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ARTICLE

On the Chromatic Number of Matching Kneser Graphs

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Abstract

In an earlier paper, the present authors (2015) introduced the *altermatic number* of graphs and used Tucker's lemma, an equivalent combinatorial version of the Borsuk–Ulam theorem, to prove that the altermatic number is a lower bound for chromatic number. A matching Kneser graph is a graph whose vertex set consists of all matchings of a specified size in a host graph and two vertices are adjacent if their corresponding matchings are edge-disjoint. Some well-known families of graphs such as Kneser graphs, Schrijver graphs and permutation graphs can be represented by matching Kneser graphs. In this paper, unifying and generalizing some earlier works by Lovász (1978) and Schrijver (1978), we determine the chromatic number of a large family of matching Kneser graphs by specifying their altermatic number. In particular, we determine the chromatic number of these matching Kneser graphs in terms of the generalized Turán number of matchings.

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1. Introduction

1.1 Main results

In a breakthrough [23], proving a conjecture by Kneser [14], Lovász determined the chromatic number of Kneser graphs. In his proof, he mainly used algebraic topological tools for the first time to solve a problem in combinatorics. This marked the beginning of the use of algebraic topology in combinatorics. Nowadays, the use of algebraic topology as a powerful tool to investigate graph colouring properties has received considerable attention in combinatorics [1, 4, 7, 8, 9, 10, 18, 23, 24, 26, 27, 30]. In this regard, using Tucker's lemma, an equivalent combinatorial version of the Borsuk–Ulam theorem, the present authors [1] introduced the *altermatic number* of graphs as a tight lower bound for their chromatic number generalizing the Kneser–Lovász theorem [23], the Dol'nikov–Kříž theorem [10, 18] and the Schrijver theorem [25]. This lower bound has been used to investigate the chromatic number of some families of graphs [1, 2, 3].

For a positive integer r and a graph G, the *matching Kneser graph* $\mathrm{KG}(G, rK_2)$ is a graph whose vertex set is the set of r-matchings in G and two vertices are adjacent if their corresponding matchings are edge-disjoint. Some well-known families of graphs can be represented in the form of matching Kneser graphs. For instance, the matchingKneser graphs $\mathrm{KG}(nK_2, rK_2)$, $\mathrm{KG}(C_n, rK_2)$

and $\mathrm{KG}(K_{m,n}, rK_2)$ are isomorphic to the Kneser graph $\mathrm{KG}(n, r)$, the Schrijver graph $\mathrm{SG}(n, r)$ and the permutation graph $S_r(m, n)$, respectively. Evidently, computing the chromatic number of matching Kneser graphs could be of interest. When $r = 1$, the matching Kneser graph $\mathrm{KG}(G, rK_2) = \mathrm{KG}(G, K_2)$ is isomorphic to the complete graph $K_{|E(G)|}$ and its chromatic number is equal to $|E(G)|$. Henceforth, we only deal with $r \geqslant 2$. In general, using a simple greedy argument, one can see that $\chi(\mathrm{KG}(G, rK_2)) \leqslant |E(G)| - \mathrm{ex}\,(G, rK_2)$, where $\mathrm{ex}\,(G, rK_2)$ denotes the largest number of edges in G avoiding a matching of size r. To see this, let $A \subseteq E(G)$ be a set of edges avoiding a matching of size r. Set $E(G) \setminus A = \{e_1, \ldots, e_t\}$. To any vertex v of $\mathrm{KG}(G, rK_2)$, which is a matching of size r in G, assign the minimum integer $i \in [t]$ as its colour so that v contains e_i. By the well-known result of Schrijver [25], we know that the chromatic number of Schrijver graph $\mathrm{SG}(n, r) \simeq \mathrm{KG}(C_n, rK_2)$ is $n - \min\{n, 2(r-1)\} = |E(C_n)| - \mathrm{ex}\,(C_n, rK_2)$, which implies that the aforementioned upper bound is tight when G is a cycle. As an interesting question, one may ask for the graphs G for which $\chi(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rk_2)$. As the main objective of this paper, we set out to explore this question more precisely. In this regard, we determine the chromatic number of matching Kneser graph $\mathrm{KG}(G, rK_2)$ provided that G satisfies some certain properties (*e.g.* G is a large dense graph). To simplify the reading of the paper, we unify the main results of the paper in the next theorem. However, due to the need for some technical definitions and also for brevity, we postpone the statement of some other interesting results.

Theorem 1.1. *For any positive integer $r \geqslant 2$, the following assertions hold.*

(i) **(Dense graphs)** *There exist constants $\alpha = \alpha(r)$, $\beta = \beta(r)$ and $n_0 = n(r)$ such that, for any graph G with $n \geqslant n_0$ vertices and $\delta(G) \geqslant (1/2 - \alpha)n + \beta$,*

$$\chi(\mathrm{KG}(G, rK_2)) = \zeta(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rK_2).$$

(ii) **(Spanning bipartite dense subgraphs)** *There exist constants $q = q(r)$ and $n_0 = n(r)$ such that, for all $n \geqslant n_0$, if G is a graph with $2n$ vertices having a bipartite subgraph $H = (U, V, E)$ such that $|U| = |V| = n$ and $\delta(H) \geqslant n/2 + q$, then*

$$\chi(\mathrm{KG}(G, rK_2)) = \zeta(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rK_2).$$

(iii) **(Regular even graphs)** *If G is a connected k-regular graph with odd-girth g, where $k \geqslant 1$ is even, $|V(G)| \geqslant 2r$ and $2r < g$, then*

$$\chi(\mathrm{KG}(G, rK_2)) = \zeta(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rK_2) = |E(G)| - k(r-1).$$

(For bipartite graphs, the odd-girth is defined to be infinite.)

(iv) **(Expander graphs)** *Let ν, τ and η be real numbers, where $0 < \nu \leqslant \tau \ll \eta < 1$. If n is sufficiently large, then, for any robust (ν, τ)-expander graph G with n vertices and $\delta(G) \geqslant \eta n$,*

$$\chi(\mathrm{KG}(G, rK_2)) = \zeta(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rK_2).$$

(v) **(Random graphs)** *If $0 < p \leqslant 1$ and $\gamma > 0$, then asymptotically almost surely, for any subgraph G' of any $G \in G(n, p)$ with $\delta(G') \geqslant p(1/2 + \gamma)n$,*

$$\chi(\mathrm{KG}(G', rK_2)) = \zeta(\mathrm{KG}(G, rK_2)) = |E(G')| - \mathrm{ex}\,(G', rK_2).$$

The quantities $\zeta(\mathrm{KG}(G, rK_2))$ and $\mathrm{ex}\,(G, rK_2)$ appearing in the statement of Theorem 1.1 refer, respectively, to the 'altermatic number of graph $\mathrm{KG}(G, rK_2)$' and the 'generalized Turán number of r-matching in G', which will be explicitly introduced in Sections 2.1 and 2.2. Also, the definition of 'robust (ν, τ)-expander graph' can be found in Section 3.2. It should be mentioned that some of the previously mentioned results will be stated in stronger forms later in the paper.

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It is worth noting that Theorem 1.1 provides a sufficient condition for a graph G to satisfy the equality $\chi(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rK_2)$. A question which naturally arises is how far this result could potentially be generalized. Note that when G is disconnected there are simple counter-examples to this conjecture. For instance, for $n \geqslant 2r$ and $G = nK_2$, by Lovász's theorem [23], $\chi(nK_2, rK_2) = n - 2(r-1)$ while $|E(G)| - \mathrm{ex}\,(nK_2, rK_2) = n - (r-1)$. We believe that the connectivity of G would be sufficient for the aforementioned equality.

Conjecture 1.2. *For any connected graph G and positive integer $r \geqslant 2$,*

$$\chi(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rK_2).$$

Theorem 1.1 and also some other results stated later in the paper give strong evidence in favour of Conjecture 1.2.

1.2 Plan

This paper is organized as follows. In Section 2 we set up notation and terminology. In particular, we will be concerned with the definition of Kneser graphs, altermatic number, and strong altermatic number of hypergraphs, and we mention some results on them. Next, as a generalization of the generalized Turán number, we introduce two types of alternating Turán number of graphs which will be used for presenting some lower bounds for the chromatic number of some general Kneser graphs. These lower bounds will be used for the proof of Theorem 1.1 and also some other results in the paper. In Section 3, we will determine the chromatic number of matching Kneser graphs provided some technical conditions are satisfied. Below, as a generalization of the well-known result of Schrijver, we will specify the chromatic number of a large family of matching Kneser graphs in terms of the generalized Turán number of matchings. In particular, we determine the chromatic number of large permutation graphs.

2. Notation and terminology

In this section, we set up some notation and terminology. Hereafter, the symbol $[n]$ stands for the set $\{1, 2, \ldots, n\}$. A *hypergraph* $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a set of elements called *vertices* and $E(\mathcal{H})$ is a family of non-empty subsets of $V(\mathcal{H})$ called *edges*. Unless otherwise stated, we consider simple hypergraphs, that is, $E(\mathcal{H})$ is a family of distinct non-empty subsets of $V(\mathcal{H})$. A subset $T \subseteq V(\mathcal{H})$ which meets every edge of $\mathcal{H}$ is called a *vertex cover* of $\mathcal{H}$. Also, a *k-colouring* of $\mathcal{H}$ is a mapping $h : V(\mathcal{H}) \longrightarrow [k]$ such that, for any edge e, we have $|\{h(v) : v \in e\}| \geqslant 2$, that is, no edge is monochromatic. The minimum k for which $\mathcal{H}$ admits a k-colouring is called the *chromatic number* of $\mathcal{H}$ and is denoted by $\chi(\mathcal{H})$. Note that if $\mathcal{H}$ has some edge with cardinality 1, then it has no k-colouring for each $k \in \mathbb{N}$. We define the chromatic number of such a hypergraph to be infinite. The hypergraph $\mathcal{H}$ is called *k-uniform* if $|e| = k$ for each $e \in E(\mathcal{H})$. Throughout the paper, a 2-uniform hypergraph is simply called a graph. Let $o(G)$ denote the number of odd components of a graph G. For brevity, we use $G \simeq H$ to denote that there is an isomorphism between two graphs G and H. Also, if $G \simeq H$, then we say G and H are *isomorphic*. A *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \longrightarrow V(H)$ which preserves the adjacency, that is, if $xy \in E(G)$, then $f(x)f(y) \in E(H)$. For brevity, we use $G \rightarrow H$ to denote that there is a homomorphism from G to H. If we have both $G \rightarrow H$ and $H \rightarrow G$, then we say G and H are *homomorphically equivalent* and show this by $G \leftrightarrow H$. Note that $\chi(G)$ is the minimum integer k for which there is a homomorphism from G to the complete graph K_k. For a subgraph H of G, the two subgraphs $G \setminus H$ and $G - H$ are obtained from G by removing the edges and the vertices of H, respectively. Note that $G \setminus H$ is a spanning subgraph of G while $V(G - H) = V(G) \setminus V(H)$.

