Proceedings of the Royal Society of Edinburgh, **142A**, 1027–1042, *2012*

Critical exponents for a semilinear parabolic equation with variable reaction

R. Ferreira
Departamento de Matemática Aplicada,
Universidad Complutense de Madrid, 28040 Madrid, Spain
(raul_ferreira@ucm.es)

A. de Pablo
Departamento de Matemáticas, Universidad Carlos III de Madrid,
28911 Leganés, Spain (arturop@math.uc3m.es)

M. Pérez-LLanos
Departamento de Matemáticas, Universidad Autónoma de Madrid,
28049 Madrid, Spain (mayte.perez@uam.es)

J. D. Rossi*
Departamento de Matemática,
Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires, Ciudad Universitaria,
Pabellón 1, 1428 Buenos Aires, Argentina
(jrossi@dm.uba.ar)

(MS received 11 March 2010; accepted 30 August 2011)

We study the blow-up phenomenon for non-negative solutions to the following parabolic problem:

$$u_t(x,t) = \Delta u(x,t) + (u(x,t))^{p(x)} \quad \text{in } \Omega \times (0,T),$$

where $0 < p_- = \min p \leqslant p(x) \leqslant \max p = p_+$ is a smooth bounded function. After discussing existence and uniqueness, we characterize the critical exponents for this problem. We prove that there are solutions with blow-up in finite time if and only if $p_+ > 1$.

When $\Omega = \mathbb{R}^N$ we show that if $p_- > 1 + 2/N$, then there are global non-trivial solutions, while if $1 < p_- \leqslant p_+ \leqslant 1 + 2/N$, then all solutions to the problem blow up in finite time. Moreover, in the case when $p_- < 1 + 2/N < p_+$, there are functions $p(x)$ such that all solutions blow up in finite time and functions $p(x)$ such that the problem possesses global non-trivial solutions.

When Ω is a bounded domain we prove that there are functions $p(x)$ and domains Ω such that all solutions to the problem blow up in finite time. On the other hand, if Ω is small enough, then the problem possesses global non-trivial solutions regardless of the size of $p(x)$.

*Present address: Departamento de Análisis Matemático, Universidad de Alicante, Apartado de correos 99, 03080 Alicante, Spain.

© 2012 The Royal Society of Edinburgh

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

1. Introduction

We consider non-negative solutions to the following parabolic semilinear problem with a reaction given by a variable exponent:

$$\left.\begin{aligned} u_t(x,t) &= \Delta u(x,t) + (u(x,t))^{p(x)} && \text{in } \Omega \times (0,T), \\ u(x,0) &= u_0(x) && \text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

where $u_0(x)$ and $p(x)$ are two non-negative continuous, bounded functions. For future reference let us define

$$p_- = \inf_x p(x) \quad \text{and} \quad p_+ = \sup_x p(x). \tag{1.2}$$

We study both cases $\Omega = \mathbb{R}^N$ or Ω a bounded smooth domain, in which case we impose Dirichlet boundary conditions to our problem

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{1.3}$$

Existence of a solution can easily be achieved, but uniqueness is subtle due to the fact that $p(x)$ can be less than 1 in some region of Ω. The difficulty comes from the non-Lipschitz character of the reaction [1, 10, 15]. Nevertheless, in this case we can prove existence of a maximal and a minimal solution. Moreover, a comparison principle among maximal solutions and among minimal solutions can easily be obtained. In the case $p_- < 1$ we show that these solutions are always different for the initial value $u_0 \equiv 0$ (and hence we have non-uniqueness). When $p_- \geqslant 1$, uniqueness is standard. We shall discuss these issues in the next section.

When dealing with a parabolic problem there are several interesting features to analyse, but the first step is to identify the so-called *critical exponents*. For p constant we have that there are solutions to (1.1) with $T < \infty$ (T is the maximal existence time) if and only if $p > 1$. In this case, we have

$$\lim_{t \nearrow T} \|u(\cdot,t)\|_\infty = +\infty,$$

a phenomenon that is called *blow-up* in the literature and has attracted great interest (see, for instance, [5, 7, 18, 22, 28, 29] and the references therein). However, the case of a reaction given by a power with variable exponent is much less known in relation with blow-up. In fact, only a brief mention is included in [24], where existence of blow-up solutions is shown provided that $p_- > 1$.

Hence, the first critical exponent one has to look for in a parabolic problem is the *blow-up exponent*, an exponent such that there are solutions with blow-up if and only if $p > p_{\mathrm{b}}$. When $\Omega = \mathbb{R}^N$ and p is constant we have $p_{\mathrm{b}} = 1$. Moreover, in this case there exists a second critical exponent, called the *Fujita exponent* [14, 17, 19, 22, 31]. We refer the reader to [12, 16, 20, 23, 25–27, 30] for more references concerning Fujita exponents in other related problems. For $p > p_{\mathrm{F}} = 1 + 2/N$ there are solutions with blow-up and global solutions, while for $1 < p \leqslant p_{\mathrm{F}}$ every non-trivial solution blows up. Thus, the Fujita exponent separates regions of parameters for which all non-trivial solutions blow up and regions where there are both global and blow-up solutions.

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

In the Dirichlet case, we also have that the blow-up exponent is $p_{\mathrm{b}} = 1$ and there is no Fujita-type exponent, since for $p > 1$ there are always both global and blow-up solutions.

Our main aim in this paper is to find conditions on the variable exponent function $p(x)$, analogous to the above for constant p, in order to have existence or nonexistence of global solutions and/or blow-up solutions. These conditions are called *blow-up conditions* or *Fujita-type conditions*.

We prove a sharp result concerning the blow-up occurrence (theorem 1.1), and two conditions of Fujita type in $\mathbb{R}^N$ (theorem 1.2) that are complemented with two examples. We also present a new phenomenon of Fujita type in bounded domains: roughly speaking, if $p(x) < 1$ in some large set and $p(x) > 1$ in some other set, also large, then every solution blows up (see theorem 1.3).

First, let us look for the critical blow-up condition. In this case we have that $p_+ = 1$ is the critical condition for (1.1) in both $\mathbb{R}^N$ and Ω bounded with (1.3). Indeed, it is enough for $p(x)$ to be larger than 1 in a small ball to have existence of blow-up solutions.

