) = z_0 + 2\rho \mathrm{e}^{\mathrm{i}2\pi t}.$$

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Furthermore, Fubini's theorem gives

$$\begin{aligned}\mathbb{E}\Big(\sup_{z\in D(z_0,\rho)}|H_n^s(z,\theta)|\Big) &\le \mathbb{E}\Big(2\int_{[0,1]}|H_n^s(\zeta(t),\theta)|\,\mathrm{d}t\Big)\\ &\le 2\int_{[0,1]}\mathbb{E}|H_n^s(\zeta(t),\theta)|\,\mathrm{d}t\\ &\le 2\exp\bigg(-\frac{\varepsilon}{4}\sum_{k=1}^{n}\varepsilon_k\eta_k^2\bigg).\end{aligned}$$

Finally, we obtain

$$\mathbb{E}\Big(\sup_{q\in K}\mu_q^s(E^1_{b,n,s,\varepsilon})\Big)\le 2\exp\bigg(-\frac{\varepsilon}{4}\sum_{k=1}^{n}\varepsilon_k\eta_k^2\bigg),$$

and then, under Hypothesis 1, we obtain (2.1), which completes the proof of Proposition 3.

2.4. Proof of Propostion 4

Let K be a compact subset of $\mathcal{J}$. For $a>1$, $q\in K$, and $n\ge 1$, set

$$E^+_{n,a}=\{t\in\partial\mathsf{T}\colon Y^s(q,t_{|n})>a^n\}$$

and

$$E^-_{n,a}=\{t\in\partial\mathsf{T}\colon Y^s(q,t_{|n})<a^{-n}\}.$$

It is sufficient to show that, for $E\in\{E^+_{n,a},E^-_{n,a}\}$,

$$\mathbb{E}\bigg(\sup_{q\in K}\sum_{n\ge 1}\mu_q^s(E)\bigg)<\infty. \tag{2.2}$$

Indeed, if this holds then, with probability 1, for each $q\in K$ and $E\in\{E^+_{n,a},E^-_{n,a}\}$, $\sum_{n\ge 1}\mu_q^s(E)<\infty$; hence, by the Borel–Cantelli lemma, for μ_q^s-almost every $t\in\partial\mathsf{T}$, if n is large enough, we have

$$-\log a\le\liminf_{n\to\infty}\frac{1}{n}\log Y^s(t_{|n},q)\le\limsup_{n\to\infty}\frac{1}{n}\log Y^s(t_{|n},q)\le\log a.$$

Letting a tend to 1 along a countable sequence yields the result.

Let us prove (2.2) for $E=E^+_{n,a}$ (the case $E=E^-_{n,a}$ is similar). At first we have

$$\begin{aligned}\sup_{q\in K}\mu_q^s(E^+_{n,a}) &= \sup_{q\in K}\sum_{u\in\mathsf{T}_n}\mu_q^s([u])\mathbf{1}_{\{Y^s(q,u)>a^n\}}\\ &= \sup_{q\in K}\sum_{u\in\mathsf{T}_n}Y^s(q,u)\prod_{k=1}^{n}\exp\left(\psi_k(q)X(u)-\tau(\psi_k(q))\right)\mathbf{1}_{\{Y^s(q,u)>a^n\}}\\ &\le \sup_{q\in K}\sum_{u\in\mathsf{T}_n}(Y^s(q,u))^{1+\nu}\prod_{k=1}^{n}\exp\left(\psi_k(q)X_u-\tau((\psi_k(q)))a^{-\nu}\right),\\ &\le \sup_{q\in K}\sum_{u\in\mathsf{T}_n}M^s(u)^{1+\nu}\prod_{k=1}^{n}\exp\left(\psi_k(q)X_u-\tau(\psi_k(q))\right)a^{-\nu},\end{aligned}$$

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where $M^s(u) = \sup_{q\in K} Y^s(q, u)$ and $\nu > 0$ is an arbitrary parameter. For $q \in K$ and $\nu > 0$, we set $L_n(q, \nu) = \sum_{u\in \mathsf{T}_n} M^s(u)^{1+\nu} \prod_{k=1}^n \exp(\psi_k(q)X_u - \tau(\psi_k(q)))a^{-\nu}$.

