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Formal classification of unfoldings of parabolic diffeomorphisms

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Abstract. We provide a complete system of invariants for the formal classification of unfoldings $\varphi(x, x_1, \ldots, x_n) = (f(x, x_1, \ldots, x_n), x_1, \ldots, x_n)$ of complex analytic germs of diffeomorphisms at $(\mathbb{C}, 0)$ that are tangent to the identity. We reduce the formal classification problem to solving a linear differential equation. Then we describe the formal invariants; their nature depends on the position of the fixed points set $\mathrm{Fix}\,\varphi$ with respect to the regular vector field $\partial/\partial x$. We get invariants specifically attached to higher dimension $(n \geq 3)$, although generically they are analogous to the one-dimensional ones.

1. *Introduction*

We provide a complete system of invariants for the formal classification of n-parameter unfoldings of complex analytic germs of diffeomorphisms at $(\mathbb{C}, 0)$ that are tangent to the identity. Consider coordinates $(x, x_1, \ldots, x_n)$ in $\mathbb{C}^{n+1}$. Denote by $\mathrm{Diff}(\mathbb{C}^{n+1}, 0)$ the group of complex analytic germs of diffeomorphisms at $(\mathbb{C}^{n+1}, 0)$. We define the group

$$\mathrm{Diff}_p(\mathbb{C}^{n+1}, 0) = \{\varphi \in \mathrm{Diff}(\mathbb{C}^{n+1}, 0) : x_j \circ \varphi = x_j \text{ for any } 1 \leq j \leq n\}$$

of parameterized diffeomorphisms. The group

$$\mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0) = \left\{\varphi \in \mathrm{Diff}_p(\mathbb{C}^{n+1}, 0) : \frac{\partial(x \circ \varphi)}{\partial x}(0, \ldots, 0) = 1\right\}$$

is the set of unipotent elements of $\mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$. If $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ then we say that φ is a unipotent parameterized diffeomorphism (or up-diffeomorphism for brevity). Let $\varphi \in \mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$. We have that $(\partial(x \circ \varphi)/\partial x)\,(0) = 1$ if and only if the unperturbed diffeomorphism $\varphi_{|x_1=\cdots=x_n=0} \in \mathrm{Diff}(\mathbb{C}, 0)$ is tangent to the identity. Thus the elements of $\mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ are exactly the n-parameter unfoldings of germs of diffeomorphisms that are tangent to the identity.

The complex analytic germs of diffeomorphisms in one complex variable are well known. Those germs whose linear part is not periodic are formally linearizable. On the one hand they are analytically linearizable if the linear part is not a rotation. On the other hand we find 'small divisor problems' [**3, 13, 18, 20**], leading to very complicated dynamics if we deal with non-linearizable diffeomorphisms whose fixed point is of indifferent type.

The study of diffeomorphisms with periodic linear part can be reduced to that of diffeomorphisms that are tangent to the identity where we know the formal, topological [**4, 8**] and analytical [**9, 19**] classifications. The only topological invariant is the order of contact with the identity; this discrete invariant plus a numerical invariant called residue (cf. §5.1) make up a complete system of formal invariants. The analytical classification is more complicated; we can express the invariants as a collection (changes of charts) of one-variable germs of diffeomorphisms [**11**]. The number of changes of charts is twice the order of contact with the identity.

A natural generalization of germs of diffeomorphisms at $(\mathbb{C}, 0)$ are their unfoldings. We are interested in the formal classification of unfoldings, i.e. elements of $\mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$. We denote the fixed points set of a diffeomorphism φ by $\mathrm{Fix}\,\varphi$. Consider $\varphi \in \mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$ such that $(\partial(x \circ \varphi)/\partial x)\,(0)$ is not a root of unity. The function linear part $\partial(x \circ \varphi)/\partial x : \mathrm{Fix}\,\varphi \to \mathbb{C}$ is the only formal invariant attached to φ as in the one-dimensional case. Thus the task of obtaining a formal classification in $\mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$ can be reduced to exhibiting a complete system of formal invariants for up-diffeomorphisms.

We denote by $\widehat{\mathrm{Diff}}(\mathbb{C}^{n+1}, 0)$, $\widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$ and $\widehat{\mathrm{Diff}}_{up}(\mathbb{C}^{n+1}, 0)$ the formal completions of $\mathrm{Diff}(\mathbb{C}^{n+1}, 0)$, $\mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$ and $\mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ respectively.

A unipotent $\varphi \in \mathrm{Diff}(\mathbb{C}^{n+1}, 0)$ is the exponential of a unique formal nilpotent vector field (see §3 for definitions), the so-called infinitesimal generator, that we denote by $\log \varphi$. Consider φ in $\mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$; we have that $\log \varphi$ is of the form $(x \circ \varphi - x)\hat{u}\,\partial/\partial x$ where $\hat{u} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ is a unit. The infinitesimal generator of φ can be extended to $\mathrm{Fix}\,\varphi$, more precisely.

PROPOSITION 1.1. *Let* $\varphi = \exp((x \circ \varphi - x)\hat{u}\,\partial/\partial x) \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. *Then* $\hat{u}$ *belongs to the projective limit* $\lim_{\leftarrow} \mathbb{C}\{x, x_1, \ldots, x_n\}/(x \circ \varphi - x)^j$.

In other words, for any $j \in \mathbb{N}$ there exists $u_j \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\hat{u} - u_j$ belongs to the ideal $(x \circ \varphi - x)^j$.

We say that a germ of analytic variety at $(\mathbb{C}^{n+1}, 0)$ is *fibered* if it is a union of orbits of $\partial/\partial x$. By definition $\hat{\rho} \in \widehat{\mathrm{Diff}}(\mathbb{C}^{n+1}, 0)$ is *normalized* with respect to $f = 0$ (for $f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$) if $\hat{\rho} \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$ and $\hat{\rho}_{|\gamma} \equiv \mathrm{Id} \Leftrightarrow x \circ \hat{\rho} - x \in I(\gamma)$ for any non-fibered irreducible component γ of $f = 0$.

PROPOSITION 1.2. *Let* $\varphi_1, \varphi_2 \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. *Assume that* φ_1 *and* φ_2 *are formally conjugated. Then there exist* $\sigma \in \mathrm{Diff}(\mathbb{C}^{n+1}, 0)$ *and a normalized* $\hat{\sigma} \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$ *(with respect to* $x \circ \varphi_2 - x = 0$*) such that* $(\hat{\sigma} \circ \sigma) \circ \varphi_1 = \varphi_2 \circ (\hat{\sigma} \circ \sigma)$.

The last proposition implies that up to analytic change of coordinates every pair of formally conjugated up-diffeomorphisms are conjugated by a normalized element of $\widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$. We study the equivalence relation $\sim_*$ in $\mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ given by $\varphi_1 \sim_* \varphi_2$ if φ_1 and φ_2 are formally conjugated by a normalized transformation with respect to

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Fix φ_1. In particular $\varphi_1 \sim_* \varphi_2$ implies Fix $\varphi_1 =$ Fix φ_2. Moreover, the ideals $(x \circ \varphi_1 - x)$ and $(x \circ \varphi_2 - x)$ of the fixed points sets of φ_1 and φ_2 coincide.

Every class of equivalence is contained in a set

$$\mathcal{D}_f = \{\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0) : (x \circ \varphi - x)/f \text{ is a unit}\}.$$

The set $\mathcal{D}_f$ is composed by the unfoldings of parabolic diffeomorphisms with fixed points set $f = 0$. Let us remark that we do not suppose that $f = 0$ is reduced; for instance we have $\mathcal{D}_x \cap \mathcal{D}_{x^2} = \emptyset$. The classes of the equivalence relation $\sim_*$ are connected sets in the compact-open topology. As a consequence, to determine whether or not there exists a formal normalized conjugation between up-diffeomorphisms can be reduced to solving a linear problem. More precisely, we can associate to $\varphi_1, \varphi_2 \in \mathcal{D}_f$ the homological equation

$$\frac{\partial \alpha}{\partial x} = \frac{1}{f}\left(\frac{1}{\hat{u}_1} - \frac{1}{\hat{u}_2}\right),$$

where $\log \varphi_j = \hat{u}_j f\, \partial/\partial x$ for $j \in \{1, 2\}$. Let $\prod_{j=1}^{p} f_j^{l_j} \prod_{j=1}^{q} F_j^{m_j}$ be the irreducible decomposition of f. By choice $F_j = 0$ is fibered for $1 \le j \le q$ whereas $f_k = 0$ is non-fibered for $1 \le k \le p$. We say that the homological equation is special (with respect to f) if there exists a solution of the form $\alpha = \hat{\beta}/(\prod_{j=1}^{p} f_j^{l_j - 1} \prod_{j=1}^{q} F_j^{m_j})$ where $\hat{\beta} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$. Such a solution is also called special. We have the following proposition.

PROPOSITION 1.3. *Let $\varphi_1, \varphi_2 \in \mathcal{D}_f \subset \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. Then φ_1 and φ_2 are formally conjugated by a normalized transformation if and only if the homological equation associated to φ_1 and φ_2 is special.*

Let $\varphi = \exp(\hat{u} f\, \partial/\partial x) \in \mathcal{D}_f$. The formal 1-form $dx/(\hat{u} f)$ is the dual of $\log \varphi$ in the relative cohomology of the vector field $\partial/\partial x$. Moreover there exists u in $\mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\hat{u} - u \in (f)$ by Proposition 1.1. Therefore we obtain

$$\frac{dx}{\hat{u} f} - \frac{dx}{u f} = \frac{1}{\hat{u} u} \frac{u - \hat{u}}{f}\, dx.$$

Since the right-hand side does not have poles then the formal properties of $dx/(\hat{u} f)$ and $dx/(uf)$ are the same. The only formal invariant of $\varphi \in \mathcal{D}_f \subset \mathrm{Diff}_{up}(\mathbb{C}, 0)$ for the normalized conjugation is the residue of $dx/(uf)$ at 0. The generalization of this invariant in the higher dimensional case is the collection of residues of $dx/(uf)$ at Fix φ. This collection defines a meromorphic function in every non-fibered irreducible component of $f = 0$.

There are other invariants which are purely related to higher dimension. For a non-zero $f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ we define the additive group $Fr(f)$ of homological equations $\partial \alpha/\partial x = A/f$ (for $A \in \mathbb{C}\{x, x_1, \ldots, x_n\}$) such that $(A/f)\, dx$ has vanishing residues. It can be considered as an additive subgroup of $\mathbb{C}\{x, x_1, \ldots, x_n\}$. Moreover we denote by $Sp(f)$ the subgroup of $Fr(f)$ of special equations.

THEOREM 1.1. *A complete system of formal invariants for the normalized conjugation in $\mathcal{D}_f \subset \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ is composed by the residue functions plus the complex vector space $Fr(f)/Sp(f)$.*

For $\mathcal{D}_f \subset \text{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ with $n \leq 1$ the only invariants are the residues; in other words we have $Fr(f)/Sp(f) = 0$. The situation is different in higher dimension; for instance for $f_0 = (x_2 - xx_1)^2$ and $\mathcal{D}_{f_0} \subset \text{Diff}_{up}(\mathbb{C}^3, 0)$ we have that $\dim_{\mathbb{C}} Fr(f_0)/Sp(f_0) = 1$. Moreover we have $\dim_{\mathbb{C}} Fr(f)/Sp(f) < +\infty$ for $\mathcal{D}_f \subset \text{Diff}_{up}(\mathbb{C}^3, 0)$. Thus besides the residue functions there are only finitely many linear invariants. In spite of that $Fr(f_0)/Sp(f_0) \sim \mathbb{C}\{x_3, \ldots, x_n\}$ is infinite dimensional if $\mathcal{D}_{f_0}$ is considered as a subset of $\text{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ for $n + 1 \geq 4$.

The nature of $Fr(f)/Sp(f)$ depends on the *evil set* $S(f)$ of f. This set is the union of the orbits of $\partial/\partial x$ contained in non-fibered irreducible components γ of $f = 0$ such that $f \in I(\gamma)^2$. For instance we have $S((x_2 - xx_1)^2) = \{x_1 = x_2 = 0\}$.

Consider the set

$$K(n) = \{f \in \mathbb{C}\{x, x_1, \ldots, x_n\} : f(0) = (\partial f/\partial x)(0) = 0\},$$

endowed with the Krull topology. The set

$$E(n) = \{f \in K(n) : f(x, 0, \ldots, 0) \not\equiv 0\}$$

is open and dense in $K(n)$. Moreover we have $S(f) = \emptyset$ for any $f \in E(n)$. The reciprocal is not true; for instance we have $S(x_1) = S(x_1^2) = \emptyset$ since $x_1 = 0$ is fibered and $S(x_2 - xx_1) = \emptyset$ since $x_2 - xx_1 \notin (x_2 - xx_1)^2$.

PROPOSITION 1.4. *Let* $0 \neq f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ *such that* $S(f) = \emptyset$. *Then we have* $Fr(f)/Sp(f) = 0$.

THEOREM 1.2. Fix $n \in \mathbb{N}$. *There exists a dense open subset* $E(n)$ *of* $K(n)$ *such that for* $f \in E(n)$ *the residue functions provide a complete system of formal invariants for the normalized conjugation in* $\mathcal{D}_f \subset \text{Diff}_{up}(\mathbb{C}^{n+1}, 0)$.

Theorem 1.1 and Proposition 1.4 imply that the residue functions provide a complete system of formal invariants for the unfoldings $\varphi \in \text{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ of parabolic diffeomorphisms $\varphi_{|x_1=\cdots=x_n=0}$ of finite codimension. Theorem 1.2 just reflects the fact that such unfoldings are generic.

A large part of this paper is devoted to dealing with the infinite codimension case, i.e. unfoldings of the identity map. Why such a large generality? In order to understand the properties of an unfolding, we can try to desingularize it by using blow-ups. Even if we consider an unfolding φ of a finite codimension parabolic diffeomorphism we obtain unfoldings of the identity map throughout the desingularization process. This strategy can be successfully used in the following two settings.

Topological. In [**14**] a complete system of topological invariants is provided for one-parameter unfoldings of parabolic germs of diffeomorphisms. The invariants are analogous in the finite and infinite codimension cases. The desingularization techniques are an ingredient in the proof of the completeness of the invariants.

Analytic. In [**10**] Mardesic, Roussarie and Rousseau provide a complete system of analytic invariants for unfoldings of generic parabolic diffeomorphisms. Such a result was generalized by the author for unfoldings of finite codimension parabolic diffeomorphisms [**15**]. Moreover, we provide a geometric interpretation of the system of invariants [**15**]. It is based on the notion of infinitesimal stability of the unfolding, whose definition and properties are obtained via a desingularization approach.

Voronin classifies analytically germs of diffeomorphisms $\varphi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $\varphi_{|y=0} \equiv \mathrm{Id}$ and $j^k\varphi$ is of the form $(x + y^k, y)$ for some $k \in \mathbb{N}$ (see [**6**]). The diffeomorphism φ is not necessarily an unfolding. He points out that both his invariants and the EcalleVoronin ones are based on the lack of pseudo-convexity of the space of orbits. This property is shared by the unfoldings of finite codimension parabolic germs. Then, the existence of a unified approach for the analytic classification of unfoldings of parabolic diffeomorphisms is possible regardless of whether or not they are of finite codimension. Even more, as Voronin's paper suggests, such an approach can be fruitful for general unipotent germs of diffeomorphisms.

2. *Notation and definitions*

We deal with complex analytic germs of diffeomorphisms defined at $(\mathbb{C}^{n+1}, 0)$. Consider coordinates $(x, x_1, \ldots, x_n)$. Consider a set $W \subset \mathbb{C}^{n+1}$. We define the ring G_W of germs of holomorphic functions in a neighborhood of W.

We say that a variety $\beta \subset \mathbb{C}^{n+1}$ is *fibered* if it is a union of orbits of $\partial/\partial x$, or in other words if there exists a system of generators of the ideal $I(\beta)$ not depending on x.

We define the group

$$\mathrm{Diff}_p(\mathbb{C}^{n+1}, 0) = \{\varphi \in \mathrm{Diff}(\mathbb{C}^{n+1}, 0) : x_j \circ \varphi = x_j \ \forall\ 1 \leq j \leq n\}$$

of parameterized diffeomorphisms. We denote by $\mathrm{Diff}_u(\mathbb{C}^{n+1}, 0)$ the subgroup of $\mathrm{Diff}(\mathbb{C}^{n+1}, 0)$ of unipotent germs of diffeomorphisms. An element φ of $\mathrm{Diff}(\mathbb{C}^{n+1}, 0)$ is unipotent if its linear part is unipotent, in other words if $j^1\varphi$ has the unique eigenvalue 1. Let us remark that $\mathrm{Diff}_u(\mathbb{C}, 0)$ is the group of diffeomorphisms in one variable that are tangent to the identity. An element φ of $\mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$ is unipotent if and only if $(\partial(x \circ \varphi)/\partial x)\,(0) = 1$. We will study the elements in the group

$$\mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0) \stackrel{\mathrm{def}}{=} \mathrm{Diff}_p(\mathbb{C}^{n+1}, 0) \cap \mathrm{Diff}_u(\mathbb{C}^{n+1}, 0)$$

of unipotent parameterized diffeomorphisms. For the sake of simplicity we will usually replace the expression 'unipotent parameterized diffeomorphism' with the shorter *up-diffeomorphism*. The groups that we have just defined have formal completions; we will denote them $\widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$, $\widehat{\mathrm{Diff}}_u(\mathbb{C}^{n+1}, 0)$ and $\widehat{\mathrm{Diff}}_{up}(\mathbb{C}^{n+1}, 0)$.

Given $f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ we consider the set

$$\mathcal{D}_f = \{\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0) : (x \circ \varphi - x)/f \text{ is a unit}\}$$

of up-diffeomorphisms whose fixed points set is $f = 0$.

Let $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. Given $P \in \mathrm{Fix}\,\varphi$ we define φ_P as the one-dimensional germ $\varphi_{|\bigcap_{j=1}^{n}\{x_j = x_j(P)\}}$ considered in a neighborhood of P.

The unipotent germs of diffeomorphisms are related with nilpotent vector fields. We denote by $\mathcal{X}(\mathbb{C}^{n+1}, 0)$ the set of germs of complex analytic vector fields which are singular at 0. We denote by $\mathcal{X}_N(\mathbb{C}^{n+1}, 0)$ the subset of $\mathcal{X}(\mathbb{C}^{n+1}, 0)$ of nilpotent vector fields, i.e. vector fields whose first jet has the unique eigenvalue 0. The formal completions of these spaces are denoted by $\hat{\mathcal{X}}(\mathbb{C}^{n+1}, 0)$ and $\hat{\mathcal{X}}_N(\mathbb{C}^{n+1}, 0)$ respectively.

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3. *Basic properties of the unipotent parameterized diffeomorphisms*

We denote by $\exp(tX)$ the flow of the vector field X. It is the unique solution of the differential equation

$$\frac{\partial}{\partial t}\exp(tX) = X(\exp(tX))$$

with initial condition $\exp(0X) = \mathrm{Id}$. The flow can be developed in power series. Such a property allows one to define the formal flow for formal vector fields. We define $X^0(g) = g$ and $X^{j+1}(g) = X^j(X(g))$ for any $j \geq 0$. The flow of X can be expressed in the form

$$\exp(tX) = \left(\sum_{j=0}^{\infty} t^j \frac{X^j(x)}{j!}, \sum_{j=0}^{\infty} t^j \frac{X^j(x_1)}{j!}, \ldots, \sum_{j=0}^{\infty} t^j \frac{X^j(x_n)}{j!}\right).$$

For any formal nilpotent vector field $\hat{X}$ the sums defining $\exp(t\hat{X})$ converge in the Krull topology. The exponential application is by definition $\exp(1\hat{X})$. The next proposition is classical; it relates formal nilpotent vector fields and formal unipotent transformations.

PROPOSITION 3.1. *The exponential application induces a bijective mapping from* $\hat{\mathcal{X}}_N(\mathbb{C}^n, 0)$ *onto* $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$. *Moreover, if* $\hat{X} \in \hat{\mathcal{X}}_N(\mathbb{C}^n, 0)$ *then every component of* $\exp(t\hat{X})$ *belongs to* $\mathbb{C}[t]\,[[x_1, \ldots, x_n]]$.

Definition 3.1. We call *infinitesimal generator* of a formal $\varphi \in \mathrm{Diff}_u(\mathbb{C}^n, 0)$ the only formal nilpotent vector field $\hat{X}$ whose exponential is φ. We denote $\hat{X}$ by $\log \varphi$.

The formal classifications of up-diffeomorphisms and of their infinitesimal generators are equivalent tasks. Indeed we can express the formal properties of φ in terms of $\log \varphi$.

3.1. *Infinitesimal generator of a up-diffeomorphism and fixed points.*

A up-diffeomorphism preserves the fibration $dx_1 = \cdots = dx_n = 0$. Somehow its infinitesimal generator has to preserve the same fibration too.

PROPOSITION 3.2. *Let* $\varphi \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^{n+1}, 0)$. *Then* $\varphi \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$ *if and only if* $\log \varphi$ *can be expressed in the form* $\hat{f}\,\partial/\partial x$ *for some* $\hat{f} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$.

Remark 3.1. Since the infinitesimal generator $\hat{f}\,\partial/\partial x$ of $\varphi \in \widehat{\mathrm{Diff}}_{up}(\mathbb{C}^{n+1}, 0)$ is nilpotent then $\hat{f}(0) = 0$ and $(\partial \hat{f}/\partial x)\,(0) = 0$.

Proof of Proposition 3.2. The implication $\Leftarrow$ is a direct consequence of the formula defining $\exp(\log \varphi)$.

Let us prove the implication $\Rightarrow$. The components of

$$\exp(t \log \varphi) = (\varphi_0(t, x, x_1, \ldots, x_n), \ldots, \varphi_n(t, x, x_1, \ldots, x_n))$$

belong to $\mathbb{C}[t]\,[[x, x_1, \ldots, x_n]]$ for $0 \leq j \leq n$. Since $\widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$ is a group then $\varphi_j(t, x, x_1, \ldots, x_n) = x_j$ for all $t \in \mathbb{Z}$ and $1 \leq j \leq n$. For $1 \leq j \leq n$ the power series $\varphi_j - x_j$ is identically 0 because it is 0 for $t \in \mathbb{Z}$ and it is polynomial in t. Since

$$(\log \varphi)\,(x_j) = \lim_{t \to 0} \frac{x_j \circ \exp(t \log \varphi) - x_j}{t} = \lim_{t \to 0} 0 = 0$$

for $1 \leq j \leq n$, then $\log \varphi$ is of the form $\hat{f}\partial/\partial x$. □

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A up-diffeomorphism φ has a fixed points set $x \circ \varphi - x = 0$; it is a germ of a hypersurface if $\varphi \neq \mathrm{Id}$. We prove next that the fixed points set of φ and the singular set of $\log \varphi$ coincide.