THEOREM 1.1. *For problem (1.1) in $\mathbb{R}^N$ or in a bounded domain with (1.3), we have the following:*

(i) *if $p_+ > 1$, then there are solutions that blow up in finite time;*

(ii) *if $p_+ \leqslant 1$, then every solution is global.*

The existence of blow-up solutions in the case $p_- > 1$ is proved in [24].

Next we look for the Fujita condition. For $\Omega = \mathbb{R}^N$ we have the following result, which says that the value $p_{\mathrm{F}} = 1+2/N$ plays a role. If $p(x)$ lies above p_{F} everywhere, then there are global solutions and if $p(x)$ lies below p_{F} everywhere, then every solution blows up, while there are functions $p(x)$ crossing the value p_{F} that show that in this case we can have both situations.

THEOREM 1.2.

(i) *If $p_- > 1 + 2/N$, then problem (1.1) possesses global non-trivial solutions.*

(ii) *If $1 < p_- < p_+ \leqslant 1 + 2/N$, then all solutions to problem (1.1) blow up in finite time.*

(iii) *If $p_- < 1 + 2/N < p_+$, then there are functions $p(x)$ such that problem (1.1) possesses global non-trivial solutions and functions $p(x)$ such that all solutions blow up.*

In a bounded domain with Dirichlet boundary conditions we find the surprising fact that there is also a Fujita-type phenomenon. In fact, we can have that every non-trivial solution to (1.1) with Dirichlet boundary conditions (1.3) blows up. This has to be contrasted with the case when p is constant, in which there are always non-trivial global solutions. On the other hand, if the domain is small enough, then there are global solutions regardless of the function $p(x)$. Both situations constitute the core of the following theorem.

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

Theorem 1.3.

(i) *There are functions $p(x)$ and domains Ω such that all solutions to problem (1.1)–(1.3) blow up in finite time.*

(ii) *If $\Omega \subset B_r(x_0)$ for some $x_0 \in \mathbb{R}^N$ and $r < \sqrt{2N}$ then problem (1.1)–(1.3) possesses global non-trivial solutions regardless the exponent $p(x)$.*

(iii) *If $p_- > 1$ then there are global solutions regardless the size of the domain Ω.*

The paper is structured as follows: in the following section we deal with the questions of existence, comparison and uniqueness for the solutions of our problems; in § 3 we perform the study of the blow-up phenomenon.

2. Existence and uniqueness

To begin our analysis, we discuss briefly existence and uniqueness of solutions to problem (1.1).

If $p_- \geqslant 1$, then the reaction term $f(x,s) = s^{p(x)}$ is continuous in both variables and locally Lipschitz in the second one. Therefore, there exists a unique classical solution, at least for small times, for any bounded initial datum [13]. Moreover, a comparison principle also holds: if $u_0 \geqslant v_0$ (and in addition $u \geqslant v$ on $\partial\Omega$ in the Dirichlet problem case), then $u(x,t) \geqslant v(x,t)$ whenever they are bounded. See [13] for Ω bounded and [1] for comparison in the whole space.

If $p_- < 1$, we still have existence of a solution but uniqueness is not true in general. For instance, when $p(x)$ is constant $p(x) \equiv p < 1$ and $\Omega = \mathbb{R}^N$, the function

$$U(t) = c_* t^{1/(1-p)}, \quad c_* = (1-p)^{1/(1-p)},$$

is a non-trivial solution with zero initial datum.

In the general case we can construct a maximal solution just by taking the limit

$$\bar{u} = \lim_{\varepsilon \to 0} u^{(\varepsilon)},$$

where $u^{(\varepsilon)}$ is the unique solution to our problem with initial condition $u^{(\varepsilon)}(x,0) = u_0(x) + \varepsilon$, and with the reaction $f(x,s) = s^{p(x)}$ replaced by

$$f_{(\varepsilon)}(x,s) = \begin{cases} s^{p(x)} & \text{if } s \geqslant \varepsilon \text{ or } p(x) \geqslant 1, \\ \varepsilon^{p(x)-1} s & \text{if } s < \varepsilon \text{ and } p(x) < 1 \end{cases} \tag{2.1}$$

(see [9]). In the Dirichlet case we also replace the boundary condition by $u^{(\varepsilon)} = \varepsilon$ on $\partial\Omega$. Since the problem for $u^{(\varepsilon)}$ satisfy a comparison principle, we get a non-increasing sequence of positive functions. The existence time is then uniformly bounded from below. We also deduce in the limit a comparison result for maximal solutions. A minimal solution is obtained by taking limits for Lipschitz problems that approximate (1.1) from below. More precisely, let

$$\underline{u} = \lim_{\varepsilon \to 0} u_{(\varepsilon)},$$

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

where $u_{(\varepsilon)}$ is the unique solution to the problem (1.1) with f replaced by $f_{(\varepsilon)}$ and with the same initial data. Clearly, we have, for any solution u to problem (1.1), the inequalities

$$0 \leqslant \underline{u} \leqslant u \leqslant \bar{u}.$$

Each solution has its own maximal existence time, and the comparison is true whenever they are defined. Furthermore, any supersolution z to (1.1) satisfies $z \geqslant \underline{u}$, though comparison with $\bar{u}$ does not necessarily hold, but it does hold if z is strictly positive, $z \geqslant \mu > 0$. An analogous property is true for subsolutions. When $p(x)$ is constant, $p(x) \equiv p < 1$ and $u_0 \equiv 0$, we have that the minimal and maximal solutions are

$$\underline{u}(x,t) \equiv 0, \qquad \bar{u}(x,t) = U(t) = c_* t^{1/(1-p)},$$

while a continuous family of different solutions between $\underline{u}$ and $\bar{u}$ exists, namely $u(x,t) = U(t-\tau)$ for $t > \tau > 0$ and $u(x,t) = 0$ for $0 \leqslant t \leqslant \tau$.

We now prove that the same phenomenon occurs if only $p_- < 1$: a non-trivial solution exists when $u_0 \equiv 0$. Therefore, in this case $\underline{u} \neq \bar{u}$. We remark that in the case when p is a constant less than 1, it has been proved in [1] (see also [10,15]) that $u_0 \equiv 0$ is the only initial value that produces non-uniqueness, a phenomenon that is denoted by *almost uniqueness* in [8], where a more general diffusion is treated. We conjecture that almost uniqueness also holds for our problem with variable exponent provided $p_- < 1$.

