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Entrainment and mixing dynamics of surface-stress-driven stratified flow in a cylinder

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We extend previous work of Boyer, Davies & Guo (*Fluid Dyn. Res.*, vol. 21, 1997, pp. 381–401) to consider the evolution of an initially two-layer stratified fluid in a cylindrical tank which is driven by a horizontal rotating disk. The turbulent motions induced by the disk drive entrainment at the interface, and similarly to the results of Boyer *et al.* (1997), the layer nearer to the disk deepens. Through high-frequency conductivity probe measurements, we establish that the deepening layer is very well-mixed, and the thickness of the interface between the two evolving layers appears to be approximately constant. Under certain circumstances, we find that the rate of increase in depth of the deepening layer decreases with time, at variance with the results of Boyer *et al.* (1997), and implying that the characteristic velocity in the deepening layer decreases as the upper layer deepens. We propose that such time-dependent deepening, and the associated weakening of the upper-layer velocities, occurs naturally because of the combined power requirements of entrainment and layer homogenization which inhibit, when the stratification is very strong, the characteristic velocities of the deepening layer approaching the (constant) velocities of the driving disk, as assumed by Boyer *et al.* (1997).

Key words: stratified flows, stratified turbulence, turbulent mixing

1. Introduction

Turbulent mixing in the presence of shear in a stratified fluid is a key small-scale process in many geophysical situations, and is a particularly important component of the heat budget within the oceans (Wunsch & Ferrari 2004; Ivey, Winters & Koseff 2008). The energy budget of sheared and turbulent stratified flows is subtle, and for parameterizations to be robust it is clearly beneficial to consider the mixing properties of a range of model flows (see the review of Fernando 1991). A particularly important class of flows (relevant for example to the deepening of the oceanic mixed layer) arises when turbulent motions, interacting with large-scale shear, occur in the

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vicinity of a relatively well-defined interface between two layers of different density. Shear-driven turbulence in the upper layer, due to some external forcing from wind, tides and breaking waves then has two distinct, yet inherently interconnected effects. Firstly, the turbulence entrains relatively dense fluid across the interface causing the deepening of the upper layer. Secondly, the turbulence also homogenizes the upper layer, redistributing the newly entrained fluid throughout the upper layer. Both these processes clearly have related energy costs, as the potential energy of the total system inevitably increases (see Peltier & Caulfield 2003, for a discussion).

Indeed, there have been many experimental and theoretical studies of both entrainment and the subsequent homogenization, using a wide variety of forcing mechanisms, as reviewed by Fernando (1991). A key challenge in interpreting experimental measurements and applying them to real flows is the character of the turbulence in the immediate vicinity of the density interface, as that determines the character of the entrainment that actually occurs (see for example Turner 1986). Also important is how that turbulence can be related both to the external forcing, and to the turbulence throughout the interior of the fluid, as it is that interior turbulence which transports the entrained fluid throughout the layer, and homogenizes the density distribution of the layer.

As discussed recently in Woods *et al.* (2010), building on previous work by a wide range of authors (including Turner 1968; Kato & Phillips 1969; Crapper & Linden 1974; Fernando & Long 1988; Guyez, Flor & Hopfinger 2007), there are two different competing mechanisms for shear-driven entrainment. When the density difference across the interface is relatively weak compared to the prevailing shear, overturning at the interface is possible, reminiscent of the classic Kelvin–Helmholtz instability. In this case the entrainment and mixing is 'diffusive' in some sense, and the region over which the density varies substantially has perhaps small yet non-trivial thickness. In such circumstances (as discussed in more detail by Spigel, Imberger & Rayner 1986 for example), this thickness can play a critical role in the flow dynamics. Conversely, in situations where the density jump across the interface is relatively strong, the interface cannot overturn, and a 'scouring' of wisps of fluid into the layer occurs by means of turbulent eddies, impinging vortices and other energetic disordered motions, as discussed in more detail in Linden (1979). Entrainment continues to occur, but the physical process is different.

Modelling of these observed different physical processes has naturally led to a range of different parameterizations. Upward entrainment of relatively dense fluid and subsequent homogenization inevitably leads to an increase in the potential energy of the system, as on average such a process raises the centre of mass of the fluid. One major class of models assumes that the rate of increase of the potential energy associated with the net upward flux of dense fluid is proportional to the cube of some characteristic (turbulent) velocity u_T^3 within the flow, as originally proposed by Turner (1968) to explain grid-turbulence-driven mixing, and reinforced by the observations of Kato & Phillips (1969) in a different stress-driven mixing flow. Equivalently, it can be assumed to be proportional to the dissipation rate of turbulent kinetic energy which, on dimensional grounds, would be expected to scale as u_T^3/l_T for some characteristic turbulent length scale l_T.

Conventionally, this assumption is often posed in terms of an 'entrainment velocity' u_e. In the simplest case of an interface between an upper layer with density $\rho(t)$ and (constant) depth h and a lower layer with density $\rho_0 + \Delta\rho = \rho_L$ (for some reference density ρ_0 where $\rho_0 \geqslant \rho(t)$ and $\rho_0 \gg \Delta\rho$ so that the Boussinesq approximation may be made), the increase in the potential energy per unit area of the upper layer is

$$\frac{\mathrm{d}}{\mathrm{d}t}[g(\rho_L - \rho)h] \equiv u_e g(\rho_L - \rho) = u_e g' \rho_0, \tag{1.1}$$

where g is the acceleration due to gravity, thus defining the reduced gravity g' of the entraining layer. Therefore, if the rate of increase of the potential energy is proportional to u_T^3 (and so independent of the present value of density jump across the interface)

$$u_e g' \propto \frac{u_T^3}{l_T}, \tag{1.2}$$

then the entrainment coefficient E is defined as

$$E \equiv \frac{u_e}{u_T} \propto \frac{u_T^2}{g' l_T} \propto \frac{1}{Ri_T}, \tag{1.3}$$

where the (turbulent) 'Richardson number' quantifies the relative strength of the potential energy to the kinetic energy in the turbulent flow. Such a scaling (see Fernando 1991 for a review) suggests both that the various molecular diffusivities in the flow play no dynamic role, and that the density flux across the interface is independent of the present value of the density jump across that interface, i.e. stratification does not modify the entrainment dynamics. It also implies that the ratio of the density flux to the dissipation rate of turbulent kinetic energy, commonly referred to as the 'flux coefficient' Γ, is close to constant, consistent with the classic modelling assumptions of Osborn (1980). Such behaviour was demonstrated recently by Woods *et al.* (2010) in stratified Taylor–Couette flow, where turbulent mixing in an annular two-layer stratified flow was forced by a rotating inner cylinder, leading typically to 'scouring' in the vicinity of the density interface.

However, particularly for flows with relatively weak stratification subject to the shear-instability-induced overturning dynamics described above, an alternative modelling approach assumes that stratification switches off mixing above a certain threshold value of Richardson number, as presented for example by Pollard, Rhines & Thompson (1973) and Spigel *et al.* (1986) and also embedded in the commonly used 'K-profile parameterization' (KPP) of eddy diffusivities described in Large, McWilliams & Doney (1994). Such a suppression of mixing suggests that the entrainment process should be a stronger (decaying) function of the Richardson number, and there is certainly some evidence to support that argument, as reviewed by Linden (1979), Fernando (1991) and more recently in the context of oceanic overflows in Wells, Cenedese & Caulfield (2010). However, numerical and experimental evidence (see for example Canuto *et al.* 2008) suggests that turbulence, and hence some mixing continues even to arbitrarily high values of characteristic Richardson numbers.

