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FROM TOPOLOGIES OF A SET TO SUBRINGS OF ITS POWER SET

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Abstract

Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. The aim of this paper is to identify the unital subrings of $\mathcal{P}(X)$ and to compute its cardinality when it is finite. It is proved that any topology τ on X such that $\tau = \tau^c$, where $\tau^c = \{U^c \mid U \in \tau\}$, is a unital subring of $\mathcal{P}(X)$. It is also shown that X is finite if and only if any unital subring of $\mathcal{P}(X)$ is a topology τ on X such that $\tau = \tau^c$ if and only if the set of unital subrings of $\mathcal{P}(X)$ is finite. As a consequence, if X is finite with cardinality $n \geq 2$, then the number of unital subrings of $\mathcal{P}(X)$ is equal to the nth Bell number and the supremum of the lengths of chains of unital subalgebras of $\mathcal{P}(X)$ is equal to $n - 1$.

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1. Introduction

The aim of this paper is to build links between two apparently different unsolved problems. The first is the determination of certain constants related to the set of intermediate rings in ring extensions with finiteness conditions. The second is the computation of the number of topologies on a finite set X of cardinality n.

Let $A \subseteq B$ be a ring extension. The set of all A-subalgebras of B (that is, rings C such that $A \subseteq C \subseteq B$) is denoted by $[A, B]$. As defined in [7], the ring extension $A \subseteq B$ is said to have FIP (the 'finitely many intermediate algebra property') if $[A, B]$ is finite. A chain of A-subalgebras of B is a set of elements of $[A, B]$ that are pairwise comparable with respect to inclusion. The ring extension $A \subseteq B$ has FCP (the 'finite chain property') if each chain of A-subalgebras of B is finite. It is clear that each extension that satisfies FIP must also satisfy FCP, but the converse is false, as can be seen most easily via field-theoretic examples, such as $\mathbb{F}_2(x^2, y^2) \subset \mathbb{F}_2(x, y)$.

For any ring extension $A \subseteq B$, the length of $[A, B]$, denoted by $\ell[A, B]$, is the supremum of the lengths of chains of A-subalgebras of B. The pair (A, B) of integral

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domains is called *a normal pair*, as defined in [6], if each intermediate ring is integrally closed in B. If $A \subseteq B$ satisfies FIP and A is integrally closed in B, then (A, B) is a normal pair (see [13]).

There are several characterisations of FIP and FCP extensions (see [7, 14]). Approximations for the number of intermediate rings in FIP extensions were obtained in [11]. The exact number was computed for the first time for principal ideal domains in [10]. An algorithm for counting the number of intermediate rings for a ring extension $A \subseteq B$ was established in [12] for the case where A is an integrally closed domain and B is the quotient field of A. This was extended to the case where (A, B) is a normal pair and B is not necessarily the quotient field of A (see [1, 2]) and for not necessarily integrally closed domains A (see [3]). But the question remains open in the general context, especially for integral extensions and for commutative rings with zero divisors.

The topological computing problem asks for the total number $T(n)$ of all labelled topologies that can be defined on an n-element set X. Until now, there is no reasonable explicit or recursive counting formula for $T(n)$. For the origins of the problem see [16] and for some related results see [4, 5].