2.1 Altermatic number

For a hypergraph $\mathcal{H}$, *the general Kneser graph* $\mathrm{KG}(\mathcal{H})$ is a graph with vertex set $E(\mathcal{H})$ and two vertices are adjacent if their corresponding edges are disjoint. For a graph G, if $\mathrm{KG}(\mathcal{H})$ and G are isomorphic, $\mathcal{H}$ is termed a *Kneser representation of* G. It is not difficult to verify that any graph has infinitely many Kneser representations.

The sequence $x_1, \ldots, x_m \in \{-1, +1\}$ is said to be an *alternating sequence* if any two consecutive terms of this sequence are different. For any $X = (x_1, \ldots, x_n) \in \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\}$, the *alternation number* of X, denoted by $\mathrm{alt}\,(X)$, is the length of a longest alternating subsequence of non-zero terms of $(x_1, \ldots, x_n)$. Note that we only consider non-zero entries to determine the alternation number of X. Also, we set $\mathrm{alt}\,(0, \ldots, 0) = 0$. Let $V = \{v_1, \ldots, v_n\}$ be a set of size n and L_V be the set of all linear orderings (or orderings for brevity) of V, that is, $L_V = \{v_{i_1} < \cdots < v_{i_n} : (i_1, \ldots, i_n) \in S_n\}$, where S_n is the permutation group on $[n]$. For any ordering $\sigma : v_{i_1} < \cdots < v_{i_n} \in L_V$ and $1 \leqslant j \leqslant n$, define $\sigma(j) = v_{i_j}$. The ordering σ can be represented by the permutation $\sigma = (v_{i_1}, \ldots, v_{i_n})$ as well. We use these two kinds of representations of any ordering interchangeably. For any $X = (x_1, \ldots, x_n) \in \{-1, 0, +1\}^n$, set

$$X^+_\sigma = \{\sigma(j) : x_j = +1\} = \{v_{i_j} : x_j = +1\} \quad \text{and} \quad X^-_\sigma = \{\sigma(k) : x_k = -1\} = \{v_{i_k} : x_k = -1\}.$$

For any hypergraph $\mathcal{H} = (V, E)$ and $\sigma \in L_V$, where $|V| = n$, we define $\mathrm{alt}_\sigma\,(\mathcal{H})$ to be the largest integer k for which there exists an $X \in \{-1, 0, +1\}^n$ with $\mathrm{alt}\,(X) = k$ such that neither X^+_σ nor X^-_σ contains an edge of $\mathcal{H}$, that is, $E(\mathcal{H}[X^+_\sigma]) = E(\mathcal{H}[X^-_\sigma]) = \varnothing$. Similarly, define $\mathrm{salt}_\sigma\,(\mathcal{H})$ to be the largest integer k for which there is an $X \in \{-1, 0, +1\}^n$ with $\mathrm{alt}\,(X) = k$ such that at most one of X^+_σ and X^-_σ contains some edge of $\mathcal{H}$. Note that if each singleton is an edge of $\mathcal{H}$, then $\mathrm{alt}\,(\mathcal{H}) = 0$. Also, $\mathrm{alt}_\sigma\,(\mathcal{H}) \leqslant \mathrm{salt}_\sigma\,(\mathcal{H})$ and equality can hold. Now, set $\mathrm{alt}\,(\mathcal{H}) = \min\{\mathrm{alt}_\sigma\,(\mathcal{H}) : \sigma \in L_V\}$ and $\mathrm{salt}\,(\mathcal{H}) = \min\{\mathrm{salt}_\sigma\,(\mathcal{H}) : \sigma \in L_V\}$. Define the *altermatic number* and the *strong altermatic number* of a graph G, respectively, as follows:

$$\zeta(G) = \max_{\mathcal{H}}\{|V(\mathcal{H})| - \mathrm{alt}\,(\mathcal{H}) : \mathrm{KG}(\mathcal{H}) \longleftrightarrow G\}$$

and

$$\zeta_s(G) = \max_{\mathcal{H}}\{|V(\mathcal{H})| + 1 - \mathrm{salt}\,(\mathcal{H}) : \mathrm{KG}(\mathcal{H}) \longleftrightarrow G\}.$$

It was proved in [1, 2] that each altermatic number and strong altermatic number provides a tight lower bound for the chromatic number of graphs.

Theorem 2.1 (Theorem 2 of [1]). *For any graph G, we have*

$$\chi(G) \geqslant \max\{\zeta(G), \zeta_s(G)\}.$$

2.2 Alternating Turán number

Let G be a graph and let $\mathcal{F}$ be a family of graphs. A subgraph of G is called an *$\mathcal{F}$-subgraph* if it is isomorphic to a member of $\mathcal{F}$. *The general Kneser graph* $\mathrm{KG}(G, \mathcal{F})$ has all the $\mathcal{F}$-subgraphs of G as its vertex set and two vertices are adjacent if the corresponding $\mathcal{F}$-subgraphs are edge-disjoint. A graph G is celled *$\mathcal{F}$-free* if it has no subgraph isomorphic to a member of $\mathcal{F}$. For a graph G, the *generalized Turán number of $\mathcal{F}$ in G*, denoted by $\mathrm{ex}\,(G, \mathcal{F})$, is the maximum number of edges of an $\mathcal{F}$-free spanning subgraph of G. An $\mathcal{F}$-free spanning subgraph of G is called *$\mathcal{F}$-extremal* if it has $\mathrm{ex}\,(G, \mathcal{F})$ edges. We denote the family of all $\mathcal{F}$-extremal subgraphs of G by $\mathrm{EX}\,(G, \mathcal{F})$. It is usually a hard problem to determine the exact value of $\mathrm{ex}\,(G, \mathcal{F})$. The concept of Turán number was generalized to alternating Turán number in [2], where it was used to determine the chromatic number of some families of general Kneser graphs.

Hereafter, by abuse of language, for a given 2-colouring (with two colours red and blue) of a subset of edges of G, if red or blue colour is assigned to an edge, then it is respectively called a *red*

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edge or a *blue edge*. If no colour is assigned to an edge, then it is called a *neutral edge*. The spanning subgraph G^R (resp. G^B) of G whose edge-set consists of all red (resp. blue) edges is termed the *red subgraph* (resp. *blue subgraph*). Let $E(G) = \{e_1, e_2, \ldots, e_m\}$. For any ordering $\sigma = (e_{i_1}, e_{i_2}, \ldots, e_{i_m})$ of edges of G, a 2-colouring of a subset of edges of G is said to be *alternating* (with respect to the ordering σ) if any two consecutive coloured edges (with respect to the ordering σ) receive different colours (note that we may assign no colour to some edges of G). In other words, in view of the ordering σ, we assign two colours red and blue alternately to a subset of edges of G. The *length* of an alternating 2-colouring of $E(G)$ is the number of coloured edges, that is, $|E(G^R)| + |E(G^B)|$. It is worth noting that $||E(G^R)| - |E(G^B)|| \leqslant 1$ for any alternating 2-colouring of $E(G)$. We use the notation $\mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}, \sigma)$ to denote the maximum length of an alternating 2-colouring of $E(G)$ with respect to σ for which G^R and G^B are both $\mathcal{F}$-free. Also, we define $\mathrm{ex}_{\mathrm{salt}}(G, \mathcal{F}, \sigma))$ to be the maximum length of an alternating 2-colouring of $E(G)$ with respect to σ for which at least one of G^R and G^B is $\mathcal{F}$-free. Note that if we have an alternating 2-colouring with respect to σ and of length more than $\mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}, \sigma)$ (resp. $\mathrm{ex}_{\mathrm{salt}}(G, \mathcal{F}, \sigma)$), then at least one of G^R and G^B (resp. each of G^R and G^B) contains a member of $\mathcal{F}$ as a subgraph.

Now, we are in a position to define the *alternating Turán number* $\mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F})$ and the *strong alternating Turán number* $\mathrm{ex}_{\mathrm{salt}}(G, \mathcal{F})$ as follows:

$$\mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}) = \min\{\mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}, \sigma); \sigma \in L_{E(G)}\}$$

and

$$\mathrm{ex}_{\mathrm{salt}}(G, \mathcal{F}) = \min\{\mathrm{ex}_{\mathrm{salt}}(G, \mathcal{F}, \sigma); \sigma \in L_{E(G)}\}.$$

For a graph G, let F be a member of $\mathrm{EX}(G, \mathcal{F})$ and let σ be an arbitrary ordering of $E(G)$. If we colour the edges of F alternately with two colours with respect to the ordering σ, no colour class has a member of $\mathcal{F}$ as its subgraph; therefore, $\mathrm{ex}(G, \mathcal{F}) \leqslant \mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}, \sigma)$. Also, it is clear that if we assign two colours to more than $2\,\mathrm{ex}(G, \mathcal{F})$ edges, then a colour class has more than $\mathrm{ex}(G, \mathcal{F})$ edges, and accordingly, it contains some member of $\mathcal{F}$ as its subgraph. It implies $\mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}, \sigma) \leqslant 2\,\mathrm{ex}(G, \mathcal{F})$. Consequently

$$\mathrm{ex}(G, \mathcal{F}) \leqslant \mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}) \leqslant 2\,\mathrm{ex}(G, \mathcal{F}).$$

The next lemma was proved in [2]. For convenience, we briefly repeat the proof, thus making our exposition self-contained.