PROPOSITION 3.3. *Let $\varphi = \exp(\hat{f}\,\partial/\partial x)$ be a up-diffeomorphism. There exists a formal unit $\hat{u} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ such that $x \circ \varphi - x = \hat{u}\hat{f}$.*

Proof. We define

$$h_0 = x,\ h_1 = \hat{f}, \ldots, h_{j+1} = \hat{f}\partial h_j/\partial x \quad \text{for all } j > 0.$$

We have $\varphi = (\sum_{j=0}^{\infty} h_j/j!, x_1, \ldots, x_n)$. Since $\hat{f}\,\partial/\partial x$ is nilpotent then $h_j/\hat{f}$ belongs to the maximal ideal of $\mathbb{C}[[x, x_1, \ldots, x_n]]$ for $j \geq 2$. We define $\hat{u} = 1 + \sum_{j=2}^{\infty} h_j/(j!\hat{f})$. Clearly the series $\hat{u}$ is the unit we are looking for. □

4. *Formal transversality of the infinitesimal generator*

The infinitesimal generator of a up-diffeomorphism can be extended to the fixed points set. Roughly speaking, the infinitesimal generator is convergent in the tangent directions to the fixed points set but it can diverges in the transverse direction. We introduce some definitions to make this very simple idea rigorous.

The formal completion of a complex space $(U, \Theta(U))$ (U is a topological space and $\Theta(U)$ is its sheaf of analytic functions) along a subvariety V given by a sheaf of ideals I is the space $(U, \hat{\Theta}_I(U))$ where

$$\hat{\Theta}_I(U) = \lim_{\leftarrow} \frac{\Theta(U)}{I^j}.$$

Throughout this paper we consider three types of formal transversality.

Definition 4.1. Let $V \subset (\mathbb{C}^{n+1}, 0)$ be a germ of analytic variety given by an ideal $I(V)$. A series $\hat{g} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ is:

- transversally formal along V (or equivalently, and in short, t.f. along V) if $\hat{g} \in \lim_{\leftarrow} \mathbb{C}\{x, x_1, \ldots, x_n\}/I(V)^j$;
- uniformly transversally formal along V (or u.t.f. along V) if $\hat{g}$ belongs to $\lim_{\leftarrow} \mathcal{O}(U)/I(V)^j$ for some neighborhood of the origin U; or
- uniformly semi-meromorphic along V (or u.s.m. along V) if $\hat{g}$ belongs to $\lim_{\leftarrow} (\mathcal{O}(U))_{I(V)}/I(V)^j$ for some neighborhood of the origin U. Note that $(\mathcal{O}(U))_{I(V)}$ is the localized ring of holomorphic functions in U with respect to the ideal $I(V)$.

For the u.s.m. definition we suppose that V is irreducible. In general we say that $\hat{g}$ is u.s.m. along V if $\hat{g}$ is u.s.m. along every irreducible component of V.

The analytic spaces that we complete are $(0, \mathbb{C}\{x, x_1, \ldots, x_n\})$, $(U, \mathcal{O}(U))$ and $(U, (\mathcal{O}(U))_{I(V)})$ respectively. We say that $X \in \hat{\mathcal{X}}(\mathbb{C}^{n+1}, 0)$ is t.f. along V if all the functions $X(x), X(x_1), \ldots, X(x_n)$ are t.f. along V. The other kinds of formal transversality for formal vector fields are defined in an analogous way.

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Definition 4.2. Denote the fixed points set of a diffeomorphism φ by $\mathrm{Fix}\,\varphi$. Let φ be a up-diffeomorphism and consider an irreducible component γ of $\mathrm{Fix}\,\varphi$. We say that γ is *unipotent* with respect to φ if $\partial(x \circ \varphi)/\partial x \equiv 1$ in γ.

If γ is unipotent then the germ of φ at P is unipotent for any $P \in \gamma$. Later on we prove that its infinitesimal generator depends analytically on $P \in \gamma$ or more precisely that $\log \varphi$ is u.t.f. along γ.

The germ φ_P (§2) is embeddable in a formal flow except if $(\partial(x \circ \varphi)/\partial x)\,(P)$ is a root of the unit different from 1 and φ_P is not periodic. As a consequence there is no hope in general for $\log \varphi$ to be u.t.f. along the non-unipotent components of $\mathrm{Fix}\,\varphi$. Nevertheless, if φ_P can be embedded in a formal flow for any $P \in \gamma$ contained in an irreducible component γ of $\mathrm{Fix}\,\varphi$, then $\log \varphi$ is u.t.f. along γ. We will make clear and prove the previous assertions.

4.1. *One-dimensional results.* The next results are well known and they are included here for the sake of clarity.

PROPOSITION 4.1. *Let $\tau \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $j^1\tau \neq \mathrm{Id}$. Then τ is the exponential of a formal vector field if and only if it is formally linearizable.*

Proof. Let $\hat{X} = (\sum_{j=1}^{\infty} a_j x^j)\,\partial/\partial x$ be a formal vector field such that $\tau = \exp(\hat{X})$. Since $j^1\tau = e^{a_1}x$ and $j^1\tau \neq \mathrm{Id}$ then $a_1 \neq 0$. The linear part is a complete system of invariants for the elements of $\hat{\mathcal{X}}(\mathbb{C}, 0)$ with no vanishing linear part. As a consequence there exists $\hat{\sigma} \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ such that $\hat{\sigma}_*(a_1 x\,\partial/\partial x) = \hat{X}$. By taking exponentials we obtain $\hat{\sigma} \circ j^1\tau = \tau \circ \hat{\sigma}$.

Suppose that $\hat{\sigma} \circ j^1\tau = \tau \circ \hat{\sigma}$. Since $j^1\tau = \exp(cx\,\partial/\partial x)$ for any $c \in \mathbb{C}$ such that $e^c = (\partial\tau/\partial x)\,(0)$ then $\tau = \exp(\hat{\sigma}_*(cx\,\partial/\partial x))$. □

PROPOSITION 4.2. *Let $\tau \in \mathrm{Diff}(\mathbb{C}, 0)$ with $j^1\tau$ not periodic. Assume that we have $\tau = \exp(\hat{X}) = \exp(\hat{Y})$ for $\hat{X}, \hat{Y} \in \hat{\mathcal{X}}(\mathbb{C}, 0)$ such that $j^1\hat{X} = j^1\hat{Y}$. Then $\hat{X} = \hat{Y}$.*

Proof. We can suppose that $\tau = j^1\tau$ by Proposition 4.1. It suffices to prove that $\hat{X} = j^1\hat{X}$. Since $j^1\hat{X} \neq 0$ there exists $\hat{\sigma} \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ such that $\hat{\sigma}_*(j^1\hat{X}) = \hat{X}$. By taking exponentials we obtain $\hat{\sigma} \circ \tau = \tau \circ \hat{\sigma}$. Since τ is linear and not periodic the center of τ in $\widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ is the linear group. Therefore $\hat{\sigma}$ is linear, which implies that $j^1\hat{X} = \hat{\sigma}_*(j^1\hat{X}) = \hat{X}$. □

PROPOSITION 4.3. *Let $\tau \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $j^1\tau$ is periodic and τ is linearizable. Suppose we have $h \circ \tau \circ h^{-1}(x) - j^1\tau(x) = O(x^{k+1})$ for some $h \in \mathrm{Diff}(\mathbb{C}, 0)$ and some $k \geq 1$. Then there exists $\sigma \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $j^k\sigma = j^k h$ and $\sigma \circ \tau \circ \sigma^{-1} = j^1\tau$.*

Proof. It suffices to prove that if $\tau - j^1\tau = O(x^{k+1})$ there exists $\sigma \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $\sigma \circ \tau = j^1\tau \circ \sigma$ and $\sigma(x) - x = O(x^{k+1})$. We denote $j^1\tau = ax$ and the period of $j^1\tau$ by q. Since τ^q is formally conjugated to $(j^1\tau)^q = \mathrm{Id}$, then $\tau^q = \mathrm{Id}$. We are done by defining $\sigma(x) = (\sum_{j=0}^{q-1} \tau^j(x)/a^j)/q$. □

4.2. *Division neighborhoods, and convergence by restriction.* We work in domains in $\mathbb{C}^{n+1}$ in which the components of $\operatorname{Fix}\varphi$ behave like their germs at 0. Consider $g \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. We say that a domain U containing the origin is a *division neighborhood* for g if:

- there is a decomposition $g = g_1^{l_1} \cdots g_r^{l_r}$ of g into irreducible factors in the ring $\mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $g_j \in \mathcal{O}(U)$ for $1 \leq j \leq r$; and
- the regular part of $g_j = 0$ is connected in U for any $1 \leq j \leq r$.

The definition of division neighborhood is intended to extend the division of germs to bigger domains. More precisely, if U is a division neighborhood for g, then

$$a = gb \text{ where } a \in \mathcal{O}(U) \text{ and } b \in \mathbb{C}\{x, x_1, \ldots, x_n\} \quad \Longrightarrow \quad b \in \mathcal{O}(U).$$

The following results can be immediately deduced from the definition of division neighborhood.

LEMMA 4.1. *Let U be a division neighborhood for g and consider a domain $V \subset U$ such that $U \cap \{g = 0\} = V \cap \{g = 0\}$. Then V is a division neighborhood for g.*

LEMMA 4.2. *Let U be a division neighborhood for g. Consider an analytic set $S \not\ni 0$. Then $U \setminus S$ is a division neighborhood for g.*

The last lemma is trivial since we cannot break the connectedness of the regular parts by removing sets of real codimension at least 2.

It is not difficult to find a division neighborhood for $f = x \circ \varphi - x$. Let $(y_1, \ldots, y_{n+1})$ be a set of coordinates such that any irreducible component γ of $\operatorname{Fix}\varphi$ can be expressed as the vanishing set of a monic Weierstrass polynomial in the variable y_1. Let $(c_1, \ldots, c_{n+1}) \in \mathbb{R}^+ \times \cdots \times \mathbb{R}^+$. Every polydisk $\bigcap_{j=1}^{n+1}\{|y_j| < c_j\}$ small enough such that

$$\{|y_1| = c_1\} \cap \bigcap_{j=2}^{n+1}\{|y_j| < c_j\} \cap \{f = 0\} = \emptyset$$

is a division neighborhood for f.

We define a new concept, which is simpler to handle than the formal transversality; it is a sort of formal transversality at the 0-level.

Definition 4.3. We say that $h \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ *converges by restriction to* γ if there exists $h' \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $h - h'$ belongs to $I(\gamma)$ where $I(\gamma)$ is the ideal of γ. If $V \subset \gamma$ is a neighborhood of 0 in γ we say that *h converges by restriction to γ in V* if h' can be chosen holomorphic in a neighborhood of V.

Let $\varphi = \exp((x \circ \varphi - x)\hat{u}\, \partial/\partial x) \in \operatorname{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. The proof of the formal transversality of $\log\varphi$ along an irreducible component $g = 0$ of $\operatorname{Fix}\varphi$ is based on an induction process and the analysis of the Taylor expansion of the exponential mapping. In every induction step it suffices to prove the convergence of a certain formal power series by restriction to $g = 0$. More precisely, suppose that there exists $u_\alpha \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\hat{u} = u_\alpha + g^\alpha \hat{v}$ for some $\hat{v} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$. Then the existence of $u_{\alpha+1} \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\hat{u} - u_{\alpha+1} \in (g^{\alpha+1})$ is equivalent to the convergence of $\hat{v}$ by

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restriction to $g = 0$. The next couple of lemmas are of technical type and they are used to carry out this approach.

LEMMA 4.3. *Let h be an irreducible element of $\mathbb{C}\{x, x_1, \ldots, x_n\}$. Consider series $\hat{c} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ and $d \in \mathbb{C}\{x, x_1, \ldots, x_n\} \setminus (h)$. Suppose that $\hat{c}d$ converges by restriction to $h = 0$. Then $\hat{c}$ converges by restriction to $h = 0$.*

There is also a uniform version of the previous lemma but we have to be careful with the setting. We keep the notation in the previous lemma. We consider coordinates $(y_1, \ldots, y_{n+1})$ such that

$$h = v(y_1, \ldots, y_{n+1})\,(y_1^l + y_1^{l-1} a_{l-1}(y_2, \ldots, y_{n+1}) + \cdots + a_0(y_2, \ldots, y_{n+1})),$$

for some unit v and some functions a_j $(0 \le j \le l-1)$. For a generic point $(y_2^0, \ldots, y_{n+1}^0)$ the number of points in $h(y_1, y_2^0, \ldots, y_{n+1}^0) = 0$ is l. We enumerate them $\alpha_1(y_2, \ldots, y_{n+1}), \ldots, \alpha_l(y_2, \ldots, y_{n+1})$. We define

$$\Delta(h, d) = \left(\prod_{j=1}^{l} (d \circ \alpha_j) \prod_{1 \le j < k \le l} (\alpha_j - \alpha_k)^2 \right) (y_2, \ldots, y_{n+1}).$$

The function $\Delta(h, d)$ is well defined since it is symmetric in $\alpha_1, \ldots, \alpha_l$; it is holomorphic in $\Delta(h, d) \neq 0$ and continuous in a neighborhood of the origin. By Riemann's theorem $\Delta(h, d)$ is holomorphic in a neighborhood of the origin. It is a type of discriminant function. Given a holomorphic function c_0 defined in a neighborhood of $D \cap \{h = 0\}$ and the remainder c_1 of the Weierstrass division of c_0/d over h, we have that $\Delta(h, d)c_1$ is still holomorphic in a neighborhood of $D \cap \{h = 0\}$. The function Δ is an auxiliary tool to prove Lemmas 4.3 and 4.4.

There exists a division neighborhood $D = D_1 \times D_{2,\ldots,n+1} \subset \mathbb{C} \times \mathbb{C}^n$ for h such that $D_{2,\ldots,n+1}$ is a division neighborhood for Δ. Given any neighborhood of the origin W the set D can be chosen to be contained in W. Suppose that d converges in a neighborhood of $D \cap \{h = 0\}$. In this context we prove the following result.

LEMMA 4.4. *If $\hat{c}d$ converges by restriction to $h = 0$ in $D \cap \{h = 0\}$ and $d \notin (h)$ then $\hat{c}$ converges by restriction to $h = 0$ in $D \cap \{h = 0\}$.*

Proof of Lemmas 4.3 and 4.4. It is clear that Lemma 4.4 implies Lemma 4.3. Consider a holomorphic function c_0 defined in the neighborhood of $D \cap \{h = 0\}$ such that $\hat{c}d - c_0 \in (h)$. The next step is using the Weierstrass division. We want to divide c_0/d by h. The remainder of that division is

$$R = \sum_{j=1}^{l} \frac{c_0(\alpha_j(y_2, \ldots, y_{n+1}))}{d(\alpha_j(y_2, \ldots, y_{n+1}))} \frac{\prod_{k \neq j}(y_1 - \alpha_k(y_2, \ldots, y_{n+1}))}{\prod_{k \neq j}(\alpha_j(y_2, \ldots, y_{n+1}) - \alpha_k(y_2, \ldots, y_{n+1}))}.$$

The function ΔR is holomorphic in a neighborhood of the origin and since Δ does not depend on y_1 then ΔR is the remainder of the Weierstrass division $[(\Delta c_0)/d]/h$. We define $\hat{R}$ to be the remainder of the Weierstrass division $\hat{c}/h$; it is a formal power series. We have

$$\Delta c_0 - \Delta R d \in (h) \quad \text{and} \quad \Delta \hat{c} d - \Delta \hat{R} d \in (h).$$

Since $d \notin (h)$ we obtain $\Delta R - \Delta \hat{R} \in (h)$. Both ΔR and $\Delta \hat{R}$ are polynomials in the variable y_1 whose degree is less than or equal to $l-1$. Therefore, we have $\Delta R \equiv \Delta \hat{R}$ by uniqueness of the Weierstrass division. The function Δ divides the l coefficients of the polynomial ΔR. As a consequence R is convergent in the neighborhood of the origin. Since $D_{2,\ldots,n+1}$ is a division neighborhood for Δ then R is defined in $\mathbb{C} \times D_{2,\ldots,n+1}$. The series $\hat{c}$ converges by restriction to $h=0$ in $D \cap \{h=0\}$ since $\hat{c} - R = \hat{c} - \hat{R} \in (h)$. □

We relate formal transversality along an analytic hypersurface with formal transversality along its irreducible components.

LEMMA 4.5. *Let f be an element of $\mathbb{C}\{x, x_1, \ldots, x_n\}$. Then $\hat{u}$ in $\mathbb{C}[[x, x_1, \ldots, x_n]]$ is t.f. (respectively u.t.f.) along $f=0$ if and only if $\hat{u}$ is t.f. (respectively u.t.f.) along every irreducible component of $f=0$.*

Proof. The implication $\Rightarrow$ is obvious. Let us prove the implication $\Leftarrow$.

Suppose we are in the u.t.f. case; the proof for the t.f. case is simpler. Let $f = f_1^{n_1} \cdots f_l^{n_l}$ be the decomposition of f into irreducible factors in $\mathbb{C}\{x, x_1, \ldots, x_n\}$. We choose coordinates $(y_1, \ldots, y_{n+1})$ such that $f_j=0$ can be expressed up to a unit as a monic Weierstrass polynomial in the variable y_1 for any $1 \leq j \leq l$. We can choose a polydisk $D = D_1 \times D_{2,\ldots,n+1}$ such that $(\partial D_1 \times D_{2,\ldots,n+1}) \cap \{f=0\} = \emptyset$. Moreover, we can suppose that D is a division neighborhood for every f_j and $D_{2,\ldots,n+1}$ is a division neighborhood for every $\Delta(f_j, f_k)$ with $j \neq k$. Finally we suppose that $\hat{u} \in \lim_{\leftarrow} \mathcal{O}(D)/(f_j)^k$ for any $1 \leq j \leq l$.

Let $F = \prod_{k=1}^{l} f_k^{a_k}$. Suppose we have $u \in \mathcal{O}(D)$ such that $\hat{u} - u \in (F)$. Fix $j \in \{1, \ldots, l\}$; it suffices to prove the existence of a function $v \in \mathcal{O}(D)$ such that $\hat{u} - v \in (f_j F)$. We claim that $(\hat{u}-u)/f_j^{a_j}$ converges by restriction to $f_j=0$. There exists $u_{a_j} \in \mathcal{O}(D)$ such that $\hat{u} - u_{a_j}$ belongs to $({f_j}^{a_j+1})$ by hypothesis. Denote $w = (u_{a_j} - u)/f_j^{a_j}$; we have $(\hat{u}-u)/f_j^{a_j} - w \in (f_j)$. By Lemma 4.4 the series $(\hat{u}-u)/F$ converges by restriction to $f_j=0$ in $D \cap \{f_j=0\}$. As a consequence there exists $b(y_1, \ldots, y_{n+1}) \in \mathcal{O}(\mathbb{C} \times D_{2,\ldots,n+1}) \subset \mathcal{O}(D)$ such that $(\hat{u}-u)/F - b \in (f_j)$. This implies that $\hat{u} - (u+bF) \in (f_j F)$. □

The next lemma provides a handy characterization of u.t.f. functions. Lemmas 4.5 and 4.6 allow one to simplify the calculations intended to prove formal transversality of the infinitesimal generator of a up-diffeomorphism φ.

LEMMA 4.6. *Let $V \subset \mathbb{C}^{n+1}$ be a germ of an analytic hypersurface at 0. Then a series $\hat{g} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ is u.t.f. along V if and only if $\hat{u}$ belongs to $\lim_{\leftarrow} G_{V \cap W}/I(V)^j$ (see §2) for some neighborhood W of the origin.*

Proof. The implication $\Rightarrow$ is obvious.

Consider coordinates $(y_1, \ldots, y_{n+1})$ such that $I(V) = (h)$ for some monic Weierstrass polynomial $h \in \mathbb{C}[y_1][[y_2, \ldots, y_{n+1}]]$ in the variable y_1. Choose a polydisk $D = D_1 \times D_{2,\ldots,n+1} \subset \mathbb{C} \times \mathbb{C}^n$ in the variables $(y_1, \ldots, y_{n+1})$ such that $(\partial D_1 \times D_{2,\ldots,n+1}) \cap V = \emptyset$. Moreover, we choose D such that it is a division neighborhood for h and $D \cap V \subset W$. We denote $\pi(y_1, \ldots, y_{n+1}) = (y_2, \ldots, y_{n+1})$. There exists a function $b_j \in G_{V \cap W}$ such that $\hat{g} - b_j \in (h^j)$ for any $j \in \mathbb{N}$. Since h^j is a monic polynomial in y_1 we can

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consider the remainder g_j of the Weierstrass division b_j/h^j. Since $b_j \in G_{D\cap V}$ then $g_j \in \mathcal{O}(\pi^{-1}(\pi(D \cap V))) \subset \mathcal{O}(D)$ for any $j \in \mathbb{N}$. Clearly $\hat{g}$ is u.t.f. along V. □

Remark 4.1. The results in this section can also be set out in the uniform semi-meromorphic case with minor adjustments. For the sake of simplicity we omit such a formulation.

4.3. *Main results.* Let us fix $\varphi = \exp((x \circ \varphi - x)\hat{u}\, \partial/\partial x) \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ and an irreducible component γ of $\mathrm{Fix}\,\varphi$. Denote $f = x \circ \varphi - x$. Consider an irreducible equation $g = 0$ of γ.

Definition 4.4. We call $m(\gamma)$ the *multiplicity* of γ in $f = 0$, i.e. the greatest $m \in \mathbb{N}$ such that g^m divides f.

Suppose γ is unipotent (see Definition 4.2). The germs φ_P (see §2) are tangent to the identity and then embedded in a formal flow. It turns out that such a formal flow depends holomorphically on $P \in \gamma$ or more precisely that $\log \varphi$ is u.t.f. along γ (Proposition 4.4).

The situation is more interesting if γ is non-unipotent. Denote $T = \partial(x \circ \varphi)/\partial x$. Given $P \in \gamma$ such that $T(P) \notin e^{2\pi i \mathbb{Q}} \setminus \{1\}$ there exists a formal vector field X_P satisfying $\varphi_P = \exp(X_P)$ (Proposition 4.1). We choose $X_0 = \log \varphi_0$, hence X_0 has vanishing linear part. Indeed X_P is well defined for $P \in \gamma \setminus T^{-1}\{e^{2\pi i \mathbb{Q}} \setminus \{1\}\}$ in a neighborhood of 0 if we require that the linear parts of X_P tend to 0 when P tends to 0. Roughly speaking X_P is analytic on P for $P \in \gamma \setminus T^{-1}(\mathbb{S}^1)$ (Lemma 4.7). The singularities of the mapping $P \to X_P$ defined in γ are meromorphic or more precisely $\log \varphi$ is u.s.m. along $\mathrm{Fix}\,\varphi$ (Proposition 4.5). Proposition 4.7 implies that, whenever φ_P is embedded in a formal flow for any $P \in \gamma$, then $P \to X_P$ can be extended analytically to the whole γ; in other words $\log \varphi$ is u.t.f. along $\mathrm{Fix}\,\varphi$.