Let X be a nonempty set and let $S := \mathcal{P}(X)$ denote the power set of X and $\tau(X)$ the set of all topologies on X. It is well known that $(S, \Delta, \cap)$ is a Boolean ring with identity X usually denoted by 1 and zero element $\emptyset$ usually denoted by 0. Thus, S is a commutative ring with characteristic two. It is obvious that $R := \{\emptyset, X\}$ is the smallest subring of S. Thus, the unital subrings of S are exactly the intermediate rings between R and S. It is not difficult to check that $R \subseteq S$ is an *integral ring extension*, since $a^2 = a \cap a = a$ for any element $a \in S$, so a is a root of the monic polynomial $Y^2 - Y \in R[Y]$. Thus, (R, S) is not a normal pair. Our main purpose here is to determine conditions under which the ring extension $R \subseteq S$ has FIP or FCP and to calculate the cardinality of $[R, S]$. In Lemma 2.1, we show that any topology τ on X such that $\tau = \tau^c$, where $\tau^c := \{U^c \mid U \in \tau\}$ and $U^c = X \setminus U$ is the complement of U in X, is an intermediate ring between R and S. In Theorem 2.3, we prove that $R \subseteq S$ satisfies FIP if and only if it satisfies FCP if and only if X is finite if and only if $[R, S]$ consists of all topologies τ on X such that $\tau = \tau^c$. That is, the unital subrings of $\mathcal{P}(X)$ coincide with the topologies on X for which complements of open sets are open, that is, the open and closed sets are the same. In fact, these topologies correspond to equivalence relations, where the open sets are the unions of equivalence classes. As a consequence, we demonstrate that if $|X| = n \geq 2$, then $|[R, S]|$ is equal to the nth Bell number B_n. In Theorem 2.5, we establish that if X is finite with cardinality $n \geq 2$, then $\ell[R, S] = n - 1$.

All rings considered in this paper are assumed to be commutative and unital and all ring inclusions are unital. We will as usual sometimes denote the first operation by addition and the second operation by multiplication. Any unexplained terminology is standard as in [9, 15].

2. Main results

To avoid unnecessary repetition, we fix the following notation for the remainder of the paper. Let X be a nonempty set X and let $R \subseteq S$ be a ring extension, where $R := \{\emptyset, X\}$ and $S := \mathcal{P}(X)$. Note that $R = S$ if and only if $|X| = 1$. Thus, from now on, we assume that $|X| \geq 2$.

A topology τ on the set X is a subset of $\mathcal{P}(X)$ that contains $\emptyset$ and X and is closed under union and finite intersection. We start with the following lemma.

Lemma 2.1. *With the above notation,* $\{\tau \in \tau(X) \mid \tau = \tau^c\} \subseteq [R, S]$.

Proof. Suppose that $\tau \in \tau(X)$ and $\tau = \tau^c$. We need to show that τ is stable under intersection and symmetric difference. The fact that τ is stable under intersection is clear since τ is a topology. Let $U, V \in \tau$. As $\tau = \tau^c$, we have $U^c, V^c \in \tau$ and $U \cap V^c, V \cap U^c \in \tau$. As τ is stable under union, $U \Delta V = (U \cap V^c) \cup (V \cap U^c) \in \tau$. □

Now, we establish our main theorem that relates several finiteness conditions on X and the set of intermediate rings $[R, S]$. In what follows, we focus on the computation of the number of intermediate rings. We are going to show that for any finite set X, the number of unital subrings of S is exactly the number of partitions of X. But, first, let us recall the definition of Stirling numbers of the second kind and the nth Bell number.

Definition 2.2. Let X be a finite set of cardinality n.

(1) The number of partitions of X into k blocks is the Stirling number of the second kind:
$$S(n, k) = \frac{1}{k!} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j} j^k.$$

(2) The number of partitions of X is the nth Bell number:
$$B_n = \sum_{1 \leq k \leq n} S(n, k).$$

Theorem 2.3. *The following statements are equivalent:*

(1) *X is finite;*
(2) *$R \subseteq S$ satisfies FIP;*
(3) *$R \subseteq S$ satisfies FCP;*
(4) $[R, S] = \{\tau \in \tau(X) \mid \tau = \tau^c\}$.

If, moreover, the set X has cardinality $n \geq 2$, then $|[R, S]| = B_n$.

Proof. (1)⇒(2) If X is a finite set, then clearly $|[R, S]| \leq 2^{|S|} = 2^{2^{|X|}} < \infty$.

(2)⇒(3) This is trivial from the definitions.