Essentially, the central question is what is an appropriate model for entrainment and homogenization in a stratified fluid as a function of the strength of the overall stratification, shear and turbulence? Though this has been widely considered, both from a fundamental fluid dynamical viewpoint (see Fernando 1991) and an oceanographic viewpoint (see for example Niiler & Kraus 1977; Sherman, Imberger & Corcos 1978; Wunsch & Ferrari 2004; Ferrari & Wunsch 2009), it is fair to say that there is as yet no consensus. A particularly important issue is that if the 'mixing' (more precisely, the horizontally averaged vertical density flux) is a non-monotonic function of the stratification, then Phillips (1972) and Posmentier (1977) established that an initially linear stratification will inevitably become layered, with relatively well-mixed regions being separated by relatively thin regions (or interfaces) of markedly increased density gradient. However, if the mixing is simply a decaying function of overall stratification for sufficiently strong stratification, it is possible to establish that the problem becomes ill-posed, with the 'strength' of the interfacial density gradients being unbounded. Barenblatt *et al.* (1993) demonstrated that this problem can be regularized by assuming that there was a time lag between the dissipation and the mixing within the flow. This idea is appealing, as it seems plausible that thorough homogenization of a fluid with varying density will not be an instantaneous process.

However, there are of course other ways in which a regularization may be achieved. One influential suggestion was made by Balmforth, Llewellyn Smith & Young (1998), who developed a reduced-mixing-length model for stratified mixing forced at some characteristic velocity scale U and length scale d. They assumed that the integral scale of the turbulent motions would reduce from d if the flow was very strongly stratified, and also that, in the absence of stratification, the characteristic turbulent velocity scale would relax towards that of the forcing device, an assumption which they referred to as 'equipartition'. In combination, these assumptions lead to a model in which the vertical density flux increases, decreases, then increases again with overall stratification, thus regularizing the problem such that the density gradient remains bounded. Guyez *et al.* (2007), also considering stratified Taylor–Couette flow but at a somewhat lower Reynolds number than Woods *et al.* (2010), observed such a non-monotonic flux curve with overall stratification with an increase of flux at high stratification. Woods *et al.* (2010) only observed a constant finite value of flux at high overall stratification in a two-layer flow, associated with scouring at the interface. Furthermore, they observed that the typical interfacial thickness remained approximately constant over much of the flow evolution, with an initial sharpening due to establishment of scouring, and a final transient thickening when the stratification became sufficiently weak to allow overturning, similarly to the 'life-cycle' of measurements of Guyez *et al.* (2007). The quasi-steady approximately constant interfacial thickness is suggestive that the flow dynamics is regularized, as the maximum local density gradient at the interface remained bounded.

Another specific experimental geometry with certain attractive features for the study of entrainment and homogenization in a stratified flow is a cylindrical tank, filled with fluid driven by a horizontal (impulsively started and then steadily rotating) disk at one of the boundaries. Boyer, Davies & Guo (1997, referred to herein as BDG97) considered the evolution of an initial two-layer density distribution in such a flow geometry, building on earlier work by Davies *et al.* (1995) who considered a flow with the same forcing applied to a linearly stratified fluid. As noted by BDG97, the flow induced by this forcing is inherently three-dimensional, and extremely complex. There is both a larger-scale circulation within the flow, leading to a radial distribution in the large-scale shear between a developing well-mixed layer and the stationary fluid, and also an active smaller-scale turbulent velocity field. However, the simple experimental measurement of the rate of deepening of the mixed layer is highly useful, as it allows the inference of the properties of the turbulent flow in the mixed layer, which are typically difficult to measure and model. Just to take one example, the modelling and parameterization of the viscous dissipation rate is especially challenging in layered stratified shear flows where there are non-trivial spatial inhomogeneities. A common approach is that presented by Niiler & Kraus (1977), who assumed that the dissipation rate will scale in a way proportional to the various processes which are generating the turbulence. In the flow considered by BDG97, this corresponds to assuming that the properties of the turbulence scale with the disk-driving mechanism.

Indeed, this is essentially what BDG97 assumed, since they supposed that the turbulent velocities characteristic both of the entrainment at the interface and the interior homogenization were constant, and scaled with the characteristic azimuthal velocity ΩR, where Ω is the angular frequency of the disk (measured in rad s^{-1}) of radius R. If the turbulent velocities are indeed constant, it seems at least plausible that the rate of change of the depth of the mixed layer is constant, and BDG97 showed convincing experimental evidence supporting this hypothesis. They hypothesized that the dominant mixing processes were associated with shear-driven overturning at the density interface, yet they typically had relatively few density profiles available to investigate any time-dependence in the variation of the interfacial thickness. Also this hypothesis supposed that larger-scale shear dominated the entrainment and mixing processes through 'overturning' compared to smaller-scale turbulence-induced 'scouring' at the density interface. In the light of the various theoretical models discussed above, their observations are consistent with the central assumption of Turner (1968) and the observations of Kato & Phillips (1969). The characteristic entrainment velocities scale with the characteristic forcing velocities, and the actual density flux across the interface is constant and hence independent of the present value of the density difference between the undisturbed and the (turbulent) mixed layer. This suggests that in this regime there is no variation of the entrainment and mixing properties with stratification, although it is important to appreciate that the flow regime which they considered has sufficiently weak overall stratification such that only overturning dynamics is expected.

From an energy viewpoint however, such a situation of constant increase in depth of the mixed layer cannot continue indefinitely. The power injected by the rotating disk in contact with the mixed layer certainly forces entrainment at the interface between the two layers. However, it also must contribute to the mobilization or agitation of the entrained fluid to the characteristic velocities of the upper layer, to the small-scale motions required to homogenize the mixed layer (and hence to increase the potential energy of the whole system) and to the enhanced viscous dissipation associated with the deepening turbulent layer. It seems reasonable to expect that as the mixed layer deepens, the forcing associated with the rotating disk becomes insufficient to sustain a 'constant' characteristic flow velocity within the mixed layer, as the energy requirement for mobilization of the fluid (and hence the attendant viscous dissipation) continues to increase as the layer increases in depth. Therefore, the entraining velocity should decrease as the mixed layer deepens, leading to a variable and reducing rate of increase of the depth of the mixed layer.

Such a change in the rate of increase does not rely on any change in the entrainment dynamics at the interface, which would be associated with a change in the local interfacial balance between buoyancy and inertia, quantified by some appropriate interfacial Richardson number, Ri_I, defined as

$$Ri_I = \frac{g \Delta\rho d_I}{\rho_0 u_I^2}, \tag{1.4}$$

where, as before g is the acceleration due to gravity, ρ_0 is a reference density, d_I is the thickness of the interface over which there is a density jump of $\Delta\rho$ and u_I is a characteristic velocity of the turbulent mixed layer in the vicinity of the interface with the quiescent layer. Assuming (for simplicity, though it is consistent with the experimental evidence of BDG97, and as presented below) that d_I stays approximately constant, the model of BDG97 implies that Ri_I decreases monotonically, as mixing will continually decrease the density jump $\Delta\rho$, and u_I remains constant. On the other hand, if the characteristic velocities in the mixed layer decrease due to the energy constraints discussed above, it is not immediately clear how Ri_I evolves, and so the implications of any model of such flows for the properties of the interfacial entrainment must be considered carefully.