Denote $f/g^{m(\gamma)}$ by h. We choose a set of coordinates $(y_1, \ldots, y_{n+1})$ and a polydisk $D = D_1 \times D_{2,\ldots,n+1} \subset \mathbb{C} \times \mathbb{C}^n$ in the variables $(y_1, \ldots, y_{n+1})$ such that $(\partial D_1 \times D_{2,\ldots,n+1}) \cap \gamma = \emptyset$ and:

- φ is holomorphic in D;
- up to a multiplicative unit g is of the form
$$g = y_1^l + y_1^{l-1} a_{l-1}(y_2, \ldots, y_{n+1}) + \cdots + a_0(y_2, \ldots, y_{n+1}),$$
where $a_j(0) = 0$ for any $0 \leq j \leq l-1$;
- D is a division neighborhood for $f = 0$; and
- $D_{2,\ldots,n+1}$ is a division neighborhood for $\Delta(g, h)$.

If γ is a non-unipotent component (see Definition 4.2) there are two more conditions:

- $D_{2,\ldots,n+1}$ is a division neighborhood for $\Delta(g, \partial f/\partial x)$; and
- $\ln(\partial(x \circ \varphi)/\partial x)$ is a holomorphic function defined in D such that the set $\{\partial(x \circ \varphi)/\partial x = 1\} \cap (D \cap \gamma)$ is connected.

This notation is fixed throughout this section.

PROPOSITION 4.4. *Let φ be a up-diffeomorphism and let γ be a unipotent irreducible component of* $\mathrm{Fix}\,\varphi$*. Then* $\log \varphi$ *is u.t.f. along γ.*

Proof. It suffices to prove that $\hat{u}$ belongs to $\lim_{\leftarrow} G_{D\cap\gamma}/I(\gamma)^j$ (see §2) by Lemma 4.6.

We have $(\log\varphi)^j(x) \in (g^{m(\gamma)+1}h)$ (see Definition 4.4) for $j \geq 2$. It is a consequence of g dividing $\partial g/\partial x$ if γ is fibered or $m(\gamma) > 1$ in the non-fibered case. By the Taylor expansion for the exponential mapping there exists a formal series $\hat{w}$ such that

$$\varphi = (x + \hat{u}g^{m(\gamma)}h + \hat{w}g^{m(\gamma)+1}h, x_1, \ldots, x_{n+1}).$$

Hence $u_1 = 1 = (x \circ \varphi - x)/f \in \mathcal{O}(D)$ satisfies $\hat{u} - u_1 \in (g)$.

We proceed by induction. Suppose that $\hat{u} = u_\alpha + g^\alpha \hat{v}$ where $u_\alpha \in G_{D\cap\gamma}$; the result is already proved for $\alpha = 1$. We want to find a similar expression for $\alpha + 1$. It suffices to prove the existence of $v \in G_{D\cap\gamma}$ such that $\hat{v} - v \in (g)$ since then $u_{\alpha+1} = u_\alpha + g^\alpha v$ satisfies $\hat{u} - u_{\alpha+1} \in (g^{\alpha+1})$. We have

$$x \circ \varphi = x \circ \exp(u_\alpha f\,\partial/\partial x) + g^{\alpha+m(\gamma)}h\hat{v} + g^{\alpha+m(\gamma)+1}h\hat{A},$$

for a certain formal series $\hat{A}$. The vector field $u_\alpha f\,\partial/\partial x$ is defined in a neighborhood of $D \cap \gamma$ and it vanishes in γ. Therefore, its exponential is defined in a neighborhood of $D \cap \gamma$. We define

$$d = (x \circ \varphi - x \circ \exp(u_\alpha f\,\partial/\partial x))/g^{\alpha+m(\gamma)}.$$

We obtain that $d \in G_{D\cap\gamma}$ by Lemma 4.1 since D is a division neighborhood for f and then for g. We have $h\hat{v} - d \in (g)$, thus $h\hat{v}$ converges by restriction to γ in $D \cap \gamma$. By applying Lemma 4.4 we get that $\hat{v}$ converges by restriction to γ in $D \cap \gamma$. Then there exists $v \in G_{D\cap\gamma}$ such that $\hat{v} - v \in (g)$. □

Let us fix $\varphi = \exp((x \circ \varphi - x)\hat{u}\,\partial/\partial x) \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ and an irreducible component γ of Fix φ. We can express $\hat{u}$ in the form $\sum_{j=0}^{\infty} u_j^\gamma g^j$ where $u_j^\gamma \in \mathbb{C}[y_1][[y_2, \ldots, y_{n+1}]]$ is a polynomial in y_1 such that $\deg_{y_1} u_j^\gamma \leq l - 1$ for any $j \geq 0$. These properties characterize the sequence $\{u_j^\gamma\}$ since u_0^γ is the remainder of the Weierstrass division $\hat{u}/g$, the series u_1^γ is the Weierstrass remainder of $[(\hat{u} - u_0^\gamma)/g]/g$, and so on.

We define $E_0^\gamma = E_1^\gamma = \emptyset$ and

$$E_k^\gamma = \left\{Q \in \gamma : \exists\, 1 < j \leq k \text{ s.t. } \left(\frac{\partial(x \circ \varphi)}{\partial x}\right)^j (Q) = 1\right\} \Big\backslash \left\{\frac{\partial(x \circ \varphi)}{\partial x} = 1\right\},$$

for any $k \geq 2$. We define $\pi(y_1, \ldots, y_{n+1}) = (y_2, \ldots, y_{n+1})$. The obstruction for φ to be u.t.f. along γ is the obstruction for the linearizability of the germs φ_P (see §2) for $P \in \gamma \setminus \{\partial(x \circ \varphi)/\partial x = 1\}$ (see Proposition 4.1). Since the lack of linearizability of φ_P implies that $(\partial(x \circ \varphi)/\partial x)\,(P) \in e^{2\pi i \mathbb{Q}}$ for any $P \in \gamma$, it is natural that the singularities of u_k^γ are located over $\pi(E_k^\gamma)$.

LEMMA 4.7. *(Generic expression) Let* $\varphi = \exp(\hat{u} f\,\partial/\partial x)$ *be a up-diffeomorphism and let* γ *be a non-unipotent irreducible component of* Fix φ. *Then* u_j^γ *is holomorphic in* $\pi^{-1}(D_{2,\ldots,n+1} \setminus \pi(E_j^\gamma))$ *for any* $j \geq 0$. *In particular* $u_j^\gamma \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ *for any* $j \geq 0$. *Moreover, we have*

$$\varphi = \exp\left(\left(\sum_{j=0}^{\infty} u_j^\gamma g^j\right) gh \frac{\partial}{\partial x}\right)$$

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in the neighborhood of every point in

$$\gamma \cap \left[D \setminus \pi^{-1}\left(\overline{\bigcup_{j \geq 0} \pi(E_j^{\gamma})} \right) \right].$$

Proof. The proof is based on the analysis of the Taylor expansion of the exponential mapping. We have

$$x \circ \varphi - \left(x + \sum_{k=1}^{\infty} \frac{1}{k!} (u_0^{\gamma})^k h^k (\partial g/\partial x)^{k-1} g \right) \in (g^2), \tag{1}$$

and then $\partial(x \circ \varphi)/\partial x - e^{u_0^{\gamma} h \partial g/\partial x} \in (g)$. We deduce that $u_0^{\gamma} h\, \partial g/\partial x$ converges by restriction to γ in $D \cap \gamma$. By Lemma 4.4 there exists $u_0' \in G_{D\cap\gamma}$ (see §2) such that $u_0^{\gamma} - u_0' \in (g)$. Since u_0^{γ} coincides with the remainder of the Weierstrass division u_0'/g then u_0^{γ} is holomorphic in $\pi^{-1}(D_{2,\ldots,n+1})$. We denote $L = e^{u_0^{\gamma} h \partial g/\partial x}$.

We are going to prove the result by induction on j. The result is true for u_0^{γ}. Suppose that u_j^{γ} is holomorphic in $\pi^{-1}(D_{2,\ldots,n+1} \setminus \pi(E_j^{\gamma}))$ for any $j < \alpha$. We will prove that u_α^{γ} is holomorphic in $\pi^{-1}(D_{2,\ldots,n+1} \setminus \pi(E_\alpha^{\gamma}))$. Let $u = \sum_{j=0}^{\alpha-1} u_j^{\gamma} g^j$; we denote $u_0 = u_0^{\gamma}$ and $\hat{v} = (\hat{u} - u)/g^{\alpha}$.

Let $\hat{B}_0 = H_0 = x$; we define $\hat{B}_{j+1} = \hat{u} g h\, \partial \hat{B}_j/\partial x$ and $H_{j+1} = ugh\, \partial H_j/\partial x$ for $j \geq 0$. The next step in the proof is proving by induction that $\hat{B}_j$ $(j \geq 0)$ can be expressed in the form

$$\hat{B}_j = H_j + \hat{v} g^{\alpha+1} h^j u_0^{j-1} (\partial g/\partial x)^{j-1} C_j + g^{\alpha+2} h \hat{D}_j,$$

where $\hat{D}_j$ belongs to $\mathbb{C}[[x, x_1, \ldots, x_n]]$ and $C_j \in \mathbb{C}$. The relations defining the sequence are $C_0 = 0$ and $C_j = (\alpha + 1)C_{j-1} + 1$ for any $j > 0$. The induction result is immediate for $j = 0$ and $j = 1$. We can develop

$$\hat{B}_{j+1} = (\partial \hat{B}_j/\partial x)\, (u + g^{\alpha} \hat{v}) gh$$

to obtain

$$\hat{B}_{j+1} \equiv \left[\frac{\partial H_j}{\partial x} + \frac{\partial}{\partial x}\left(\hat{v} g^{\alpha+1} h^j u_0^{j-1} \left(\frac{\partial g}{\partial x} \right)^{j-1} C_j \right) \right] (ugh + g^{\alpha+1} h \hat{v}),$$

modulo $(g^{\alpha+2} h)$. By using $u - u_0 \in (g)$ and equation (1) we get

$$\partial H_j/\partial x - u_0^j h^j (\partial g/\partial x)^j \in (g).$$

This leads us to the desired expression

$$\hat{B}_{j+1} - [H_{j+1} + u_0^j h^{j+1} (\partial g/\partial x)^j g^{\alpha+1} \hat{v} (1 + C_j(\alpha + 1))] \in (g^{\alpha+2} h)$$

for $\hat{B}_{j+1}$. We note that $C_j = ((\alpha + 1)^j - 1)/\alpha$. We have

$$x \circ \varphi \equiv x \circ \exp(ugh\, \partial/\partial x) + \sum_{j=1}^{\infty} \frac{C_j u_0^{j-1} h^{j-1} (\partial g/\partial x)^{j-1}}{j!} h g^{\alpha+1} \hat{v},$$

modulo $(g^{\alpha+2} h)$. We simplify to obtain

$$x \circ \varphi - \left[x \circ \exp(ugh\, \partial/\partial x) + \frac{(L^{\alpha+1} - L) h g^{\alpha+1} \hat{v}}{\alpha u_0 h\, \partial g/\partial x} \right] \in (g^{\alpha+2} h).$$

Since u is defined in a neighborhood of $(D \cap \gamma) \setminus \pi^{-1}(\pi(E_{\alpha-1}^{\gamma}))$ and $ugh\, \partial/\partial x$ vanishes on γ, then $x \circ \exp(ugh\, \partial/\partial x)$ is defined in a neighborhood of $(D \cap \gamma) \setminus \pi^{-1}(\pi(E_{\alpha-1}^{\gamma}))$. We have the inclusion

$$\{\partial(x \circ \varphi)/\partial x = 1\} \cap D \cap \gamma = \{L = 1\} \cap D \cap \gamma \subset \{u_0 h\, \partial g/\partial x = 0\},$$

since otherwise $\{\partial(x \circ \varphi)/\partial x = 1\} \cap D \cap \gamma$ is not connected. As a consequence we obtain that $(L-1)/(u_0 h\, \partial g/\partial x)$ is never vanishing in $D \cap \gamma$. The function

$$K \stackrel{\text{def}}{=} \frac{x \circ \varphi - x \circ \exp(ugh\, \partial/\partial x)}{1 + L + \cdots + L^{\alpha-1}} \frac{\alpha}{L} \frac{u_0 h\, \partial g/\partial x}{L-1}$$

is holomorphic in a neighborhood of $(D \cap \gamma) \setminus (\pi^{-1}(\pi(E_{\alpha-1}^{\gamma})) \cup E_{\alpha}^{\gamma})$. The polydisk D is a division neighborhood for g. By Lemmas 4.2 and 4.1 the series $K' = K/g^{\alpha+1}$ is a holomorphic function in a neighborhood of $(D \cap \gamma) \setminus (\pi^{-1}(\pi(E_{\alpha-1}^{\gamma})) \cup E_{\alpha}^{\gamma})$.

Since $(K'/h)h$ converges by restriction to γ in $(D \cap \gamma) \setminus \pi^{-1}(\pi(E_{\alpha}^{\gamma}))$ then so does K'/h by Lemmas 4.4 and 4.2. The series u_{α}^{γ} is the Weierstrass remainder of the division $[K'/h]/g$. Therefore u_{α}^{γ} is holomorphic in $\pi^{-1}(D_{2,\ldots,n+1} \setminus \pi(E_{\alpha}^{\gamma}))$.

Since $x \circ \varphi - x \circ \exp((\sum_{j=0}^{\alpha-1} u_j^{\gamma} g^{\alpha-1}) gh\, \partial/\partial x) \in (g^{\alpha+1})$ we can apply Lemmas 4.2 and 4.1 to prove the existence of a holomorphic function M_{α} defined in a neighborhood of $(D \cap \gamma) \setminus \pi^{-1}(\pi(E_{\alpha-1}^{\gamma}))$ and such that

$$x \circ \varphi - x \circ \exp\left(\left(\sum_{j=0}^{\alpha-1} u_j^{\gamma} g^{\alpha-1}\right) gh\, \partial/\partial x\right) = g^{\alpha+1} M_{\alpha}.$$

The previous equality running on $\alpha \in \mathbb{N}$ implies that

$$\varphi = \exp\left(\left(\sum_{j=0}^{\infty} u_j^{\gamma} g^{j}\right) gh \frac{\partial}{\partial x}\right),$$

in the neighborhood of every point in

$$\gamma \cap \left[D \setminus \pi^{-1}\left(\overline{\bigcup_{j \geq 0} \pi(E_j^{\gamma})}\right)\right]. \qquad \square$$

Remark 4.2. There is not always uniform formal transversality. The infinitesimal generator of $\varphi = (x + y - x^2, y)$ is not u.t.f. along $\gamma \equiv \{y = x^2\}$. We claim that φ_P (see §2) cannot be embedded in a formal flow if $P \in \bigcup_{j \geq 0} E_j^{\gamma} \subset \operatorname{Fix} \varphi$. Otherwise φ_P is formally conjugated to $j^1 \varphi_P$ (Proposition 4.1) and then periodic; that is not possible since φ_P is nonlinear and polynomial. We obtain that $\log \varphi$ is not u.t.f. along γ since $0 \in \overline{\bigcup_{j \geq 0} E_j^{\gamma}}$.

Consider $\varphi = \exp(\hat{u} f\, \partial/\partial x) \in \operatorname{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ and let γ be a non-unipotent (see Definition 4.2) irreducible component of $\operatorname{Fix} \varphi$. When calculating the coefficients of the linearizing mapping of φ_P for $P \in \gamma$ we obtain fractions whose denominator is of the form $((\partial(x \circ \varphi)/\partial x)^p - 1)(P)$ for some $p \in \mathbb{N}$. The obstructions to the linearizability of φ_P for $P \in \gamma$ lead to the lack of u.t.f. character of the infinitesimal generator of φ along γ (see Proposition 4.1). It is then natural to think that $\log \varphi$ has meromorphic nature. Indeed, this is the case; the functions u_j^{γ} are meromorphic for any $j \geq 0$.

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PROPOSITION 4.5. *Let* $\varphi = \exp(\hat{u} f\, \partial/\partial x) \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ *and let* γ *be a non-unipotent irreducible component of* Fix φ. *The coefficients of* u_j^{γ} *in* $\mathbb{C}\{y_2, \ldots, y_{n+1}\}$ *are meromorphic in* $D_{2,\ldots,n+1}$ *for any* $j \geq 0$.

Proof. We denote $L = e^{u_0^{\gamma} h \partial g/\partial x}$; we know that $L - \partial(x \circ \varphi)/\partial x \in (g)$. Let P be a point of $(D \cap \gamma) \setminus \{L = 1\}$. Since $\{L = 1\} \cap \gamma = \{\partial f/\partial x = 0\} \cap \gamma$ then $f = 0$ is transversal to $\partial/\partial x$ in P. In particular γ is smooth at P and it is the only irreducible component of Fix φ passing through P. We can find new coordinates $(z, x_1, \ldots, x_n)$ in the neighborhood of P such that $\partial/\partial x$ and $g = 0$ become $\partial/\partial z$ and $z = 0$ respectively. In these coordinates φ is of the form

$$\varphi(z, x_1, \ldots, x_n) = (a_1^1(x_1, \ldots, x_n)z + a_2^1(x_1, \ldots, x_n)z^2 + \ldots, x_1, \ldots, x_n).$$

Consider a small enough open neighborhood $W(P)$ of P in $D \cap \gamma$, which is parameterized by $(x_1, \ldots, x_n)$. We ask that $a_1^1 - 1$ never vanishes in $W(P)$; this is possible since $L(P) \neq 1$. We claim that there exists σ such that $\sigma^{-1} \circ \varphi \circ \sigma = (a_1^1 z, z_1, \ldots, z_n)$ of the form

$$\sigma = \left(z + \sum_{j=2}^{\infty} \sigma_j(x_1, \ldots x_n)z^j, x_1, \ldots, x_n\right),$$

where σ_j is holomorphic in $W(P) \setminus E_{j-1}^{\gamma}$ and meromorphic in $W(P)$ for any $j \geq 2$. We are going to construct a sequence of diffeomorphisms (φ_k) such that $\varphi_1 = \varphi$ and φ_k is of the form

$$\varphi_k = \left(a_1^1(x_1, \ldots, x_n)z + \sum_{j=k+1}^{\infty} a_j^k(x_1, \ldots, x_n)z^j, x_1, \ldots, x_n\right),$$

where a_j^k is holomorphic in $W(P) \setminus E_{k-1}^{\gamma}$ and meromorphic in $W(P)$ for any $j \geq 2$. There exists $\tau_k = (z + b_{k+1}(x_1, \ldots, x_n)z^{k+1}, x_1, \ldots, x_n)$ such that

$$z \circ \tau_k^{-1} \circ \varphi_k \circ \tau_k - a_1^1(x_1, \ldots, x_n)z \in (z^{k+2}).$$

Moreover, τ_k is unique and $((a_1^1)^{k+1} - a_1^1)b_{k+1} = a_{k+1}^k$. By hypothesis, b_{k+1} is holomorphic in $W(P) \setminus E_k^{\gamma}$ and meromorphic in $W(P)$. We define the diffeomorphism $\varphi_{k+1} = \tau_k^{-1} \circ \varphi_k \circ \tau_k$. Consider $\sigma = \lim_{j\to\infty} \tau_1 \circ \cdots \circ \tau_j$ where the limit is considered in the Krull topology. Clearly σ_j is holomorphic in $W(P) \setminus E_{j-1}^{\gamma}$ and meromorphic in $W(P)$ for any $j \geq 2$. Then we define $Y_P = (u_0^{\gamma} h(\partial g/\partial x))\,(0, x_1, \ldots, x_n)z\, \partial/\partial z$. We have $\varphi = \exp(\sigma_* Y_P)$ since $(a_1^1 z, x_1, \ldots, x_n) = \exp(Y_P)$. We obtain

$$\sigma_* Y_P = \left(\left[\left(u_0^{\gamma} h \frac{\partial g}{\partial x}\right)(0, x_1, \ldots, x_n)\left(1 + \sum_{j=2}^{\infty} j\sigma_j z^{j-1}\right)z\right] \circ \sigma^{-1}\right)\frac{\partial}{\partial z}.$$

We can change coordinates to obtain

$$\sigma_* Y_P = \left(u_0^{\gamma} + \sum_{j=1}^{\infty} v_j^P(x_1, \ldots, x_n)g^j\right)gh\frac{\partial}{\partial x},$$

where $v_j^P \in \mathcal{O}(W(P) \setminus E_j^{\gamma})$ and is meromorphic in $W(P)$ for any $j \geq 1$.

Fix $Q \in W(P)$ such that $\pi(Q) \notin \pi(\overline{\bigcup_{j \in \mathbb{N}} E_j^\gamma} \cup \{L=1\})$. There exists an open neighborhood W' of Q in $W(P)$ such that $\pi(\overline{\bigcup_{j \in \mathbb{N}} E_j^\gamma} \cup \{L=1\})$ and $\pi(W')$ are disjoint. This implies that the series $u_0^\gamma + \sum_{j=1}^\infty u_j^\gamma g^j$ and $u_0^\gamma + \sum_{j=1}^\infty v_j^P g^j$ belong to $\lim_{\leftarrow} G_{W'}/I(W')^r$. By Proposition 4.2 applied to the fibers of $dx_1 = \cdots = dx_n = 0$ we obtain

$$u_0^\gamma + \sum_{j=1}^\infty u_j^\gamma g^j = u_0^\gamma + \sum_{j=1}^\infty v_j^P g^j. \tag{2}$$

We deduce that $u_1^\gamma - v_1^P = 0$ contains W'. Thus we can extend the meromorphic function $(u_1^\gamma)_{|\gamma}$ to $W(P)$. By considering every point $P \notin \gamma \cap \{\partial f/\partial x = 0\}$ we get that $(u_1^\gamma)_{|\gamma}$ is meromorphic in

$$(D \cap \gamma) \setminus (E_1^\gamma \cap \{\partial f/\partial x = 0\}) = D \cap \gamma.$$

We can apply the Weierstrass division theorem to obtain that u_1^γ is meromorphic in $\pi^{-1}(D_{2,\ldots,n+1})$. By using equation (2) and an induction process we obtain that $(u_j^\gamma)_{|\gamma}$ is meromorphic in

$$(D \cap \gamma) \setminus (E_j^\gamma \cap \{\partial f/\partial x = 0\}) = D \cap \gamma,$$

for $j \geq 1$. Thus u_j^γ is meromorphic in $\pi^{-1}(D_{2,\ldots,n+1})$ for any $j \in \mathbb{N}$. □

PROPOSITION 4.6. *Let $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. Then* $\log \varphi$ *is t.f. along* Fix φ. *Let γ be an irreducible component of* Fix φ. *If γ is unipotent (see Definition 4.2) then* $\log \varphi$ *is u.t.f. along γ. Anyway, it is always u.s.m. along γ.*

Proof. The results on γ are a consequence of Propositions 4.4 and 4.5 and Lemma 4.7. Then $\log \varphi$ is t.f. along Fix φ by Lemma 4.5. □

We claim that the obstruction for $\log \varphi$ to be u.t.f. along a non-unipotent component γ is the existence of germs φ_P for $P \in \gamma$ (see §2) which cannot be embedded in a formal flow.