Lemma 2.2. ([2]). *For any graph G and family $\mathcal{F}$ of graphs,*

$$|E(G)| - \mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}) \leqslant \chi(\mathrm{KG}(G, \mathcal{F})) \leqslant |E(G)| - \mathrm{ex}(G, \mathcal{F}),$$
$$|E(G)| + 1 - \mathrm{ex}_{\mathrm{salt}}(G, \mathcal{F}) \leqslant \chi(\mathrm{KG}(G, \mathcal{F})) \leqslant |E(G)| - \mathrm{ex}(G, \mathcal{F}).$$

In particular, if $\mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}) = \mathrm{ex}(G, \mathcal{F})$, *then*

$$\chi(\mathrm{KG}(G, \mathcal{F})) = \zeta(\mathrm{KG}(G, \mathcal{F})) = |E(G)| - \mathrm{ex}(G, \mathcal{F}),$$

and if $\mathrm{ex}_{\mathrm{salt}}(G, \mathcal{F}) = \mathrm{ex}(G, \mathcal{F}) + 1$, *then*

$$\chi(\mathrm{KG}(G, \mathcal{F})) = \zeta_s(\mathrm{KG}(G, \mathcal{F})) = |E(G)| - \mathrm{ex}(G, \mathcal{F}).$$

Proof. Let $K \in \mathrm{EX}(G, \mathcal{F})$. Clearly, any vertex of $\mathrm{KG}(G, \mathcal{F})$ contains at least one edge of $E(G) \setminus E(K)$. This implies $\chi(\mathrm{KG}(G, \mathcal{F})) \leqslant |E(G)| - \mathrm{ex}(G, \mathcal{F})$. On the other hand, consider the hypergraph $\mathcal{H}$ whose vertex set is $E(G)$ and its edge-set consists of all subgraphs of G isomorphic to some member of $\mathcal{F}$. Note that $\mathrm{KG}(\mathcal{H})$ is isomorphic to $\mathrm{KG}(G, \mathcal{F})$. One can check that $\mathrm{alt}(\mathcal{H}) = \mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F})$ and $\mathrm{salt}(\mathcal{H}) = \mathrm{ex}_{\mathrm{salt}}(G, \mathcal{F})$. Now, by Theorem 2.1, the assertion holds. □

This lemma suggests an approach to determine the chromatic number of $\mathrm{KG}(G, \mathcal{F})$. Indeed, if we present an appropriate ordering σ of the edges of G such that $\mathrm{ex}_{\mathrm{alt}}(G, \mathcal{F}) = \mathrm{ex}(G, \mathcal{F})$ or

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$\text{ex}_{\text{salt}}(G, \mathcal{F}) - 1 = \text{ex}(G, \mathcal{F})$, then we would be able to conclude that

$$\chi(\text{KG}(G, \mathcal{F})) = |E(G)| - \text{ex}(G, \mathcal{F}).$$

Using this observation, the chromatic number of several families of graphs was determined in [2, 3].

3. Matching Kneser graphs

The matching Kneser graphs can be viewed as a generalization of Kneser, Schrijver and permutation graphs since $\text{KG}(nK_2, rK_2)$, $\text{KG}(C_n, rK_2)$ and $\text{KG}(K_{m,n}, rK_2)$ are isomorphic to Kneser, Schrijver and permutation graphs, respectively. Hence, as a generalization of Lovász's theorem [23] and Schrijver's theorem [25] and to unify their results, it would be of interest to study the chromatic number of matching Kneser graphs. We recall the following classical generalization of Tutte's theorem [28] by Berge [5], which will play an essential role in the proofs of some results in this section.

Theorem 3.1 (Tutte–Berge formula [5, 28]). *For a graph G, the maximum number of vertices of a matching in G is*

$$\min_{S \subseteq V(G)} \{|V(G)| - o(G - S) + |S|\},$$

where $o(G - S)$ is the number of odd components in $G - S$.

The rest of this section mainly concerns the proofs of the main results. In Section 3.1 we will study the chromatic number of the matching Kneser graph $\text{KG}(G, rK_2)$ when G is a sparse graph. In contrast, in Section 3.2 we will determine the chromatic number of $\text{KG}(G, rK_2)$ provided G is a large dense graph.

3.1 A generalization of Schrijver's theorem

The *odd-girth* of a graph G is the length of a shortest odd-cycle contained in G. For a bipartite graph, its odd-girth is set to be infinite. Also, a graph is called *even* if each of its vertices has an even degree.

Theorem 3.2. *Let $r \geqslant 2$ be an integer and let G be either a connected graph or a disconnected graph with no even component. Also, assume that G has odd-girth at least g, vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and degree sequence $\deg_G(v_1) \geqslant \deg_G(v_2) \geqslant \cdots \geqslant \deg_G(v_n)$. Moreover, suppose that*

$$r \leqslant \max\left\{\frac{g}{2}, \frac{\deg_G(v_{r-1}) + 1}{4}\right\}$$

and $U = \{v_1, \ldots, v_{r-1}\}$ forms an independent set. If $\deg_G(v_{r-1})$ is an even integer or

$$\deg_G(v_{r-1}) > \deg_G(v_r),$$

then

$$\chi(\text{KG}(G, rK_2)) = |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i).$$

Proof. Consider the spanning subgraph of G whose edges meet $U = \{v_1, \ldots, v_{r-1}\}$ which is clearly an rK_2-free subgraph with $\sum_{i=1}^{r-1} \deg_G(v_i)$ edges. This concludes that $\text{ex}(G, rK_2) \geqslant$

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$\sum_{i=1}^{r-1} \deg_G(v_i)$. Thus, by Lemma 2.2, we would have

$$\chi(\mathrm{KG}(G, rK_2)) \leqslant |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i).$$

Hence, to prove the theorem, it suffices to show that

$$\chi(\mathrm{KG}(G, rK_2)) \geqslant |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i).$$

To this end, in what follows, we introduce an ordering σ of $E(G)$ for which either

$$\mathrm{ex}_{\mathrm{alt}}\,(G, rK_2, \sigma) \leqslant \sum_{i=1}^{r-1} \deg_G(v_i)$$

or

$$\mathrm{ex}_{\mathrm{salt}}\,(G, rK_2, \sigma) \leqslant 1 + \sum_{i=1}^{r-1} \deg_G(v_i),$$

leading us to the desired inequality by using Lemma 2.2.

Let s be the number of vertices with odd degree in $\{v_1, \ldots, v_{r-1}\}$. If G is an even graph, set $H = G$; otherwise, add a new vertex w and join it to every odd vertex of G to obtain the graph H. Clearly, since G is either connected or disconnected with no even component, H is clearly a connected even graph and hence an Eulerian graph. Let $e'_1, e'_2, \ldots, e'_m$ be an Eulerian tour of H such that if G is an even graph, then it starts with v_n; otherwise, it starts with w. Now, consider the ordering $(e'_1, e'_2, \ldots, e'_m)$ and remove all edges incident with w from this ordering to obtain the ordering σ of $E(G)$. In other words, if we traverse $e_i \in E(G)$ before $e_j \in E(G)$ in the Eulerian tour, then in the ordering σ we have $e_i < e_j$. Now, consider an alternating colouring (with colours blue and red) of edges of G with respect to the ordering σ and of length t, where if $s \neq 0$, then $t = 1 + \sum_{i=1}^{r-1} \deg_G(v_i)$; otherwise, $t = 2 + \sum_{i=1}^{r-1} \deg_G(v_i)$. Recall that G^R and G^B are the spanning subgraphs of G whose edge-sets are the sets of red and blue edges respectively. We show that if $s = 0$ then both of G^R and G^B have an r-matching, and consequently

$$\mathrm{ex}_{\mathrm{salt}}\,(G, rK_2, \sigma) \leqslant 1 + \sum_{i=1}^{r-1} \deg_G(v_i),$$

and if $s \neq 0$ then G^R or G^B has an r-matching, and consequently

$$\mathrm{ex}_{\mathrm{alt}}\,(G, rK_2, \sigma) \leqslant \sum_{i=1}^{r-1} \deg_G(v_i).$$

Hence, by using Lemma 2.2,

$$\chi(\mathrm{KG}(G, rK_2)) \geqslant |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i),$$

as desired.

To do this, first note that, in view of the definition σ, for each vertex $x \in V(G) \setminus \{v_n\}$, each of colours red and blue is assigned to at most $\lceil \deg_G(x)/2 \rceil$ edges incident with x. For v_n, this amount can be increased by at most one whenever G is an even graph and t is thus odd. Indeed, in this case, if neither the first nor the last edge in the ordering σ is neutral, then they both have the same colour. Since each of these two edges is incident with v_n, the number of red (resp. blue) edges incident with v_n is at most

$$2+\frac{\deg_G(v_n)-2}{2}=\frac{\deg_G(v_n)}{2}+1.$$

Now, the proof falls into the two following parts.

Case 1: $r\leqslant g/2$. Note that if G^j is an rK_2-free subgraph for a $j\in\{R,B\}$, then, in view of the Tutte–Berge formula (Theorem 3.1), there exists an $S^j\subseteq V(G^j)=V(G)$ such that

$$|V(G^j)|-o(G^j-S^j)+|S^j|\leqslant 2r-2.$$

Let $O_1^j,O_2^j,\ldots,O_{t_j}^j$ be the components of G^j-S^j. Since $t_j\geqslant o(G^j-S^j)$,

$$\begin{aligned}\sum_{i=1}^{t_j}(|V(O_i^j)|-1)&\leqslant |V(G^j)|-|S^j|-t_j\\&\leqslant |V(G^j)|-|S^j|-o(G^j-S^j)\\&\leqslant 2r-2-2|S^j|,\end{aligned}$$

which concludes $|V(O_i^j)|\leqslant 2r-2|S^j|-1\leqslant g-1$ for each $j\in\{R,B\}$ and $1\leqslant i\leqslant t_j$. Therefore, every component O_i^j contains no odd cycle, and it is thus a bipartite graph. Set $O_i^j=(X_i^j,Y_i^j,E_i^j)$ such that $|X_i^j|\leqslant|Y_i^j|$ (X_i^j might be an empty set). Set

$$X^j=\bigcup_{i=1}^{t_j}X_i^j.$$

It is clear that any edge of G^j intersects $Y^j=X^j\cup S^j$. Note that

$$\begin{aligned}|X^j|&\leqslant\sum_{i=1}^{t_j}\left\lfloor\frac{|V(O_i^j)|}{2}\right\rfloor\\&=\frac{|V(G^j)|-|S^j|}{2}-\frac{o(G^j-S^j)}{2}\\&\leqslant\frac{|V(G^j)|-|S^j|}{2}-\frac{|V(G^j)|+|S^j|-2r+2}{2}\\&=r-|S^j|-1,\end{aligned}$$

which concludes $|Y^j|\leqslant r-1$ for each $j\in\{R,B\}$. In the following, we will discuss the two cases $s=0$ and $s\neq 0$ separately.