Therefore, the two primary objectives of this paper are to develop quantitative models for both a flow with a constant rate of change of depth of the mixed layer, and for a flow with a decreasing rate of change of the depth of the mixed layer, and then to compare the predictions of these models to experimental measurements of the flow evolution. Our modelling objective is not to capture every aspect of the flow, but to understand in bulk terms what appear to be the key processes driving the deepening through entrainment and homogenization of the mixed layer. To address these objectives, the rest of the paper is organized as follows. In § 2, we describe the evolution of a typical experiment, focusing on observations of the key flow dynamics, and then develop competing models for the flow evolution. In § 3 we present a quantitative comparison between these competing models and our experimental measurements, showing categorically that the assumption that the entraining velocity is constant cannot always be valid. Finally, in § 4, we draw our conclusions.

2. Experimental observations

A cylindrical tank of total depth 30 cm and radius $R_t = 15$ cm is filled initially with two layers of fluid with density ρ_L and $\rho_U(t)$ and depth $h_L(t)$ and $h_U(t) = H - h_L(t)$, where $H = 27$ cm and the initial (filled) depth of the upper layer h_0 was varied between 6 and 20.6 cm. A horizontal disk of radius $R = 12$ cm was located at the top of the tank, just below the free surface. The angular frequency of the disk varied in the range $1 \leqslant \Omega \leqslant 6$ rad s^{-1}, where Ω is the angular frequency, and remained constant during an experiment. A conductivity probe traversed the full depth of the fluids throughout the entirety of an experiment in the thin gap between the edge of the disk and the tank wall, with a single up or down pass taking approximately 67.5 s. We took measurements at 20 Hz, thus yielding a measurement every 0.1 mm. Because of the inevitable wake effect, we only took measurements on the down stroke of the probe, and so we chose to take profiles at precisely three minute intervals. Therefore, the number of profiles in an experiment ranged between four and 80. The vertical position of the probe was reproducible from one profile to the next, with an error of the order of ± 1 mm.

All experimental fluids were stored in the (temperature-controlled) laboratory for several days before use, and so their temperature was stable throughout every experiment. Also, such storage ensured that the experimental fluids were adequately de-aerated. To calibrate the conductivity probe, we measured the conductivity of fluid samples from the initial lower layer, the initial upper layer, and an equal-volume mixture of these two fluids both before and after the experiment. Typically there was negligible difference between these two calibrations. The densities of these reference samples were determined with a DMA58 Anton Paar densitometer with an accuracy of 10^{-5} g cm^{-3}.

This experimental set-up is somewhat different from that considered by BDG97, in which the driving disk was at the bottom of the tank, but the dynamical behaviour of the flow is very similar. The key large-scale parameter for the flow is the bulk

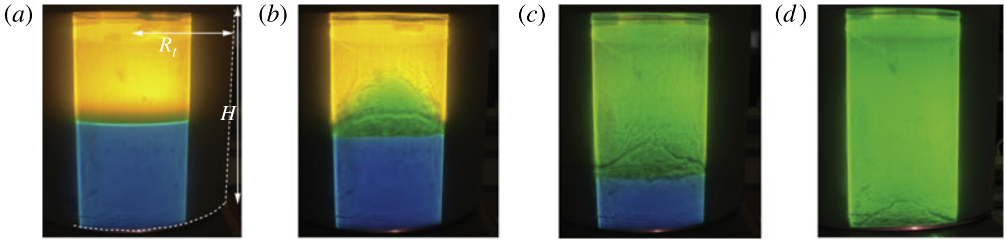

FIGURE 1. Snapshots of an experiment with $\Omega = 3$ rad s^{-1}, $Ri_B = 0.237$ at: (*a*) $\tau = \Omega t = 0$; (*b*) $\tau = 375$; (*c*) $\tau = 2700$; and (*d*) $\tau = 7950$. Note both the doming of the lower layer during the initial spin-up <180 s, and the fact that the upper layer appears to be largely well-mixed by the induced turbulent flow. The dashed white line indicates the bottom right margins of the tank. The illuminated cross-section of the circular tank is distorted and looks narrower than it is in reality.

Richardson number Ri_B defined as

$$Ri_B = \frac{g(\rho_L - \overline{\rho_U})h_U}{\rho_L \Omega^2 R^2} = \frac{g'_U h_U}{\Omega^2 R^2}, \quad \overline{\rho_U} = \frac{1}{h_U}\int_0^{h_U} \rho_U \, \mathrm{d}z, \tag{2.1}$$

where the vertical coordinate z is directed downwards from the free surface at $z = 0$, and $g'_U(t)$ is the (average, though time-dependent) reduced gravity of the upper layer. Although the upper layer depth h_U is also time-dependent, conservation of mass within the tank implies that $g'_U h_U$ is independent of time, and so this quantity is a natural measure of the overall balance between the strength of the density stratification, and the intensity of the forcing. However, Ri_B is not necessarily the best measure of the actual entrainment processes at the density interface, where it is appropriate to consider more local measures of the relative strength of the stratification and the shear (or more generally the potential energy required to overturn the interface and the kinetic energy available in the flow), such as the interfacial Richardson number Ri_I defined above in (1.4).

In figure 1, we show four different stages of a typical experiment, with $Ri_B = 0.237$. The upper layer is dyed yellow, and the lower layer is dyed blue. As described in more detail in BDG97, the spinning disk sets up a large-scale circulation which leads to a radial pressure gradient in the upper-layer fluid which induces a characteristic doming of the lower-layer fluid, as is apparent in figure 1(*b*). There is also clear evidence of shear-driven mixing at the interface between the forced upper layer and the largely quiescent lower layer.

This statement can be made more quantitative by consideration of the evolution of the density profiles measured by the conductivity probe, as shown in figure 2, for an experiment with $Ri_B = 0.53$. In figure 2(*a*), we plot the normalized density $\hat{\rho}(\zeta, \tau)$ against ζ, where

$$\hat{\rho}(\zeta, \tau) = \frac{\rho - \rho_U(0)}{\rho_L - \rho_U(0)}, \quad \zeta = z/H, \quad \tau = \Omega t, \tag{2.2}$$

for eight different times. Over much of the flow evolution, to a very good approximation the upper layer is well-mixed, and the interface between the two layers remains sharp, with the same, very thin, characteristic thickness. Shear-driven entrainment processes lead to an erosion of the lower layer through entrainment

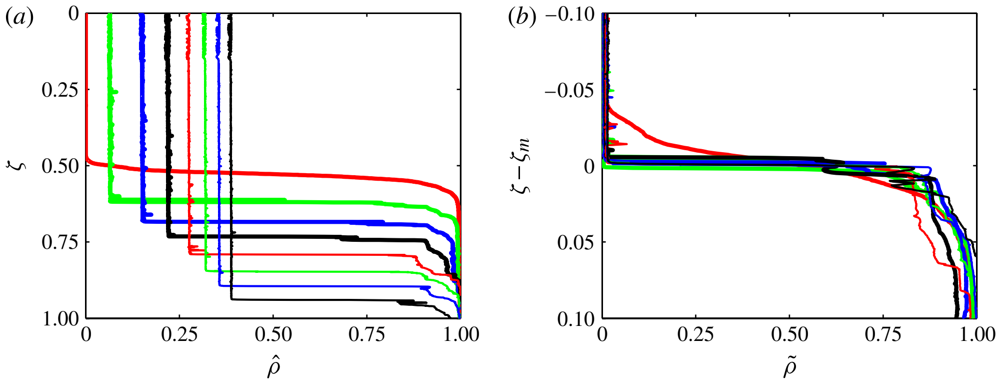

FIGURE 2. Plots of (*a*) $\hat{\rho}$ against ζ as defined in (2.2), and (*b*) $\tilde{\rho}$ against $\zeta - \zeta_m$ as defined in (2.3) for an experiment with $\Omega = 2$ rad s^{-1}, $Ri_B = 0.53$, at times $\tau = 3600n$ for: $n = 0$ (plotted with a thick red line); $n = 1$ (thick green line); $n = 2$ (thick blue line); $n = 3$ (thick black line); $n = 4$ (thin red line); $n = 5$ (thin green line); $n = 6$ (thin blue line); $n = 7$ (thin black line).

of denser fluid into the upper layer in which turbulent motions in turn rapidly homogenize the density to a steadily increasing value.