(3)⇒(1) Suppose that X is infinite and let
$$Y_1 = \{x_1\} \subsetneq Y_2 = \{x_1, x_2\} \subsetneq \cdots \subsetneq Y_n = \{x_1, x_2, \ldots, x_n\} \subsetneq \cdots$$

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be an infinite sequence of subsets of X. Then

$$R + \mathcal{P}(Y_1) \subsetneq R + \mathcal{P}(Y_2) \subsetneq \cdots \subsetneq R + \mathcal{P}(Y_n) \subsetneq \cdots$$

is clearly an infinite sequence of $[R, S]$.

(1)⇒(4) The fact that $\{\tau \in \tau(X) \mid \tau = \tau^c\} \subseteq [R, S]$ is guaranteed by Lemma 2.1. For the reverse inclusion, let T be an intermediate ring between R and S. We need to show that T is a topology on X and that $T = T^c$. As X is finite, it is enough to prove that for any $A, B \in T$, one has $A \cup B, A \cap B$ and $A^c \in T$. The fact that $A \cap B \in T$ is obvious since T is a ring, so it is closed under intersection. Now, $A^c = X \Delta A \in T$, because A and X are in T and T is stable under the symmetric difference. It remains to show that $A \cup B \in T$. Since T is stable under complement, it suffices to show that $(A \cup B)^c \in T$. But $(A \cup B)^c = A^c \cap B^c \in T$ since T is stable under intersection and complement.

(4)⇒(1) If X is infinite, we can find an intermediate ring between R and S which is not even stable under union. To this end, let

$$I = \mathrm{Fin}(X) = \{A \subseteq X : |A| < \infty\}.$$

One can easily check that I is a proper ideal of S since X is infinite. Now, let

$$T = R + I = \{A + B : A \in R, B \in I\} = \{B, B^c : B \in I\}.$$

As X is infinite, there exists $Y \subsetneq X$, where both Y and Y^c are infinite. Now, for every $y \in Y$, we have $\{y\} \in T$, but $Y = \bigcup_{y \in Y} \{y\} \notin T$.

Now, we prove the final statement of the theorem. It follows from (4) that

$$|[R, S]| = |\{\tau \in \tau(X) \mid \tau = \tau^c\}|.$$

Thus, it suffices to calculate $|\{\tau \in \tau(X) \mid \tau = \tau^c\}|$ and we show that this quantity is equal to the number of partitions of the set X. To this end, let $\mathbf{P} = \{A_1, A_2, \ldots, A_k\}$ be a partition of X into k blocks and let $\tau_{\mathbf{P}}$ be the topology on X generated by $\mathbf{P}$. The topology $\tau_{\mathbf{P}}$ consists of $\emptyset$ and the open sets of the form $A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}$, where $A_{i_1}, A_{i_2}, \ldots, A_{i_k} \in \mathbf{P}$. It is easy to see that $\tau_{\mathbf{P}}^c = \tau_{\mathbf{P}}$. Now, let τ be a topology on $X = \{x_1, x_2, \ldots, x_n\}$ such that $\tau = \tau^c$. For each $x_i \in X$, let O_{x_i} be the smallest open set of τ containing x_i. We can assume, after a suitable renumbering of the x_i and after eliminating repetition in the O_{x_i}, that $\{O_{x_1}, O_{x_2}, \ldots, O_{x_q}\}$, where $q \leq n$ is the collection of the smallest open sets containing all the x_i. It is obvious that $\bigcup_{1 \leq i \leq q} O_{x_i} = X$. We claim that $O_{x_i} \cap O_{x_j} = \emptyset$ for $i \neq j$ with $1 \leq i, j \leq q$.