Case 1A: $s=0$. Note that in this case G is Eulerian, so $t=2+\sum_{i=1}^{r-1}\deg_G(v_i)$ is even. Thus, $|E(G^R)|=|E(G^B)|=1+t/2$. Clearly, if G^j is rK_2-free for a $j\in\{R,B\}$, then, using the facts that $X^j\cap S^j=\varnothing$, $|X^j|+|S^j|\leqslant r-1$, each edge $e\in E(G^j)$ meets $X^j\cup S^j$, and $\deg_{G^j}(x)\leqslant\deg_G(x)/2$ for each $x\in V(G)$, we have

$$\begin{aligned}1+\sum_{i=1}^{r-1}\frac{\deg_G(v_i)}{2}&=1+\frac{t}{2}\\&=|E(G^j)|\\&\leqslant\sum_{x\in X^j\cup S^j}\deg_{G^j}(x)\end{aligned}$$

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$$\leqslant \sum_{x \in X^j} \frac{\deg_G(x)}{2} + \sum_{x \in S^j} \frac{\deg_G(x)}{2}$$
$$\leqslant \sum_{i=1}^{r-1} \frac{\deg_G(v_i)}{2},$$

which is impossible. This means that

$$\mathrm{ex}_{\mathrm{salt}}\,(G, rK_2, \sigma) \leqslant 1 + \sum_{i=1}^{r-1} \deg_G(v_i).$$

Accordingly, by Lemma 2.2,

$$\chi(\mathrm{KG}(G, rK_2)) = \zeta_s(\mathrm{KG}(G, rK_2)) = |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i).$$

Case 1B: $s \neq 0$. For a contradiction, suppose that neither G^R nor G^B has an r-matching. It is not difficult to verify the formula

$$2|(Y^R \cup Y^B) \cap U| - |(Y^R \Delta Y^B) \cap U| + |Y^R \setminus U| + |Y^B \setminus U| = |Y^R| + |Y^B| \leqslant 2(r-1),$$

which will be used in the proof of the next inequality. Since each edge $e \in E(G^j)$ meets $Y^j = X^j \cup S^j$ and $|Y^R|, |Y^B| \leqslant r - 1$,

$$\begin{aligned}
1 + \sum_{i=1}^{r-1} \deg_G(v_i) =& |E(G^R)| + |E(G^B)| \\
&\leqslant \sum_{j \in \{R,B\}} \sum_{x \in Y^j} \deg_{G^j}(x) \\
&\leqslant \sum_{x \in Y^R \cap Y^B \cap U} \deg_G(x) + \sum_{x \in (Y^R \Delta Y^B) \cap U} \left\lceil \frac{\deg_G(x)}{2} \right\rceil \\
&\quad + (|Y^R \setminus U| + |Y^B \setminus U|) \left\lceil \frac{\deg_G(v_r)}{2} \right\rceil \\
&\leqslant \sum_{i=1}^{|Y^R \cap Y^B \cap U|} \deg_G(v_i) + \sum_{i=|Y^R \cap Y^B \cap U|+1}^{|(Y^R \cup Y^B) \cap U|} \left\lceil \frac{\deg_G(v_i)}{2} \right\rceil \\
&\quad + (|Y^R \setminus U| + |Y^B \setminus U|) \left\lceil \frac{\deg_G(v_r)}{2} \right\rceil \\
&\leqslant \sum_{i=1}^{|(Y^R \cup Y^B) \cap U|} \deg_G(v_i) - \sum_{i=|Y^R \cap Y^B \cap U|+1}^{|(Y^R \cup Y^B) \cap U|} \left\lfloor \frac{\deg_G(v_i)}{2} \right\rfloor \\
&\quad + (|Y^R \setminus U| + |Y^B \setminus U|) \left\lceil \frac{\deg_G(v_r)}{2} \right\rceil \\
&\leqslant \sum_{i=1}^{|(Y^R \cup Y^B) \cap U|} \deg_G(v_i) - |(Y^R \Delta Y^B) \cap U| \left\lfloor \frac{\deg_G(v_{r-1})}{2} \right\rfloor \\
&\quad + (|Y^R \setminus U| + |Y^B \setminus U|) \left\lfloor \frac{\deg_G(v_{r-1})}{2} \right\rfloor
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{i=1}^{r-1} \deg_G(v_i) - 2(r-1-|(Y^R \cup Y^B)\cap U|)\left\lfloor \frac{\deg_G(v_{r-1})}{2} \right\rfloor \\
&\qquad - |(Y^R \Delta Y^B)\cap U|\left\lfloor \frac{\deg_G(v_{r-1})}{2} \right\rfloor + (|Y^R \setminus U| + |Y^B \setminus U|)\left\lfloor \frac{\deg_G(v_{r-1})}{2} \right\rfloor \\
&\leqslant \sum_{i=1}^{r-1} \deg_G(v_i),
\end{aligned}$$

a contradiction. Therefore, G^R or G^B has an r-matching, and consequently

$$\mathrm{ex}_{\mathrm{alt}}\,(G, rK_2, \sigma) \leqslant \sum_{i=1}^{r-1} \deg_G(v_i).$$

Accordingly, by Lemma 2.2,

$$\chi(\mathrm{KG}(G, rK_2)) = \zeta(\mathrm{KG}(G, rK_2)) = |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i),$$

completing the proof in this case.

Case 2: $r \leqslant (\deg_G(v_{r-1})+1)/4$. For $j \in \{R, B\}$, if G^j is rK_2-free, in view of the Tutte–Berge formula (Theorem 3.1), there exists $S^j \subseteq V(G^j)$ such that $|V(G^j)| - o(G^j - S^j) + |S^j| \leqslant 2r-2$. Note that $|S^j| \leqslant |V(G)| - o(G^j - S^j)$ and hence $|S^j| \leqslant r-1$. Let $O_1^j, O_2^j, \ldots, O_{t_j}^j$ be the components of $G^j - S^j$, where $t_j \geqslant o(G^j - S^j)$. We discuss the two different cases $s=0$ and $s>0$ separately.

Case 2A: $s=0$. Note that in this case G is Eulerian, so $t = 2 + \sum_{i=1}^{r-1} \deg_G(v_i)$ is even. Thus, $|E(G^R)| = |E(G^B)| = 1 + t/2$. In what follows, we prove that each of G^R and G^B has a matching of size r. By similarity, we just prove that G^r has an rK_2 subgraph. For a contradiction, suppose that G^R is rK_2-free. It is easy to see that

$$\binom{a}{2} + \binom{b}{2} \leqslant \binom{a+b-1}{2}.$$

We used this inequality in the second succeeding inequality. Since each red edge either intersects S^R or is in $E(O_i^R)$ for some i, we have

$$\begin{aligned}
1 + \frac{t}{2} &= |E(G^R)| \\
&\leqslant \sum_{x \in S^R} \deg_{G^R}(x) + \sum_{i=1}^{t_R} \binom{|V(O_i^R)|}{2} \\
&\leqslant \sum_{x \in S^R} \deg_{G^R}(x) + \binom{\sum_{i=1}^{t_R} |V(O_i^R)| - (t_R - 1)}{2} \\
&\leqslant \sum_{x \in S^R} \deg_{G^R}(x) + \binom{(|V(G)| - |S^R|) - (o(G^R - S^R) - 1)}{2} \\
&\leqslant \sum_{x \in S^R} \deg_{G^R}(x) + \binom{2r - 2|S^R| - 1}{2} \\
&\leqslant \sum_{x \in S^R} \frac{\deg_{G^R}(x)}{2} + (r - |S^R| - 1)\frac{\deg_{G^R}(v_{r-1})}{2}
\end{aligned}$$

$$\leqslant \frac{1}{2}\sum_{i=1}^{r-1}\deg_G(v_i)$$
$$= \frac{t}{2},$$

which is impossible. Thus

$$\mathrm{ex}_{\mathrm{salt}}\,(G, rK_2, \sigma) \leqslant 1 + \sum_{i=1}^{r-1}\deg_G(v_i)$$

and, by Lemma 2.2,

$$\chi(\mathrm{KG}(G, rK_2)) = \zeta_s(\mathrm{KG}(G, rK_2)) = |E(G)| - \sum_{i=1}^{r-1}\deg_G(v_i),$$

completing the proof in this case.

Case 2B: $s \neq 0$. In what follows, we will see that G^R or G^B has a matching of size r. On the contrary, suppose that neither G^R nor G^B has an r-matching. Note that

$$1 + \sum_{i=1}^{r-1}\deg_G(v_i) = t = |E(G^R)| + |E(G^B)|.$$

On the other hand, as in Case 2A, since for each $j \in \{R, B\}$, each edge of colour j either intersects S^j or is in $E(O_i^j)$ for some i, we have

$$\begin{aligned}
t &= |E(G^R)| + |E(G^B)| \\
&\leqslant \sum_{j\in\{R,B\}}\sum_{x\in S^j}\deg_{G^j}(x) + \sum_{j\in\{R,B\}}\sum_{i=1}^{t_j}\binom{|V(O_i^j)|}{2} \\
&\leqslant \sum_{j\in\{R,B\}}\sum_{x\in S^j}\deg_{G^j}(x) + \sum_{j\in\{R,B\}}\binom{\sum_{i=1}^{t_j}|V(O_i^j)| - (t_j-1)}{2} \\
&\leqslant \sum_{j\in\{R,B\}}\sum_{x\in S^j}\deg_{G^j}(x) + \sum_{j\in\{R,B\}}(r-|S^j|-1)(2r-2|S^j|-1) \\
&\leqslant \sum_{j\in\{R,B\}}\sum_{x\in S^j\cap U}\deg_{G^j}(x) + \sum_{j\in\{R,B\}}\sum_{x\in S^j\setminus U}\left\lceil\frac{\deg_G(v_r)}{2}\right\rceil \\
&\quad + \sum_{j\in\{R,B\}}(r-|S^j|-1)\frac{\deg_G(v_{r-1})-1}{2} \\
&\leqslant \sum_{x\in S^R\cap S^B\cap U}\deg_G(x) + \sum_{x\in(S^R\Delta S^B)\cap U}\left\lceil\frac{\deg_G(x)}{2}\right\rceil \\
&\quad + (|S^R\setminus U| + |S^B\setminus U|)\left\lceil\frac{\deg_G(v_r)}{2}\right\rceil + \sum_{j\in\{R,B\}}(r-|S^j|-1)\frac{\deg_G(v_{r-1})-1}{2}
\end{aligned}$$