That the upper layer is well-mixed, and the interface remains sharp is made clear in figure 2(*b*), where we plot an alternatively scaled density $\tilde{\rho}(\zeta, \tau)$ against $\zeta - \zeta_m$, where $\zeta_m(\tau)$ is the height at which $\hat{\rho} = \hat{\rho}_m = 1/2$, i.e. the (constant) mean value of the fluid in the tank, and $\tilde{\rho}$ is defined as

$$\tilde{\rho}(\zeta, \tau) = \frac{\hat{\rho}(\zeta, \tau) - \hat{\rho}(0, \tau)}{\hat{\rho}_L - \hat{\rho}(0, \tau)}, \tag{2.3}$$

where $\hat{\rho}_L = \hat{\rho}(1, \tau) = 1$, and so $0 = \tilde{\rho}(0, \tau) \leqslant \tilde{\rho}(\zeta, \tau) \leqslant \tilde{\rho}(1, \tau) = 1$ for all time.

There are three aspects of these profiles which are important to appreciate. The first is that the upper and lower layers seem to be (to a very good approximation) very close to constant in density. This indicates that, although the profile took 67.5 s to be completed, it is fair to assume that it was 'instantaneous' as there is no evidence of migration of the density in either layer in one profile. Secondly, by comparing in particular the first (thick red) profile from before the experiment started with the subsequent profiles, it is clear that the entrainment and mixing processes lead to a 'sharpening' of the interface as observed by Guyez *et al.* (2007), and also reminiscent of the 'scouring' observed in sufficiently strongly stratified Taylor–Couette flow studied by Woods *et al.* (2010). Thirdly, the interfacial thickness after the experiment started remains very close to constant (noting how figure 2(*b*) is expanded substantially in the vertical).

A major flaw of our experimental procedure is that we only measured the density distribution at a single radial distance (relatively close to the cylinder wall). Therefore, there is no direct measurement of the radial variation of the interfacial thickness. From figure 1, there is clearly variation in the vertical location of the interface, due to the induced radial pressure gradient, as discussed in much more detail in BDG97. However, BDG97 observed little radial variation in the thickness of the (primary) density interface, although the substantial radial variation in the velocity distribution (involving in particular non-trivial vertical velocities in some parts of the flow due to recirculation) suggests a very complicated structure for the interfacial Richardson number Ri_I, if u_I is allowed to depend on radial location as well as time. Consistently with their experiments, we observed Kelvin–Helmholtz-like shear-driven overturning over only some of the interface, with mixing processes near the tank centre and the tank wall being apparently more associated with impinging (or 'scouring') turbulent eddies.

Nevertheless, the fact that the interfacial thickness is approximately constant with time is highly suggestive of the assumption that the interfacial thickness is not a strong function of radial position. Any such radial variation in thickness would be likely to be detected (at least transiently) during the evolution of the flow. Furthermore, this constant interfacial thickness (as observed also by Woods *et al.* 2010) suggests that the overall entrainment processes (which appear to be radially dependent and complex) at the interface between the two layers do not change substantially with time. These observations simplify the modelling of the evolution of the flow substantially compared to, for example, Spigel *et al.* (1986), who tracked carefully the time-dependent evolution of the interfacial region and modelled the large-scale shear-induced mixing separately. In the present experiments, since both overturning and scouring appear to occur simultaneously at different radial locations, a combined, bulk parameterization seems most appropriate.

Therefore, it is reasonable to use the mean reduced gravity g'_U as a measure of the density jump across the interface (since the upper layer is well-mixed) and to suppose that the characteristic length scale of the interface d_I is roughly constant. If the entrainment at the interface is governed by a local measure of the relative strength of the stratification and the shear (in the form of a Richardson number) the natural form for that measure is Ri_I, as defined in (1.4), which can be related to Ri_B by

$$Ri_I = \frac{g'_U d_I}{u_I^2} = Ri_B \frac{d_I \Omega^2 R^2}{h_U u_I^2}, \tag{2.4}$$

where u_I is the characteristic (presumably turbulent) velocity scale at the interface. It is important to appreciate that this velocity scale is some appropriately radially averaged value, as a non-trivial radial variation is expected in the actual time-averaged velocity components.

The essential requirements for a simplified, yet still useful model of the flow may then be reduced to two inter-related questions concerning aspects of Ri_I. Firstly, how does the forcing determine u_I? Secondly, how does mixing, i.e. the entrainment (or equivalently the deepening of the upper layer) and the homogenization of the density distribution, depend on Ri_I, or equivalently the upper-layer depth h_U and the characteristic velocity scale u_I?

2.1. *Constant-velocity 'V' model*

The simplest possible model for the flow evolution is the model proposed in BDG97. They assumed that the characteristic velocity throughout the upper well-mixed layer was constant. We refer to this as the 'V' model, as it postulates constant velocity. Since the bottom layer is stationary the power $\mathscr{P}_I$ supplied at the interface in terms of the interfacial stress σ is (within the Boussinesq approximation)

$$\mathscr{P}_I = \frac{\mathrm{d}\mathscr{W}}{\mathrm{d}t} = \pi R^2 u_I \sigma = \pi R^2 c_D \rho_L u_I^3, \tag{2.5}$$

where c_D is some empirically determined drag coefficient, and $\mathscr{W}$ is the work done by the disk forcing at the interface.

BDG97 further hypothesized that a fixed proportion of the rate of working led to increasing potential energy. For convenience, we define the (total) potential energy

$\mathscr{PE}$ to be zero when $h_U = 0$, and so the tank is completely filled with lower-layer fluid of density ρ_L, and so in general

$$\mathscr{PE} = \pi R^2 \left[-g \int_0^{h_U} \rho_u z \, dz - g \int_{h_U}^{H} \rho_L z \, dz + \frac{g \rho_L H^2}{2} \right] = (\pi R^2 \rho_L) \frac{g'_U h_U^2}{2}, \quad (2.6a)$$

$$\frac{d}{dt} \mathscr{PE} = \lambda \mathscr{P}_I = \lambda \frac{d}{dt} \mathscr{W}, \quad (2.6b)$$

where λ is an empirical constant.

Since, as noted above, $g'_U h_U$ is a constant, (2.6*b*) is effectively an evolution equation for the velocity of the interface,

$$\frac{d}{dt} h_U = (2\lambda c_D) \frac{u_I^3}{g'_U h_U} = (\Omega R)(2\lambda c_D) \frac{1}{Ri_B} \left(\frac{u_I}{\Omega R} \right)^3, \quad (2.7)$$

using the definition for the bulk Richardson number (2.1). We choose to scale the layer depth with its initial value, the time with the angular frequency Ω, and the interfacial velocity u_I with ΩR, and so

$$\hat{h}_U = \frac{h_U}{h_0}, \quad \hat{u}_I = \frac{u_I}{\Omega R}. \quad (2.8)$$

Under the (constant-velocity) assumption that $\hat{u}_I = \beta$ a constant (BDG97 made the simplest assumption that $\beta = 1$), (2.7) is easily integrated to yield

$$\hat{h}_U = 1 + \frac{R}{h_0} \frac{c_V}{Ri_B} \tau, \quad c_V = 2 c_D \lambda \beta^3, \quad (2.9)$$

where c_V is a combination of empirical constants.