PROPOSITION 4.7. *Let $\varphi = \exp(\hat{u} f\, \partial/\partial x) \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. Consider a non-unipotent irreducible component γ of* Fix φ. *Then* $\log \varphi$ *is u.t.f. along γ if and only if there exists a neighborhood of the origin U such that φ_P (see §2) is embedded in a formal flow for any $P \in \gamma \cap U$.*

Proof. The implication $\Rightarrow$ is trivial. We denote $L = e^{u_0^\gamma h \partial g/\partial x}$. We define the set $F = \gamma \cap [\{dL \wedge dg = 0\} \cup \{L = 1\}]$. Since the function L is non-constant in γ then F is a proper analytic subset of γ. We can suppose that the origin belongs to every irreducible component of $F \cap D$ by shrinking D. Moreover we can suppose that φ_Y is embedded in a formal flow for any $Y \in \gamma \cap D$. Next, we use the strategy in the proof of Proposition 4.5 in order to show that u_j^γ is analytic in $\pi^{-1}(D_{2,\ldots,n+1} \setminus \pi(F))$ for any $j \geq 0$. Since u_j^γ is analytic in $\pi^{-1}(D_{2,\ldots,n+1} \setminus \pi(E_j^\gamma))$ by Lemma 4.7 we deduce that u_j^γ is analytic in $\pi^{-1}(D_{2,\ldots,n+1} \setminus [\pi(E_j^\gamma) \cap \pi(F)])$ for any $j \geq 0$. The end of the proof is an application of Riemann's theorem.

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Let P be a point in $(D \cap \gamma) \setminus F$. There exists a system of coordinates $(z, x_1, \ldots, x_n)$ in a neighborhood of P (see proof of Proposition 4.5) such that $g = 0$ and $\partial/\partial x$ become $z = 0$ and $\partial/\partial z$ respectively. Consider a neighborhood $W(P)$ of P in $D \cap \gamma$ such that $W(P) \cap F = \emptyset$.

Consider the notation in the proof of Proposition 4.5. Let $r \in \mathbb{N}$; suppose a_j^k and b_q are holomorphic in $W(P)$ for all $k \leq r$, $j \geq k+1$ and $1 < q \leq r$. We claim that b_{r+1} is holomorphic in $W(P)$ and then $a_j^{r+1} \in \mathcal{O}(W(P))$ for any $j \geq r+2$. We have $[a_1^1((a_1^1)^r - 1)]b_{r+1} = a_{r+1}^r$ by the proof of Proposition 4.5. Consider any point $Q \in W(P)$ such that $(a_1^1(Q))^r = L(Q)^r = 1$. The diffeomorphism φ_Q is linearizable by Proposition 4.1. There exists an element of $h \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $h^{-1} \circ \varphi_Q \circ h = a_1^1(Q)z$ and $j^r h = j^r((\tau_1 \circ \cdots \circ \tau_{r-1})_Q)$ by Proposition 4.3. Now

$$(\tau_1 \circ \cdots \circ \tau_{r-1})_Q^{-1} \circ h = z + C_{r+1}(Q)z^{r+1} + O(z^{r+2})$$

for some $C_{r+1}(Q) \in \mathbb{C}$. By construction we obtain

$$[a_1^1(Q)\,((a_1^1(Q))^r - 1)]C_{r+1}(Q) = a_{r+1}^r(Q).$$

Therefore the set $a_{r+1}^r = 0$ contains the set $(a_1^1)^r = 1$. We have

$$\emptyset = F \cap W(P) \supset \{da_1^1 \wedge dz = 0\} \cap W(P).$$

Since a_1^1 does not depend on z then da_1^1 never vanishes in $W(P)$. Therefore any hypersurface $a_1^1 = cte$ is locally irreducible in $W(P)$. Since $a_{r+1}^r \in I(a_1^1 - \mu)$ for every r-root of the unit μ then

$$b_{r+1} = \frac{a_{r+1}^r}{a_1^1((a_1^1)^r - 1)} = \frac{a_{r+1}^r}{a_1^1 \prod_{\lambda^r=1}(a_1^1 - \lambda)}$$

is analytic in $W(P)$. By proceeding as in the proof of Proposition 4.5 we obtain that u_j^γ is analytic in $\pi^{-1}(D_{2,\ldots,n+1} \setminus [\pi(E_j^\gamma) \cap \pi(F)])$ for any $j \geq 0$. Since all the irreducible components of F adhere to 0 and $0 \notin \pi(E_j^\gamma)$ then the codimension of $\pi(F) \cap \pi(E_j^\gamma) \subset D_{2,\ldots,n+1}$ is greater than or equal to 2. By Hartogs' theorem u_j^γ is analytic in $\pi^{-1}(D_{2,\ldots,n+1})$ for $j \geq 0$. Hence $\hat{u} = \sum_{j=0}^\infty u_j^\gamma g^j$ is u.t.f. along γ. □

5. *Formal classification*

The goal of this section is to provide a complete system of invariants for the formal classification of up-diffeomorphisms. More precisely, we describe the formal moduli modulo analytic change of coordinates.

5.1. *Nature of the residue functions.*

The formal invariants attached to the germs φ_P (see §2) for $P \in \mathrm{Fix}\,\varphi$ are included in the formal invariants of the up-diffeomorphism φ. In this subsection we describe the nature of such invariants.

Let $\tau = \exp(\hat{h}(x)\,\partial/\partial x) \in \mathrm{Diff}_u(\mathbb{C}, 0)$. We define the *order of contact* $\nu(\tau)$ between τ and Id as the order of the function $\tau(x) - x$ at 0; it does not depend on the choice of the coordinate x. There exists $X = h(x)\,\partial/\partial x \in \mathcal{X}_N(\mathbb{C}, 0)$ such that $\tau(x) - \exp(h\,\partial/\partial x)\,(x)$

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belongs to the ideal $(x^{2\nu(\tau)})$. We consider the *dual form* of $\log \tau$, i.e. the unique formal meromorphic form $\hat{\omega}$ such that $\hat{\omega}(\log \tau) = 1$. The form $\hat{\omega} = dx/\hat{h}(x)$ can be expressed as $\omega + \hat{a}(x)\, dx$ where ω is the dual form of X and $\hat{a} \in \mathbb{C}[[x]]$. We define the residues $\mathrm{Res}(\tau)$ and $\mathrm{Res}(\log \tau)$ as the residue of $\hat{\omega}$ at 0. These residues coincide with the residue of ω at 0 since $\hat{\omega} - \omega$ has no poles. Moreover, the couple $(\nu(\tau), \mathrm{Res}(\tau))$ provides a complete system of formal invariants in $\mathrm{Diff}_u(\mathbb{C}, 0)$. These definitions can be extended for $\tau \in \mathrm{Diff}_u(\mathbb{C}, x_0)$ by applying them to $(x - x_0) \circ \tau \circ (x + x_0) \in \mathrm{Diff}_u(\mathbb{C}, 0)$.

Fix $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. Let γ be a unipotent non-fibered irreducible component of $\mathrm{Fix}\, \varphi$. We have $\varphi_P \in \mathrm{Diff}_u(\mathbb{C}, P)$ for any $P \in \gamma$ but $Q \mapsto \mathrm{Res}(\varphi_Q)$ is not continuous in γ. An example is provided by $\varphi = \exp(x^2(x-y)^2\, \partial/\partial x)$, we have that $\mathrm{Res}(\varphi_{(0,y)}) = 2/y^3$ if $y \neq 0$ whereas $\mathrm{Res}(\varphi_{(0,0)}) = 0$. We define

$$S = \left\{ P \in \gamma : \nu(\varphi_P) > \min_{Q \in \gamma} \nu(\varphi_Q) \right\}.$$

The set S is a proper analytic subset of γ. The function $\mathrm{Res}_\gamma(\varphi) : \gamma \setminus S \to \mathbb{C}$ defined by $\mathrm{Res}_\gamma(\varphi)\,(P) = \mathrm{Res}(\varphi_P)$ is analytic. Later on we prove that $\mathrm{Res}_\gamma(\varphi)$ can be extended meromorphically to the whole γ as in the example. Let γ be a non-unipotent irreducible component of $\mathrm{Fix}\, \varphi$; we define $\mathrm{Res}_\gamma(\varphi) : \gamma \to \mathbb{C}$ given by

$$\mathrm{Res}_\gamma(\varphi)\,(Q) = \frac{1}{\ln([\partial(x \circ \varphi)/\partial x](Q))},$$

where we choose the determination of log such that $\log 1 = 0$. This definition makes sense, since for $Q = (x_0, x_1^0, \ldots, x_n^0) \in \gamma$ such that $(\partial(x \circ \varphi)/\partial x)\,(Q) \notin e^{2\pi i \mathbb{Q}}$ we have that the residue of $(\log \varphi)_{|\bigcap_{j=1}^n (x_j = x_j^0)} \in \hat{\mathcal{X}}(\mathbb{C}, x_0)$ is equal to $\mathrm{Res}_\gamma(\varphi)\,(Q)$.

PROPOSITION 5.1. *Let* $\varphi = \exp(\hat{u} f\, \partial/\partial x) \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ *and let* γ *be a non-fibered (see §2) irreducible component of* $\mathrm{Fix}\, \varphi$*. Suppose that* γ *is transversal to* $\partial/\partial x$ *in a neighborhood of* 0*. Then* $\mathrm{Res}_\gamma(\varphi)$ *is a meromorphic function of* γ*.*

Proof. Up to a change of coordinates $\sigma = (x + h(x_1, \ldots, x_n), x_1, \ldots, x_n)$ we can suppose that $\gamma \equiv \{x = 0\}$. Since $\hat{u}$ is convergent by restriction to γ then $\hat{u}(0, x_1, \ldots, x_n)$ belongs to $\mathbb{C}\{x_1, \ldots, x_n\}$. We can express f in the form $\sum_{j=\nu}^{\infty} f_j(x_1, \ldots, x_n) x^j$ where $\nu = \min_{P \in \gamma} \nu(\varphi_P)$; in fact we have that $f_\nu(P) \neq 0$ if and only if $\nu(\varphi_P) = \nu$ for $P \in \gamma$. Thus $\mathrm{Res}_\gamma(\varphi)$ is holomorphic in $\gamma \setminus \{f_\nu = 0\}$.

We consider the transformation

$$\chi(z, x_1, \ldots, x_n) = (z f_\nu(x_1, \ldots, x_n), x_1, \ldots, x_n).$$

Then χ is a change of coordinates outside of $f_\nu = 0$. We define the diffeomorphism $\tilde{\varphi} = \chi^{-1} \circ \varphi \circ \chi$; we obtain

$$\tilde{\varphi} = \exp\left((\hat{u} \circ \chi)\, (f_\nu)^\nu \left(z^\nu + \sum_{j=\nu+1}^{\infty} f_j (f_\nu)^{j-(\nu+1)} z^j \right) \frac{\partial}{\partial z} \right).$$

We define $\hat{\varphi} = \exp((\log \tilde{\varphi})/f_\nu^\nu)$. Since $(\hat{u} \circ \chi - \hat{u})\,(0, x_1, \ldots, x_n) \equiv 0$ then the function $(x_1, \ldots, x_n) \mapsto \nu(\hat{\varphi}_{(0,x_1,\ldots,x_n)})$ is equal to the constant ν. Hence the function $\mathrm{Res}_\gamma(\hat{\varphi}) :$

$\gamma \to \mathbb{C}$ is holomorphic in a neighborhood of the origin. Note that f_ν does not depend on z; thus we obtain

$$\mathrm{Res}_\gamma(\varphi) = \mathrm{Res}_\gamma(\tilde{\varphi}) = \mathrm{Res}_\gamma(\hat{\varphi})/f_\nu^\nu.$$

Clearly $\mathrm{Res}_\gamma(\varphi)$ is a meromorphic function of γ. □

PROPOSITION 5.2. *Let* $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ *and let* γ *be a non-fibered irreducible component of* $\mathrm{Fix}\,\varphi$. *Then* $\mathrm{Res}_\gamma(\varphi)$ *is a meromorphic function of* γ.

Proof. Suppose that γ does not contain orbits of $\partial/\partial x$. As a consequence there exists an irreducible Weierstrass polynomial

$$H = x^k + a_{k-1}(x_1, \ldots, x_n)x^{k-1} + \cdots + a_0(x_1, \ldots, x_n)$$

such that $\gamma \equiv \{H = 0\}$. We denote $\pi(x, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. Let D be the critical locus of the projection $\pi_{|\gamma}$. We denote by DD the critical locus of the projection $\pi_{|D}$. The analytic set DD has codimension at least 2 in γ. By Proposition 5.1 the function $\mathrm{Res}_\gamma(\varphi)$ is meromorphic in $\gamma \setminus D$. Suppose that $\mathrm{Res}_\gamma(\varphi)$ is meromorphic in $\gamma \setminus DD$. Then there exists a polynomial $R = \sum_{j=0}^{k-1} b_j(x_1, \ldots, x_n)x^j$ such that $\mathrm{Res}_\gamma(\varphi) \equiv R_{|\gamma}$ by the Weierstrass division. The coefficients b_j $(0 \leq j \leq k-1)$ are meromorphic in a neighborhood of the origin deprived of $\pi(DD)$. Since the codimension of $\pi(DD)$ is at least 2 in $\mathbb{C}^n$ then b_j is meromorphic in a neighborhood of the origin for any $0 \leq j \leq k-1$. Thus $\mathrm{Res}_\gamma(\varphi) = R_{|\gamma}$ is meromorphic.

Let $P \in D \setminus DD$. The set D is transversal to $\partial/\partial x$ at P and then smooth. Since $\dim D = n-1$ then there exist coordinates $(z, z_1, \ldots, z_n)$ centered at P such that $\partial/\partial x$ and D become $\partial/\partial z$ and $\{z = 0\} \cap \{z_1 = 0\}$ respectively. We consider the ramification

$$T(z, z_1, z_2, \ldots, z_n) = (z, z_1^{k!}, z_2, \ldots, z_n).$$

The set $T^{-1}(\gamma)$ is the union of at most k hypersurfaces in the neighborhood of P, all of them being transversal to $\partial/\partial z$. The function $\mathrm{Res}_\gamma(\varphi) \circ T = \mathrm{Res}_{T^{-1}(\gamma)}(T^*\varphi)$ is meromorphic in $T^{-1}(\gamma)$ by Proposition 5.1. We undo the ramification to obtain that $\mathrm{Res}_\gamma(\varphi)$ is meromorphic in a neighborhood of P and then in $\gamma \setminus DD$.

If γ contains orbits of $\partial/\partial x$ we can reduce the situation to the previous one by blowing up fibered submanifolds of γ. We are done since the field of meromorphic functions in γ is invariant by blow-up. □

Next we express the function $\mathrm{Res}_\gamma(\varphi)$ in a convenient way.

LEMMA 5.1. *Let* $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ *and let* γ *be a non-fibered irreducible component of* $\mathrm{Fix}\,\varphi$. *Then there exist* $A \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ *and* $B \in \mathbb{C}\{x_1, \ldots, x_n\}$ *such that* $\mathrm{Res}_\gamma(\varphi) = (A/B)_{|\gamma}$.

Proof. There exist $A', B' \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\mathrm{Res}_\gamma(\varphi) = (A'/B')_{|\gamma}$ by Proposition 5.2. We denote $\pi(x, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$ and $Z = \gamma \cap \{B' = 0\}$. Let $g = 0$ be an irreducible equation of γ. We have $\dim Z \leq n-1$ and then $\dim \pi(Z) \leq n-1$. Consider h in $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $h_{|\pi(Z)} \equiv 0$ but $h_{|\gamma} \neq 0$; that is possible since γ is non-fibered. We obtain

$$h \in IZ(g, B') \quad \Longrightarrow \quad h \in \sqrt{(g, B')} \quad \Longrightarrow \quad h^m = Jg + KB',$$

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for some $m \in \mathbb{N}$ and $J, K \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. The function

$$\frac{A}{B} = \frac{KA'}{KB' + Jg} = \frac{KA'}{h^m}$$

is equal to $\mathrm{Res}_\gamma(\varphi)$ in γ. □

5.2. *The homological equation.* We can linearize the problem of formal classification of up-diffeomorphisms. It can be reduced to deal with equations of the form

$$\frac{\partial \alpha}{\partial x} = \frac{A}{f},$$

where $A, f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. This equation is called the *homological equation*.

Definition 5.1. Consider the decomposition $f_1^{l_1} \cdots f_p^{l_p} F_1^{m_1} \cdots F_q^{m_q}$ of f into irreducible factors. We suppose that $f_j = 0$ is non-fibered and $F_k = 0$ is fibered for all $1 \leq j \leq p$ and $1 \leq k \leq q$. Moreover we suppose that $F_k \in \mathbb{C}\{x_1, \ldots, x_n\}$ for $1 \leq k \leq q$. Now we define $f_N = \prod_{j=1}^{p} f_j^{l_j}$ and $f_F = \prod_{k=1}^{q} F_k^{m_k}$. We say that a meromorphic germ of function α is *special* with respect to f if it can be expressed in the form

$$\alpha = \frac{\beta}{f_1^{l_1 - 1} \cdots f_p^{l_p - 1} f_F},$$

for some β in $\mathbb{C}\{x, x_1, \ldots, x_n\}$. A homological equation $\partial\alpha/\partial x = A/f$ is *special* if it has a special solution (with respect to f). Most of the time we drop the expression 'with respect to f' since it is clear from the context.

Definition 5.2. Let $f = f_F f_N \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ (see Definition 5.1). We say that the homological equation $\partial\alpha/\partial x = A/f$ is *free of residues* if the one-dimensional 1-form

$$\left(\frac{A}{f_N}(x, x_1^0, \ldots, x_n^0)\right) dx \in \Omega_1\left(\bigcap_{j=1}^{n} \{x_j = x_j^0\}\right)$$

has residue zero in the neighborhood of every point $P = (x^0, x_1^0, \ldots, x_n^0)$ such that $f_N(P) = 0$ and $f_N(x, x_1^0, \ldots, x_n^0) \not\equiv 0$.

Clearly a special homological equation is free of residues. Roughly speaking in the rest of this subsection we study the gap between the concepts of 'free of residues' and 'special' for homological equations. We determine generic conditions in which free of residues implies special (see Lemma 5.2, Corollary 5.1 and Proposition 5.5). Moreover, in the remaining cases we prove that every free of residues homological equation admits a meromorphic solution (Proposition 5.3).

Generically a free of residues homological equation $\partial\alpha/\partial x = A/f$ is special and, in particular, in the finite codimension case, i.e. $f(x, 0, \ldots, 0) \not\equiv 0$.

LEMMA 5.2. *Let $E \equiv [\partial\alpha/\partial x = A/f]$ be a free of residues homological equation. Suppose that $f_N(x, 0, \ldots, 0) \not\equiv 0$ (see Definition 5.1). Then E is special.*

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Proof. We can suppose that f_F (see Definition 5.1) is a unit since, provided a special solution α' of $\partial\alpha'/\partial x = A/f_N$, the function $\alpha = \alpha'/f_F$ is a special solution of E. Consider a small neighborhood $V' \times W$ of the origin in $\mathbb{C} \times \mathbb{C}^n$ contained in the domain of definition of E. We can suppose that V' is simply connected. Let $V \subset \mathbb{C}$ be a simply connected open set such that $0 \notin V \subset V'$. We can suppose that $(V \times W) \cap \{f_N = 0\} = \emptyset$.

The equation E has a solution $\alpha \in \mathcal{O}(V \times W)$. It can be extended to the set $(V \times W) \setminus \{f_N = 0\}$ by analytic continuation since the residues vanish. Consider a point $Q \in \{f_N = 0\}$ such that $\partial/\partial x$ is transversal to $f_N = 0$ at Q. There exist coordinates $(z, z_1, \ldots, z_n)$ centered at Q such that $\partial/\partial x$ and $f_N = 0$ become $\partial/\partial z$ and $z = 0$ respectively. By integrating with respect to z we obtain a special solution α_Q in the neighborhood of Q. Now $\partial(\alpha - \alpha_Q)/\partial x = 0$ implies that $\alpha - \alpha_Q$ is holomorphic in a neighborhood of Q. Hence α is special in a neighborhood of Q. Since $\partial/\partial x$ is transversal to $f_N = 0$ except at a set whose codimension is greater than 1, then α is a special solution of E. □

COROLLARY 5.1. *For $n = 1$ (i.e. $A, f \in \mathbb{C}\{x, x_1\}$) we have that a homological equation $\partial\alpha/\partial x = A/f$ is free of residues if and only if it is special.*

In general the vanishing of the residues does not imply the existence of a special solution. An example is given by $\partial\alpha/\partial x = 1/(z - xy)^2$. On the one hand it is free of residues since $\alpha = 1/((z - xy)y)$ is a solution. On the other hand there is no special solution $\beta/(z - xy)$ since otherwise we obtain $1 = (\partial\beta/\partial x)\,(z - xy) + \beta y \in (y, z)$.

Definition 5.3. Let $f = f_N f_F \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ (see Definition 5.1). Let $\prod_{j=1}^{p} f_j^{l_j}$ be the decomposition into irreducible factors of f_N. We define the evil set $S(f)$ of f as the union of the fibered varieties contained in $\prod_{l_j>1} f_j = 0$. The set $S(f)$ is analytic and of codimension at least 2.

We want to obtain a solution as close as possible to be special for a free of residues equation $\partial\alpha/\partial x = A/f$. The obstruction is somehow located in the evil set $S(f)$ (see Proposition 5.5). Moreover, a free of residues $\partial\alpha/\partial x = A/f$ always admits a meromorphic solution of the form β/J where β is special and J vanishes to order big enough in $S(f)$.

PROPOSITION 5.3. *Let $E \equiv [\partial\alpha/\partial x = A/f]$ be a free of residues homological equation. Let $H \in \mathbb{C}\{x_1, \ldots, x_n\} \setminus \{0\}$ vanishing in the evil set of f. Then there exists $k \in \mathbb{N} \cup \{0\}$ depending only on f such that $\partial\alpha/\partial x = (AH^k)/f$ is special.*

5.2.1. *Proof of Proposition 5.3.* The proof is based on a reduction to the case $S(f) = \emptyset$ by doing a sequence of blow-ups centered in fibered manifolds. We associate a cocycle $\delta^0(E)$ to E and then we prove that E is special if and only if the class of $\delta^0(E)$ in a certain cohomology group H^1 is zero. Anyway, the class of $\delta^0(E)$ is always vanishing when we consider the right group of homology. As a consequence we find a meromorphic solution of E.