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$$\begin{aligned}
&\leqslant \sum_{i=1}^{|S^R\cap S^B\cap U|} \deg_G(v_i) + \sum_{i=|S^R\cap S^B\cap U|+1}^{|(S^R\cup S^B)\cap U|} \left\lceil \frac{\deg_G(v_i)}{2} \right\rceil \\
&\quad + (|S^R\setminus U| + |S^B\setminus U|)\left\lceil \frac{\deg_G(v_r)}{2} \right\rceil + \sum_{j\in\{R,B\}} (r-|S^j|-1)\frac{\deg_G(v_{r-1})-1}{2} \\
&\leqslant \sum_{i=1}^{|(S^R\cup S^B)\cap U|} \deg_G(v_i) - \sum_{i=|S^R\cap S^B\cap U|+1}^{|(S^R\cup S^B)\cap U|} \left\lfloor \frac{\deg_G(v_i)}{2} \right\rfloor \\
&\quad + (|S^R\setminus U| + |S^B\setminus U|)\left\lceil \frac{\deg_G(v_r)}{2} \right\rceil + \sum_{j\in\{R,B\}} (r-|S^j|-1)\frac{\deg_G(v_{r-1})-1}{2} \\
&\leqslant \sum_{i=1}^{|(S^R\cup S^B)\cap U|} \deg_G(v_i) - |(S^R\Delta S^B)\cap U|\left\lfloor \frac{\deg_G(v_{r-1})}{2} \right\rfloor \\
&\quad + (|S^R\setminus U| + |S^B\setminus U|)\left\lfloor \frac{\deg_G(v_{r-1})}{2} \right\rfloor + \sum_{j\in\{R,B\}} (r-|S^j|-1)\left\lfloor \frac{\deg_G(v_{r-1})}{2} \right\rfloor \\
&\leqslant \sum_{i=1}^{|(S^R\cup S^B)\cap U|} \deg_G(v_i) + (2(r-1) - 2|(S^R\cup S^B)\cap U|)\left\lfloor \frac{\deg_G(v_{r-1})}{2} \right\rfloor \\
&\leqslant \sum_{i=1}^{r-1} \deg_G(v_i),
\end{aligned}$$

which is impossible. Consequently $\mathrm{ex}_{\mathrm{alt}}(G, rK_2, \sigma) \leqslant \sum_{i=1}^{r-1} \deg_G(v_i)$; accordingly, by Lemma 2.2,

$$\chi(\mathrm{KG}(G, rK_2)) = \zeta(\mathrm{KG}(G, rK_2)) = |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i),$$

which completes the proof. □

Since $\mathrm{KG}(C_n, rK_2) \simeq \mathrm{SG}(n, r)$, the aforementioned theorem can be viewed as a generalization of the Schrijver theorem [25].

Theorem 3.3 (Theorem 3 of [25]). *For any positive integers n and r, where $n \geqslant 2r$, we have* $\chi(\mathrm{SG}(n, r)) = n - 2r + 2$.

Proof of Theorem 1.1(iii) (Regular even graphs). If $G = (X, Y, E)$ is bipartite, then X is an independent set satisfying Theorem 3.2 and hence we have the desired assertion. When G is not bipartite, set C to be a minimal odd cycle in G. Note that C is an induced subgraph of G and $|V(C)| \geqslant g$. Clearly, it contains an independent set of size $\lfloor g/2 \rfloor$ and the proof thus follows from Theorem 3.2. □

3.2 Matching-dense graphs

In this subsection we determine the chromatic number of matching Kneser graphs $\mathrm{KG}(G, rK_2)$ provided that G is a large dense graph. Let H be a graph with $V(H) = \{v_1, \ldots, v_n\}$. The graph H is termed (r, c)-*locally Eulerian* if there are edge-disjoint non-trivial Eulerian connected subgraphs $H_1, \ldots, H_n$ of H, such that $v_i \in V(H_i)$ for each $i \in [n]$ and $\deg_{H_i}(v_i) \geqslant (r-1)\deg_{H_i}(u) + c$ for each

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$u \in V(H_i) \setminus \{v_i\}$. It is worth noting that the union of edge-disjoint (r, c)-locally Eulerian graphs is (r, c)-locally Eulerian. The next lemma plays a key role in the rest of the paper.

Lemma 3.4. *Let c, r and s be non-negative integers where $r \geqslant 2$ and $c \geqslant \binom{r-1}{2} + (s+3)(r-1)$. Also, let G be a graph with n vertices and $\delta(G) > \binom{r+2}{2} + (r-2)s$. If there exists an $(r+s, c)$-locally Eulerian graph H with $n+s$ vertices containing G as an induced subgraph, then*

$$\chi(\mathrm{KG}(G, rK_2)) = \zeta(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rK_2).$$

Proof. In view of Lemma 2.2, it suffices to prove the existence of an ordering σ of $E(G)$ for which $\mathrm{ex}_{\mathrm{alt}}\,(G, rK_2, \sigma) = \mathrm{ex}\,(G, rK_2)$. To this end, in what follows, we will first construct an ordering σ and then prove that this σ satisfies the aforementioned property. Let $V(G) = \{v_1, \ldots, v_n\}$ and $V(H) = \{v_1, \ldots, v_{n+s}\}$. Since H is $(r+s, c)$-locally Eulerian, there are pairwise edge-disjoint non-trivial Eulerian subgraphs $H_1, \ldots, H_{n+s}$ of H such that $v_i \in V(H_i)$ for each $i \in [n+s]$ and

$$\deg_{H_i}(v_i) \geqslant (r+s-1)\deg_{H_i}(u) + \binom{r-1}{2} + (s+3)(r-1)$$

for each $u \in V(H_i) \setminus \{v_i\}$.

Constructing the ordering σ. To define the ordering σ, add a new vertex x and join it to all vertices of H by two parallel edges (an edge with multiplicity two) to obtain the graph H'. Precisely, for any $1 \leqslant i \leqslant n+s$, join x and v_i to two distinct edges f_i and f_i'. Now, if H' has no odd vertices, then set $\bar{H} = H'$; otherwise, add a new vertex z and join it to the odd vertices of H' to obtain the graph $\bar{H}$. The graph $\bar{H}$ is an even connected graph and it is therefore Eulerian. Also, the graph $K = \bar{H} - x$ is an even graph; accordingly, each connected component of $K' = K \setminus \left(\bigcup_{i=1}^{n+s} H_i\right)$ is Eulerian as well. Let $K_1, \ldots, K_l$ be the non-trivial connected components of K'. Construct an Eulerian tour for $\bar{H}$ as follows. At the ith step, where $1 \leqslant i \leqslant n+s$, start from vertex x and traverse the edge f_i to reach v_i. Consider an arbitrary Eulerian tour of H_i starting at v_i and traverse it. Next, if there exists a K_j such that $v_i \in V(K_j)$ and the edges of this K_j is still untraversed, then consider an Eulerian tour of K_j starting at v_i and traverse it. Next, traverse the edge f_i' to reach x. If $i < n+s$, then start the $(i+1)$th step. After terminating this process, we obtain an Eulerian tour of $\bar{H}$. Henceforth, this Eulerian tour of $\bar{H}$ will be fixed and we refer to it as the 'Eulerian tour of $\bar{H}$'. Now, construct an ordering σ of $E(G)$ such that the ordering of edges in $E(G)$ corresponds to their ordering in the Eulerian tour of $\bar{H}$, that is, if we traverse the edge $e_i \in E(G)$ before the edge $e_j \in E(G)$ in the Eulerian tour of $\bar{H}$, then $e_i < e_j$ in σ.

Note that the proof of the lemma would be completed once we have proved the following claim.

Claim. $\mathrm{ex}_{\mathrm{alt}}\,(G, rK_2, \sigma) = \mathrm{ex}\,(G, rK_2)$.

Let $t \leqslant r-1$ be a positive integer. For any t-subset $T \subseteq V(G)$, since the spanning subgraph of G whose edges are the ones incident with an $(r-1)$-subset of $V(G)$ is rK_2-free, we clearly have

$$\mathrm{ex}\,(G, rK_2) \geqslant \sum_{v \in T} \deg_G(v) + (r-1-t)\delta(G) - \binom{r-1}{2}. \tag{3.1}$$

Consider an alternating 2-colouring of $E(G)$ with respect to σ and of length $\mathrm{ex}\,(G, rK_2) + 1$, that is, the colours red and blue are assigned alternately (with respect to σ) to $\mathrm{ex}\,(G, rK_2) + 1$ edges of G. Hereafter, this alternating 2-colouring of $E(G)$ is also extended to an alternating 2-colouring of $E(\bar{H})$ with respect to the ordering imposed by the Eulerian tour of $\bar{H}$ (chosen in the definition of σ), where each edge in $E(\bar{H}) \setminus E(G)$ is considered neutral. So we can speak of red, blue and neutral edges in $\bar{H}$ with no ambiguity.