The rate of increase in depth of the mixed upper layer is constant, and proportional to $1/Ri_B$. Comparing this to the parameterization of the entrainment coefficient E as defined in (1.3) requires a little care, since in the flow considered in this paper the mixed layer is also increasing in depth, and so the left-hand side of (1.1) is zero by construction. However, there are effectively two fluxes (which in this case precisely cancel) corresponding to a migration flux F_m associated with the fact that the layer is deepening, and an entrainment flux F_e across that interface, i.e.

$$\frac{d}{dt}(g'_U h_U) = 0 = F_m + F_e = g'_U \frac{d}{dt} h_U + h_U \frac{d}{dt} g'_U = g'_U \frac{d}{dt} h_U + g'_U u_e, \quad (2.10)$$

generalizing the definition of the entrainment velocity u_e from (1.1) to the case where the layer depth is varying. Remembering that the entrainment is upwards, and the coordinate system is defined so that the mixed layer increases in depth downwards, we obtain

$$u_e = -\frac{dh_U}{dt} = -(2\lambda c_D) \frac{u_I^3}{g'_U h_U} = (\Omega R)(2\lambda c_D) \frac{1}{Ri_B} \left(\frac{u_I}{\Omega R} \right)^3, \quad (2.11)$$

from (2.7). Since it is assumed that u_I remains constant, u_I must be proportional to u_T as defined in (1.3). Under the reasonable assumption that the dissipation in the evolving layer should scale as u_I^3/h_U, $h_U \propto l_T$, and so $Ri_B \propto Ri_T$. Therefore, the constant-velocity 'V' model does indeed naturally lead to the classical $1/Ri$ scaling originally suggested by Turner (1968) which is commonly encountered scaling for shear-driven turbulent entrainment (see Fernando 1991 for a fuller discussion).

Also, as u_I stays constant while h_U increases, within this model Ri_I decreases as the mixed layer deepens. Interestingly, this variation of Ri_I requires a further assumption for the constant rate of increase of the layer depth defined by (2.9) to remain valid, namely that the entrainment processes at the interface and the subsequent entrainment is actually independent of the value of Ri_I. As discussed in detail by Linden (1979), and mentioned in the Introduction, the dependence of the vertical density flux on some measure of overall stratification (such as Ri_I) is expected to be non-monotonic, with the maximum value of the order of $\Gamma \simeq 0.2$ (where Γ, the flux coefficient, is the ratio of the vertical density flux to the dissipation rate, as defined initially by Osborn (1980)) being associated with an intermediate value of $Ri_I = Ri_M \sim O(1)$, and $\Gamma \rightarrow 0$ for both smaller and larger values of Ri_I on either side of Ri_M.

Therefore, if the initial value of $Ri_I < Ri_M$, conventional models for Γ would appear to suggest that entrainment at the interface should become progressively less efficient, and thus the depth of the mixed layer should increase more slowly with time. Conversely, if $Ri_I > Ri_M$ initially, Γ should initially increase and then decrease. This non-monotonicity, while leading also to a non-constant rate of increase of the layer depth, may not be immediately apparent, because over time the initial increase and subsequent decrease in Γ may compensate for each other. Since BDG97 principally considered flows where $Ri_B < 1.5$, on the grounds that for such flows the mixing dynamics at the interface appeared to be dominated by shear-driven overturnings (and hence the local value of Ri_I had to be sufficiently small), it is not possible to distinguish between these two possibilities, but caution must be exercised in drawing the strong conclusion that the entrainment process is always the same at the interface as the interfacial Richardson number is likely to vary strongly while u_I stays constant.

There is a range of different mixing processes occurring at the interface at any particular instant, associated both with shear-driven overturning and various smaller-scale turbulent motions, and the combined effect of these different processes may be relatively insensitive to the specific value of Ri_I, which should not be interpreted as a precise measure of local sensitivity to stratified shear instability. Furthermore, the assumption that the density flux across the interface is proportional to u_I^3, and independent of the stratification at the interface, is consistent with the entrainment model and experiments of Turner (1968) and Kato & Phillips (1969). As shown in the recent experiments of Woods *et al.* (2010), such dynamics (where the flux is dependent on u_I^3 and not on the present value of the density jump across the interface) can persist for very strongly stratified flows compared to the parameter regime considered by BDG97.

2.2. *Constant-disk-power 'P' model*

Although this 'V' model is plausible, and consistent with the observed behaviour as reported by BDG97, consideration of the flow energetics suggests that a constant rate of increase of the depth of the upper mixed layer cannot continue for all times. As the layer deepens, more and more power input is required to maintain the flow with a constant characteristic velocity in the upper layer, not least because the total viscous dissipation must increase as the total amount of mobilized fluid (in the upper layer) increases due to entrainment. The only source of power for the total (fluid) system is from the rotating disk, which is driven at constant velocity. Furthermore, the entrainment of relatively dense fluid into the upper layer is only the first stage of the mixing, as the turbulent motions also act to homogenize completely the density of the upper layer. Therefore, it appears that requiring the local fluid velocity throughout the mixed layer to have a constant characteristic velocity requires a continually increasing power from the driving disk, which must ultimately be impossible to supply.

This issue is made more clear by considering an energy equation for the entire deepening upper mixed layer. The kinetic energy $\mathscr{K}_U$ of the upper layer is

$$\mathscr{K}_U = \frac{1}{2}\int_0^{h_U}\int_0^{2\pi}\int_0^R \rho|\boldsymbol{u}(r,\theta,z,t)|^2\, r\,\mathrm{d}r\,\mathrm{d}\theta\mathrm{d}z = \frac{1}{2}\rho_L \pi R^2 h_U u_U^2, \tag{2.12}$$

using cylindrical polar coordinates, the constant lower-layer density ρ_L as a reference density within the Boussinesq approximation, and thus defining u_U as a characteristic integral velocity scale of the inherently turbulent motions within the upper layer, as evidenced by the very efficient homogenization of the density within that layer. Although as shown by BDG97 there are significant radial and vertical velocities, our averaging implicitly defines a characteristic velocity u_U constant across each horizontal plane of cross-sectional area πR^2 for a layer of depth h_U.

The entire layer is turbulent, and the total dissipation of kinetic energy by those turbulent motions must depend on the characteristic velocity of the fluid motions, as well as the total amount of fluid which is in the upper layer. Therefore, the simplest scaling argument is that the turbulent dissipation should scale like the (local) kinetic energy, and so the total dissipation throughout the layer is $C_\epsilon \rho_L \pi R^2 (u_U^2 h_U/l_T)^{3/2}$ where l_T is, as before, an integral length scale of the turbulence (see for example Ivey & Imberger 1991) and C_ϵ is an empirical constant. Naturally, the actual dissipation rate varies strongly with space and time, and a full description of the flow would require integration over the whole flow domain. However, as argued by Niiler & Kraus (1977), the most natural assumption is to suppose that the (total) dissipation rate is 'composed of terms which are individually proportional to the active turbulent generating processes', and so we choose this scaling as we believe the dominant 'turbulent generating process' is the disordered motion of the deepening mixed layer itself.