First note that we cannot have both x_i and x_j in the intersection $O_{x_i} \cap O_{x_j}$. Indeed, if $x_i \in O_{x_i} \cap O_{x_j}$, then $O_{x_i} \subseteq O_{x_i} \cap O_{x_j}$ (as O_{x_i} is included in every open set containing x_i). This implies that $O_{x_i} \subseteq O_{x_j}$. In the same way, $x_j \in O_{x_i} \cap O_{x_j}$ implies that $O_{x_j} \subseteq O_{x_i}$. We would have then $O_{x_i} = O_{x_j}$, the desired contradiction. Thus, we necessarily have $x_i \notin O_{x_i} \cap O_{x_j}$ or $x_j \notin O_{x_i} \cap O_{x_j}$. If $x_i \notin O_{x_i} \cap O_{x_j}$, then $x_i \in O_{x_i} \cap O_{x_j}^c$. This means that $O_{x_i} \subseteq O_{x_i} \cap O_{x_j}^c \subseteq O_{x_j}^c$ and so $O_{x_i} \cap O_{x_j} = \emptyset$. The second possible hypothesis $x_j \notin O_{x_i} \cap O_{x_j}$ leads to the same conclusion. This ends the proof of our claim.

Therefore, $\mathbf{P} = \{O_{x_1}, \ldots, O_{x_q}\}$ is a partition of X. Clearly $\tau = \tau_{\mathbf{P}}$ is the topology generated by $\mathbf{P}$. Therefore, the number of topologies τ is equal to the number of

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partitions of X. According to Definition 2.2, the number of partitions of X into k nonempty unlabelled subsets is $S(n,k)$, the Stirling number of the second kind. Thus, the number of all partitions is $\sum_{1\leq k\leq n} S(n,k) = B_n$. □

REMARK 2.4. Here, we present an alternative proof of the final statement in the previous theorem. Finite topological spaces are exactly finite preordered sets, that is, the two categories are isomorphic. (More generally, denote by Top the category of topological spaces. The category of all preordered sets is always isomorphic to the full subcategory of Top consisting of all Alexandrov-discrete topological spaces.) In this categorical isomorphism, a finite preordered set (X,ρ) corresponds to a topological space (X,τ) such that $\tau^c = \tau$ if and only if $\rho^{op} = \rho$, that is, if and only if the preorder ρ is symmetric, that is, if and only if ρ is an equivalence relation. Now, equivalence relations and partitions are the same thing. This completes the proof.

Now, we establish the following result for the length $\ell[R,S]$.

THEOREM 2.5. *If X is finite of cardinality $n \geq 2$, then $\ell[R,S] = n-1$.*

PROOF. As X is finite, Theorem 2.3 guarantees that $R \subset S$ satisfies FIP and hence FCP. Let $R_0 = R \subset R_1 \subset \cdots \subset R_{k-1} \subset R_k = S$ be a chain of rings from R to S with length k. As S is a Boolean ring with cardinality 2^n, each ring R_i in the above chain is also Boolean with cardinality 2^{n_i} for some $n_i \leq n$. It follows that $k \leq n-1$. Therefore, $\ell[R,S] \leq n-1$.

Now, let $X = \{x_1, x_2, \ldots, x_n\}$ and consider the chain

$$A_0 = R \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} = S,$$

where $A_k = R + \mathcal{P}(\{x_1,\ldots,x_k\})$ for $1 \leq k \leq n-1$. It is clear that this chain has length $n-1$. Thus, $\ell[R,S] = n-1$. □

Recall from [8] that a ring extension $A \subset B$ is said to be *minimal* if $[A,B] = \{A,B\}$. We close the paper with the following corollary, which characterises when $R \subset S$ is a minimal ring extension.

COROLLARY 2.6. *The ring extension $R \subset S$ is minimal if and only if X consists of exactly two elements.*

PROOF. The 'only if' part follows readily from Theorem 2.5. For the 'if' part, write $X = \{a,b\}$. It is clear that the intermediate sets between R and S are R, $T_1 = \{\emptyset, X, \{a\}\}$, $T_2 = \{\emptyset, X, \{b\}\}$ and S. As T_1 and T_2 are not rings, it follows that $R \subset S$ is a minimal ring extension. □

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