To fulfil the claim, we must prove that G^R or G^B has an r-matching. For a contradiction, suppose that the red spanning subgraph G^R and the blue spanning subgraph G^B are both rK_2-free. In view of the Tutte–Berge formula (Theorem 3.1), there are two sets $T^R, T^B \subseteq V(G)$ such that

$$|V(G^j)| - o(G^j - T^j) + |T^j| \leqslant 2r - 2 \quad \text{for each } j \in \{R, B\}. \tag{3.2}$$

For each $j \in \{R, B\}$, we have $|T^j| \leqslant |V(G)| - o(G^j - T^j)$, which concludes $|T^R| \leqslant r - 1$ and $|T^B| \leqslant r - 1$. Furthermore, for each $j \in \{R, B\}$, the number of edges of G^j incident with no vertex of T^j (*i.e.* $|E(G^j - T^j)|$) is at most $\binom{2r-2|T^j|-1}{2}$. To see this, let $O_1^j, O_2^j, \ldots, O_{t_j}^j$ be the connected components of $G^j - T^j$ and note that

$$\sum_{i=1}^{t_j} |V(O_i^j)| = |V(G^j)| - |T^j| \quad \text{and} \quad t_j \geqslant o(G^j - T^j) \geqslant |V(G^j)| + |T^j| - 2r + 2,$$

resulting in

$$\begin{aligned} |E(G^j - T^j)| &\leqslant \sum_{i=1}^{t_j} \binom{|V(O_i^j)|}{2} \\ &\leqslant \binom{\sum_{i=1}^{t_j} |V(O_i^j)| - (t_j - 1)}{2} \\ &\leqslant \binom{2r - 2|T^j| - 1}{2}. \end{aligned} \tag{3.3}$$

For each vertex $u \in V(G)$ and each $j \in \{R, B\}$,

$$\deg_{G^j}(u) \leqslant \frac{\deg_{\bar{H}}(u)}{2} \leqslant \frac{1}{2}(\deg_G(u) + s + 3). \tag{3.4}$$

The second inequality is clear due to the definition of $\bar{H}$. To prove the first one, we need to estimate the number of red and blue edges incident with u. Note that the edges incident with u in $\bar{H}$ can be partitioned into $(\deg_{\bar{H}}(u))/2$ pairs of consecutive edges in the Eulerian tour of $\bar{H}$. Clearly, at most one of the two edges in each pair is red (resp. blue) which concludes the desired inequality. We consider three different cases.

Case I: $|T^R| \leqslant r - 2$ or $|T^B| \leqslant r - 2$. By similarity, assume $|T^R| \leqslant r - 2$. Note that each edge of G^R either intersects T^R or is an edge of $G^R - T^R$. This observation, along with inequalities (3.1), (3.3) and (3.4) and the assumption $\delta(G) > \binom{r+2}{2} + (r-2)s$, implies

$$\begin{aligned} |E(G^R)| &\leqslant |E(G^R - T^R)| + \sum_{u \in T^R} \deg_{G^R}(u) \\ &\leqslant \binom{2r - 2|T^R| - 1}{2} + \sum_{u \in T^R} \frac{1}{2}(\deg_G(u) + s + 3) \\ &\leqslant \binom{2r - 2|T^R| - 1}{2} + \frac{\text{ex}\,(G, rK_2)}{2} + \frac{1}{2}\binom{r-1}{2} - \frac{r - 1 - |T^R|}{2}\delta(G) + \frac{(s+3)|T^R|}{2} \\ &< \frac{\text{ex}\,(G, rK_2)}{2}. \end{aligned}$$

For the last inequality, we must prove $f(x) < 0$ for each $x \in \{0, \ldots, r-2\}$, where

$$f(x) = \binom{2r - 2x - 1}{2} + \frac{1}{2}\binom{r-1}{2} - \frac{r - 1 - x}{2}\delta(G) + \frac{(s+3)x}{2}.$$

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Note that $f(x)$ is a quadratic polynomial with a positive coefficient of x^2 and thus its maximum for $x \in [0, r-2]$ occurs in $x = 0$ or $x = r-2$. Now, using $\delta(G) > \binom{r+2}{2} + (r-2)s$, it is simple to check that $f(0) < 0$ and $f(r-2) < 0$.

Combining the inequalities $|E(G^R)| < \mathrm{ex}\,(G, rK_2)/2$ and $|E(G^B)| \leqslant |E(G^R)| + 1$, we obtain

$$\mathrm{ex}\,(G, rK_2) + 1 = |E(G^R)| + |E(G^B)| < \mathrm{ex}\,(G, rK_2) + 1,$$

which is impossible.

Case II: $|T^R| = |T^B| = r-1$ and $T^R \neq T^B$. Using Formula 3.2, it is easy to conclude that each connected component of $G^R - T^R$ and $G^B - T^B$ consists of singletons. This means that T^R and T^B, respectively, are vertex covers of G^R and G^B. Without loss of generality, assume that $|E(G^R)| \geqslant |E(G^B)|$ and choose a vertex $v_i \in T^R \setminus T^B$. Set $L^B = |E(H_i^B)|$, *i.e.* the number of blue edges of H_i. Since T^B is a vertex cover of G^B, each blue edge of H_i is incident with some vertex in T^B. Let x be an arbitrary vertex in T^B. Consider the Eulerian tour of H_i used in the definition of σ and note that it starts and ends with v_i. Since $x \neq v_i$ ($v_i \notin T^B$), the edges of H_i incident with x can be partitioned into pairs of consecutive edges in the Eulerian tour of H_i. For each pair, at most one of its edges is blue. This concludes that for each blue edge of H_i incident with x, there is an edge of H_i incident with x which is not blue (note that this edge might be out of $E(G)$). Therefore, $\deg_{H_i}(x) \geqslant 2\deg_{H_i^B}(x)$ and hence $\sum_{x \in T^B \cap V(H_i)} \deg_{H_i}(x) \geqslant 2L^B$. It implies that there is a vertex $z \in V(H_i) \setminus \{v_i\}$ whose degree in H_i is at least $2L^B/(r-1)$. Using the assumption that for $z \in V(H_i) \setminus \{v_i\}$,

$$\deg_{H_i}(v_i) \geqslant (r+s-1)\deg_{H_i}(z) + \binom{r-1}{2} + (s+3)(r-1),$$

we would have

$$\deg_{H_i}(v_i) \geqslant (r+s-1)\left\lceil \frac{2L^B}{r-1} \right\rceil + \binom{r-1}{2} + (s+3)(r-1).$$

Thus, since $s \geqslant 0$,

$$L^B \leqslant \frac{1}{2}\left(\deg_{H_i}(v_i) - \binom{r-1}{2} - (s+3)(r-1)\right). \tag{3.5}$$

Since the edges in $E(H_i) \cap E(G)$ are consecutive in σ and between any two consecutive blue (resp. red) edges there is exactly one red (resp. blue) edge, we must have $||E(H_i^R)| - |E(H_i^B)|| \leqslant 1$ and thus

$$\deg_{H_i^R}(v_i) \leqslant |E(H_i^R)| \leqslant |E(H_i^B)| + 1 \leqslant L^B + 1.$$

If there exists a $1 \leqslant j \leqslant l$ such that $v_i \in V(K_j)$ and $v_q \notin (K_j)$ for any $q < i$ (the edges of K_j are traversed in no step before the ith step), then set $H_i' = H_i \cup K_j$; otherwise, $H_i' = H_i$. Estimating the number of red edges incident with v_i in the graph H_i', we claim that

$$\deg_{H_i'^R}(v_i) \leqslant L^B + 1 + \frac{1}{2}\deg_{H_i'}(v_i) - \frac{1}{2}\deg_{H_i}(v_i). \tag{3.6}$$

If $H_i' = H_i$, then we have already proved it. So we consider the case $H \neq H'$. As in the discussion after inequality (3.4),

$$\deg_{K_j^R}(v_i) \leqslant \frac{\deg_{K_j}(v_i)}{2} + 1 \leqslant \frac{1}{2}\deg_{H_i'}(v_i) - \frac{1}{2}\deg_{H_i}(v_i) + 1.$$

However, if $\deg_{H_i^R}(v_i) = L^B + 1$, then $|E(H_i^R)| = |E(H_i^B)| + 1$ and hence the first and last coloured edges of H_i are both red. Consequently, since the edges of K_i are consecutive and the first edge of

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K_i is located immediately after the last one of H_i in the Eulerian tour of $\bar{H}$ used for the definition of σ, the first edge of K_i in σ which is indeed adjacent with v_i cannot be blue, and hence

$$\deg_{K_j^R}(v_i) \leqslant \frac{\deg_{K_j}(v_i)}{2} = \frac{1}{2}\deg_{H_i'}(v_i) - \frac{1}{2}\deg_{H_i}(v_i).$$

Thus, in view of the aforementioned discussion, we have

$$\deg_{H_i'^R}(v_i) \leqslant \deg_{H_i^R}(v_i) + \deg_{K_j^R}(v_i) \leqslant L^B + 1 + \frac{1}{2}\deg_{H_i'}(v_i) - \frac{1}{2}\deg_{H_i}(v_i),$$

as claimed. Furthermore, it is easy to see that the edges incident with v in $E(\bar{H}\setminus H_i')\setminus\{f_i, f_i'\}$ can be partitioned into $(\deg_{(\bar{H}\setminus H_i')}(v_i) - 2)/2$ pairs of consecutive edges in the Eulerian tour of $\bar{H}$. Since f_i, f_i' are neutral and at most one of the edges in each pair is red,

$$\deg_{(G\setminus H_i')^R}(v_i) \leqslant \frac{\deg_{(\bar{H}\setminus H_i')}(v_i) - 2}{2} \leqslant \frac{1}{2}(\deg_G(v_i) + s + 1 - \deg_{H_i'}(v_i)).$$

Therefore, by the previous inequality together with inequalities (3.5) and (3.6),

$$\begin{aligned}
\deg_{G^R}(v_i) &= \deg_{H_i'^R}(v_i) + \deg_{(G\setminus H_i')^R}(v_i)\\
&\leqslant L^B + 1 + \frac{1}{2}\deg_{H_i'}(v_i) - \frac{1}{2}\deg_{H_i}(v_i) + \frac{1}{2}(\deg_G(v_i) + s + 1 - \deg_{H_i'}(v_i))\\
&\leqslant \frac{1}{2}\left(\deg_G(v_i) - (s+3)(r-2) - \binom{r-1}{2}\right).
\end{aligned}$$

Finally, using this inequality, inequality (3.4) and the fact that T^R is a vertex cover of G^R, we have

$$\begin{aligned}
|E(G^R)| &\leqslant \sum_{u\in T^R} \deg_{G^R}(u)\\
&\leqslant \frac{1}{2}\left(\deg_G(v_i) - (s+3)(r-2) - \binom{r-1}{2}\right) + \sum_{u\in T^R\setminus\{v_i\}} \frac{1}{2}(\deg_G(u) + s + 3)\\
&\leqslant -\frac{1}{2}\binom{r-1}{2} + \sum_{u\in T^R} \frac{1}{2}\deg_G(u)\\
&\leqslant \#\{e\in E(G)\colon e\cap T^R \neq \varnothing\}\\
&\leqslant \frac{1}{2}\,\mathrm{ex}\,(G, rK_2),
\end{aligned}$$

which is impossible since $|E(G^R)| \geqslant |E(G^B)|$ and $|E(G^R)| + |E(G^B)| = \mathrm{ex}\,(G, rK_2) + 1$.