Recall from the proof of Proposition 2 that there exists a neighborhood $U_K \subset \mathbb{C}$ of K such that, for all $z \in U_K$ and $k \geq 1$,

$$\psi_k(z) \text{ is well defined} \quad \text{and} \quad \mathbb{E}\bigg(\sum_{i=1}^N \mathrm{e}^{\psi_k(z)X_i}\bigg) \neq 0.$$

Lemma 5. *Fix $a > 1$. For $z \in U_K$ and $\nu > 0$, let*

$$L_n(z, \nu) = \bigg[\prod_{k=1}^n \mathbb{E}\bigg(\sum_{i=1}^N \exp(\psi_k(z)X_i)\bigg)^{-1}\bigg] \sum_{u\in T_n} M^s(u)^{1+\nu} \prod_{k=1}^n \exp(\psi_k(z)X_{u_{|k}})a^{-\nu}.$$

There exists a neighborhood $V \subset \mathbb{C}^d$ of K and a positive constant C_K such that, for all $z \in V$ and all integers $n \geq 1$,

$$\mathbb{E}(|L_n(z, p_K - 1)|) \leq C_K a^{-n(p_K-1)/2},$$

where p_K is given by Proposition 2.

Proof. For $z \in U_K$ and $\nu > 0$, let

$$\widetilde{L}_1(z, \nu) = \bigg|\mathbb{E}\bigg(\sum_{i=1}^N \exp(zX_i)\bigg)\bigg|^{-1} \mathbb{E}\bigg(\sum_{i=1}^N \bigg|\exp(zX_i)\bigg|\bigg)a^{-\nu}.$$

Let $q \in K$. Since $\mathbb{E}(\widetilde{L}_1(q, \nu)) = a^{-\nu}$, there exists a neighborhood $V_q \subset U_K$ of q such that, for all $z \in V_q$, we have $\mathbb{E}(|\widetilde{L}_1(z, \nu)|) \leq a^{-\nu/2}$. By extracting a finite covering of K from $\bigcup_{q\in K} V_q$, we find a neighborhood $V \subset U_K$ of K such that, for all $z \in V$, $\mathbb{E}(|\widetilde{L}_1(z, \nu)|) \leq a^{-\nu/2}$. Without loss of generality (recall the proof of Proposition 2 and the fact that $\eta_k = \mathrm{o}(1)$), we can suppose that, for all $k \geq 1$,

$$\mathbb{E}\big(|\widetilde{L}_1(\psi_k(z), \nu)|\big) \leq a^{-\nu/2}$$

for all $z \in V$. Therefore,

$$\begin{aligned}
&\mathbb{E}(|L_n(z, \nu)|) \\
&\quad = \bigg[\prod_{k=1}^n \bigg|\mathbb{E}\bigg(\sum_{i=1}^N \exp\big(\psi_k(z)X_i\big)\bigg)\bigg|^{-1}\bigg]\mathbb{E}\bigg(\bigg|\sum_{u\in \mathsf{T}_n} M^s(u)^{1+\nu} \prod_{k=1}^n \exp\big(\psi_k(z)X(u)\big)\bigg|\bigg)a^{-n\nu} \\
&\quad \leq \bigg[\prod_{k=1}^n \bigg|\mathbb{E}\bigg(\sum_{i=1}^N \exp\big(\psi_k(z)X_i\big)\bigg)\bigg|^{-1}\bigg]\mathbb{E}\bigg(\sum_{u\in \mathsf{T}_n} M^s(u)^{1+\nu} \prod_{k=1}^n \bigg|\exp\big(\psi_k(z)X(u)\big)\bigg|\bigg)a^{-n\nu}.
\end{aligned}$$

By Proposition 2, there exists $p_K \in (1, 2]$ such that, for all $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$,

$$\mathbb{E}(M^s(u)^{p_K}) = \mathbb{E}(M^s(\varnothing)^{p_K}) = C_K < \infty.$$