THEOREM 2.1. *Let $u_0 \equiv 0$ in problem (1.1) (posed in Ω or in $\mathbb{R}^N$), and assume that the exponent satisfies $p(x) \leqslant \gamma < 1$ for every $x \in D$, an open bounded subset of Ω. Then the corresponding maximal solution satisfies $\bar{u}(x,t) > 0$ for every $x \in D$, and for any $0 < t < T$.*

Proof. We construct a non-trivial positive subsolution. To this end let $\tilde{D} \subset D$ be a smooth domain and consider the function

$$w(x,t) = a(t)\varphi_1(x),$$

where φ_1 is the first eigenfunction of the Laplacian with Dirichlet boundary condition in $\tilde{D}$, namely φ_1 satisfies $-\Delta\varphi_1 = \lambda_1\varphi_1$ in $\tilde{D}$, $\varphi_1 = 0$ on $\partial\tilde{D}$, normalized according with $\max_{\tilde{D}} \varphi = 1$. We want to choose a function $a(t)$ with $a(0) = 0$ such that w is a subsolution to (1.1). We first need, for $x \in \tilde{D}$,

$$w_t - \Delta w = a'(t)\varphi_1 + \lambda_1 a(t)\varphi_1 \leqslant a(t)^{p(x)}\varphi_1^{p(x)} = w^{p(x)}.$$

To satisfy this inequality it suffices to consider, for small t (e.g. $t \leqslant 1$), the function

$$a(t) = ct^{1/(1-\gamma)}$$

with a suitable small constant $c > 0$. Now, extending w by 0 outside $\tilde{D}$, we get that w is a subsolution to (1.1) for $0 \leqslant t \leqslant 1$. This implies $\bar{u} \geqslant w > 0$ in $\tilde{D}$ for $0 \leqslant t \leqslant 1$. Finally, for times $k < t \leqslant k+1$ we compare with $w(x,t)$ replaced by $w(x,t-k)$. □

We want to refine the proof of theorem 2.1 in order to obtain a lower estimate for every solution.

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

Lemma 2.2. *Assume that the exponent satisfies $p(x) \leqslant \gamma < 1$ for every $x \in D \subset \Omega$ and let u be any solution to problem (1.1) with initial datum $u_0(x) \not\equiv 0$. Given any compact subset $\tilde{D} \subset D$, there exists a constant $c > 0$ (depending only on N, the function p and the distance between $\tilde{D}$ and D^{c}) such that*

$$u(x,t) \geqslant ct^{1/(1-\gamma)} \tag{2.2}$$

for every $x \in \tilde{D}$, $0 \leqslant t \leqslant 1$.

Proof. First of all, since comparison is not true for general solutions, we construct a strictly positive subsolution and perform a comparison argument in a set where our solution is also strictly positive.

It is clear that, since u is a supersolution to the heat equation, we have that, given $x_0 \in D$, $r > 0$ such that $B_r(x_0) \subset D$ and $t_0 > 0$ small enough,

$$\mu = \min\{u(x,t) \colon x \in B_r(x_0),\ t_0 \leqslant t \leqslant t_0 + 1\} > 0.$$

Now, for some $\varepsilon < \mu$, $\eta > 0$ and $\alpha = 1/(1-\gamma)$, consider the function

$$\tilde{w}(x,t) = \varepsilon + \eta t^{\alpha} \varphi_1\biggl(\frac{x - x_0}{r}\biggr),$$

where φ_1 is the first eigenfunction of the Laplacian in the unit ball with $\varphi(0) = 1$. We want to compare $u(x, t + t_0)$ with $\tilde{w}(x,t)$ in $B_r(x_0) \times (0,1)$. The ingredients we need are the following.

- The problem has the comparison property: both functions satisfy $u(x, t+t_0) \geqslant \mu$, $\tilde{w}(x,t) \geqslant \varepsilon$ for $(x,t) \in B_r(x_0) \times (0,1)$.
- Comparison of the initial conditions: $\tilde{w}(x,0) = \varepsilon < \mu \leqslant u(x,t_0)$.
- Comparison of the boundary data: $\tilde{w}(x,t) = \varepsilon < \mu \leqslant u(x,t)$ for $x \in \partial B_r(x_0)$.
- An inequality for the equation: in order to substitute $\tilde{w}$ into the equation, we need $\eta > 0$ to satisfy

  $$\eta \alpha t^{\alpha-1} \varphi_1 + \eta t^{\alpha} \frac{\lambda_1}{r^2} \varphi_1 - (\eta t^{\alpha} \varphi_1 + \varepsilon)^{p(x)} \leqslant 0.$$

  This holds if we choose
  $$\eta = \biggl(\alpha + \frac{\lambda_1}{r^2}\biggr)^{-\alpha}.$$

This implies $u(x, t+t_0) \geqslant \tilde{w}(x,t)$, and thus $u(x,t+t_0) \geqslant ct^{\alpha} + \varepsilon$ for $x \in B_{r/2}(x_0)$, $0 \leqslant t \leqslant 1$, where $c = \eta \min_{B_{1/2}(0)} \varphi$. Observe that the constant η (and thus c) is independent of t_0 and ε. Therefore, letting ε and t_0 go to 0, we obtain (2.2). By the same reason, if $D = \mathbb{R}^N$, a sharp constant can be obtained by letting $r \to \infty$. □

Remark 2.3. Notice that from this proof we also obtain that, for all $t \geqslant 1$ and $x \in \tilde{D} \subset D$,

$$u(x,t) \geqslant \delta > 0$$

holds, where δ depends only on N, the function p and the distance between $\tilde{D}$ and D^{c}.

We merely compare this result with the subsolution $s(x,t) = w(x,t+1)$.

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

3. Blow-up

Now we focus our attention on the blow-up phenomenon, and consider the question of whether or not the solutions to our problems blow up. This leads to two types of results. Namely, on the one hand, the conditions on the reaction exponent $p(x)$ under which we have the existence of blow-up solutions or all solutions are globally defined. On the other hand, we also look for conditions on $p(x)$ such that every solution blows up or there exist also global solutions.

The first result concerning existence of blow-up solutions is an application of Kaplan's method of eigenfunctions if $p(x) > 1$ somewhere. We need the following version of Jensen's inequality. It uses some easy properties of the functional spaces $L^{p(x)}$ (see, for example, [11]).