Let $\prod_{j=1}^{p} f_j^{l_j}$ be the decomposition in irreducible factors of f_N (see Definition 5.1). We define the following sheafs:

- $\mathcal{O}_R(f)$ of functions α such that $\alpha f/(f_1 \ldots f_p)$ is holomorphic;
- $\mathcal{O}_D(f)$ is the subsheaf of $\mathcal{O}_R(f)$ of first integrals of $\partial/\partial x$; and
- $\mathcal{O}_Q(f)$ is the quotient sheaf $\mathcal{O}_R(f)/\mathcal{O}_D(f)$.

The sheaf $\mathcal{O}_R(f)$ is the sheaf of special functions. Note that the sheafs $\mathcal{O}_R(f)$, $\mathcal{O}_D(f)$ and $\mathcal{O}_Q(f)$ do not change if we replace f with $f_F \prod_{l_j>1} f_j^{l_j}$.

The vanishing of the residues of E implies that f_j divides A if $l_j = 1$. By replacing E with $\partial\alpha/\partial x = (A/\prod_{l_j=1} f_j)/(f_F \prod_{l_j>1} f_j^{l_j})$ we can suppose that $l_j > 1$ for any $1 \leq j \leq p$.

Consider a small polydisk $\Delta \subset \mathbb{C}^{n+1}$ such that $A \in \mathcal{O}(\Delta)$. Now for every $Y \in \Delta \setminus S(f)$ there exists a solution $\alpha_Y \in \mathcal{O}_R(f)$ of E defined in a neighborhood U_Y of Y by Lemma 5.2. For another $Y' \in \Delta \setminus S(f)$ we have $\partial(\alpha_Y - \alpha_{Y'})/\partial x = 0$ and then $\alpha_Y - \alpha_{Y'} \in \mathcal{O}_D(f)\,(U_Y \cap U_{Y'})$. Therefore E defines a unique section in $H^0(\Delta \setminus S(f), \mathcal{O}_Q(f))$. Conversely, let $B \in H^0(\Delta \setminus S(f), \mathcal{O}_Q(f))$; we have that $f\,\partial B/\partial x$ is holomorphic in $\Delta \setminus S(f)$. Since $\operatorname{cod} S(f) \geq 2$ then $f\,\partial B/\partial x$ can be extended to Δ. Thus $\partial\alpha/\partial x = (f\,\partial B/\partial x)/f$ is a free of residues homological equation defined in Δ. We have proved the following result.

LEMMA 5.3. Fix $f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. *The free of residues homological equations defined in* Δ *of the form* $\partial\alpha/\partial x = A/f$ *are in a bijective correspondence with* $H^0(\Delta \setminus S(f), \mathcal{O}_Q(f))$.

The exact sequence $0 \to \mathcal{O}_D(f) \xrightarrow{i} \mathcal{O}_R(f) \xrightarrow{p} \mathcal{O}_Q(f) \to 0$ provides the long exact sequence

$$\begin{aligned} 0 \to\ & H^0(\Delta \setminus S(f), \mathcal{O}_D(f)) \xrightarrow{i^0} H^0(\Delta \setminus S(f), \mathcal{O}_R(f)) \\ \xrightarrow{p^0}\ & H^0(\Delta \setminus S(f), \mathcal{O}_Q(f)) \xrightarrow{\delta^0} H^1(\Delta \setminus S(f), \mathcal{O}_D(f)) \to \cdots . \end{aligned}$$

The set of special equations of the form $\partial\alpha/\partial x = A/f$ is the image of p^0. We have

$$0 \to \frac{H^0(\Delta \setminus S(f), \mathcal{O}_Q(f))}{p^0(H^0(\Delta \setminus S(f), \mathcal{O}_R(f)))} \xrightarrow{\delta^0} H^1(\Delta \setminus S(f), \mathcal{O}_D(f)).$$

Since the existence of the evil set is itself an obstruction for the vanishing of $H^1(\Delta \setminus S(f), \mathcal{O}_D(f))$ we try to remove it by blow-up. Consider a sequence of blow-ups $\pi_1, \pi_2, \ldots, \pi_d$ centered at fibered varieties. In other words the center of π_1 is fibered, the center of π_2 is a union of orbits of $(\pi_1)^*(\partial/\partial x)$ and so on. We denote $\varpi = \pi_1 \circ \cdots \circ \pi_d$.

After a finite number of blow-ups we can obtain ϖ such that the strict transform $\tilde{f}_j$ of $f_j = 0$ does not contain orbits of $\varpi^*(\partial/\partial x)$ for any $1 \leq j \leq p$. Let $\prod_{j=1}^q F_j^{m_j}$ be the irreducible decomposition of f_F. The divisor $[f \circ \varpi]$ is of the form

$$[f \circ \varpi] = \sum_{j=1}^{p} l_j[\tilde{f}_j] + \sum_{j=1}^{q} m_j[\tilde{F}_j] + \sum_{j=1}^{s} c_j[H_j],$$

where $\tilde{F}_j$ is the strict transform of $F_j = 0$. We have that $\varpi^{-1}(S(f)) = \bigcup_{j=1}^s H_j$ where H_j is a fibered hypersurface and $c_j \in \mathbb{N}$ for any $1 \leq j \leq s$. We define $k = \max_{1 \leq j \leq s} c_j$. We obtain

$$[(H^k f_F) \circ \varpi] = k[\tilde{H}] + \sum_{j=1}^{q} m_j[\tilde{F}_j] + \sum_{j=1}^{s} t_j[H_j],$$

where $\tilde{H}$ is the strict transform of $H = 0$ and $t_j \geq k \geq c_j$ for any $1 \leq j \leq s$.

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Consider a polydisk Δ and the sheafs $\mathcal{O}_D(f \circ \varpi)$, $\mathcal{O}_R(f \circ \varpi)$ and $\mathcal{O}_Q(f \circ \varpi)$. We have

$$0 \to \frac{H^0(\varpi^{-1}(\Delta), \mathcal{O}_Q(f \circ \varpi))}{p^0(H^0(\varpi^{-1}(\Delta), \mathcal{O}_R(f \circ \varpi)))} \stackrel{\delta^0}{\to} H^1(\varpi^{-1}(\Delta), \mathcal{O}_D(f \circ \varpi)).$$

The polydisk Δ is of the form $\Delta_0 \times \Delta' \subset \mathbb{C} \times \mathbb{C}^n$. Moreover, the choice of ϖ implies that $\varpi^{-1}(\Delta_0 \times \Delta') = \Delta_0 \times \varpi^{-1}(\Delta')$. We obtain

$$H^1(\varpi^{-1}(\Delta), \mathcal{O}_D((H^k f_F) \circ \varpi)) \sim H^1(\varpi^{-1}(\Delta'), \mathcal{O}).$$

We can prove by using the expression of the blow-up in coordinate charts that $H^1(\varpi^{-1}(\Delta'), \mathcal{O}) = H^1(\Delta', \mathcal{O})$. This implies that

$$H^1(\varpi^{-1}(\Delta), \mathcal{O}_D((H^k f_F) \circ \varpi)) = 0.$$

Since

$$\mathcal{O}_D(f \circ \varpi) = \mathcal{O}_D\left(\sum_{j=1}^{q} m_j[\tilde{F}_j] + \sum_{j=1}^{s} c_j[H_j]\right) \subset \mathcal{O}_D((H^k f_F) \circ \varpi),$$

then for any homological equation $\partial\alpha/\partial x = A/f$ such that $A \in \mathcal{O}(\Delta)$ we can find a meromorphic solution β' in $\mathcal{O}(\varpi^{-1}(\Delta))$ such that

$$[\beta']_\infty \leq k[\tilde{H}] + \sum_{j=1}^{p}(l_j - 1)\,[\tilde{f}_j] + \sum_{j=1}^{q} m_j[\tilde{F}_j] + \sum_{j=1}^{s} t_j[H_j].$$

By blowing down we obtain a solution $\beta/(H^k f_F \prod_{1 \leq j \leq p} f_j^{l_j - 1})$ of the equation $\partial\alpha/\partial x = A/f$ for some $\beta \in \mathcal{O}(\Delta)$.

5.2.2. *Relation between free of residues and special.* Given $f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ we are interested in the properties of the quotient (free of residues/special) of homological equations and in particular under what conditions on the evil set $S(f)$ it is 0 or a finite dimensional complex vector space.

Definition 5.4. Let $f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. Let $Fr(f)$ be the set of free of residues homological equations of the form $\partial\alpha/\partial x = A/f$ for some $A \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. We define $Sp(f)$ as the subset of $Fr(f)$ of special equations. We denote the set $H^0(\Delta \setminus S(f), \mathcal{O}_Q(f))$ of free of residues homological equations defined in Δ by $Fr(f, \Delta)$ and its subset $p^0(H^0(\Delta \setminus S(f), \mathcal{O}_R(f)))$ of special equations by $Sp(f, \Delta)$.

The next proposition is straightforward.

PROPOSITION 5.4. *Let* $0 \neq f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. *We have*

$$\frac{Fr(f)}{Sp(f)} = \lim_{\to} \frac{H^0(\Delta \setminus S(f), \mathcal{O}_Q(f))}{p^0(H^0(\Delta \setminus S(f), \mathcal{O}_R(f)))}.$$

We have $Fr(f)/Sp(f) = 0$ for small evil sets.

PROPOSITION 5.5. *Suppose* $\operatorname{cod} S(f) \geq 3$. *Then a free of residues* $\partial\alpha/\partial x = A/f$ *is special.*

Proof. It suffices to show that $H^1(\Delta \setminus S(f), \mathcal{O}_D(f)) = 0$ for any polydisk Δ small enough. We have $\mathcal{O}_D(f) \sim \mathcal{O}_D(f_N) = \mathcal{O}$ where $\mathcal{O}$ is the sheaf of holomorphic functions in $\mathbb{C}^n$. The sheaf $\mathcal{O}$ is coherent and then the first homology group does not change by removing analytic sets of codimension at least 3 (see [**17**]). We obtain $H^1(\Delta \setminus S(f), \mathcal{O}_D(f)) = 0$. □

Denote by $e(\Delta)$ the canonical mapping from $Fr(f, \Delta)/Sp(f, \Delta)$ to $Fr(f)/Sp(f)$. The next proposition makes clear that $Fr(f)/Sp(f)$ behaves like a space of germs.

PROPOSITION 5.6. *There exists a fundamental system $(\Delta_j)_{j\in\mathbb{N}}$ of open neighborhoods of the origin such that $e(\Delta_j)$ is injective for any $j \in \mathbb{N}$. In particular we have*

$$\frac{Fr(f)}{Sp(f)} = \bigcup_{j\in\mathbb{N}} \frac{Fr(f, \Delta_j)}{Sp(f, \Delta_j)}.$$

Proof. We can suppose that f_F is a unit without lack of generality. Therefore we have $\mathcal{O}_D(f) = \mathcal{O}$ where $\mathcal{O}$ is the sheaf of holomorphic functions in $\mathbb{C}^n$.

Consider the subset $S'(f)$ of $S(f)$ of points at which $S(f) \subset \mathbb{C}^n$ is smooth and of codimension 2. We define $\Delta_j = V_j \times V_j^n$ such that:

- $(V_j)_{j\in\mathbb{N}}$ is a sequence of open neighborhoods of $0 \in \mathbb{C}$;
- $(V_j^n)_{j\in\mathbb{N}}$ is a sequence of open neighborhoods of $0 \in \mathbb{C}^n$; and
- all the connected components of $S'(f)$ in V_j^n adhere to 0 for $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$. We have to prove that $E \in Fr(f, \Delta_j)$ and $e(\Delta_j)(E) = 0$ imply that $E \in Sp(f, \Delta_j)$. Consider the element $\delta^0(E)$ of $H^1(V_j^n \setminus S(f), \mathcal{O})$ (see §5.2.1). For every open set $U \subset V_j^n \setminus S(f)$ we define $\delta^0(E, U)$ as the image of $\delta^0(E)$ by the canonical mapping

$$H^1(V_j^n \setminus S(f), \mathcal{O}) \to H^1(U, \mathcal{O}).$$

We define $S''(f)$ as the subset of $S'(f)$ whose elements $P \in S''(f)$ satisfy the property that there exists an open neighborhood $U_P \subset V_j^n$ of P such that $\delta^0(E, U_P \setminus S(f)) = 0$. This property is equivalent to the existence of a neighborhood of $V_j \times \{P\}$ where E has a special solution.

By definition $S''(f)$ is open in $S'(f) \cap V_j^n$. We claim that $S''(f)$ is closed in $S'(f) \cap V_j^n$. Let $Q \in \overline{S''(f)} \cap (S'(f) \cap V_j^n)$. There exists a coordinate system $(y_1, \ldots, y_n)$ centered at Q such that $S(f)$ is given by $y_1 = y_2 = 0$ in a neighborhood of Q. We denote $K_\delta = \bigcap_{k=1}^n \{|y_k| < \delta\}$; we fix $\delta > 0$ such that K_δ is contained in V_j^n. Denote $K_\delta^j = K_\delta \setminus \{y_j = 0\}$ for $j \in \{1, 2\}$; then $K_\delta^1 \cup K_\delta^2$ is a Leray covering of $K_\delta \setminus S(f)$ for $\mathcal{O}$. Thus, there exists a special solution $h_k \in \mathcal{O}_R(f)\,(V_j \times K_\delta^k)$ of E for any $k \in \{1, 2\}$; moreover $\delta^0(E, K_\delta \setminus S(f))$ is given by the function $h_1 - h_2 \in \mathcal{O}(K_\delta^1 \cap K_\delta^2)$. As a consequence $h_1 - h_2$ can be expressed in the form

$$h_1 - h_2 = \sum_{(k,l)\in\mathbb{Z}^2} a_{k,l}(y_3, \ldots, y_n) y_1^k y_2^l,$$

where $a_{k,l}$ is analytic in $\bigcap_{r=3}^n \{|y_r| < \delta\}$ for any $(k, l) \in \mathbb{Z}^2$. Since $Q \in \overline{S''(f)}$ then $a_{k,l}$ vanishes in an open set of $\bigcap_{r=3}^n \{|y_r| < \delta\}$ and then in the whole $\bigcap_{r=3}^n \{|y_r| < \delta\}$ for any

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$(k, l) \in (\mathbb{Z}_{<0})^2$. The function

$$h_1 - \sum_{k\in\mathbb{Z},l\geq 0} a_{k,l}(y_3, \ldots, y_n) y_1^k y_2^l = h_2 + \sum_{k\geq 0,l<0} a_{k,l}(y_3, \ldots, y_n) y_1^k y_2^l$$

is a special solution of E defined in $V_j \times (K_\delta \setminus S(f))$ and then in $V_j \times K_\delta$ since cod $S(f) \geq 2$. Therefore $S''(f)$ is closed in $S'(f) \cap V_j^n$.

Every connected component of $S'(f) \cap V_j^n$ adheres to 0 and then it intersects $S''(f)$ since $e(\Delta_j)(E) = 0$. We obtain $S''(f) = S'(f) \cap V_j^n$. Indeed $\delta^0(E)$ belongs to $H^1(V_j^n \setminus (S(f) \setminus S'(f)), \mathcal{O})$ by the previous discussion. The set $S(f) \setminus S'(f)$ has codimension greater than 2 and then $\delta^0(E) = 0$ by Scheja's theorem. Thus E belongs to $Sp(f, \Delta_j)$. □

In the low dimensional cases we can be even more explicit.

PROPOSITION 5.7. *Let $0 \neq f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. Suppose $n \leq 2$. Then there exists a fundamental system (Δ_j) of open neighborhoods of the origin such that $e(\Delta_j)$ is an isomorphism. Moreover $Fr(f)/Sp(f)$ is a finite dimensional complex vector space.*

Proof. For $n \leq 1$ we have $S(f) = \emptyset$. Thus $H^1(\Delta \setminus S(f), \mathcal{O}_D(f)) = 0$ for every domain Δ small enough. This implies that $Fr(f)/Sp(f) = 0$.

Let $n = 2$. We can suppose that f_F is a unit without lack of generality. Moreover we can also suppose that $S(f) \neq \emptyset$ since otherwise we proceed as for $n \leq 1$. Thus we have $S(f) = \{(0, 0)\}$ since cod $S(f) \geq 2$. Consider a sequence $\Delta_j = B(0, 1/j)^3$. We define $K_j^l = B(0, 1/j)^2 \setminus \{x_l = 0\}$ for $l \in \{1, 2\}$; the set $B(0, 1/j)^2 \setminus \{(0, 0)\}$ admits a Leray covering $K_j^1 \cup K_j^2$ for $\mathcal{O}$.

By Proposition 5.3 there exists $k \in \mathbb{N}$ such that every $E \in Fr(f, \Delta_j)$ has a solution $\alpha_{E,l}/x_l^k$ where $\alpha_{E,l}$ is special in $B(0, 1/j)^3$ for $l \in \{1, 2\}$. Now $\delta^0(E)$ (see §5.2.1) is given by the function $\alpha_{E,1}/x_1^k - \alpha_{E,2}/x_2^k \in \mathcal{O}(K_j^1 \cap K_j^2)$. By construction $\alpha_{E,1}/x_1^k - \alpha_{E,2}/x_2^k$ is of the form $h/(x_1^k x_2^k)$ where h is holomorphic in $B(0, 1/j)^2$. Since $x_1^a x_2^b$ is 0 in $H^1(B(0, 1/j)^2 \setminus \{0\}, \mathcal{O})$ if $(a, b) \notin (\mathbb{Z}_{<0})^2$ then the dimension of $Fr(f, \Delta_j)/Sp(f, \Delta_j)$ as a complex vector space is less than or equal to k^2.

The canonical mapping

$$\frac{Fr(f, \Delta_j)}{Sp(f, \Delta_j)} \to \frac{Fr(f, \Delta_{j+1})}{Sp(f, \Delta_{j+1})}$$

is injective by Proposition 5.6 for $j \gg 0$. Thus $\dim_{\mathbb{C}} Fr(f, \Delta_j)/Sp(f, \Delta_j)$ is a non-decreasing sequence from some moment on. Since it is bounded from above then $\dim_{\mathbb{C}} Fr(f, \Delta_j)/Sp(f, \Delta_j)$ is constant for any $j \geq j_0$ and some $j_0 \in \mathbb{N}$. Hence $e(\Delta_j)$ is an isomorphism for any $j \geq j_0$. Clearly $Fr(f)/Sp(f)$ is finite dimensional. □

5.3. *The residue functions are formal invariants.* Given a germ $\tau \in \mathrm{Diff}_u(\mathbb{C}, 0)$ tangent to the identity, the couple $(\nu(\tau), \mathrm{Res}(\tau))$ (see §5.1) provides a complete system of formal invariants. Since the ideal generated by $\tau(x) - x$ is $(x)^{\nu(\tau)}$ then we can replace $\nu(\tau)$ with the ideal $(\tau(x) - x)$. Given $\varphi, \eta \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ and a formal $\sigma \in \widehat{\mathrm{Diff}}(\mathbb{C}^{n+1}, 0)$ such that $\sigma \circ \varphi = \eta \circ \sigma$ we have:

- $(x_1, \ldots, x_n) \circ \sigma \in \mathbb{C}[[x_1, \ldots, x_n]]^n$, i.e. σ preserves $dx_1 = \cdots = dx_n = 0$; and
- the equality of ideals $(x \circ \eta - x) \circ \sigma = (x \circ \varphi - x)$ in $\mathbb{C}[[x, x_1, \ldots, x_n]]$.

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The first property means that a conjugation preserves the parameter space. The second property implies that the ideal $(x \circ \varphi - x)$ associated to the fixed points set of $\varphi \in \text{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ is a formal invariant. It is the generalization of the order of contact associated to elements of $\text{Diff}_u(\mathbb{C}, 0)$. It is natural to ask whether the residue functions defined in §5.1 are also formal invariants, then generalizing the residue invariant in one variable. The answer is positive and the proof is the object of this subsection.

The proof of the invariance of the residues is based on the study of the dual form. Let $\varphi = \exp(\hat{u} f\, \partial/\partial x) \in \text{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. We call *dual form* of $\log \varphi$ the dual form $dx/(\hat{u} f)$ of $\log \varphi$ in the relative cohomology of $\partial/\partial x$. Since $\hat{u}$ is t.f. along $f = 0$ then there exists $u \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\hat{u} - u \in (f)$. We define the differential $d_1 h = (\partial h/\partial x)\, dx$ relative to $\partial/\partial x$. We have

$$\frac{dx}{\hat{u} f} = \frac{dx}{uf} + \frac{u - \hat{u}}{f} \frac{1}{u\hat{u}} dx = \frac{dx}{uf} + d_1 \hat{K},$$

for some $\hat{K} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$. Thus the dual form can be decomposed as the sum of a meromorphic 1-form and a formal exact 1-form. Let $\alpha = \exp(uf\, \partial/\partial x)$; by the definitions in §5.1 the diffeomorphisms α and φ have the same residue functions. Let $f_N = \prod_{j=1}^{p} f_j^{l_j}$; by Lemma 5.1 there exist series $P_j \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ and $Q_j \in \mathbb{C}\{x_1, \ldots, x_n\}$ such that we have $\text{Res}_{f_j=0}(\varphi) = (P_j/Q_j)_{|f_j=0}$ for any $1 \leq j \leq p$. We define

$$\omega = \frac{dx}{uf} - \sum_{1 \leq j \leq p} \frac{P_j}{Q_j} \frac{\partial f_j/\partial x}{f_j} dx.$$

The form ω has vanishing residues in $f_N = 0$. We obtain $\omega = d_1(A/B)$ for some $A, B \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ by Proposition 5.3. This implies the following result.

LEMMA 5.4. *Let* $\varphi = \exp(\hat{u} f\, \partial/\partial x) \in \text{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. *We have*

$$\frac{dx}{\hat{u} f} = d_1\left(\frac{C}{D}\right) + \sum_{1 \leq j \leq p} \frac{P_j}{Q_j} \frac{\partial f_j/\partial x}{f_j} dx$$

for some $C \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ *and* $D \in \mathbb{C}\{x, x_1, \ldots, x_n\}$.

Let $\varphi_1, \varphi_2 \in \text{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. Suppose that there exists $\hat{\sigma} \in \widehat{\text{Diff}}(\mathbb{C}^{n+1}, 0)$ such that $\varphi_2 \circ \hat{\sigma} = \hat{\sigma} \circ \varphi_1$. Denote $f = x \circ \varphi_1 - x$. Consider the irreducible decomposition $f_N f_F = \prod_{j=1}^{p} f_j^{l_j} \prod_{j=1}^{q} F_j^{m_j}$ (see Definition 5.1) of f. We have that:

- $\hat{\sigma}$ preserves the fibration $dx_1 = \cdots = dx_n = 0$;
- $f_j \circ \hat{\sigma}^{-1} = 0$ is a non-fibered subvariety of Fix φ_2 for $1 \leq j \leq p$; and
- $F_j \circ \hat{\sigma}^{-1} = 0$ is a fibered analytic subset of Fix φ_2 for $1 \leq j \leq q$.