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Take $\nu = p_K - 1$ in the last calculation. It follows, from the independence of $\sigma(\{(X_{u1}, \dots, X_{uN_u}),\ u \in \mathsf{T}_{n-1}\})$ and $\sigma(\{Y^s(\cdot, u),\ u \in \mathsf{T}_n\})$ for all $n \geq 1$, that

$$\begin{aligned}
&\mathbb{E}(|L_n(z, p_K - 1)|) \\
&\quad \leq \Bigg[\prod_{k=1}^{n} \bigg| \mathbb{E}\bigg(\sum_{i=1}^{N} \exp(\psi_k(z) X_i) \bigg) \bigg|^{-1} \Bigg] \prod_{k=1}^{n} \mathbb{E}\bigg(\sum_{i=1}^{N} \big| \exp(\psi_k(z) X_i) \big| \bigg)^{n} C_K a^{-n(p_K-1)} \\
&\quad = C_K \prod_{k=1}^{n} \mathbb{E}\big(|\widetilde{L}_1(\psi_k(z), p_K - 1)| \big) \\
&\quad \leq C_K a^{-n(p_K-1)/2},
\end{aligned}$$

completing the proof. □

With probability 1, the functions $z \in V \to L_n(z, \nu)$ are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V$, $\rho > 0$, such that $D(z_0, 2\rho) \subset V$. Theorem 2 gives

$$\sup_{z \in D(z_0, \rho)} |L_n(z, p_K - 1)| \leq 2 \int_{[0,1]} |L_n(\zeta(t), p_K - 1)| \,\mathrm{d}t,$$

where, for $t \in [0, 1]$,

$$\zeta(t) = z_0 + 2\rho \mathrm{e}^{\mathrm{i}2\pi t}.$$

Furthermore, Fubini's theorem gives

$$\begin{aligned}
\mathbb{E}\Big(\sup_{z \in D(z_0, \rho)} |L_n(z, p_K - 1)| \Big) &\leq \mathbb{E}\Big(2 \int_{[0,1]} |L_n(\zeta(r), p_K - 1)| \,\mathrm{d}r \Big) \\
&\leq 2 \int_{[0,1]} \mathbb{E}|L_n(\zeta(r), p_K - 1)| \,\mathrm{d}r \\
&\leq 2 C_K a^{-n(p_K-1)/2}.
\end{aligned}$$

Since $a > 1$ and $p_K - 1 > 0$, we obtain (2.2).

Appendix A. Cauchy formula in several variables

Let us recall the Cauchy formula for holomorphic functions.

Definition 1. Let $D(\zeta, r)$ be a disc in $\mathbb{C}$ with centre ζ and radius r. The set ∂D is the boundary of D. Let $g \in \mathcal{C}(\partial D)$ be a continuous function on ∂D. We define the integral of g on ∂D as

$$\int_{\partial D} g(\zeta) d\zeta = 2\mathrm{i}\pi r \int_{[0,1]} g(\zeta(t)) \mathrm{e}^{\mathrm{i}2\pi t} \,\mathrm{d}t,$$

where $\zeta(t) = \zeta + r\mathrm{e}^{\mathrm{i}2\pi t}$.

Theorem 2. *Let $D = D(a, r)$ be a disc in $\mathbb{C}$ with radius $r > 0$, and let f be a holomorphic function in a neighborhood of D. Then, for all $z \in D$,*

$$f(z) = \frac{1}{2\mathrm{i}\pi} \int_{\partial D} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}.$$

It follows that

$$\sup_{z \in D(a, r/2)} |f(z)| \leq 2 \int_{[0,1]} |f(\zeta(t))| \,\mathrm{d}t.$$

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Appendix B. Mass distribution principle

Theorem 3. ([9, Theorem 4.2].) *Let ν be a positive and finite Borel probability measure on a compact metric space (X, d). Assume that $M \subseteq X$ is a Borel set such that $\nu(M) > 0$ and*

$$M \subseteq \left\{ t \in X, \liminf_{r \to 0^+} \frac{\log \nu(B(t, r))}{\log r} \geq \delta \right\}.$$

Then the Hausdorff dimension of M is bounded from below by δ.

Lemma 6. ([6].) *If $\{X_i\}$ is a family of integrable and independent complex random variables with $\mathbb{E}(X_i) = 0$, then $\mathbb{E}|\sum X_i|^p \leq 2^p \sum \mathbb{E}|X_i|^p$ for $1 \leq p \leq 2$.*

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