LEMMA 3.1. *Let μ be a positive measure in $B \subset \mathbb{R}^N$ with $\int_B \mathrm{d}\mu = 1$ and let $f \in L^\gamma(B, \mathrm{d}\mu)$ and $1 \leqslant \sigma \leqslant p(x) \leqslant \gamma$ for $x \in B$. Then, there exists a constant $c > 0$ such that*

$$\int_B |f|^{p(x)}\,\mathrm{d}\mu \geqslant c \min\biggl\{\biggl(\int_B |f|\,\mathrm{d}\mu\biggr)^{\sigma}, \biggl(\int_B |f|\,\mathrm{d}\mu\biggr)^{\gamma}\biggr\}.$$

Proof. Following [11], we consider the space

$$L^{p(x)}(B, \mathrm{d}\mu) = \biggl\{g \text{ measurable}\colon \int_B |g(x)|^{p(x)}\,\mathrm{d}\mu < \infty\biggr\}$$

with the norm

$$\|g\|_* \equiv \|g\|_{L^{p(x)}(B,\mathrm{d}\mu)} = \inf\biggl\{\tau > 0\colon \int_B \left|\frac{g(x)}{\tau}\right|^{p(x)}\mathrm{d}\mu \leqslant 1\biggr\}.$$

The condition on f guarantees $f \in L^{p(x)}(B, \mathrm{d}\mu)$. It is easily verified that

$$\int_B |f|^{p(x)}\,\mathrm{d}\mu \geqslant \begin{cases} \|f\|_*^{\sigma} & \text{if } \|f\|_* \geqslant 1, \\ \|f\|_*^{\gamma} & \text{if } \|f\|_* \leqslant 1. \end{cases}$$

On the other hand, an inequality of Hölder type holds in the space defined above, so we also have

$$\int_B |f|\,\mathrm{d}\mu \leqslant c_1 \|f\|_*.$$

The constant c_1 is explicit in terms of the bounds on $p(x)$, and it satisfies $c_1 < 2$ [11]. We have also used the fact that $\|1\|_* = 1$. Therefore, we have proved the statement with $c = (\frac{1}{2})^\gamma$. □

Two useful consequences are given next.

COROLLARY 3.2. *In the above hypotheses we have*

(i) $$\int_B |f|^{p(x)}\,\mathrm{d}\mu \geqslant c \min\biggl\{\biggl(\int_B |f|^{\sigma}\,\mathrm{d}\mu\biggr), \biggl(\int_B |f|^{\sigma}\,\mathrm{d}\mu\biggr)^{\gamma/\sigma}\biggr\},$$

(ii) $$\int_B |f|\,\mathrm{d}\mu \geqslant 1 \implies \int_B |f|^{p(x)}\,\mathrm{d}\mu \geqslant c\biggl(\int_B |f|\,\mathrm{d}\mu\biggr)^{\sigma}.$$

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

We are now in a position to reproduce the classical Kaplan argument for blow-up [21], and proceed with the proof of theorem 1.1.

THEOREM 3.3. *If there exists some ball $B \subset \Omega$ in which the exponent function satisfies $p(x) \geqslant \sigma > 1$, then there exist solutions to problem (1.1) (with (1.3) in a bounded domain) that blow up in finite time.*

On the other hand, if $p(x) \leqslant 1$ everywhere, then every non-trivial solution to problem (1.1) (with (1.3) in a bounded domain) is globally defined.

Proof. Take φ the first eigenfunction of $-\Delta$ with Dirichlet boundary conditions in B (with eigenvalue λ), normalized this time to have integral 1. Let

$$J(t) = \int_B u\varphi.$$

From (1.1) and the above corollary we have

$$J' = \int_\Omega u\Delta\varphi + \int_\Omega u^{p(x)}\varphi \geqslant -\lambda J + cJ^\sigma$$

whenever $J(t) \geqslant 1$. Clearly, this implies blow-up, provided that $J(0)$ is large. That is, if the initial datum is such that $J(0) \geqslant \max\{1, (2^\sigma\lambda)^{1/(\sigma-1)}\}$, then the solution blows up.

To see that solutions are global when $p(x) \leqslant 1$ in the whole Ω, it suffices to observe that the function

$$w(t) = \|u_0\|_\infty \mathrm{e}^t$$

is a strictly positive supersolution to (1.1). Hence, for any $t_0 > 0$ the maximal solution to the problem is bounded in $\mathbb{R}^N \times [0, t_0]$ and then it is global. Observe that, to use comparison arguments, it is crucial that w is strictly positive. Therefore, any solution is global. □

We next consider the so-called Fujita phenomenon. First we treat the standard case of the problem posed in the whole space.

3.1. The case when $\Omega = \mathbb{R}^N$

THEOREM 3.4. *If $p_- > 1+2/N$, then (1.1) with $\Omega = \mathbb{R}^N$ possesses global solutions.*

Proof. We only have to consider as a supersolution a global solution to the problem with constant reaction exponent p_- such that it lies always below 1 [14]. □

THEOREM 3.5. *If $1 < p_- \leqslant p_+ \leqslant 1+2/N$, then all solutions to (1.1) with $\Omega = \mathbb{R}^N$ blow up in finite time.*

Proof. Again, the proof follows the classical methods of Fujita and Weissler [14, 31] for the constant exponent case, once we have established Jensen's inequality (corollary 3.2). Assume first then that $1 < p_- \leqslant p_+ < 1 + 2/N$. Applying Kaplan's method with ϕ replaced by $\phi_\mu(x) = \mu^N\phi_1(\mu x)$, where ϕ_1 is any non-negative function satisfying

$$\int_{\mathbb{R}^N} \phi_1 = 1 \quad \text{and} \quad \Delta\phi_1 + \phi_1 \geqslant 0 \quad \text{in } \mathbb{R}^N$$

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

(for instance, a Gaussian), and using lemma 3.1 we have

$$J' \geqslant -\mu^2 J + c\min\{J^{p_-}, J^{p^+}\}. \tag{3.1}$$

Now take $\mu > 0$ small enough such that $J(0) > (\mu^2/c)^{1/(p_+-1)}$. This is possible because of the restriction on p_+. In fact,

$$\int_{\mathbb{R}^N} \mu^N \phi_1(\mu x) u_0(x)\,\mathrm{d}x = J(0) > \left(\frac{\mu^2}{c}\right)^{1/(p_+-1)}$$

means that

$$\int_{\mathbb{R}^N} \phi_1(\mu x) u_0(x)\,\mathrm{d}x > c^{-1/(p_+-1)}\mu^{2/(p_+-1)-N} \to 0.$$

Now, whenever $J(t)$ is small, (3.1) implies

$$J' \geqslant -\mu^2 J + cJ^{p^+} > \tilde{c}J^{p^+}.$$

This gives that $J(t)$ increases and for times $t > t_1$, where t_1 is such that $J(t_1) = 1$, the inequality (3.1) becomes

$$J' \geqslant -\mu^2 J + cJ^{p^-} > \tilde{c}J^{p^-}.$$

This implies blow-up, since $p_- > 1$.