Let $g_j = 0$ be an irreducible analytic equation of $f_j \circ \hat{\sigma}^{-1} = 0$ for $1 \leq j \leq p$. The dual forms of $\log \varphi_1$ and $\log \varphi_2$ can be expressed as

$$d_1\left(\frac{C_1}{D_1}\right) + \sum_{1 \leq j \leq p} \frac{P_j^1}{Q_j^1} \frac{\partial f_j/\partial x}{f_j} dx \quad \text{and} \quad d_1\left(\frac{C_2}{D_2}\right) + \sum_{1 \leq j \leq p} \frac{P_j^2}{Q_j^2} \frac{\partial g_j/\partial x}{g_j} dx,$$

respectively.

PROPOSITION 5.8. *The residue functions are formal invariants.*

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Proof. We keep the notation preceding the proposition. We define $M_j = Q_j^1(Q_j^2 \circ \hat{\sigma})$ and $N_j = P_j^1(Q_j^2 \circ \hat{\sigma}) - Q_j^1(P_j^2 \circ \hat{\sigma})$ for $1 \leq j \leq p$. Our claim is equivalent to $N_j \in (f_j)$ for any $1 \leq j \leq p$. We denote by Ω_j the dual form of $\log \varphi_j$ for j in $\{1, 2\}$. We have $\hat{\sigma}^*\Omega_2 = \Omega_1$. Note that $\hat{\sigma}^*(h\,dx) = (h \circ \hat{\sigma})\,(\partial(x \circ \hat{\sigma})/\partial x)\,dx$; we are always working in the relative cohomology with respect to $\partial/\partial x$. We have that $\hat{\sigma}^*\Omega_2 - d_1((C_2/D_2) \circ \hat{\sigma})$ is equal to

$$\sum_{1 \leq j \leq p} \left(\left(\frac{P_j^2}{Q_j^2} \frac{\partial g_j/\partial x}{g_j} \right) \circ \hat{\sigma} \right) d_1(x \circ \hat{\sigma}) = \sum_{1 \leq j \leq p} \frac{P_j^2}{Q_j^2} \circ \hat{\sigma} \frac{\partial (g_j \circ \hat{\sigma})/\partial x}{g_j \circ \hat{\sigma}}\, dx.$$

By construction $g_j \circ \hat{\sigma}$ is of the form $\hat{v}_j f_j$ for some formal unit $\hat{v}_j$ and any $1 \leq j \leq p$. We obtain

$$\hat{\sigma}^*\Omega_2 - d_1((C_2/D_2) \circ \hat{\sigma}) = \sum_{1 \leq j \leq p} \frac{P_j^2}{Q_j^2} \circ \hat{\sigma} \frac{\partial \hat{v}_j/\partial x}{\hat{v}_j} dx + \sum_{1 \leq j \leq p} \frac{P_j^2}{Q_j^2} \circ \hat{\sigma} \frac{\partial f_j/\partial x}{f_j} dx.$$

The form $\sum_{1 \leq j \leq p}((P_j^2/Q_j^2) \circ \hat{\sigma})\,((\partial \hat{v}_j/\partial x)/\hat{v}_j)\,dx$ does not have non-fibered poles and then it can be expressed in the form $d_1(J_1/J_2)$ for some $J_1 \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ and $J_2 \in \mathbb{C}[[x_1, \ldots, x_n]]$. Since $\hat{\sigma}^*\Omega_2 = \Omega_1$ then there exist $C, D \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ such that

$$d_1\left(\frac{C}{D}\right) = \sum_{1 \leq j \leq p} \frac{\partial f_j/\partial x}{f_j} \left(\frac{P_j^1}{Q_j^1} - \frac{P_j^2}{Q_j^2} \circ \hat{\sigma} \right) dx.$$

The previous expression is equivalent to

$$D^2 \sum_{j=1}^{p} \frac{\partial f_j}{\partial x} N_j \prod_{k \neq j} f_k \prod_{k \neq j} M_k = \prod_{k=1}^{p} f_k \prod_{k=1}^{p} M_k \left(\frac{\partial C}{\partial x} D - C \frac{\partial D}{\partial x} \right).$$

Let μ_r be the greatest integer such that $f_r^{\mu_r}$ divides D for $1 \leq r \leq p$. For $\mu_r > 0$ the left-hand side belongs to $(f_r^{2\mu_r})$ and the right-hand side belongs to $(f_r^{\mu_r}) \setminus (f_r^{\mu_r+1})$; this implies that $\mu_r = 0$ for $1 \leq r \leq p$. The right-hand side belongs to (f_r), and as a consequence

$$D^2 \sum_{j=1}^{p} \frac{\partial f_j}{\partial x} N_j \prod_{k \neq j} f_k \prod_{k \neq j} M_k \in (f_r) \quad \Longrightarrow \quad N_r \in (f_r),$$

for any $1 \leq r \leq p$. □

5.4. *Homological equation and formal conjugation.* In this subsection we relate the formal conjugacy problem with the solvability of homological equations.

Definition 5.5. Let $f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. We say that $\hat{\sigma} \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$ is *normalized* with respect to $f = 0$ if $x \circ \hat{\sigma} - x \in \sqrt{(f_N)}$ (see Definition 5.1).

Definition 5.6. We say that $\varphi_1, \varphi_2 \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ are *formally conjugated by a normalized transformation* if there exists a normalized $\hat{\sigma} \in \mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$ (with respect to $x \circ \varphi_1 - x = 0$) such that $\varphi_2 \circ \hat{\sigma} = \hat{\sigma} \circ \varphi_1$.

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In such a case the ideals of fixed points $(x \circ \varphi_1 - x)$ and $(x \circ \varphi_2 - x)$ coincide. Thus φ_1, φ_2 both belong to $\mathcal{D}_f$ (see §2) for some $f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $f(0) = 0$ and $(\partial f/\partial x)(0) = 0$. We restrict our study to formal normalized conjugations. Later on we will see that this point of view is complete since general formal conjugations can be reduced to the normalized setting.

We associate a homological equation $E = (\partial\alpha/\partial x = A/f)$ to any pair of elements $\varphi_1, \varphi_2 \in \mathcal{D}_f$. In this subsection we prove that φ_1 and φ_2 are formally conjugated by a normalized transformation if and only if E is special. This equivalence allows us to describe a complete system of formal invariants (§5.6) by using the results in §5.2.2.

Let $\varphi_1, \varphi_2 \in \mathcal{D}_f$. We define $\hat{u}_j = (\log \varphi_j)(x)/f$ for $j \in \{1, 2\}$. Throughout this section we fix the decomposition $\prod_{j=1}^{p} f_j^{l_j}$ of f_N (see Definition 5.1) into irreducible factors. The equation

$$\frac{\partial\alpha}{\partial x} = \frac{1}{\hat{u}_1 f} - \frac{1}{\hat{u}_2 f}$$

is called the homological equation associated to φ_1 and φ_2. We call it *special* if there exists a *special* solution $\beta/(f_F \prod_{j=1}^{p} f_j^{l_j - 1})$ where β in $\mathbb{C}[[x, x_1, \ldots, x_n]]$. Note that if $1/\hat{u}_1 - 1/\hat{u}_2 \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ then the definition of special of §5.2 implies that the solution is convergent. The two definitions are the same.

LEMMA 5.5. *Consider a homological equation $E \equiv (\partial\alpha/\partial x = A/f)$ where A, f belong to $\mathbb{C}\{x, x_1, \ldots, x_n\}$. If there exists a formal special solution then there also exists a convergent special solution.*

Proof. We can suppose that f_F is a unit without lack of generality. Consider a formal special solution $\hat{\beta}/(\prod_{1 \le j \le p} f_j^{l_j - 1})$ of E. We have

$$\frac{\partial\hat{\beta}}{\partial x} \prod_{j=1}^{p} f_j - \hat{\beta} \sum_{j=1}^{p} (l_j - 1) \frac{\partial f_j}{\partial x} \prod_{k \in \{1,\ldots,p\} \setminus \{j\}} f_k = A.$$

If $l_j = 1$ then f_j divides A. By considering $\partial\alpha/\partial x = (A/\prod_{l_j = 1} f_j)/(\prod_{l_j > 1} f_j^{l_j})$ we can suppose that $l_j > 1$ for any $1 \le j \le p$. The function A belongs to the ideal generated by $\prod_{j=1}^{p} f_j$ and $\sum_{j=1}^{p} (l_j - 1)(\partial f_j/\partial x) \prod_{k \ne j} f_k$ in $\mathbb{C}[[x, x_1, \ldots, x_n]]$. Thus it also belongs to the ideal in $\mathbb{C}\{x, x_1, \ldots, x_n\}$ sharing the same generators; in particular there exist $C, D_0 \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that

$$A = C \prod_{j=1}^{p} f_j - D_0 \sum_{j=1}^{p} (l_j - 1) \frac{\partial f_j}{\partial x} \prod_{k \in \{1,\ldots,p\} \setminus \{j\}} f_k.$$

We define $\hat{\gamma} = \hat{\beta} - D_0$. We obtain

$$\frac{\partial\hat{\gamma}}{\partial x} \prod_{j=1}^{p} f_j - \hat{\gamma} \sum_{j=1}^{p} (l_j - 1) \frac{\partial f_j}{\partial x} \prod_{k \in \{1,\ldots,p\} \setminus \{j\}} f_k = \left(C - \frac{\partial D_0}{\partial x} \right) \prod_{j=1}^{p} f_j.$$

The previous formula implies that $\hat{\gamma} \in (\prod_{j=1}^{p} f_j)$. Hence $(\hat{\gamma}/\prod_{j=1}^{p} f_j)/(\prod_{j=1}^{p} f_j^{l_j - 2})$ is a solution of

$$\frac{\partial\alpha}{\partial x} = \frac{C - \partial D_0/\partial x}{\prod_{j=1}^{p} f_j^{l_j - 1}}.$$

By induction on $\max_{1\leq j\leq p} l_j$ we can prove that there exists $D \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $(\hat{\beta} - D)/\prod_{j=1}^{p} f_j^{l_j-1}$ is a solution of the equation $\partial\alpha/\partial x = \xi$ for some $\xi \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. We choose a convergent solution $\alpha_0 \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ of the latter equation. Then $D/\prod_{j=1}^{p} f_j^{l_j-1} + \alpha_0$ is a special convergent solution of E. □

We introduce the main proposition in this subsection.

PROPOSITION 5.9. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. Then they are formally conjugated by a normalized transformation if and only if the associated homological equation is special.*

The proposition implies that the existence of formal normalized conjugation is equivalent to the solvability of a linear differential equation. The proof of Proposition 5.9 is obtained by reducing the setting to the case where $\log \varphi_j$ is convergent for $j \in \{1, 2\}$.

Definition 5.7. Denote by $\sim$ the equivalence relation given by $\varphi_1 \sim \varphi_2$ if φ_1 is conjugated to φ_2 by a normalized $\sigma \in \mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$.

PROPOSITION 5.10. *Let $\varphi_j = \exp(u_j f\, \partial/\partial x) \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ with convergent infinitesimal generator for $j \in \{1, 2\}$. Assume that the homological equation associated to φ_1 and φ_2 is special. Then we obtain $\varphi_1 \sim \varphi_2$.*

Proof. Let us use the path method (see [**16**] and [**12**]). We define

$$X_{1+z} = u_{1+z} f \frac{\partial}{\partial x} = \frac{u_1 u_2 f}{z u_1 + (1-z) u_2} \frac{\partial}{\partial x}.$$

Denote $c = u_2(0)/(u_2(0) - u_1(0))$. We have $X_{1+z_0} \in \mathcal{X}(\mathbb{C}^{n+1}, 0)$ for any $z_0 \in \mathbb{C} \setminus \{c\}$. The choice of X_{1+z} assures that the homological equation

$$\frac{\partial \alpha}{\partial x} = z\left(\frac{1}{u_1 f} - \frac{1}{u_2 f}\right)$$

associated to $\exp(X_1)$ and $\exp(X_{1+z})$ is special. It suffices to prove that $X_1 \sim X_2$ for $c \notin [0, 1]$. If $c \in [0, 1]$ we define

$$Y_{1+z}^1 = \frac{u_1 u_{1+i} f}{z u_1 + (1-z) u_{1+i}} \frac{\partial}{\partial x} \quad \text{and} \quad Y_{1+z}^2 = \frac{u_{1+i} u_2 f}{z u_{1+i} + (1-z) u_2} \frac{\partial}{\partial x}.$$

Since $u_{1+i}(0)/(u_{1+i}(0) - u_1(0))$ and $u_2(0)/(u_2(0) - u_{1+i}(0))$ do not belong to $[0, 1]$ then $X_1 \sim X_{1+i} \sim X_2$.

Suppose that $c \notin [0, 1]$. We look for $W \in \mathcal{X}(\mathbb{C}^{n+2}, 0)$ of the form

$$W = h(x, x_1, \ldots, x_n, z) f \frac{\partial}{\partial x} + \frac{\partial}{\partial z}$$

such that $[W, X_{1+z}] = 0$. We ask for $h f_F \prod_{j=1}^{p} f_j^{l_j-1}$ to be holomorphic in a connected domain $V \times V' \subset \mathbb{C}^{n+1} \times \mathbb{C}$ containing $\{0\} \times [0, 1]$. We also require hf to vanish at $\{0\} \times V'$. Suppose that such a W exists, then $\exp(W)_{|z=0}$ is a normalized mapping conjugating X_1 and X_2.

The equation $[W, X_{1+z}] = 0$ is equivalent to

$$u_{1+z} f \frac{\partial(hf)}{\partial x} - hf \frac{\partial(u_{1+z} f)}{\partial x} = \frac{\partial(u_{1+z} f)}{\partial z}.$$

By simplifying we obtain

$$u_{1+z} f \frac{\partial h}{\partial x} - hf \frac{\partial u_{1+z}}{\partial x} = \frac{\partial u_{1+z}}{\partial z} \quad \Longrightarrow \quad \frac{\partial (h/u_{1+z})}{\partial x} = \frac{1}{u_1 f} - \frac{1}{u_2 f}.$$

Let α be a special solution of the homological equation associated to φ_1 and φ_2. For $p = 0$ we can suppose that $(\alpha f)(0) = 0$ by choosing α of the form α'/f_F where $\alpha' \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ satisfies $\alpha'(0) = 0$ and $\partial\alpha'/\partial x = 1/u_1 - 1/u_2$. We are done by defining $h = u_{1+z}\alpha$. □

The reciprocal is also true.

PROPOSITION 5.11. *Let $\varphi_j = \exp(u_j f\, \partial/\partial x) \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ with convergent infinitesimal generator for $j \in \{1, 2\}$. Suppose $\varphi_1 \sim \varphi_2$ (see Definition 5.7). Then the associated homological equation is special.*

Proof. Consider $P \notin \{f_N = 0\}$. By Lemma 5.2 there exists a special solution $\psi_{j,P}$ of $\partial\psi_{j,P}/\partial x = 1/(u_j f)$ defined in a neighborhood of P for $j \in \{1, 2\}$. Denote $c = (\partial(x \circ \sigma)/\partial x)(0)$. Consider a normalized diffeomorphism σ conjugating φ_1 and φ_2. We define the diffeomorphism $\sigma_z = (1 - z)\,\mathrm{Id} + z\sigma$ for any $z \in \mathbb{C} \setminus \{(1 - c)^{-1}\}$; we have that σ_z is normalized for any $z \in \mathbb{C} \setminus \{(1 - c)^{-1}\}$. We can suppose that $(1 - c)^{-1} \notin [0, 1]$; otherwise we proceed in an analogous way as in Proposition 5.10 by considering the pairs $(\varphi_1, \sigma_i^{-1} \circ \varphi_2 \circ \sigma_i)$ and $(\sigma_i^{-1} \circ \varphi_2 \circ \sigma_i, \varphi_2)$. For $P \notin \{f_N = 0\}$ and $z \in [0, 1]$ we define

$$\gamma_z(P) = \psi_{2,P} \circ \sigma_z(P) - \psi_{2,P}(P).$$

In the previous expression $\psi_{2,P} \circ \sigma_z(P)$ is the value at $\sigma_z(P)$ of the analytical continuation of $\psi_{2,P}$ along the path $[0, z] : s \to \sigma_s(P)$. Then γ_z is by construction a special solution of the homological equation associated to $\sigma_z^{-1} \circ \varphi_2 \circ \sigma_z$ and φ_2 defined in the complement of $f_N = 0$ for any $z \in [0, 1]$. There exists a special solution α_P of the homological equation associated to φ_1 and φ_2 and defined in the neighborhood of P for $P \notin S(f)$ by Lemma 5.2. Since $\partial(\gamma_1 - \alpha_P)/\partial x = 0$ then γ_1 can be extended to the complement of $S(f)$. Moreover $\mathrm{cod}\, S(f) \geq 2$ implies that γ_1 is special in a neighborhood of the origin. □

The next proposition claims that every formal class of conjugation contains at least one convergent normal form. That will allow us to prove Proposition 5.9 by reducing the problem to the settings considered in Propositions 5.10 and 5.11.

PROPOSITION 5.12. *Let $\varphi = \exp(\hat{u} f\, \partial/\partial x) \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. There exists a germ of function $u \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that φ and $\exp(uf\, \partial/\partial x)$ are formally conjugated by a normalized transformation.*

Proof. Since $\hat{u}$ is t.f. along $f = 0$ (Proposition 4.6) then there exists $u_k \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\hat{u} - u_k \in (f^k)$ for any $k \in \mathbb{N}$. We denote $\varphi_k = \exp(u_k f\, \partial/\partial x)$. Let γ_k be a solution of the homological equation associated to φ_k and φ_1. Since $1/u_k - 1/u_1$ belongs to the ideal (f) we can choose γ_k in $(x) \cap \mathbb{C}\{x, x_1, \ldots, x_n\}$. We have $\gamma_{k+1} - \gamma_k \in \mathfrak{m}^k$ where $\mathfrak{m}$ is the maximal ideal since $\partial(\gamma_{k+1} - \gamma_k)/\partial x \in (f^{k-1})$. The diffeomorphisms φ_k and φ_1 are conjugated by

$$\sigma_k \stackrel{\text{def}}{=} \exp\left(\gamma_k \frac{u_k u_1}{z u_k + (1 - z) u_1} f \frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)_{|z=0}.$$

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Moreover σ_k converges in the Krull topology to some normalized $\hat{\sigma} \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$ conjugating φ and φ_1. □

Proof of the implication ⇒ of Proposition 5.9. Define $f = x \circ \varphi_1 - x$. We have $\log \varphi_j = \hat{u}_j f\, \partial/\partial x$ for $j \in \{1, 2\}$. Consider a unit u_j in $\mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\hat{u}_j - u_j \in (f)$. Then $\alpha_j = \exp(u_j f\, \partial/\partial x)$ is formally conjugated to φ_j by a normalized transformation for $j \in \{1, 2\}$ by the proof of Proposition 5.12. Thus α_1 and α_2 are formally conjugated by a normalized transformation $\hat{\sigma} \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$. Since

$$\left(\frac{1}{\hat{u}_1 f} - \frac{1}{\hat{u}_2 f}\right) - \left(\frac{1}{u_1 f} - \frac{1}{u_2 f}\right) \in \mathbb{C}[[x, x_1, \ldots, x_n]],$$

it suffices to prove that the homological equation associated to α_1 and α_2 is special.

Consider the decomposition $\prod_{j=1}^{p} f_j^{l_j}$ into irreducible factors of f_N (see Definition 5.1). We denote by $\hat{\mathfrak{m}}$ the maximal ideal of $\mathbb{C}[[x, x_1, \ldots, x_n]]$. For $k \geq 2$ there exists h_k in $\mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $h_k - x \in (\prod_{j=1}^{p} f_j)$ and $(x \circ \hat{\sigma} - h_k)/\prod_{j=1}^{p} f_j \in \hat{\mathfrak{m}}^k$ since $\hat{\sigma}$ is normalized. We define the normalized diffeomorphism $\sigma_k = (h_k, x_1, \ldots, x_n)$.

Suppose that $f_N(P) \neq 0$. There exists a special solution $\psi_{2,P}$ of $\partial\alpha/\partial x = 1/u_2 f$ defined in the neighborhood of P. We define $\gamma_k(P) = \psi_{2,P} \circ \sigma_k(P) - \psi_{2,P}(P)$ as in Proposition 5.11. Then γ_k extends to a special solution defined in a neighborhood of the origin of the homological equation associated to $\sigma_k^{-1} \circ \alpha_2 \circ \sigma_k$ and α_2 (see proof of Proposition 5.11). We define $\beta_k = \gamma_k f_F \prod_{j=1}^{p} f_j^{l_j - 1}$; we claim that the sequence β_k converges to some $\hat{\beta} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ in the Krull topology. That is a consequence of Taylor's formula since

$$\gamma_k - \gamma_l = \sum_{r=1}^{\infty} \frac{1}{r!} \left(\frac{\partial^r \psi_2}{\partial x^r} \circ \sigma_l\right) (x \circ \sigma_k - x \circ \sigma_l)^r$$

implies that $\beta_k - \beta_l \in \hat{\mathfrak{m}}^{\min(k,l)}$. Since $(\log(\sigma_k^{-1} \circ \alpha_2 \circ \sigma_k))(x)$ converges to $(\log \alpha_1)(x)$ in the Krull topology then $\hat{\beta}/(f_F \prod_{j=1}^{p} f_j^{l_j - 1})$ is a special solution of the homological equation associated to α_1 and α_2. □

Proof of the implication ⇐ of Proposition 5.9. We have that $\log \varphi_j$ is of the form $\hat{u}_j f\, \partial/\partial x$ for $j \in \{1, 2\}$. Let $u_j \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\hat{u}_j - u_j \in (f)$ for $j \in \{1, 2\}$. We define $\alpha_j = \exp(u_j f\, \partial/\partial x)$. The homological equation

$$\frac{\partial \alpha}{\partial x} = \left(\frac{1}{u_1 f} - \frac{1}{\hat{u}_1 f}\right) + \left(\frac{1}{\hat{u}_1 f} - \frac{1}{\hat{u}_2 f}\right) + \left(\frac{1}{\hat{u}_2 f} - \frac{1}{u_2 f}\right),$$

associated to α_1 and α_2 is the result of adding three special equations; thus it is special. The diffeomorphisms α_1 and α_2 are conjugated by a germ of normalized diffeomorphism σ (Proposition 5.10). By the proof of Proposition 5.12 we know that φ_j and α_j are conjugated by a formal normalized diffeomorphism $\hat{\sigma}_j$ for $j \in \{1, 2\}$. Then $\hat{\sigma}_2^{-1} \circ \sigma \circ \hat{\sigma}_1$ is a formal normalized diffeomorphism conjugating φ_1 and φ_2. □

5.5. *Normal forms.* Every up-diffeomorphism is formally conjugated to the exponential of a holomorphic vector field (Proposition 5.12). Thus obtaining normal forms for up-diffeomorphisms and unfoldings of elements of $\mathcal{X}_N(\mathbb{C},0)=x^2\mathbb{C}\{x\}\,\partial/\partial x$ are equivalent tasks. We generalize the classical normal forms for unfoldings of finite codimension elements of $\mathcal{X}_N(\mathbb{C},0)$ (see [**7**]) to unfoldings of finite codimension elements of $\mathrm{Diff}_u(\mathbb{C},0)$. The normal forms are considered up to normalized conjugation (Proposition 5.13), parameterized conjugation (Proposition 5.14) and general formal conjugation (Corollary 5.2).