Turning attention to the stratified mixing within the layer, the characteristic time scale for the combination of entrainment and homogenization to occur should be given by the reduced gravity and depth of the upper layer, i.e. $t_\rho = \sqrt{h_U/g'_U}$. It seems straightforward that the homogenization time should increase with the depth of the upper layer, as thorough mixing requires the transport by turbulence of relatively dense fluid (entrained at the interface) completely throughout the interior. It is not however immediately obvious that the time scale should increase with decreasing g'_U, because it would appear sensible to say that entrainment would be easier with smaller values of the density difference across the interface. However, entrainment is only part of the flow dynamics, and indeed being a relatively fast part, associated typically with either scouring of the density interface by strong vortices, or overturning at the interface associated with stratified shear flow instability (see Linden 1979; Fernando 1991; Woods *et al.* 2010 for a more detailed discussion).

The dominant component of the stratified mixing, lagging somewhat behind this initial entrainment process (as postulated by Barenblatt *et al.* 1993) is the small-scale irreversible mixing leading to homogenization of the stratified fluid in the upper layer. This thorough (and particularly efficient) mixing by overturning vortices appears to be most strongly associated with instances when relatively dense fluid is over light fluid (see e.g. Caulfield & Peltier 2000) and can indeed be shown to occur more rapidly when the overall density difference is larger, as demonstrated recently in the model problem of high-aspect-ratio Rayleigh–Taylor instability by Dalziel *et al.* (2008). It is important to appreciate that this homogenization is not localized in the vicinity of the interface, but rather extends throughout the entire mixed layer, as evidenced by the fact that the measured density distribution is very close to uniform throughout the upper mixed layer. Therefore, it seems reasonable to assume that the total amount of power $\mathscr{M}_S$ required to drive both the continual entrainment and homogenization of the upper layer should scale with $\mathscr{P}\mathscr{E}/t_\rho$.

Of course, t_ρ is not the only plausible time scale which can be constructed to describe the entrainment and homogenization of the upper layer. For example, another plausible choice is $t_{\rho u} = h_U^2 g'_U/u_U^3$. This time scale assumes that it is the characteristic velocity of the upper layer that plays a fundamental role in the homogenization process. However from the definition of the potential energy $\mathscr{P}\mathscr{E}$ (2.6*a*), choosing $t_{\rho u}$ implies that the stratified mixing power demand $\mathscr{M}_S = \mathscr{P}\mathscr{E}/t_{\rho u}$ is independent of both the reduced gravity and the depth of the upper layer, and depends only on the velocity of the upper layer, which seems inconsistent with our observations, and indeed our physical intuition.

To reiterate for clarity then, in this flow, we consider 'stratified mixing' to involve both the entrainment of new dense fluid into the upper layer, and the homogenization of the upper layer, distributing this entrained fluid evenly, and we expect this homogenization to dominate both in terms of power consumption and time scale. Fundamentally, this means that the stratified mixing dynamics of our experiments is not dominated by the interfacial dynamics, but rather by the non-local homogenization of the density field throughout the mixed layer. The rotating disk forces this mixed layer at the upper surface, with a constant angular frequency Ω, which leads to a forcing power injection $\mathscr{P}_f$. This power must be balanced by the three components described above, namely the rate of increase of the kinetic energy, $\mathscr{I}_K$, the viscous dissipation of the kinetic energy, $\mathscr{D}_K$, and the stratified mixing power demand $\mathscr{M}_S$, and so

$$\begin{aligned}\mathscr{P}_f = & \quad \mathscr{I}_K \quad + \mathscr{D}_K \quad + \mathscr{M}_S \\ = & \rho_L \pi R^2 \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u_U^2 h_U}{2}\right) + C_l\,(u_U^2 h_U)^{3/2} + C_P\frac{(g'_U h_U)^{3/2}}{2}\right],\end{aligned} \tag{2.13}$$

where C_l and C_P are empirically determined constants associated with $\mathscr{D}_K$ and $\mathscr{M}_S$ respectively, and hence $C_l = C_\epsilon/l_T^{3/2}$. Indeed, this scaling for the dissipation rate $\mathscr{D}_K$ is the natural generalization to a cylindrical geometry of that used in the more complicated model due to Spigel *et al.* (1986). It is important to note that our model (unlike for example that presented by Pollard *et al.* 1973) does not assume *a priori* any balance between the rate of change of the turbulent kinetic energy and the dissipation rate within the flow.

Since $g'_U h_U$ is constant, within this simple model, the stratified mixing requires a steady power input of energy. Also, to maintain the characteristic velocity in the ever-deepening layer at a constant value, as assumed in BDG97, it is apparent that the kinetic energy of the layer must then increase linearly with time, which also requires a steady rate of power input. (As an aside, even if we chose to use the alternative time scale $t_{\rho u}$ in our definition of $\mathscr{M}_S$, the assumption that u_U stays constant would also imply that $\mathscr{M}_S$ remained constant.) However, as the kinetic energy increases, the associated turbulent dissipation increases too, as it is completely natural that it takes more power input to spin up completely a larger volume of fluid, and so unless $\mathscr{P}_f$ increases continually (within this model like $h_U^{3/2}$), eventually it becomes impossible to maintain the velocity of the ever-deepening layer at a constant value. Therefore, the BDG97 model does not seem to be appropriate for all times, particularly as the layer deepens substantially. If the characteristic velocity of the mixed layer u_U does not stay constant, the velocity u_I at the interface driving the entrainment definitely is not constant, and so from (2.7) the rate of increase in the mixed-layer depth is not constant. This observation is also consistent with the modelling approach of Balmforth *et al.* (1998), as their 'equipartition' is predicated on the assumption that stratified mixing will lead to a mismatch between the characteristic forcing velocity and the characteristic velocity of the turbulent motions.

Indeed, under the simplest assumption that the power input $\mathscr{P}_f$ is a constant, there is a more general possible behaviour of the flow, which for sufficiently small times agrees with the 'V' model discussed above. We shall refer to the model based around this assumption as the 'P' model (for constant power). As the upper layer deepens, it must eventually become impossible to sustain the constant rate of increase of the layer's kinetic energy, and so $\mathscr{I}_K$ reduces towards zero. Therefore, the rate of increase of the turbulent dissipation also decreases, until the flow comes into a (stable) balance. The kinetic energy of the mixed layer is constant, and thus the dissipation is also constant, although not in precise balance, since there is always a non-zero stratified mixing power demand $\mathscr{M}_S$. Indeed, the steady forcing balances both the dissipation and the overall stratified mixing, and $u_U^2 h_U$ approaches a constant value, which we choose to define by

$$u_U^2 h_U \rightarrow \varOmega^2 R^2 h_0 C_\infty^2, \quad \hat{u}_U \rightarrow \frac{C_\infty}{\hat{h}_U^{1/2}}, \tag{2.14}$$

for some constant C_∞, using the natural scales for the velocity and the layer depth. Within this 'P' model, as the layer deepens, the characteristic velocity reduces in a particularly simple way. This has a very interesting implication for the time evolution of the layer depth, particularly if it is assumed that the velocity u_I at the interface can be linearly related to the characteristic velocity $u_I = C_I u_U$ for some constant C_I. Therefore, (2.7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} h_U = (2\varGamma c_D C_I^3) \frac{u_U^3}{g'_U h_U} = (\varOmega R)(2 c_D \varGamma C_I^3 C_\infty^3) \frac{1}{Ri_B} \hat{h}_U^{-3/2}, \tag{2.15}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \hat{h}_U = \frac{R}{h_0} c_P \frac{1}{Ri_B} \hat{h}_U^{-3/2}, \quad \hat{h}_U = \left(1 + \frac{5}{2} \frac{R}{h_0} \frac{c_P}{Ri_B} \tau\right)^{2/5}, \tag{2.16}$$

and so the rate of increase of the depth of the well-mixed layer decreases as time progresses. Analogously to before c_P is a combined empirical constant.