Now, let $p_+ = 1 + 2/N$. Here it suffices to show that the integral

$$\int_{\mathbb{R}^N} u(\cdot, t)\,\mathrm{d}x$$

is large for t large. Without loss of generality we may assume, as in [31], that the initial datum is above some Gaussian,

$$u_0(x) \geqslant cG_\varepsilon(x) = c(4\pi\varepsilon)^{-N/2}\exp\left(-\frac{|x|^2}{4\varepsilon}\right).$$

Therefore, $u(x,t) \geqslant cG_{t+\varepsilon}(x)$ for every $t \geqslant 0$. Now, by the integral representation of the solution of the semilinear equation, we have

$$u(x,t) \geqslant c\int_0^t G_{t-s} * u^{p(x)}(x,s)\,\mathrm{d}s \geqslant c\int_0^t G_{t-s} * G_{s+\varepsilon}^{p(x)}(x)\,\mathrm{d}s.$$

Integrating in $\mathbb{R}^N$ and using corollary 3.2 we obtain

$$\int_{\mathbb{R}^N} u(x,t)\,\mathrm{d}x \geqslant c\int_0^t \min\left\{\left(\int_{\mathbb{R}^N} G_{s+\varepsilon}^{p_-}(x)\,\mathrm{d}x\right), \left(\int_{\mathbb{R}^N} G_{s+\varepsilon}^{p_-}(x)\,\mathrm{d}x\right)^{p_+/p_-}\right\}\mathrm{d}s.$$

But a simple computation shows that

$$\int_{\mathbb{R}^N} cG_{s+\varepsilon}^{p_-}(x)\,\mathrm{d}x = c\int_{\mathbb{R}^N} (s+\varepsilon)^{(1-p_-)N/2} G_{(s+\varepsilon)/p_-}(x)\,\mathrm{d}x = c(s+\varepsilon)^{(1-p_-)N/2}.$$

Since $p_- \leqslant 1 + 2/N$, we have $(1-p_-)N/2 \geqslant -1$, and thus the integral of the latter term diverges as $t \to \infty$. □

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

Next we show some examples for the intermediate case, that is, for functions $p(x)$ with $1 < p_- < 1 + 2/N < p_+$. In the first one every solution blows up and in the second one there are global solutions.

Example 3.6. Consider (1.1) in $\mathbb{R}$ and let $p(x)$ be any function such that $p(x)$ agrees with its minimum, which we fix between 1 and 2, in the half-line,

$$p(x) = p_- \in (1, 2) \quad \text{for } x \geqslant 0.$$

In this example, we construct a subsolution with finite time blow-up. First, we note that the solution u to (1.1) is positive for all $t > 0$. Therefore, we can take a non-trivial function $w_0(x) \leqslant u(x, t_0)$ with $w_0(0) = 0$.

Now we consider the following problem:

$$\begin{aligned} w_t &= w_{xx} + w^{p_-} && \text{in } \mathbb{R}^+ \times (0, \tilde{T}), \\ w(0, t) &= 0 && \text{in } (0, \tilde{T}), \\ w(x, 0) &= w_0(x) && \text{in } \mathbb{R}^+. \end{aligned}$$

It is clear from the above that u is a supersolution to this problem. On the other hand, it is known (see [23]) that for this problem the blow-up and the Fujita exponents are given by $p_{\mathrm{b}} = 1$ and $p_{\mathrm{F}} = 2$, respectively.

Therefore, any solution to our problem with the chosen reaction exponent blows up.

Example 3.7. First, we consider a discontinuous exponent. We take

$$r(x) = \begin{cases} p_+, & |x| > R, \\ p_-, & |x| \leqslant R, \end{cases}$$

where $p_+ > N/(N-2) > 1 + 2/N > p_- > 1$.

In this case, we construct a stationary supersolution. In the region $|x| > R$ we consider the explicit radial solution

$$u(r) = cr^{-\alpha}, \quad \alpha = \frac{2}{p_+ - 1}, \quad c = (\alpha(N - 2 - \alpha))^{1/(p_+ - 1)}.$$

In the inner region we consider a radial solution of

$$\begin{aligned} \Delta v + v^{p_-} &= 0, && x \in B_R(0), \\ v &= cR^{-\alpha}, && x \in \partial B_R(0). \end{aligned}$$

The existence of such a v is equivalent to the existence of a positive solution to

$$\begin{aligned} \Delta w + (w + cR^{-\alpha})^{p_-} &= 0, && x \in B_R(0), \\ w &= 0, && x \in \partial B_R(0), \end{aligned}$$

which can be obtained using a mountain-pass argument (see, for example, [2]) considering the functional

$$L(u) = \frac{1}{2} \int_{B_R(0)} |\nabla w|^2 - \frac{1}{p_- + 1} \int_{B_R(0)} (w + cR^{-\alpha})^{p_- + 1}$$

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

in $H_0^1(B_R(0))$. Note that p_- is subcritical and hence we have compactness of the inclusion $H_0^1(B_R(0)) \hookrightarrow L^{p_-+1}(B_R(0))$.

Notice that the function

$$\bar{U}(x) = \begin{cases} v(|x|), & x \in B_R(0), \\ u(|x|), & x \in \mathbb{R}^N \setminus B_R(0), \end{cases}$$

is a supersolution in the whole space if and only if $|v'(R)| \leqslant |u'(R)|$. In order to estimate $v'(r)$, we consider the function

$$w(r) = \frac{v(Rr)}{cR^{-\alpha}},$$

which solves the problem

$$\begin{aligned} \Delta w + c^{p-1} R^{2+\alpha(1-p_-)} w^p &= 0, \quad x \in B_1(0), \\ w &= 1, \quad x \in \partial B_1(0). \end{aligned}$$

Observe that for $R = 0$ we obtain the constant solution $w(r) = 1$. It is easy to check that for R small enough we have

$$w(r) = 1 + o(1) \quad \text{and} \quad w'(r) = o(1).$$

This gives us $v'(r) = o(R^{-\alpha-1})$. On the other hand, u verifies that $u'(R) = -c\alpha R^{-\alpha-1}$; then, taking R small enough, the function $\bar{U}$ is a supersolution to our problem with $r(x)$ as exponent.