Case III: $|T^R| = |T^B| = r-1$ and $T = T^R = T^B$. In this case T is a vertex cover for each of G^R and G^B. Hence, the number of blue and red edges of G, *i.e.* $|E(G^R)| + |E(G^B)|$, is at most the number of edges of G incident with T. Note that the set of edges incident with T is rK_2-free, which implies

$$\mathrm{ex}\,(G, rK_2) + 1 = |E(G^R)| + |E(G^B)| \leqslant \mathrm{ex}\,(G, rK_2),$$

a contradiction. □

Let G be a graph. A G-decomposition of a graph H is a set $\{G_1, \ldots, G_t\}$ of pairwise edge-disjoint subgraphs of H such that, for each $1 \leqslant i \leqslant t$, the graph G_i is isomorphic to G; moreover, the edge-sets of the G_i partition the edge-set of H. A G-decomposition of H is called *monogamous* if any distinct pair of vertices of H appear in at most one copy of G in the decomposition. Note that any K_t-decomposition of a graph is clearly monogamous. A necessary and sufficient condition for a complete bipartite graph $K_{m,n}$ to have a monogamous C_4-decomposition was provided in [22].

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Theorem 3.5 (Theorem 2.7 of [22]). *Let m and n be positive even integers. The complete bipartite graph $K_{m,n}$ has a monogamous C_4-decomposition if and only if $(m, n) = (2, 2)$ or $6 \leqslant n \leqslant m \leqslant 2n - 2$.*

Using this theorem, in the following lemma we present a sufficient condition for $K_{t,t'}$ to be locally Eulerian.

Lemma 3.6. *Let r, t and t' be positive integers, where $11 \leqslant t \leqslant t' \leqslant 2t - 2$. If c is a non-negative integer and $t \geqslant 8r + 4c + 2$, then the complete bipartite graph $K_{t,t'}$ is (r, c)-locally Eulerian.*

Proof. Let $t = 2p + q$ and $t' = 2p' + q'$, where $0 \leqslant q \leqslant 1$ and $0 \leqslant q' \leqslant 1$. Extend the complete bipartite graph $G = K_{t,t'}$ to the complete bipartite graph $H = K_{T,T'}$, where $T = t + q$ and $T' = t' + q'$. In view of Theorem 3.5, consider a monogamous C_4-decomposition of H. Call any C_4 of this decomposition a block if it is entirely in G. Construct a bipartite graph with the vertex set (U, V) where U consists of $\lfloor (t - 3)/8 \rfloor$ copies of each vertex of $K_{t,t'}$ and V consists of all blocks. Join a vertex of U to a vertex of V if the corresponding vertex of $K_{t,t'}$ is a vertex of the corresponding block. Since the C_4-decomposition is monogamous, the degree of each vertex in the part U is at least $\lceil (t - 3)/2 \rceil$ and the degree of any vertex in the part V is $4(\lfloor (t - 3)/8 \rfloor)$. Using Hall's theorem, this bipartite graph has a matching saturating all the vertices in U. Consider such a matching. For any vertex $v \in K_{t,t'}$, define H_v to be the subgraph of $K_{t,t'}$ formed by the union of $\lceil (t - 3)/8 \rceil$ blocks assigned to v through this matching. Again, since these blocks came from a monogamous C_4-decomposition, one can see that $\deg_{H_v}(v) = 2(\lfloor (t - 3)/8 \rfloor)$, while the degree of any other vertex of H_v is 2. In view of these H_v, clearly $K_{t,t'}$ is an (r, c)-locally Eulerian graph, as desired. □

For a family of graphs $\mathcal{F}$, we say a graph G has an *$\mathcal{F}$-factor* if there are vertex-disjoint subgraphs $H_1, H_2, \ldots, H_t$ of G such that each H_i is a member of $\mathcal{F}$ and $\bigcup_{i=1}^{t} V(H_i) = V(G)$. Note that if a graph G has an $\mathcal{F}$*-factor*, where each member of $\mathcal{F}$ is an (r, c)-locally Eulerian graph, then G is also (r, c)-locally Eulerian. Hence, by Lemma 3.6, if a graph G has an $\mathcal{F}$-factor, where the family $\mathcal{F}$ consists of all the $K_{t,t'}$ satisfying the condition of Lemma 3.6, then G is (r, c)-locally Eulerian. Now, we review sufficient conditions for a graph to have a $K_{t,t'}$-factor.

Graph expansion has been studied extensively in the literature. Let G be a graph with n vertices and $0 < \nu \leqslant \tau < 1$. For $S \subseteq V(G)$, the *ν-robust neighbourhood of S*, $RN_{\nu,G}(S)$, is the set of vertices $v \in V(G)$ for which $|N_G(v) \cap S| \geqslant \nu n$. A graph G is called a *robust (ν, τ)-expander* if $|RN_{\nu,G}(S)| \geqslant |S| + \nu n$ for any $S \subseteq V(G)$ with $\tau n \leqslant |S| \leqslant (1 - \tau)n$. For more about robust (ν, τ)-expanders see [15]. Throughout this section, we write $0 < a \ll b \ll c$ to mean that we can choose the constants a, b and c from right to left. More precisely, there are two increasing functions f and g such that, given c, we can find some $b \leqslant g(c)$ and $a \leqslant f(b)$. A graph G with n vertices has *bandwidth* at most b if there exists a bijective assignment $l : V(G) \longrightarrow [n]$ such that, for every edge $uv \in E(G)$, we have $|l(u) - l(v)| \leqslant b$.

Theorem 3.7 (Theorem 1.8 of [15]). *Let ν, τ and η be real numbers, where $0 < \nu \leqslant \tau \ll \eta < 1$, and let Δ be a positive integer. There exist constants $\beta > 0$ and n_0 such that the following holds. Let H be a bipartite graph on $n \geqslant n_0$ vertices with $\Delta(H) \leqslant \Delta$ and bandwidth at most βn. If G is a robust (ν, τ)-expander with n vertices and $\delta(G) \geqslant \eta n$, then G contains a copy of H.*

Proof of Theorem 1.1(iv) (Expander graphs). In view of Lemma 3.4, it is sufficient to show that the graph G is an (r, c)-locally Eulerian graph, where $c = \binom{r-1}{2} + 3(r - 1)$. Set $t = 2r^2 + 14r - 6$ and let k and t' be integers such that $n = 2tk + t'$ and $2t \leqslant t' \leqslant 4t - 1$. Now, set H to be a bipartite graph on n vertices with $k + 1 = 1 + (n - t')/2t$ connected components such that one component is isomorphic to $K_{\lceil t'/2 \rceil, \lfloor t'/2 \rfloor}$ and any other component is isomorphic to $K_{t,t}$. Note that, using Theorem 3.7, if n is sufficiently large, then H is a spanning subgraph of G. By Lemma 3.6, one

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can see that $K_{\lceil t'/2\rceil,\lfloor t'/2\rfloor}$ and $K_{t,t}$ are both (r,c)-locally Eulerian graphs; consequently G is an (r,c)-locally Eulerian graph as well, and thus, by Lemma 3.4, the theorem follows. □

For a graph G, if we replace each of its edges with two opposite-direction edges, then we obtain a digraph whose in-degree and out-degree sequences are the same as the degree sequence of G. The following lemma is therefore an immediate consequence of Lemma 13 in [21].

Lemma 3.8 (Lemma 13 of [21]). *For positive constants $\tau \ll \eta < 1$, there exists an integer n_0 such that if G is a graph with $n \geqslant n_0$ vertices and the degree sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ such that for any $i < n/2$, $d_i \geqslant i + \eta n$ or $d_{n-i-\lfloor \eta n\rfloor} \geqslant n - i$, then $\delta(G) \geqslant \eta n$ and G is a robust (τ^2, τ)-expander.*

In view of the previous lemma and Theorem 1.1(iv), we have the next corollary.

Corollary 3.9. *For any positive constant $\gamma < 1$, there is an integer n_0 such that for any $n \geqslant n_0$ we have the following. If G is a connected graph with n vertices and the degree sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ such that for each $i < n/2$ we have $d_i \geqslant i + \gamma n$ or $d_{n-i-\lfloor \gamma n\rfloor} \geqslant n - i$, then $\chi(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rK_2)$.*

For a graph property $\mathcal{P}$, we say $G(n,p)$ possesses $\mathcal{P}$ *asymptotically almost surely*, or a.a.s. for brevity, if the probability that $G \in G(n,p)$ possesses the property $\mathcal{P}$ tends to 1 as n tends to infinity. As noted in [15], one can see that, for constants $0 < \nu \ll \tau \ll p < 1$, a.a.s. any graph G in $G(n,p)$ is a robust (ν,τ)-expander graph with minimum degree at least $pn/2$ and maximum degree at most $2np$. This observation and Theorem 1.1(iv) imply that a.a.s. for any graph G in $G(n,p)$ we have $\chi(\mathrm{KG}(G, rK_2)) = |E(G)| - \mathrm{ex}\,(G, rK_2)$. Moreover, Huang, Lee and Sudakov [13] have proved a more general theorem. Here, we state it in a special case.