The total power requirement for the two aspects of stratified mixing (entrainment and homogenization) is constant within this 'P' model, but since the layer is deepening, the power required for homogenization is increasing, and so there is less available to drive entrainment, manifested by the decreasing velocity at the interface. However, since the local kinetic energy density (per unit area) $u_I^2 h_U$ at the interface remains constant within this model, the interfacial Richardson number Ri_I defined by (2.4) remains constant. Therefore, although it is important to remember that the interfacial Richardson number is effectively a characteristic value of the local balance between stratification and shear in the vicinity of the interface, it is reasonable to suppose that the overall properties of the entrainment and mixing dynamics at

h_0 (cm)	Ω (rad s^{-1})	g'_0 (cm s^{-2})	Ri_B	$c_V = c_P$ ($\times 10^5$)	Symbol
13.5	4	22.7	0.130	2.93	+
8.7	3	22.7	0.150	5.62	○
13.5	5.4	73.2	0.234	2.23	∗
13.5	3	22.7	0.237	2.71	×
6.1	2	22.7	0.241	6.97	□
18.5	3	17.9	0.260	2.47	◇
17.8	3	22.6	0.310	3.38	△
13.5	2	22.7	0.530	2.68	▽
20.6	1.8	22.7	1.000	1.34	▷
13.5	1	22.7	2.100	3.09	◁

TABLE 1. Experimental parameters.

the interface remain similar as the layer deepens. This supposition is more firmly grounded within this model than with the model originally presented in BDG97, which has an implied variation of Ri_I. Indeed, we now have two models with distinguishable predictions for how the upper-layer depth should vary with time, and thus what is implied for the time evolution of the characteristic shear velocity u_I at the interface.

3. Experimental results

The best way to distinguish between the two models is to consider the results of a sequence of laboratory experiments. Ten different experiments were conducted for a range of initial layer depths, initial density differences, and disk angular frequencies. For simplicity, we typically used filtered sea water as the dense-layer fluid, and fresh water for the light-layer fluid, and so $g'_0 = 22.7$ cm s^{-2} in most experiments. As already noted, we stored experimental fluids in the temperature-controlled laboratory for several days so that the temperature of the fluids played no significant role in the flow dynamics. The range of bulk Richardson numbers varied from 0.13 to 2.1, with all but one being less than 1.5, the parameter regime where BDG97 postulated that the mixing had the same typical character, dominated by shear-driven overturnings. As already noted, the flow properties were measured by continual traces of the vertical profiles of the density, and unsurprisingly, this confirmed that $g'_U h_U$ remained constant with time. We list the various flow parameters in table 1.

To compare the quality of the two models, we therefore plot the time dependence of $\hat{h}_U$ in figure 3 for the ten different experiments. Each experiment is represented with a different symbol, as listed in table 1. On these axes, the uncertainty in the location of the interface (due to possible alignment issues with our traverse mechanism) and the time measurement (due to the finite duration of each down pass by the probe) is substantially smaller than the typical sizes of the symbols we have used. The key empirical constant c_V for the V-model is determined from the initial slope of the experimental data (appropriately scaled by the initial aspect ratio R/h_0 and the bulk Richardson number Ri_B), assuming that the initial (relatively rapid) spin-up of the upper layer (see BDG97 for a more detailed discussion) has already occurred, and that the layer is thus deepening at a constant rate at least initially. For each experiment, the initial spin-up of the fluid in the upper layer, after which shearing and entrainment of the fluid at the interface occurred, was assumed to be completed after three minutes. This time scale typically involved multiple rotations of the disk, so that the upper layer

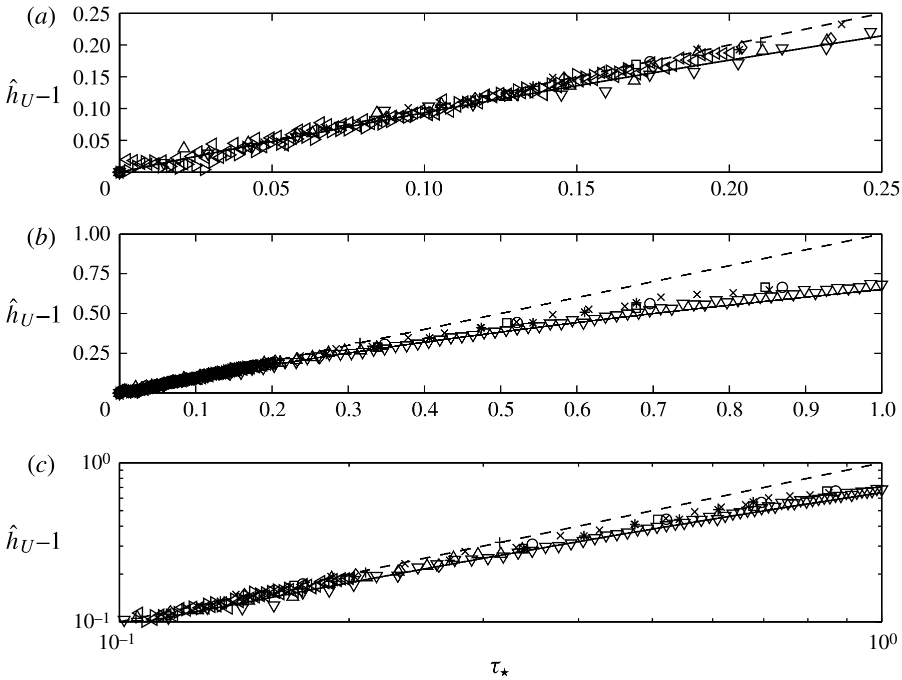

FIGURE 3. Plots of experimental measurements (using symbols as listed in table 1) of $\hat{h}_U - 1$ against $\tau_\star$ (as defined in (3.1)) compared with the prediction of the 'V' model as defined in (2.9) (plotted with a dashed line) and the prediction of the 'P' model as defined in (2.16) (solid line) for: (*a*) small values of $\tau_\star$; (*b*) large values of $\tau_\star$; (*c*) large values of $\tau_\star$ with logarithmic axes.

was adequately energised, turbulence was well-developed, and (at least quasi-) steady. We therefore define the time origin as three minutes after we started driving the disk. (Typically, this meant that $h_U(0) > h_0$, and $g'_U(0) < g'_0$ by some small amount, such that nevertheless $g'_U(0)h_U(0) = g'_0 h_0$, and hence Ri_B took the value as listed in table 1.) Analogously, we determine the constant c_P by assuming the flow is at least initially evolving consistently with the small-t leading-order form of (2.16), which implies that $c_P = c_V$. For these experiments, the constant $c_V = c_P$ varied somewhat, particularly with initial layer depth. The value of this empirically determined constant is also listed in the table.

To plot all the experiments on one figure, we use the rescaled time variable $\tau_\star$, defined as

$$\tau_\star = \frac{R}{h_0}\frac{c_V}{Ri_B}\tau, \tag{3.1}$$

Therefore, the 'V' and the 'P' models (remembering that $c_V = c_P$) as defined in (2.9) and (2.16) reduce to

$$\hat{h}_U = 1 + \tau_\star, \quad \hat{h}_U = (1 + 5\tau_\star/2)^{2/5}, \tag{3.2}$$

respectively, and we plot these two models with a dashed ('V') and solid ('P') line.