Now, we want to modify $r(x)$ to obtain a continuous exponent $p(x)$ such that $\bar{U}$ is still a supersolution to the problem with $p(x)$.

Since R is small and $\bar{U}|_{\partial B_R(0)} = cR^{-\alpha}$ there exists a small $\delta > 0$ such that $\bar{U} > 1$ in the annulus $B_{R+\delta}(0) \setminus B_R(0)$. Now, we just consider $p(x)$ any continuous function that verifies $p(x) = p_-$ in $B_R(0)$, $p(x) = p_+$ in $\mathbb{R}^N \setminus B_{R+\delta}(0)$ and $p_- \leqslant p(x) \leqslant p_+$ in $B_{R+\delta}(0) \setminus B_R(0)$. We observe that $\bar{U}$ is a supersolution to our problem with $p(x)$. In fact, by our previous calculations, we only have to take care of points in the annulus $B_{R+\delta}(0) \setminus B_R(0)$ and for those points we have

$$\Delta \bar{U}(x) + (U(x))^{p(x)} \leqslant \Delta \bar{U}(x) + (U(x))^{p_+} = \Delta u(x) + (u(x))^{p(x)} = 0.$$

3.2. The case when Ω is bounded

Our next aim is to study the occurrence of a Fujita-type phenomenon in a bounded domain. Actually, we find sufficient conditions ensuring that every solution to problem (1.1)–(1.3) corresponding to a non-trivial non-negative initial datum u_0, with Ω bounded, blows up. Note that this is an important difference with respect to the problem with a constant exponent in the reaction posed in a bounded domain. To build such examples we argue as follows: first we need a large region in which $p(x)$ lies below 1 (this will force the solution to grow in the whole Ω) and a large region where $p(x)$ is above 1 (this is necessary for blow-up to occur; see theorem 1.1).

We begin with a preliminary lemma.

Lemma 3.8. *If there exists some ball $B_R(x_0) \subset \Omega' \subset \Omega$ in which the exponent function satisfies $0 < \sigma \leqslant p(x) \leqslant \gamma < 1$ for $x \in \Omega'$, then any solution to problem*

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

(1.1)–(1.3) verifies for every $x \in B_{R/2}(x_0)$ that $u(x,t) \geqslant cR^{2/(1-\sigma)}$, from some time $t > t_0 > 0$, with $c = c(N, p(x)) > 0$ independent of R.

Proof. Without loss of generality let us suppose that the ball in the hypothesis is centred at the origin.

Since $p(x) \leqslant \gamma < 1$ in Ω', we can apply remark 2.3 to obtain that there exists $\delta > 0$ such that u is a supersolution to the problem

$$\left.\begin{aligned} v_t &= \Delta v + v^{p(x)} && \text{in } B_R(0) \times (1,\infty), \\ v &= \delta && \text{on } \partial B_R(0) \times (1,\infty), \\ v &= \delta && \text{for } t = 1. \end{aligned}\right\} \tag{3.2}$$

Observe that $\underline{w}(x,t) = \delta$ is a subsolution. Therefore, we can replace the reaction term by a Lipschitz continuous function without changing the problem (see (2.1)), and then we have uniqueness and comparison. On the other hand, taking A large, we have that

$$\bar{w}(x,t) = A - A^{\gamma}\frac{|x|^2}{2N}$$

is a supersolution of (3.2). This implies that v is uniformly bounded. Moreover, we have a Lyapunov functional given by

$$F(v) = \frac{1}{2}\int_{B_R(0)} |\nabla v|^2 - \int_{B_R(0)} \frac{|v|^{p(x)+1}}{p(x)+1},$$

which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}F(v)(t) = -\int_{B_R(0)} |v_t|^2(x,t)\,\mathrm{d}x \leqslant 0.$$

We conclude in a rather standard way (see, for example, [4]) that, for every sequence $t_j \to \infty$, we can extract a subsequence, still denoted by t_j, such that

$$\lim_{j\to\infty} v(x,t_j) = V(x)$$

in $L^2(B_R(0))$, where $V(x) \geqslant \delta$ is a stationary solution of (3.2). The uniqueness of the stationary solution follows in the same way as for the case of constant exponent [6]. Therefore, the above limit holds for every sequence of times.

Now, in order to get rid of the dependence upon R, we pass to the unit ball with the change of scales

$$V(x) = R^{2/(1-\sigma)}\tilde{V}(x/R),$$

where $\tilde{V}$ satisfies

$$\left.\begin{aligned} -\Delta\tilde{V} &\geqslant \tilde{V}^{p(Rx)} && \text{in } B_1(0), \\ \tilde{V} &= \delta R^{-2/(1-\sigma)} && \text{on } \partial B_1(0). \end{aligned}\right\} \tag{3.3}$$

By [3, proposition 1], there exists a unique, positive, classical solution ϕ of

$$\left.\begin{aligned} -\Delta\phi &= \phi^{\gamma} && \text{in } B_1(0), \\ \phi &= 0 && \text{on } \partial B_1(0). \end{aligned}\right\} \tag{3.4}$$

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

Set $z = \eta\phi$ with $\eta = \min(1; \|\phi\|_\infty^{-1})$. Then z satisfies $z \leqslant 1$ and $-\Delta z \leqslant z^\gamma$ in $B_1(0)$. We claim that $\tilde{V} \geqslant z$ in $B_1(0)$. Assume the contrary. Arguing similarly to [3, p. 383], we set

$$\tau_0 = \sup\{\tau \geqslant 1;\ \tau\tilde{V} - z \text{ takes some negative values in } B_1(0)\}.$$

Since $\tilde{V} > 0$ in $B_1(0)$ it is clear that $\tau_0 \in (1, \infty)$. Moreover, $w = \tau_0\tilde{V} - z$ is greater than or equal to zero in $B_1(0)$ and attains a null minimum at some point of $B_1(0)$ (note that $w > 0$ on $\partial B_1(0)$). On the other hand, we have

$$\begin{aligned} -\Delta w &\geqslant \tau_0\tilde{V}^{p(Rx)} - z^\gamma \geqslant (\tau_0 - \tau_0^{p(Rx)})\tilde{V}^{p(Rx)} + (\tau_0\tilde{V})^{p(Rx)} - z^{p(Rx)} \\ &\geqslant (\tau_0 - \tau_0^{p(Rx)})\tilde{V}^{p(Rx)} \\ &> 0 \end{aligned}$$

due to $z \leqslant 1$, $p(Rx) \leqslant \gamma < 1$, $\tau_0 > 1$, $w \geqslant 0$ and $\tilde{V} > 0$. But this contradicts the maximum principle.