PROPOSITION 5.13. *Let $\varphi\in\mathrm{Diff}_{up}(\mathbb{C}^{n+1},0)$ such that $\varphi_{|x_1=\cdots=x_n=0}\neq\mathrm{Id}$. There exist functions $a_0,\ldots,a_\nu,b_0,\ldots,b_\nu\in\mathbb{C}\{x_1,\ldots,x_n\}$ such that $a_j(0)=0$ for any $j\in\{0,\ldots,\nu\}$, $b_0(0)\neq 0$ and φ is conjugated to*

$$\exp\left(\frac{x^{\nu+1}+a_\nu(x_1,\ldots,x_n)x^\nu+\cdots+a_0(x_1,\ldots,x_n)}{b_0(x_1,\ldots,x_n)+\cdots+b_\nu(x_1,\ldots,x_n)x^\nu}\frac{\partial}{\partial x}\right),$$

by a formal normalized transformation.

The number ν is the codimension of the element $\varphi_{|x_1=\cdots=x_n=0}$ of $\mathrm{Diff}_u(\mathbb{C},0)$. It satisfies $\nu=\nu(\varphi_{|x_1=\cdots=x_n=0})-1$ (see §5.1). The previous proposition provides a normal form (up to normalized conjugation) for unfoldings of finite codimension one-dimensional diffeomorphisms that are tangent to the identity.

Proof. By Proposition 5.12 we can suppose that $\log\varphi$ is a holomorphic vector field. Denote $f=(\log\varphi)(x)$; then we have $f\in\mathbb{C}\{x,x_1,\ldots,x_n\}$. The finite codimension hypothesis implies that $f(x,0,\ldots,0)\not\equiv 0$. We obtain that f is of the form

$$[x^{\nu+1}+a_\nu(x_1,\ldots,x_n)x^\nu+\cdots+a_0(x_1,\ldots,x_n)]/v(x,x_1,\ldots,x_n),$$

for some unit $v\in\mathbb{C}\{x,x_1,\ldots,x_n\}$ by the Weierstrass division theorem. Consider the remainder $\sum_{j=0}^{\nu}b_j(x_1,\ldots,x_n)x^j$ of the Weierstrass division $v/(x^{\nu+1}+\sum_{j=0}^{\nu}a_jx^j)$. Denote $g=(x^{\nu+1}+\sum_{j=0}^{\nu}a_jx^j)/(\sum_{j=0}^{\nu}b_jx^j)$. Since $g-f$ belongs to the ideal (f^2) then the homological equation associated to $\exp(f\,\partial/\partial x)$ and $\exp(g\,\partial/\partial x)$ is of the form $\partial\alpha/\partial x=A$ for some $A\in\mathbb{C}\{x,x_1,\ldots,x_n\}$. Such an equation is special, and thus φ and $\exp(g\,\partial/\partial x)$ are conjugated by a normalized diffeomorphism (Proposition 5.10). □

By relaxing the conditions on the conjugating mappings we obtain simpler normal forms. The next proposition provides a normal form which can be interpreted as a generalization of the one provided by Proposition 5.13 in which we remove half of the coefficients. The second normal form is interesting since it is polynomial in x.

PROPOSITION 5.14. *Let $\varphi\in\mathrm{Diff}_{up}(\mathbb{C}^{n+1},0)$ with $\varphi_{|x_1=\cdots=x_n=0}\neq\mathrm{Id}$. There exist functions $a_0,\tilde{a}_0,\ldots,a_{\nu-1},\tilde{a}_{\nu-1},b,\tilde{b}\in\mathbb{C}\{x_1,\ldots,x_n\}$ such that $a_j(0)=\tilde{a}_j(0)=0$ for any $j\in\{0,\ldots,\nu-1\}$ and φ is conjugated to*

$$\exp\left(\frac{x^{\nu+1}+\sum_{j=0}^{\nu-1}a_jx^j}{1+bx^\nu}\frac{\partial}{\partial x}\right)\quad\text{and}\quad\exp\left(\left(\tilde{b}x^{2\nu+1}+x^{\nu+1}+\sum_{j=0}^{\nu-1}\tilde{a}_jx^j\right)\frac{\partial}{\partial x}\right)$$

by elements of $\widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1},0)$.

Proof. By Proposition 5.12 we can suppose that $\log\varphi$ is a holomorphic vector field. Denote $f = (\log\varphi)(x)$. The finite codimension hypothesis implies that there exists $h \in \mathrm{Diff}(\mathbb{C}, 0)$ conjugating $f(x, 0, \ldots, 0)\,\partial/\partial x$ and $(cx^{2\nu+1} + x^{\nu+1})\,\partial/\partial x$ for some $\nu \in \mathbb{N}$ and $c = -\mathrm{Res}(\varphi_0)$. Up to replacing $f\,\partial/\partial x$ with $(h(x), x_1, \ldots, x_n)_*(f\,\partial/\partial x)$ we can suppose that $f(x, 0, \ldots, 0) = cx^{2\nu+1} + x^{\nu+1}$. The family

$$\left(\beta x^{2\nu+1} + x^{\nu+1} + \sum_{j=0}^{\nu-1}\alpha_j x^j\right)\partial/\partial x \quad \text{with } \alpha_0, \alpha_1, \ldots, \alpha_{\nu-1}, \beta \in \mathbb{C}$$

is the versal deformation of $(cx^{2\nu+1} + x^{\nu+1})\,\partial/\partial x$ (see [**7**]). Therefore $f\,\partial/\partial x$ is conjugated by an element of $\mathrm{Diff}_p(\mathbb{C}^{n+1}, 0)$ to a vector field $(\tilde{b}x^{2\nu+1} + x^{\nu+1} + \sum_{j=0}^{\nu-1}\tilde{a}_j x^j)\,\partial/\partial x$ for some $\tilde{a}_0, \ldots, \tilde{a}_{\nu-1}, \tilde{b} \in \mathbb{C}\{x_1, \ldots, x_n\}$.

The normal form $\exp[(1 + bx^\nu)^{-1}(x^{\nu+1} + \sum_{j=0}^{\nu-1} a_j x^j)\,\partial/\partial x]$ is obtained in an analogous way since $f(x, 0, \ldots, 0)\,\partial/\partial x$ is analytically conjugated to $x^{\nu+1}/(1 - cx^\nu)\,\partial/\partial x$ and $(1 + \beta x^\nu)^{-1}(x^{\nu+1} + \sum_{j=0}^{\nu-1}\alpha_j x^j)\,\partial/\partial x$ with $\alpha_0, \ldots, \alpha_{\nu-1}, \beta \in \mathbb{C}$ is the versal deformation of $x^{\nu+1}/(1 - cx^\nu)\,\partial/\partial x$. □

Given a codimension ν element ϕ of $\mathrm{Diff}_u(\mathbb{C}, 0)$ we consider a general position unfolding φ of ϕ. In fact φ is a ν-parameter unfolding; it belongs to $\mathrm{Diff}_{up}(\mathbb{C}^{\nu+1}, 0)$. This case admits a specially simple normal form in which the low degree terms of the normal form can be replaced by coordinate functions. The proof is a consequence of Proposition 5.14 (see [**7**]).

COROLLARY 5.2. *Let $\phi \in \mathrm{Diff}_u(\mathbb{C}, 0)$ of codimension ν. Consider a general position unfolding $\varphi \in \mathrm{Diff}_{up}(\mathbb{C}^{\nu+1}, 0)$ of ϕ. There exist functions $b, \tilde{b} \in \mathbb{C}\{x_1, \ldots, x_\nu\}$ such that φ is formally conjugated to*

$$\exp\left(\frac{x^{\nu+1} + \sum_{j=0}^{\nu-1} x_{j+1}x^j}{1 + bx^\nu}\frac{\partial}{\partial x}\right) \quad \text{and} \quad \exp\left(\left(\tilde{b}x^{2\nu+1} + x^{\nu+1} + \sum_{j=0}^{\nu-1} x_{j+1}x^j\right)\frac{\partial}{\partial x}\right)$$

by elements of $\widehat{\mathrm{Diff}}(\mathbb{C}^{\nu+1}, 0)$.

5.6. *Theorem of formal normalized conjugation.* Next we describe the nature of the invariants for the formal normalized conjugation. Given $\varphi_1, \varphi_2 \in \mathcal{D}_f$ we associate to them a homological equation E. Then E belongs to $Sp(f)$ (see Definition 5.4) if and only if φ_1 and φ_2 are conjugated by a formal normalized transformation (Proposition 5.9). Indeed the residue functions of φ_1 and φ_2 coincide if and only if $E \in Fr(f)$ (see Definition 5.4). It is natural to think that a complete system of invariants is composed of the residue functions and the space $Fr(f)/Sp(f)$.

THEOREM 5.1. *The residue functions associated to the non-fibered irreducible components of $f = 0$ and the complex vector space $Fr(f)/Sp(f)$ are a complete system of formal invariants for the normalized conjugation in $\mathcal{D}_f$.*

Let us make explicit how $Fr(f)/Sp(f)$ can be interpreted as a space of formal normalized invariants. Let $\varphi \in \mathcal{D}_f$. We define the subset $\mathcal{D}_f(\varphi)$ of $\mathcal{D}_f$ whose elements τ

satisfy the property that $\mathrm{Res}_\gamma(\varphi) \equiv \mathrm{Res}_\gamma(\tau)$ for any irreducible component γ of $f_N = 0$ (see Definition 5.1). The theorem claims the existence of $\mathrm{Inv}_f : \mathcal{D}_f(\varphi) \to Fr(f)/Sp(f)$ such that $\mathrm{Inv}_f(\varphi_1) = \mathrm{Inv}_f(\varphi_2)$ if and only if φ_1 and φ_2 are formally conjugated by a normalized transformation.

Consider $\tau = \exp(\hat{u}_\tau f\, \partial/\partial x) \in \mathcal{D}_f$. Since $\hat{u}_\tau$ is t.f. along $f = 0$ there exists a unit $u_\tau \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\hat{u}_\tau - u_\tau \in (f)$. We define a mapping $\mathrm{Inv}_f^\varphi : \mathcal{D}_f(\varphi) \to Fr(f)/Sp(f)$ given by

$$\mathrm{Inv}_f^\varphi(\tau) = \left[\frac{\partial\alpha}{\partial x} = \frac{1}{f}\left(\frac{1}{u_\tau} - \frac{1}{u_\varphi}\right)\right] + Sp(f).$$

The value $\mathrm{Inv}_f^\varphi(\tau)$ is independent of the choices of u_φ and u_τ. The mapping Inv_f^φ is not the only choice for Inv_f since $\mathrm{Inv}_f^\varphi \neq \mathrm{Inv}_f^\tau$ if $\mathrm{Inv}_f^\varphi(\tau) \neq 0$. Thus $Fr(f)/Sp(f)$ is a classifying space for the formal normalized conjugation but the mapping Inv_f is not canonical.

We say that $Fr(f)$ contains units if there exists $[\partial\alpha/\partial x = A/f]$ in $Fr(f)$ for some unit $A \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. Next we prove that there are no redundant invariants in $Fr(f)/Sp(f)$.

LEMMA 5.6. *The mapping* Inv_f^φ *is surjective except if* $Fr(f)$ *contains units but* $Sp(f)$ *does not. In such a case* $[\partial\alpha/\partial x = A/f] + Sp(f)$ *belongs to* $\mathrm{Inv}_f^\varphi(\mathcal{D}_f(\varphi))$ *if and only if* $A(0) \neq -1/u_\varphi(0)$. *Anyway* $Fr(f)/Sp(f)$ *is the complex vector space generated by* $\mathrm{Inv}_f^\varphi(\mathcal{D}_f(\varphi))$.

Proof. Fix a homological equation $E = [\partial\alpha/\partial x = A/f] \in Fr(f)$.

Suppose that $1/u_\varphi(0) \neq -A(0)$. The formula $1/u = 1/u_\varphi + A$ defines a unit $u \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ such that $\mathrm{Inv}_f^\varphi(\exp(uf\,\partial x)) = E + Sp(f)$. Then we suppose from now on that $1/u_\varphi(0) = -A(0)$. We have that $\lambda E + Sp(f) \in \mathrm{Inv}_f^\varphi(\mathcal{D}_f(\varphi))$ for any $\lambda \in \mathbb{C} \setminus \{1\}$. Note that $\lambda E = [\partial\alpha/\partial x = \lambda A/f]$ for $\lambda \in \mathbb{C}$.

Suppose that both $Fr(f)$ and $Sp(f)$ contain units. Since there exists an equation $[\partial\alpha/\partial x = B/f] \in Sp(f)$ such that $B(0) \neq 0$ then

$$E + Sp(f) = [\partial\alpha/\partial x = (A + B)/f] + Sp(f) \in \mathrm{Inv}_f^\varphi(\mathcal{D}_f(\varphi)).$$

If $Fr(f)$ contains units but $Sp(f)$ does not then there does not exist a special $[\partial\alpha/\partial x = B/f]$ such that $1/u_\varphi(0) + A(0) + B(0) \neq 0$. As a consequence $E + Sp(f)$ does not belong to $\mathrm{Inv}_f^\varphi(\mathcal{D}_f(\varphi))$. □

The next results are a direct consequence of the analogous ones on the homological equation.

COROLLARY 5.3. *Let* $f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. *Suppose that either* $\mathrm{cod}\, S(f) \geq 3$ *or* $n \leq 1$. *Then the residue functions are a complete system of formal invariants for the normalized conjugation in* $\mathcal{D}_f \subset \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$.

COROLLARY 5.4. *Let* $f \in \mathbb{C}\{x, x_1, x_2\}$. *Then the residue functions plus a finite number of linear invariants are a complete system of formal invariants for the normalized conjugation in* $\mathcal{D}_f \subset \mathrm{Diff}_{up}(\mathbb{C}^3, 0)$.

5.6.1. *The example* $f=(x_2-xx_1)^2 \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. We describe the space $Fr(f)/Sp(f)$. Moreover in the case $n=2$ where $\dim_{\mathbb{C}} Fr(f)/Sp(f)=1$ we express the invariant in $Fr(f)/Sp(f)$ associated to $\varphi \in \mathcal{D}_f$ in terms of the coefficients of φ.

LEMMA 5.7. *Let* $f=(x_2-xx_1)^2$. *Consider* $E=[\partial\alpha/\partial x = A/f]$ *in* $Fr(f)$. *Then* $E \in Sp(f)$ *if and only if* $A \in (x_1, x_2)$.

Proof. If there exists a special solution $\alpha=\beta/(x_2-xx_1)$ of E then

$$A=(\partial\beta/\partial x)(x_2-xx_1)+\beta x_1 \in (x_1, x_2).$$

If $A \in (x_1, x_2)$ we obtain $A=(x_2-xx_1)C+x_1 D$ for some C, D in $\mathbb{C}\{x, x_1, \ldots, x_n\}$. Denote $E'=[\partial\alpha/\partial x=(C-\partial D/\partial x)/(x_2-xx_1)]$. Since we have

$$\frac{C-\partial D/\partial x}{x_2-xx_1}=\frac{A}{f}-\frac{\partial}{\partial x}\left(\frac{D}{x_2-xx_1}\right)$$

then $E' \in Fr(f)$. We deduce that $C-\partial D/\partial x \in (x_2-xx_1)$; thus E has a special solution of the form $D/(x_2-xx_1)+\gamma$ where γ is a holomorphic solution of E'. □

Consider $E=[\partial\alpha/\partial x=A/f] \in Fr(f)$. There exists a solution of E of the form $\beta_j/((x_2-xx_1)x_j)$ for some $\beta_j \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ and any $j \in \{1, 2\}$ by Lemma 5.7. Then $\delta^0(E)$ (see §5.2.1) is given by $\beta_1/((x_2-xx_1)x_1)-\beta_2/((x_2-xx_1)x_2)$. This function is of the form $\beta(x_1, \ldots, x_n)/(x_1x_2)$ for some $\beta \in \mathbb{C}\{x_1, \ldots, x_n\}$.

Consider $E_0=[\partial\alpha/\partial x=1/f]$. We have

$$\frac{\partial}{\partial x}\left(\frac{1}{x_1(x_2-xx_1)}\right)=\frac{\partial}{\partial x}\left(\frac{x}{x_2(x_2-xx_1)}\right)=\frac{1}{(x_2-xx_1)^2}.$$

Then $\delta^0(E_0)$ is given by the function $1/(x_1x_2)$. This implies the following result.

LEMMA 5.8. *Let* $f=(x_2-xx_1)^2 \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. *Then the space* $Fr(f)/Sp(f)$ *is equal to* $\mathbb{C}\{x_3, \ldots, x_n\}E_0$.

Suppose from now on that $n=2$; this implies that $Fr(f)/Sp(f) \sim \mathbb{C}$. Therefore for $\varphi_1=\exp(\hat{u}_1 f\,\partial/\partial x)$ and $\varphi_2=\exp(\hat{u}_2 f\,\partial/\partial x)$ such that $\varphi_2 \in \mathcal{D}_f(\varphi_1)$ there is a unique $\lambda \in \mathbb{C}$ such that $1/\hat{u}_1-1/\hat{u}_2-\lambda \in (x_1, x_2)$. Then φ_1 and φ_2 are conjugated by a formal normalized transformation if and only if $\lambda=0$; this is equivalent to $\hat{u}_1(0)=\hat{u}_2(0)$. Note that $2\hat{u}_j(0)=\partial^2(x\circ\varphi_j)/\partial {x_2}^2(0)$. We deduce that

$$\left(\mathrm{Res}_{x_2-xx_1=0}(\varphi), \frac{\partial^2(x\circ\varphi)}{\partial {x_2}^2}(0,0,0)\right)$$

is a complete system of formal normalized invariants in $\mathcal{D}_f \subset \mathrm{Diff}_{up}(\mathbb{C}^3, 0)$.

We can provide a geometrical interpretation for the non-residual invariant. Consider $\varphi_1=\exp(\hat{u}_1 f\,\partial/\partial x)$, $\varphi_2=\exp(\hat{u}_2 f\,\partial/\partial x) \in \mathcal{D}_f$ such that their associated homological equation $\partial\alpha/\partial x=A/f$ is free of residues. Let $P=(x^0, x_1^0, x_2^0)$ an element of $\{f=0\} \setminus \{x_1=x_2=0\}$. Denote $w=x_2^0-x_1^0 x$; then $\varphi_j=\exp(\hat{u}_j f\,\partial/\partial x)$ implies that

$$\varphi_{j,P}=\exp\left((-x_1^0\hat{u}_j(P)w^2+O(w^3))\frac{\partial}{\partial w}\right)$$

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in the neighborhood of $w = 0$ for any $j \in \{1, 2\}$. Let us point out that $\hat{u}_j(P)$ is well defined since $\hat{u}_j$ is t.f. along $f = 0$ (Proposition 4.6). The one-dimensional germs $\varphi_{1,P}$ and $\varphi_{2,P}$ are formally conjugated by a transformation whose linear part is $(\hat{u}_1(P)/\hat{u}_2(P))w$, no other linear part is possible. Since a normalized conjugation restricted to $x_1 = x_2 = 0$ is the identity, then the existence of a formal normalized conjugation at the 1-jet along $f = 0$ level implies $A\hat{u}_1 = 1 - \hat{u}_1/\hat{u}_2 \in (x_1, x_2)$ and then $A \in (x_1, x_2)$. By Lemma 5.7 we have that vanishing of residues plus 1-jet compatibility is equivalent to the existence of a formal normalized conjugation.

6. *Convergent actions*

We have restricted our study to formal normalized conjugations. The goal of this section is linking the equivalence relations 'being formally conjugated' and 'being formally conjugated by a normalized transformation'. The main result is the following.

THEOREM 6.1. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ be formally conjugated. Then φ_1 and φ_2 are formally conjugated by a transformation of the form $\hat{\sigma} \circ \sigma$ where σ belongs to* $\mathrm{Diff}(\mathbb{C}^{n+1}, 0)$ *and $\hat{\sigma} \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$ is normalized.*

Let us remark that in the theorem $\hat{\sigma}$ is normalized with respect to $x \circ \varphi_2 - x = 0$.

In general a formal conjugation is not of the form $\hat{\sigma} \circ \sigma$. The action induced by $\hat{\sigma} \circ \sigma$ in the non-fibered components of $x \circ \varphi_1 - x = 0$ is the one induced by σ since $\hat{\sigma}$ is normalized. Thus it is convergent. Now consider $\varphi_1 = \varphi_2 = (x/(1 - x), y)$. We have that φ_1 and φ_2 are conjugated by $\hat{\tau} = (x, \sum_{j=1}^{\infty} j! y^j)$. The action of $\hat{\tau}$ on $x = 0$ is not convergent, and therefore $\hat{\tau}$ cannot be expressed in the form $\hat{\sigma} \circ \sigma$ where $\sigma \in \mathrm{Diff}(\mathbb{C}^2, 0)$ and $\hat{\sigma} \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^2, 0)$ is normalized.

In order to prove Theorem 6.1 it suffices to show the following.

PROPOSITION 6.1. *Let φ_j be an element of* $\mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ *with convergent infinitesimal generator for $j \in \{1, 2\}$. Suppose that φ_1 and φ_2 are formally conjugated. Then they are analytically conjugated.*

Let us explain why Proposition 6.1 implies Theorem 6.1. Let φ_1, φ_2 be elements of $\mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ which are formally conjugated by $\hat{\tau}$. By Proposition 5.12 there exists $\alpha_j \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$ such that $\log \alpha_j$ is convergent and α_j is conjugated to φ_j by a normalized $\hat{H}_j \in \widehat{\mathrm{Diff}}_p(\mathbb{C}^{n+1}, 0)$ for $j \in \{1, 2\}$. We obtain that α_1 and α_2 are formally conjugated and then conjugated by some $\tau \in \mathrm{Diff}(\mathbb{C}^{n+1}, 0)$. We define

$$\hat{\sigma} = \hat{H}_2 \circ (\tau \circ \hat{H}_1^{-1} \circ \tau^{-1}) \quad \text{and} \quad \sigma = \tau.$$

Clearly $\hat{\sigma} \circ \sigma$ conjugates φ_1 and φ_2. Moreover $\hat{\sigma}$ is normalized and σ is convergent.

The next proposition is a sort of preparation theorem.