Several aspects of the experimental results are immediately apparent. Firstly (as shown in figure 3*a*), for 'early times' (defined as sufficiently small values of $\tau_\star$), it is difficult to distinguish between the two models. At early times, the interface descends at a constant rate to a good approximation, and so the linear model proposed in BDG97 seems perfectly reasonable, and the power supplied by the driving disk appears adequate to ensure a constant rate of increase of the depth of the upper layer. This issue also applies to the experiments where the initial layer depth h_0 was large (e.g. the experiment with $Ri_B = 1.0$), as in those cases the entrainment processes definitely did not last long enough for the two models to be distinguishable.

However, for experiments where the descent of the interface continued for a sufficiently large depth or equivalently to a sufficiently large value of $\tau_\star$ (the best two examples being the experiments for $Ri_B = 0.241$, which lasts until $\tau_\star \sim 6$, and is plotted with a square and for $Ri_B = 0.53$, plotted with a downward pointing triangle) it is clear that the 'P' model is substantially superior. The rate of increase of $\hat{h}_U$ noticeably and measurably decreases with time, with a rate very well-modelled by (2.16), as is particularly apparent on figure 3(*c*), where we have plotted the experimental results with logarithmic axes. This is very suggestive that the quasi-steady state discussed in the previous section occurs in these experiments. This phenomenon occurs for the entire range of Ri_B considered here and identification of the fact that the flow is in this regime relies only on the depth of the evolving layer changing by a sufficiently large amount for the sub-linear growth to be identifiable and distinguishable from the 'V' model. From our experiments, this critical amount appears to be approximately one quarter of the original layer depth, or equivalently, when $\tau_\star > 1/4$.

From (3.2), $\tau_\star > 1/4$ corresponds approximately to the time when the two models differ by 3–4 %. At such a time, the dimensional deviation between the two model predictions is of the order of a centimetre, and so is definitely distinguishable using the density profile measurements. Since at early times the two predictions are identical (as is apparent from calculating the Taylor series expansion of the 'P' model prediction in powers of $\tau_\star$) there are two different flow evolutions which are consistent with the observed early-time linear increase in depth, and the late-time slower sub-linear increase in depth of the mixed layer. Firstly, the evidence is consistent with the flow initially being in the constant-velocity 'V' model regime, with a 'cross-over' to the constant-power 'P' model flow regime at some later time (which appears to be a time $\tau_\star \lesssim 1/4$) when the power demand of the mixed layer exceeds that which can be supplied by the rotating disk. Secondly, the evidence is also consistent with the flow being in the constant-power 'P' flow regime from very early times, due to the fact that at early times the 'P' model prediction reduces to the constant increase in depth prediction of the 'V' model.

Although this second behaviour is physically unlikely, as it seems plausible that there will be an initial 'spin up' phase where the turbulent kinetic energy of the mixed layer will inevitably increase, the unfortunate consequence is that we are unable to distinguish between these two different flow evolutions. In particular, there is no way to determine a specific 'cross-over' time between the two flow regimes, just that the 'P' model regime certainly appears to occur for $\tau_\star > 1/4$. From the definitions of τ and $\tau_\star$ (i.e. (2.2) and (3.1)), this corresponds to the dimensional critical time t_c

$$t > t_c = \frac{Ri_B h_0}{4R\Omega c_V}, \tag{3.3}$$

which is intuitively reasonable since t_c increases with stratification and initial layer depth, and decreases with disk angular frequency.

Therefore, we believe we have established categorically two key results. Firstly for sufficiently large changes in the layer depth (and hence long enough periods of flow evolution) the rate of change of layer depth is definitely sub-linear, and so the modelling assumption of BDG97 that the fluid layer is 'spun up' at a constant velocity which scales with the angular frequency is demonstrably not true for all time. Secondly, and somewhat more surprisingly because of the sweeping nature of the underlying assumptions, the data appear to be consistent with the simple 'P' model, suggesting that the dynamics in the evolving flow is in a subtle, yet appealing energy balance. The external forcing mechanism is injecting a certain amount of power, and this power is being partitioned into two parts. One part leads to the homogenization of the evolving upper layer, redistributing the entrained fluid throughout the upper layer. The other part is 'lost' to viscous dissipation, in just the right proportion to maintain the total kinetic energy of the upper layer at a constant value, and thus implying that the characteristic velocities in that upper layer (which drive the erosion of the lower layer for example) decay proportionally to $\hat{h}^{-1/2}$, thus inevitably leading to a reduction in the rate of increase of the upper-layer depth. The entrainment dynamics at the interface (since it appears to have a constant, and relatively small thickness throughout the flow evolution) appears not to be dominated by a turbulent diffusion-type overturning dynamics, but rather a continual 'scouring' dynamics as considered recently by Woods *et al.* (2010) in a different Taylor–Couette geometry. The data are very suggestive of the supposition that the rotating disk at the surface is ultimately unable to 'spin up' the entire well-mixed layer, and that this imperfect spin-up leads to decaying (as the mixed layer deepens) characteristic velocities, definitely different from the characteristic forcing velocities of the forcing disk.

4. Conclusions

We have considered experimentally and theoretically the mixing induced by a rotating disk at the surface of a cylindrical tank containing an initially two-layer density distribution. The turbulent motions induced by this disk prove to be very efficient at both entraining fluid from the lower layer, and thoroughly homogenizing the deepening upper layer. We find that the interface between the two layers (at least to leading order) remains of the same thickness. However, the rate at which the upper layer deepens does not remain constant with time, at variance with the model previously presented for this flow in BDG97. We present a model that captures this key aspect of the flow evolution, essentially assuming that it is not the characteristic velocity of the upper layer which stays constant during the evolution of the flow, but rather the total kinetic energy of the upper layer.

This assumption has two corollaries. Firstly, the disk is eventually unable to 'spin up' an arbitrarily deep layer of fluid due to the increased dissipation associated with this layer, even though it is conceivable that the disk can continue to energise the initially static fluid from the lower layer that is entrained. Secondly, there appears to be a constant amount of power required to entrain and (more significantly) homogenize the density of the upper layer (i.e. the stratified mixing power demand $\mathscr{M}_S$ as defined in (2.13)) thoroughly distributing the entrained denser fluid into the upper layer. The experimental evidence is consistent with the assumption that the characteristic velocity u_U of the entraining layer eventually decreases such that $u_U^2 h_U$ remains constant, particularly when the layer has deepened by a sufficiently large amount compared to its original depth.

Finally, there is no evidence that the entrainment processes (in particular the buoyancy flux) across the interface vary during the flow's evolution. However, since the (local) Richardson number Ri_I (as defined in (2.4)) at the interface is predicted to remain constant, it is moot whether the entrainment invariance with (relatively strong) stratification as identified by Woods *et al.* (2010) is occurring, though the relative sharpness of the observed interfacial thickness is suggestive that 'scouring' rather than strong shear-induced overturning is the dominant entrainment mechanism. The natural way to investigate this issue within this flow geometry is to repeat this experimental approach with an initially linearly stratified layer in the tank. Such an initial density distribution will lead inevitably to a non-constant interfacial Richardson number, even if the characteristic length scale of the interface between the deepening mixed layer and the remaining original linearly stratified layer remains constant. It is an interesting open question whether our core assumption (i.e. that the kinetic energy of the deepening mixed layer remains essentially constant) remains valid in such a qualitatively different flow, particularly in the light of the investigations of closely related flows by Munro & Davies (2006) and Munro, Foster & Davies (2010). We plan to report on the results of this study in due course.

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