Summing up, we get that, for $x \in B_{R/2}(0)$ and $t > t_0$, it holds that

$$u(x,t) \geqslant \tfrac{1}{2}V(x) = \tfrac{1}{2}R^{2/(1-\sigma)}\tilde{V}(x/R) \geqslant \tfrac{1}{2}R^{2/(1-\sigma)} \min_{x \in B_{1/2}(0)} z(x) = cR^{2/(1-\sigma)}.$$

□

We are now ready to state sufficient conditions ensuring blow-up occurrence for every solution to problem (1.1)–(1.3). As stated in § 1, if $p(x) < 1$ in some large set and $p(x) > 1$ in some other set, also large, then every solution blows up. In fact, the first condition, together with lemma 3.8, makes the solution grow, which implies, together with the second condition and theorem 3.3, that the solution blows up. We make this argument rigorous in the next result.

THEOREM 3.9. *Let q be any arbitrary continuous function defined in the unit ball $B_1(0)$ verifying that $q(x) - 1$ changes sign. Then there exist two large positive constants $M > L > 0$ such that if the ball $B_M(x_1) \subset \Omega$ for some x_1, then the solution to the problem (1.1)–(1.3) with the reaction exponent satisfying $p(x) \equiv q((x - x_1)/L)$ for every $x \in B_L(x_1)$ blows up in finite time for any non-trivial non-negative initial datum u_0.*

Proof. Since $q(x) - 1$ changes the sign, there exist two balls in $B_1(0)$ such that $q(x) - 1$ has different signs in each of them. For simplicity, we assume that $q(x) - 1 < 0$ in a ball centred at $x = 0$. This allows us to choose two large constants $R_1, R_2 < L$ such that

(i) $0 < \sigma \leqslant p(x) \leqslant \gamma < 1$ for every $x \in B_{R_1}(x_1)$,

(ii) $p(x) \geqslant \mu > 1$ for every $x \in B_{R_2}(x_2) \subset B_L(x_1)$.

The specific sizes of the R_1 and R_2 needed will be made precise later on. From (i) we are working under the hypothesis of lemma 3.8, which yields that for any $t \geqslant t_0$ we have $u(x,t) \geqslant A \equiv cR_1^{2/(1-\sigma)}$ for any $x \in B_{R_1/2}(x_1)$. This implies that u is a

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

supersolution to the following problem:

$$\begin{aligned} \omega_t &= \Delta\omega && \text{in } B_M(x_1)\setminus B_{R_1/2}(x_1),\\ \omega &= 0 && \text{on } \partial B_M(x_1),\\ \omega &= A && \text{on } \partial B_{R_1/2}(x_1),\\ \omega(x,0) &= u(x,t_0) && \text{in } B_M(x_1)\setminus B_{R_1/2}(x_1). \end{aligned}$$

By classical theory, we know that ω converges uniformly to the unique stationary solution given by

$$r(x) = A\frac{\Gamma(M)-\Gamma(|x-x_1|)}{\Gamma(M)-\Gamma(R_1/2)},$$

where Γ is the fundamental solution of the Laplacian. Then, there exist $t_1 > t_0$ and M large enough (in fact, $M - L$ large) such that for all $x \in B_L(x_1)\setminus B_{R_1/2}(x_1)$ we have

$$u(x,t) \geqslant \omega(x,t) \geqslant r(x) - \varepsilon \geqslant \tfrac{1}{2}A - \varepsilon.$$

This means that we can take R_1 large (which means A is large) in order to get $u(x,t) \geqslant 2$ in the whole ball $B_{R_2}(x_2)$. We have now reached an appropriate point to give the precise meaning of R_2 being large. If we take a look at the proof of theorem 3.3, we observe that, defining

$$J(t) = \int_{B_{R_2}(x_2)} \varphi_1 u \, \mathrm{d}x,$$

where φ_1 is the first eigenfunction of the Laplacian in $B_{R_2}(x_2)$, normalized to have integral 1, a sufficient condition to have blow-up in finite time is given by

$$J(t) > \max\{1, (2^\mu\lambda_1)^{1/(\mu-1)}\} \tag{3.5}$$

for some $t \geqslant 0$, where λ_1 is the first eigenvalue associated to φ_1. Since the above calculations imply $J(t_1) \geqslant 2$, the blow-up condition (3.5) is achieved by taking R_2 large enough in order to get λ_1 small. Indeed $\lambda_1 < \frac{1}{2}$ is sufficiently small. This completes the proof. □

Now we prove that, when Ω is contained in a small ball, there are global solutions regardless of the size of $p(x)$.

THEOREM 3.10. *If there exists some ball $B_r(x_0) \supset \Omega$ with $r < \sqrt{2N}$, then there are global solutions to (1.1) with Dirichlet boundary conditions (1.3) for every $p(x) \geqslant 0$.*

Proof. We only need to observe that the function

$$w(x) = \frac{2N - |x-x_0|^2}{2N}$$

is a supersolution of (1.1). Indeed, since $r < \sqrt{2N}$, we have that $w(x) > 0$ at $\partial\Omega$. Moreover, $w(x) \leqslant 1$; hence, $w(x)^{p(x)} \leqslant 1 = -\Delta w(x)$. □

THEOREM 3.11. *Let $p_- > 1$. Then there is a global non-trivial solution to (1.1)–(1.3).*

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

Proof. Let
$$z(x,t) = \varepsilon \mathrm{e}^{-\lambda t}\varphi_1(x),$$
where φ_1 is the first eigenfunction of the Laplacian in Ω with Dirichlet boundary conditions, normalized with $\max_x \varphi_1(x) = 1$.