PROPOSITION 6.2. *Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$. Suppose that φ_1 and φ_2 are formally conjugated. Then for any $\nu \in \mathbb{N}$ there exists ρ_ν in* $\mathrm{Diff}(\mathbb{C}^{n+1}, 0)$ *such that the diffeomorphism $\varphi_{2,\nu} = \rho_\nu^{-1} \circ \varphi_2 \circ \rho_\nu$ satisfies:*

- $\varphi_{2,\nu} \in \mathcal{D}_{x \circ \varphi_1 - x} \subset \mathrm{Diff}_{up}(\mathbb{C}^{n+1}, 0)$*;*
- $\mathrm{Res}_\gamma(\varphi_1) \equiv \mathrm{Res}_\gamma(\varphi_{2,\nu})$ *for any non-fibered component γ of* $\mathrm{Fix}\, \varphi_1$*; and*
- $x \circ \varphi_1 - x \circ \varphi_{2,\nu} \in (I(S(x \circ \varphi_1 - x)) + (x))^{\nu+1}$.

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Proof. Denote $f = x \circ \varphi_1 - x$. Let $f_N f_F = \prod_{j=1}^{p} f_j^{l_j} \prod_{j=1}^{q} F_j^{m_j}$ be the decomposition of f into irreducible factors. Let $\hat{\rho} \in \widehat{\text{Diff}}(\mathbb{C}^{n+1}, 0)$ be the transformation conjugating φ_1 and φ_2. Since $\hat{\rho}(\text{Fix}\,\varphi_1) = \text{Fix}\,\varphi_2$ there exist functions $g_j \in \mathbb{C}\{x, x_1, \ldots, x_n\}$ and $G_k \in \mathbb{C}\{x_1, \ldots, x_n\}$ such that $(g_j \circ \hat{\rho})/f_j$ and $(G_k \circ \hat{\rho})/F_k$ are formal units for all $1 \leq j \leq p$ and $1 \leq k \leq q$. Consider a function P_j^1/Q_j^1 such that its restriction to $f_j = 0$ is the function $\text{Res}_{f_j=0}(\varphi_1)$ for any $1 \leq j \leq p$ (see Lemma 5.1). In an analogous way we consider a function P_j^2/Q_j^2 such that its restriction to $g_j = 0$ is the function $\text{Res}_{g_j=0}(\varphi_2)$ for any $1 \leq j \leq p$. We obtain the system

$$\begin{cases} g_j \circ \hat{\rho} = \hat{v}_j f_j & \text{for any } 1 \leq j \leq p, \\ G_j \circ \hat{\rho} = \hat{w}_j F_j & \text{for any } 1 \leq j \leq q, \\ P_j^1(Q_j^2 \circ \hat{\rho}) - Q_j^1(P_j^2 \circ \hat{\rho}) = \hat{r}_j f_j & \text{for any } 1 \leq j \leq p. \end{cases} \tag{X}$$

The third set of equations is a consequence of the invariance of the residues. We have that $\hat{v}_j$ and $\hat{w}_k$ are formal units whereas $\hat{r}_j$ is just a power series.

The ideal $I(S(f))$ associated to the evil set has a system $L_1, \ldots, L_d$ of generators composed by elements of $\mathbb{C}\{x_1, \ldots, x_n\}$. Let us study the equation $\hat{\rho} \circ \varphi_1 = \varphi_2 \circ \hat{\rho}$. The transformation $\hat{\rho}$ is of the form

$$\hat{\rho} = \left(\sum_{j=0}^{2\nu} \hat{a}_j x^j + x^{2\nu+1} \hat{A}, \hat{\rho}_1, \ldots, \hat{\rho}_n \right),$$

where $\hat{a}_0, \ldots, \hat{a}_{2\nu}$, $\hat{\rho}_1, \ldots, \hat{\rho}_n \in \mathbb{C}[[x_1, \ldots, x_n]]$ and $\hat{A} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$. We denote by $\hat{\rho}'$ the transformation obtained by replacing $\hat{A}$ with 0 in the expression of $\hat{\rho}$. We want to compare the coefficients of x^b $(b \leq \nu)$ of $x \circ \hat{\rho}' \circ \varphi_1$ and $x \circ \varphi_2 \circ \hat{\rho}'$. We have

$$\frac{\partial^b (x \circ \varphi_2 \circ \hat{\rho})}{\partial x^b}(0, x_1, \ldots, x_n) = \frac{\partial^b (x \circ \varphi_2 \circ \hat{\rho}')}{\partial x^b}(0, x_1, \ldots, x_n)$$

and

$$\left(\frac{\partial^b (x \circ \hat{\rho} \circ \varphi_1)}{\partial x^b} - \frac{\partial^b (x \circ \hat{\rho}' \circ \varphi_1)}{\partial x^b} \right)(0, x_1, \ldots, x_n) \in I(S(f))^{\nu+1},$$

for any $0 \leq b \leq \nu$. The coefficient of x^b of $x \circ \hat{\rho}' \circ \varphi_1$ can be expressed in the form $C_b(x_1, \ldots, x_n, \hat{a}_0, \ldots, \hat{a}_{2\nu})$ for some holomorphic C_b. Conversely the coefficient of x^b of $x \circ \varphi_2 \circ \hat{\rho}'$ is of the form $D_b(\hat{\rho}_1, \ldots, \hat{\rho}_n, \hat{a}_0, \ldots, \hat{a}_{2\nu})$ for some holomorphic function D_b. We have

$$C_b - D_b = \sum_{k_1+\cdots+k_d=\nu+1} \hat{K}_{k_1,\ldots,k_d} L_1^{k_1} \ldots L_d^{k_d} \quad \text{for any } 0 \leq b \leq \nu, \tag{Y}$$

where $\hat{K}_{k_1,\ldots,k_d} \in \mathbb{C}[[x, x_1, \ldots, x_n]]$ for $k_1 + \cdots + k_d = \nu + 1$. By Artin's theorem **[1]** we can find a solution $(a_0, \ldots, a_{2\nu}, A, \rho_1, \ldots, \rho_n)$ satisfying both the systems (X) and (Y) and such that

$$\{a_0, \ldots, a_{2\nu}, \rho_1, \ldots, \rho_n\} \subset \mathbb{C}\{x_1, \ldots, x_n\}, \quad A \in \mathbb{C}\{x, x_1, \ldots, x_n\}.$$

Moreover, we can suppose that $j^1 a_k = j^1 \hat{a}_k$ for any $0 \leq k \leq 2\nu$ and $j^1 \rho_k = j^1 \hat{\rho}_k$ for any $1 \leq k \leq n$. We define

$$\rho_\nu = \left(\sum_{j=0}^{2\nu} a_j x^j + x^{2\nu+1} A, \rho_1, \ldots, \rho_n \right).$$

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By construction we have that $\rho_\nu \in \mathrm{Diff}(\mathbb{C}^{n+1}, 0)$ and $\varphi_{2,\nu} \in \mathcal{D}_{x\circ\varphi_1 - x}$. The invariance of the residues by ρ_ν implies that $\varphi_{2,\nu}$ satisfies the second condition in the statement of the proposition. Now since $(\sum_{j=0}^{2\nu} a_j x^j, \rho_1, \ldots, \rho_n)$ is a solution of system (Y) then

$$x \circ \rho_\nu \circ \varphi_1 - x \circ \varphi_2 \circ \rho_\nu \in (x^{\nu+1}) + I(S(f))^{\nu+1}.$$

We deduce that $x \circ \varphi_1 - x \circ \varphi_{2,\nu} \in ((x) + I(S(f)))^{\nu+1}$. □

We intend to prove that the homological equation associated to φ_1 and $\varphi_{2,\nu}$ is special for $\nu \gg 0$. We will construct special solutions in the neighborhood of every point outside of a set of codimension greater than or equal to 3. The next lemmas are of technical interest.

LEMMA 6.1. *Let I, J be ideals of a Noetherian ring R. Then there exists $\nu_0 \in \mathbb{N}$ such that we have $J^\nu \cap I \subset J^{\nu-\nu_0} I$ for any $\nu \geq \nu_0$.*

Proof. The equation $J^\nu \cap I = J^{\nu-\nu_0}(J^{\nu_0} \cap I)$ is a consequence of the Artin–Rees lemma (see [**2**, Corollary 10.10]); this implies that $J^\nu \cap I \subset J^{\nu-\nu_0} I$. □

LEMMA 6.2. *Let R be a domain of integrity. Consider an element g in $R \setminus \{0\}$ and an ideal $J \subset A$. Then there exists $\nu_0 \in \mathbb{N}$ such that $(J^\nu : g) \subset J^{\nu-\nu_0}$ for any $\nu \geq \nu_0$.*

Proof. We define $I = (g)$. Consider $h \in (J^\nu : g)$, we have $hg \in I \cap J^\nu$. By Lemma 6.1 there exists $\nu_0 \in \mathbb{N}$ such that $J^\nu \cap I \subset J^{\nu-\nu_0} I$ for $\nu \geq \nu_0$. Therefore h belongs to $J^{\nu-\nu_0}$. □

Let I be an ideal of $\mathbb{C}\{x, x_1, \ldots, x_n\}$. Fix a set $L = \{L_1, \ldots, L_d\}$ of generators of I. There exists a neighborhood W_L of the origin such that $L_j \in \mathcal{O}(W_L)$ for $1 \leq j \leq d$. For $P = (x^0, x_1^0, \ldots, x_n^0) \in W_L$ we define the ideal I_P contained in the ring $\mathbb{C}\{x - x^0, x_1 - x_1^0, \ldots, x_n - x_n^0\}$ and generated by $L_1, \ldots, L_d$. The definition of I_P does not depend on the system of generators. For a different finite system of generators L' there exists a neighborhood of the origin $W_{LL'} \subset W_L \cap W_{L'}$ where both definitions of I_P coincide for any $P \in W_{LL'}$.

LEMMA 6.3. *Let $0 \neq f \in \mathbb{C}\{x, x_1, \ldots, x_n\}$. There exists $\nu_0 \in \mathbb{N}$ such that for any $\nu \geq \nu_0$ we have an open set $U_\nu \ni 0$ satisfying that, for all $P \in U_\nu$ and A in $((x) + I(S(f)))_P^\nu$ such that $\partial\alpha/\partial x = A/f$ is special in a neighborhood of P, there exists a special solution $\beta_P/(f_F \prod_{j=1}^p f_j^{l_j-1})$ with $\beta_P \in ((x) + I(S(f)))_P^{\nu-\nu_0}$.*

Proof. Let $f_N = \prod_{j=1}^p f_j^{l_j}$ (see Definition 5.1). Denote $J = (x) + I(S(f))$. The proof is by induction on $l = \max_{j=1}^p l_j$. If $l = 0$ we can choose $\beta \in J^{\nu+1}$.

There exists $\nu_1 \in \mathbb{N}$ such that $(J^\nu : f_j) \subset J^{\nu-\nu_1}$ for all $\nu \geq \nu_1$ and $1 \leq j \leq p$ by Lemma 6.2. Indeed we obtain $(J_P^\nu : f_j) \subset J_P^{\nu-\nu_1}$ for any P in some open set $U_\nu^1 \ni 0$ by Oka's coherence theorem (see [**5**, p. 67]).

Denote $E = [\partial\alpha/\partial x = A/f]$ and $U_\nu^2 = \bigcap_{j=1}^p U^1_{\nu-(j-1)\nu_1}$. We can suppose that $l_j \neq 1$ for $1 \leq j \leq p$ since otherwise we replace E with

$$\frac{\partial\alpha}{\partial x} = \frac{A/\prod_{l_j=1} f_j}{f_F \prod_{l_j \neq 1} f_j^{l_j}},$$

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where $A/\prod_{l_j=1} f_j \in J_P^{\nu-p\nu_1}$ for all $P \in U_\nu^2$ and $\nu \geq p\nu_1$. A special solution $\beta'/(f_F \prod_{j=1}^p f_j^{l_j-1})$ of E is characterized by

$$\frac{\partial \beta'}{\partial x}\prod_{j=1}^{p} f_j - \beta' \sum_{j=1}^{p}(l_j-1)\frac{\partial f_j}{\partial x}\prod_{k\in\{1,\ldots,p\}\setminus\{j\}} f_k = A. \tag{3}$$

We define the ideal

$$I = \left(\prod_{j=1}^{p} f_j,\ \sum_{j=1}^{p}(l_j-1)\frac{\partial f_j}{\partial x}\prod_{k\in\{1,\ldots,p\}\setminus\{j\}} f_k\right).$$

Since E is special in the neighborhood of P then $A \in I_P$. By Lemma 6.1 and Oka's theorem there exist $\nu_3 \in \mathbb{N}$ such that $I_P \cap J_P^\nu \subset I_P J_P^{\nu-\nu_3}$ for all P in some open set $U_\nu^3 \ni 0$ and $\nu \geq \nu_3$. As a consequence there exist $B_P \in J_P^{\nu-p\nu_1-\nu_3}$ and $C_P \in J_P^{\nu-p\nu_1-\nu_3}$ such that

$$B_P\prod_{j=1}^{p} f_j - C_P \sum_{j=1}^{p}(l_j-1)\frac{\partial f_j}{\partial x}\prod_{k\in\{1,\ldots,p\}\setminus\{j\}} f_k = A \tag{4}$$

for all $P \in U_\nu^2 \cap U_{\nu-p\nu_1}^3$ and $\nu \geq p\nu_1+\nu_3$. By subtracting equations (3) and (4) we obtain $\beta' - C_P \in (\prod_{j=1}^p f_j)$. Therefore the function

$$\eta = \frac{(\beta'-C_P)/\prod_{j=1}^p f_j}{f_F\prod_{j=1}^p f_j^{l_j-2}}$$

is a special solution of

$$\frac{\partial\alpha}{\partial x} = \frac{B_P - \partial C_P/\partial x}{f_F\prod_{j=1}^p f_j^{l_j-1}} \tag{5}$$

(see proof of Lemma 5.5). We have that $B_P - \partial C_P/\partial x \in J_P^{\nu-p\nu_1-\nu_3-1}$ for any $\nu \geq p\nu_1+\nu_3+1$. By the hypothesis of induction there exists $\nu_4 \in \mathbb{N}$ and a special solution $\gamma_P/(f_F\prod_{j=1}^p f_j^{l_j-2})$ of equation (5) such that γ_P belongs to $J_P^{\nu-p\nu_1-\nu_3-1-\nu_4}$ for all $\nu \geq p\nu_1+\nu_3+\nu_4+1$ and $P \in U_\nu^4$ for some open set $U_\nu^4 \ni 0$. We define $\nu_0 = p\nu_1+\nu_3+\nu_4+1$ and $U_\nu = U_\nu^2 \cap U_{\nu-p\nu_1}^3 \cap U_\nu^4$. The function $\beta_P = C_P + \gamma_P\prod_{j=1}^p f_j$ belongs to $J_P^{\nu-\nu_0}$. Moreover $\beta_P/(f_F\prod_{j=1}^p f_j^{l_j-1})$ is a special solution of E in the neighborhood of P if $P \in U_\nu$. □

Next we prove Proposition 6.1. The proof is based on the fact that in the neighborhood of the generic points of $S(f)$ the quotient 'free of residues homological equations/special equations' generates a finite dimensional vector space over the meromorphic functions in $S(f)$.

Proof of Proposition 6.1. Let $\varphi_{2,\nu}$ be the diffeomorphism and U_ν be the open set given by Proposition 6.2 for any $\nu \in \mathbb{N}$. It suffices to prove that there exists $\nu_0 \in \mathbb{N}$ such that $\varphi_1 \sim \varphi_{2,\nu}$ (see Definition 5.7) for any $\nu \geq \nu_0$. Denote $f = x\circ\varphi_1 - x$. Let $f_N = \prod_{j=1}^p f_j^{l_j}$. We have $\varphi_1 = \exp(u_1 f\,\partial/\partial x)$ and $\varphi_{2,\nu} = \exp(u_{2,\nu} f\,\partial/\partial x)$. Consider the homological equation $E_\nu = [\partial\alpha/\partial x = A_\nu/f]$ associated to φ_1 and $\varphi_{2,\nu}$. The equation

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E_ν is free of residues by Proposition 6.2. Denote $J = (x) + I(S(f))$; we claim that $u_1 f - u_{2,\nu} f \in J^{\nu+1}$. Otherwise we have $u_1 f - u_{2,\nu} f \in J^a \setminus J^{a+1}$ for some $a < \nu + 1$. Note that since $S(f) \subset \{\prod_{l_j \geq 2} f_j = 0\}$ then $f \in I(S(f))^2$; this implies that $f \in J^2$ and $a \geq 2$. This property can be used to prove that

$$(u_1 f\, \partial/\partial x)^j(x) - (u_{2,\nu} f\, \partial/\partial x)^j(x) \in J^{a+1},$$

for any $j \geq 2$. As a consequence we obtain that $x \circ \varphi_1 - x \circ \varphi_{2,\nu} \notin J^{a+1}$, and that is impossible since $a + 1 \leq \nu + 1$. Since $A_\nu = (u_{2,\nu} f - u_1 f)/(u_1 u_{2,\nu} f)$ then $A_\nu \in J^{\nu-\nu_1}$ for any $\nu \geq \nu_1$ and some $\nu_1 \in \mathbb{N}$. The function A_ν is defined in some open set $U_\nu^1 \ni 0$.

Let $T(f)$ be the set of points of $S(f)$ where $S(f)$ is smooth and of local codimension 2. Consider $P = (0, x_1^0, \ldots, x_n^0) \in T(f)$; there exists $k(P) \in \mathbb{N}$ such that

$$\frac{\partial \alpha}{\partial x} = \frac{H^{k(P)} A_\nu}{f}$$

is special in the neighborhood of P for every $H \in \mathbb{C}\{x_1 - x_1^0, \ldots, x_n - x_n^0\}$ vanishing in $S(f)$ and any $\nu \in \mathbb{N}$. Moreover, a review of the proof of Proposition 5.3 implies that we can choose the same $k = k(P)$ for any $P \in \{x = 0\} \cap (T(f) \setminus F(f))$ where $F(f) \subset S(f)$ is a fibered analytic variety such that $\operatorname{cod}(F(f)) \geq 3$.

Fix $P = (0, x_1^0, \ldots, x_n^0) \in T(f) \setminus F(f)$. We can find new coordinates $(y_1, \ldots, y_n)$ centered at $(x_1, \ldots, x_n) = (x_1^0, \ldots, x_n^0)$ such that $S(f) = \{y_1 = y_2 = 0\}$. Suppose that $P \in U_\nu \cap U_\nu^1$; by Lemma 6.3 there exists $\nu_2 \in \mathbb{N}$ such that the equation E_ν has a solution

$$\alpha_{\nu,P,j} = \frac{\beta_{\nu,P,j}}{y_j^k f_F \prod_{r=1}^p f_r^{l_r - 1}},$$

where $\beta_{\nu,P,j} \in J_P^{\nu-\nu_1-\nu_2}$ for all $\nu \geq \nu_1 + \nu_2$ and $j \in \{1, 2\}$. Consider the set

$$K_\delta = \{|x| < \delta\} \cap \bigcap_{j=1}^{n} \{|y_j| < \delta\}$$

for some $\delta = \delta(\nu, P) > 0$ small enough. The element $\delta^0(E_\nu, K_\delta \setminus S(f))$ (see §5.2.1) of $H^1(K_\delta \setminus S(f), \mathcal{O}_D(f))$ is given by the function $\alpha_{\nu,P,1} - \alpha_{\nu,P,2}$. We obtain

$$(\alpha_{\nu,P,1} - \alpha_{\nu,P,2}) f_F = \frac{y_2^k \beta_{\nu,P,1} - y_1^k \beta_{\nu,P,2}}{y_1^k y_2^k \prod_{j=1}^p f_j^{l_j - 1}}.$$

Since $\partial(\alpha_{\nu,P,1} - \alpha_{\nu,P,2})/\partial x = 0$ then $\prod_{j=1}^p f_j^{l_j-1}$ divides $y_2^k \beta_{\nu,P,1} - y_1^k \beta_{\nu,P,2}$. Hence there exists an open set $U_\nu^2 \ni 0$ such that for $P \in U_\nu^2$ the function $(\alpha_{\nu,P,1} - \alpha_{\nu,P,2}) f_F$ can be expressed in the form $h_{\nu,P}(y_1, \ldots, y_n)/(y_1^k y_2^k)$ where $h_{\nu,P} \in J_P^{\nu-\nu_1-\nu_2-\nu_3}$ for any $\nu \geq \nu_1 + \nu_2 + \nu_3$ and some $\nu_3 \in \mathbb{N}$. We define $\nu_0 = \nu_1 + \nu_2 + \nu_3 + (2k - 1)$. The set $J_P^b \cap \mathbb{C}\{y_1, \ldots, y_n\}$ is contained in $I(S(f))_P^b \subset (y_1, y_2)^b$ for any $b \in \mathbb{N}$, and then for $\nu \geq \nu_0$ the function $h_{\nu,P}/(y_1^k y_2^k)$ is of the form

$$\frac{h_{\nu,P}}{y_1^k y_2^k} = H + \sum_{0<j\leq k} \frac{c_j(y_2, \ldots, y_n)}{y_1^j} + \sum_{0<j\leq k} \frac{d_j(y_1, y_3, \ldots, y_n)}{y_2^j},$$

where H, c_j and d_j are holomorphic in K_δ for any $0 < j \leq k$. The function

$$\alpha_{\nu,P} \stackrel{\text{def}}{=} \alpha_{\nu,P,1} - \Big(H + \sum_{-k\leq j<0} c_j y_1^j\Big) \Big/ f_F = \alpha_{\nu,P,2} + \Big(\sum_{-k\leq j<0} d_j y_2^j\Big) \Big/ f_F$$

is a special solution of E_ν in K_δ for $\nu \geq \nu_0$.

Consider a polydisk $0 \in \Delta_\nu$ in the variables $(x, x_1, \ldots, x_n)$ contained in the set $U_\nu \cap U_\nu^1 \cap U_\nu^2$. We denote $\pi(x, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. For all $\nu \geq \nu_0$ and $P \in \{x = 0\} \cap (T(f) \setminus F(f)) \cap \Delta_\nu$ there exists a polydisk $\Delta_{\nu,P} \subset \Delta_\nu$ centered at P and a special solution $\alpha_{\nu,P}$ of E_ν defined in $\Delta_{\nu,P}$. By using the homological equation we can extend $\alpha_{\nu,P}$ to $\Delta_\nu \cap \pi^{-1}(\Delta_{\nu,P} \setminus S(f))$ and then to $\Delta_\nu \cap \pi^{-1}(\Delta_{\nu,P})$ since cod $S(f) \geq 2$. We obtain special solutions of E_ν $(\nu \geq \nu_0)$ in the neighborhood of every point not in $(S(f) \setminus T(f)) \cup F(f)$. Therefore we have

$$\delta^0(E_\nu) \in H^1(\Delta_\nu \setminus [(S(f) \setminus T(f)) \cup F(f)], \mathcal{O}_D(f)).$$

Since the codimension of $(S(f) \setminus T(f)) \cup F(f)$ is greater than or equal to 3 then $\delta^0(E_\nu) = 0$ for $\nu \geq \nu_0$. We deduce that $E_\nu \in Sp(f)$ and then $\varphi_1 \sim \varphi_{2,\nu}$ for $\nu \geq \nu_0$ by Proposition 5.10. □

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