Theorem 3.10 (Theorem 1.1 of [13]). *For positive integers r, Δ and reals $0 < p \leqslant 1$ and $\gamma > 0$, there exists a constant $\beta > 0$ such that a.a.s. any spanning subgraph G' of any $G \in G(n,p)$ with minimum degree $\delta(G') \geqslant p(1/2+\gamma)n$ contains every n-vertex bipartite graph H that has maximum degree at most Δ and bandwidth at most βn.*

Proof of Theorem 1.1(v) (Random graphs). Using the previous theorem, the proof follows similarly to the proof of Theorem 1.1(iv). □

Let H be a graph with h vertices and $\chi(H) = l$. Set $\mathrm{cr}(H)$ to be the size of the smallest colour class over all proper l-colourings of H. In [16], the *critical chromatic number* $\chi_{\mathrm{cr}}(H)$ is defined as $(l-1)h/(h - \mathrm{cr}(H))$. One can check that $\chi(H) - 1 < \chi_{\mathrm{cr}}(H) \leqslant \chi(H)$, and equality holds in the upper bound if and only if, in any l-colouring of H, all colour classes have the same size. Suppose that H has k connected components $C_1, C_2, \ldots, C_k$. Define $\mathrm{hcf}_c(H)$ to be the highest common factor of integers $|C_1|, |C_2|, \ldots, |C_k|$. Let f be an l-colouring of H such that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_l$ are the sizes of colouring classes in f. Set $D(f) = \{x_{i+1} - x_i \mid 1 \leqslant i \leqslant l-1\}$ and

$$D(H) = \bigcup D(f),$$

where the union ranges over all l-colourings f of H. Now, define $\mathrm{hcf}_\chi(H)$ to be the highest common factor of the members of $D(H)$. If $D(H) = \{0\}$, then we define $\mathrm{hcf}_\chi(H) = \infty$. We say that

$$H \text{ is in class 1 if } \begin{cases} \mathrm{hcf}_\chi(H) = 1 & \text{when } \chi(H) \neq 2, \\ \mathrm{hcf}_\chi(H) \leqslant 2 \text{ and } \mathrm{hcf}_c(H) = 1 & \text{when } \chi(H) = 2, \end{cases}$$

otherwise, H is in class 2; for more details, see [20]. Generalizing a result by Komlós, Sárközy and Szemerédi [17], Kühn and Osthus [20] proved the next theorem.

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Theorem 3.11 (Theorem 4 of [20] and Theorem 1 of [17]). *For every graph H on h vertices, there are integers c and m_0 such that, for all integers $m \geqslant m_0$, if G is a graph on $n = mh$ vertices, then the following holds. If*

$$\delta(G) \geqslant \begin{cases} \left(1 - \dfrac{1}{\chi_{\text{cr}}(H)}\right)n + c & H \text{ is in class } 1, \\ \left(1 - \dfrac{1}{\chi(H)}\right)n + c & H \text{ is in class } 2, \end{cases}$$

then G has an H-factor.

Proof of Theorem 1.1(i) (Dense graphs). Define $t = 2r^2 + 14r - 6$. Set H to be a bipartite graph with two connected components C_1 and C_2 isomorphic to $K_{t,t}$ and $K_{t+1,t}$, respectively. One can check that H is in class 1. Hence, by Theorem 3.11, there are integers c_1 and m_1 such that, if $|V(G')| \geqslant m_1$, $|V(H)|$ divides $|V(G')|$ and

$$\delta(G') \geqslant \left(1 - \frac{1}{\chi_{\text{cr}}(H)}\right)|V(G')| + c_1,$$

then the graph G' has an H-factor. Let T be an integer such that $4t + 1 \leqslant T < 8t + 2$ and $4t + 1 | n - T$. It is known that if n is sufficiently large and $\delta(G) \geqslant \eta n$, where $0 < \eta < 1$, then G contains a copy of the complete bipartite graph $K_{\lceil T/2\rceil,\lfloor T/2\rfloor}$. Note that

$$\frac{1}{\chi_{\text{cr}}(H)} = \frac{1}{2} + \frac{1}{8t+2}.$$

Set $\alpha = 1/(8t + 2)$ and $\beta = c_1 + 8t - 1$. If $\delta(G) \geqslant (1/2 - \alpha)n + \beta$ and n is sufficiently large, then G contains $K_{\lceil T/2\rceil,\lfloor T/2\rfloor}$ and also the graph $G \setminus K_{\lceil T/2\rceil,\lfloor T/2\rfloor}$ has an H-factor. Hence, G can be decomposed into the complete bipartite graphs $K_{t,t}$, $K_{t+1,t}$ and $K_{\lceil T/2\rceil,\lfloor T/2\rfloor}$. In view of Lemma 3.6, these graphs are (r, c)-locally Eulerian graphs with $c = \binom{r-1}{2} + 3(r - 1)$. Therefore, G is an (r, c)-locally Eulerian graph, and consequently, by Lemma 3.4, the assertion holds. □

3.3 Permutation graphs

Let m, n, r be positive integers, where $r \leqslant m, n$. For an r-subset $A \subseteq [m]$ and an injective map $f : A \longrightarrow [n]$, the ordered pair (A, f) is said to be an r-partial permutation [11]. Let $S_r(m, n)$ denote the set of all r-partial permutations. Two partial permutations (A, f) and (B, g) are said to be intersecting if there exists an $x \in A \cap B$ such that $f(x) = g(x)$. Note that $S_n(n, n)$ is the set of n-permutations. The permutation graph $S_r(m, n)$ has all r-partial permutations (A, σ) as its vertex set and two r-partial permutations are adjacent if and only if they are not intersecting. Note that $S_r(m, n) \simeq S_r(n, m)$, and therefore, for simplicity, we assume that $m \geqslant n$ for all permutation graphs. One can see that the permutation graph $S_r(m, n)$ is isomorphic to $\text{KG}(K_{m,n}, rK_2)$.

The next theorem gives a sufficient condition for a balanced bipartite graph to be decomposable into complete bipartite subgraphs.

Theorem 3.12 (Theorem 1.2 of [29]). *For any integer $q \geqslant 2$, there exists a positive integer m_0 such that, for all $m \geqslant m_0$, the following holds. If $G = (X, Y, E)$ is a balanced bipartite graph on $2n = 2mq$ vertices, i.e. $|X| = |Y| = mq$, with*

$$\delta(G) \geqslant \begin{cases} \frac{n}{2} + q - 1 & \text{if } m \text{ is even}, \\ \frac{n+3q}{2} - 2 & \text{if } m \text{ is odd}, \end{cases}$$

then G has a $K_{q,q}$-factor.

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Proof of Theorem 1.1(ii) (Spanning bipartite dense subgraphs). In view of Lemma 3.4, it is sufficient to show that the graph H (and thus G) is an (r, c)-locally Eulerian graph, where $c = \binom{r-1}{2} + 3(r-1)$. Set $t = 2r^2 + 14r - 6$. By Theorem 3.12, there are integers q_1 and m_1 such that if $n \geqslant m_1$ and $t|n$, then any balanced bipartite graph H' with $2n$ vertices and $\delta(H') \geqslant n/2 + q_1$ has a $K_{t,t}$-factor.

Let t' be an integer, where $t \leqslant t' < 2t$ and $t|n - t'$. It is known that if n is sufficiently large and $\delta(H) \geqslant \eta n$, where $0 < \eta < 1$, then H contains a copy of the complete bipartite graph $K_{t',t'}$. Define $q = q_1 + 2t - 1$. Note that if n is sufficiently large, then H contains a copy of $K_{t',t}$ and also, in view of Theorem 3.12, $H \setminus K_{t',t'}$ has a $K_{t,t}$-factor. This implies that H can be decomposed into complete bipartite graphs $K_{t',t'}$ and $K_{t,t}$. In view of Lemma 3.6, these graphs are (r, c)-locally Eulerian graphs, where $c = \binom{r-1}{2} + 3(r-1)$. Therefore, H and consequently G are (r, c)-locally Eulerian graphs. Therefore, by Lemma 3.4, the assertion holds. □

In the rest of paper, we focus on the chromatic number of general Kneser graph $\mathrm{KG}(K_{m,n}, rK_2)$. In particular, we determine the chromatic number of any permutation graph $S_r(m, n)$ provided that m is either even or large enough. For more about permutation graphs, see [6, 12, 19].

Corollary 3.13. *Let m, n, r be positive integers, where $m \geqslant n \geqslant r$. If m is large enough, then*

$$\chi(\mathrm{KG}(K_{m,n}, rK_2)) = \zeta(\mathrm{KG}(K_{m,n}, rK_2)) = m(n - r + 1).$$

Proof. Using Hall's theorem, any maximal rK_2-free subgraph of $K_{m,n}$ has $(r-1)m$ edges. Hence, in view of Lemma 2.2, we have $\chi(\mathrm{KG}(K_{m,n}, rK_2)) \leqslant m(n - r + 1)$. In view of Theorem 1.1(ii), if m is sufficiently large, then $\chi(\mathrm{KG}(K_{m,m}, rK_2)) = m(m - r + 1)$. Now, we show that for any positive integer $n < m$, if m is sufficiently large, then $\chi(\mathrm{KG}(K_{m,n}, rK_2)) = m(n - r + 1)$. To see this, on the contrary, suppose that

$$f\colon V(\mathrm{KG}(K_{m,n}, rK_2)) \longrightarrow \{1, 2, \ldots, \chi(\mathrm{KG}(K_{m,n}, rK_2))\}$$

is a proper colouring of $\mathrm{KG}(K_{m,n}, rK_2)$, where $\chi(\mathrm{KG}(K_{m,n}, rK_2)) < m(n - r + 1)$. Add $m - n$ new vertices to the small part of $K_{m,n}$ and join them to all vertices in the other part to construct $K_{m,m}$, and call the new edges $e_1, \ldots, e_{(m-n)m}$. Extend the colouring f to a proper colouring g for $\mathrm{KG}(K_{m,m}, rK_2)$ as follows. If a matching M is a subset of $K_{m,n}$, then set $g(M) = f(M)$; otherwise, suppose that i is the smallest positive integer such that $e_i \in M$, and in this case set $g(M) = i + \chi(\mathrm{KG}(K_{m,n}, rK_2))$. This provides a proper colouring for $\mathrm{KG}(K_{m,m}, rK_2)$ with less than $m(m - r + 1)$ colours, which is a contradiction. □

Let $s \geqslant t$ be positive integers and let $G = G(X, Y)$ be a connected (s, t)-regular connected bipartite graph. Theorem 3.2 implies that if s is an even integer, then for any $r \leqslant |X|$ we have $\chi(\mathrm{KG}(G, rK_2)) = s(|X| - r + 1)$. Setting $G = K_{m,n}$, this result indicates $\chi(\mathrm{KG}(K_{m,n}, rK_2)) = \chi(S_r(m, n)) = m(n - r + 1)$ provided that m is an even integer and $m \geqslant n \geqslant r$. However, when m is a small odd value the chromatic number of the permutation graph $S_r(m, n)$ is unknown, we conjecture that its chromatic number is $m(n - r + 1)$.

Corollary 3.14. *Let m, n, r be positive integers, where $m \geqslant n \geqslant r$. If m is even, then*

$$\chi(\mathrm{KG}(K_{m,n}, rK_2)) = \zeta(\mathrm{KG}(K_{m,n}, rK_2)) = m(n - r + 1).$$

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