Then we have that z is a supersolution, provided that λ and ε are small. In fact, we have
$$z_t(x,t) = -\lambda z(x,t)$$
and
$$\begin{aligned}\Delta z(x,t) + z^{p(x)}(x,t) &= -\lambda_1 z(x,t) + (\varepsilon \mathrm{e}^{-\lambda t}\varphi_1(x))^{p(x)} \\ &\leqslant -\lambda_1 z(x,t) + (\varepsilon \mathrm{e}^{-\lambda t}\varphi_1(x))^{p_-}.\end{aligned}$$
And hence it suffices that
$$\lambda z(x,t) \leqslant \lambda_1 z(x,t) - z^{p_-}(x,t),$$
that is,
$$(\lambda_1 - \lambda) z(x,t) \geqslant z^{p_-}(x,t),$$
which holds on choosing λ and ε small enough, since $p_- > 1$. □

Acknowledgements

R.F., A.d.P. and M.P. are partly supported by Grant no. MTM2008-06326-C02-02 (MEC Spain). R.F. is also partly supported by Grant no. CCG07-UCM/ESP-2393 (UCM-Com. de Madrid). J.D.R. is partly supported by Project no. MTM2004-02223 (MEC, Spain) and by UBA X066 and CONICET, Argentina. We thank the anonymous referee for comments, which helped us to improve the manuscript.

References

1 J. Aguirre and M. Escobedo. A Cauchy problem for $u_t - \Delta u = u^p$ with $0 < p < 1$: asymptotic behaviour of solutions. *Annales Fac. Sci. Toulouse Math.* **8** (1986), 175–203.

2 A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Analysis* **14** (1973), 349–381.

3 D. G. Aronson and L. A. Peletier. Large time behaviour of solutions of the porous medium equation in bounded domains. *J. Diff. Eqns* **39** (1981), 378–412.

4 D. G. Aronson, M. G. Crandall and L. A. Peletier. Stabilization of solutions of a degenerate nonlinear diffusion problem. *Nonlin. Analysis* **6** (1982), 1001–1022.

5 C. Bandle and H. Brunner. Blow-up in diffusion equations: a survey. *J. Computat. Appl. Math.* **97** (1998), 3–22.

6 H. Brézis and L. Oswald. Remarks on sublinear elliptic equations. *Nonlin. Analysis* **10** (1986), 55–64.

7 K. Deng and H. A. Levine. The role of critical exponents in blow-up theorems: the sequel. *J. Math. Analysis Applic.* **243** (2000), 85–126.

8 A. de Pablo and J. L. Vázquez. Balance between strong reaction and slow diffusion. *Commun. PDEs* **15** (1990), 159–183.

9 A. de Pablo and J. L. Vázquez. Travelling waves and finite propagation in a reaction–diffusion equation. *J. Diff. Eqns* **93** (1991), 19–61.

10 M. Escobedo and M. A. Herrero. Semilinear parabolic systems in a bounded domain. *Annali Mat. Pura Appl.* **165** (1993), 315–336.

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press

11 X. Fan and D. Zhao. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Analysis Applic.* **263** (2001), 424–446.

12 R. Ferreira, A. de Pablo and J. L. Vázquez. Classification of blow-up with nonlinear diffusion and localized reaction. *J. Diff. Eqns* **231** (2006), 195–211.

13 A. Friedman. *Partial differential equations of parabolic type* (Englewood Cliffs, NJ: Prentice-Hall, 1964).

14 H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo* **13** (1966), 109–124.

15 H. Fujita and S. Watanabe. On the uniqueness of solutions of initial value problems for some quasi-linear parabolic equations. *Commun. Pure Appl. Math.* **21** (1968), 631–652.

16 V. A. Galaktionov and H. A. Levine. On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary. *Israel J. Math.* **94** (1996), 125–146.

17 V. A. Galaktionov and H. A. Levine. A general approach to critical Fujita exponents in nonlinear parabolic problems. *Nonlin. Analysis* **34** (1998), 1005–1027.

18 V. Galaktionov and J. L. Vázquez. The problem of blow-up in nonlinear parabolic equations. *Discrete Contin. Dynam. Syst.* A **8** (2002), 399–433.

19 K. Hayakawa. On the nonexistence of global solutions of some semilinear parabolic equations. *Proc. Jpn Acad.* A **49** (1973), 503–525.

20 W. Huang, J. Yin and Y. Wang. On critical Fujita exponents for the porous medium equation with a nonlinear boundary condition. *J. Math. Analysis Applic.* **286** (2003), 369–377.

21 S. Kaplan. On the growth of solutions of quasilinear parabolic equations. *Commun. Pure Appl. Math.* **16** (1963), 305–330.

22 H. A. Levine. The role of critical exponents in blowup theorems. *SIAM Rev.* **32** (1990), 262–288.

23 N. Mizoguchi and E. Yanagida. Critical exponents for the blowup of solutions with sign changes in a semilinear parabolic equation. II. *J. Diff. Eqns* **145** (1998), 295–331.

24 J. P. Pinasco. Blow-up for parabolic and hyperbolic problems with variable exponents. *Nonlin. Analysis* **71** (2009), 1094–1099.

25 Y. W. Qi. The critical exponents of parabolic equations and blow-up in $\mathbb{R}^n$. *Proc. R. Soc. Edinb.* A **128** (1998), 123–136.

26 Y. W. Qi and H. A. Levine. The critical exponent of degenerate parabolic systems. *Z. Angew. Math. Phys.* **44** (1993), 249–265.

27 F. Quirós and J. D. Rossi. Blow-up sets and Fujita type curves for a degenerate parabolic system with nonlinear boundary conditions. *Indiana Univ. Math. J.* **50** (2001), 629–654.

28 P. Quittner and P. Souplet. *Superlinear parabolic problems: blow-up, global existence and steady states* (Basel: Birkhäuser, 2007).

29 A. Samarski, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikailov. *Blow-up in quasilinear parabolic equations* (Berlin: Walter de Gruyter, 1995).

30 C. Wang and S. Zheng. Critical Fujita exponents of degenerate and singular parabolic equations. *Proc. R. Soc. Edinb.* A **136** (2006), 415–430.

31 F. B. Weissler. Existence and nonexistence of global solutions for a semilinear heat equation. *Israel J. Math.* **38** (1981), 29–40.

(*Issued 5 October 2012*)

https://doi.org/10.1017/S0308210510000399 Published online by Cambridge University Press