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A bifurcation diagram of solutions to an elliptic equation with exponential nonlinearity in higher dimensions

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We consider the following semilinear elliptic equation:

$$\left.\begin{aligned} -\Delta u &= \lambda \exp(u^p) && \text{in } B_1, \\ u &= 0 && \text{on } \partial B_1, \end{aligned}\right\} \qquad (*)$$

where B_1 is the unit ball in $\mathbb{R}^d$, $d \geqslant 3, \lambda > 0$ and $p > 0$. Firstly, following Merle and Peletier, we show that there exists an eigenvalue $\lambda_{p,\infty}$ such that $(*)$ has a solution $(\lambda_{p,\infty}, W_p)$ satisfying $\lim_{|x|\to 0} W_p(x) = \infty$. Secondly, we study a bifurcation diagram of regular solutions to $(*)$. It follows from the result of Dancer that $(*)$ has an unbounded bifurcation branch of regular solutions that emanates from $(\lambda, u) = (0, 0)$. Here, using the singular solution, we show that the bifurcation branch has infinitely many turning points around $\lambda_{p,\infty}$ when $3 \leqslant d \leqslant 9$. We also investigate the Morse index of the singular solution in the $d \geqslant 11$ case.

Keywords: exponential nonlinearity; singular solution; bifurcation diagram; Morse index

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1. Introduction

In this paper, we study the following semilinear elliptic equation:

$$\left.\begin{aligned} -\Delta u &= \lambda \exp(u^p) && \text{in } B_1, \\ u &= 0 && \text{on } \partial B_1, \end{aligned}\right\} \qquad (1.1)$$

where B_1 is the unit ball in $\mathbb{R}^d$, $d \geqslant 3$, $\lambda > 0$ and $p > 0$.

The aim of this paper is to study the existence of a singular solution and a bifurcation diagram of regular solutions to (1.1) for a general power $p > 0$. By a singular solution, we mean a positive regular solution to (1.1) in $B_1 \setminus \{0\}$ that tends to ∞ at the origin ($x = 0$). For example, setting $\lambda_{1,\infty} = 2(d-2)$ and

$W_1(x) = -2\log|x|$, we see that $(\lambda_{1,\infty}, W_1)$ is a singular solution to (1.1) in the $p = 1$ case.

Several studies have been made on (1.1) in the $p = 1$ case (see [1, 3, 5, 6, 9, 10, 15–17] and the references therein). We recall some of them. Gel′fand [6] showed that when $d = 3$ (1.1) has infinitely many solutions at $\lambda = \lambda_{1,\infty}$. Then, Joseph and Lundgren [10] gave a complete classification of solutions to (1.1). More precisely, they showed that (1.1) has infinitely many solutions at $\lambda = \lambda_{1,\infty}$ when $3 \leqslant d \leqslant 9$ and has a unique solution for $0 < \lambda < \lambda_{1,\infty}$ and no solution for $\lambda > \lambda_{1,\infty}$ when $d \geqslant 10$. See [9] for a survey of this problem.

In this paper, we will treat a general power $p > 0$ and show that (1.1) has a singular solution when $p > 0$ and $d \geqslant 3$. In addition, we shall show that (1.1) has infinitely many regular solutions when $p > 0$ and $3 \leqslant d \leqslant 9$.

First, we focus our attention on the existence of a singular solution. As we mentioned above, in the $p = 1$ case, (1.1) has the explicit singular solution $(\lambda_{1,\infty}, W_1)$. The singular solution plays an important role in the bifurcation analysis of regular solutions to (1.1). However, we encounter difficulties when we seek a singular solution if the power $p \neq 1$. Therefore, it is worthwhile investigating the existence of a singular solution for general power $p > 0$. Concerning this, we obtain the following.

Theorem 1.1. *Assume that $d \geqslant 3$ and $p > 0$. Then, there exists a unique eigenvalue $\lambda_{p,\infty} > 0$ such that the equation (1.1) has a singular solution $(\lambda_{p,\infty}, W_p)$ satisfying*

$$W_p(x) = \left[-2\log|x| - \left(1 - \frac{1}{p}\right)\log(-\log|x|)\right]^{1/p} + o((\log|x|)^{-1+1/p}) \tag{1.2}$$

as $|x| \to 0$.

Once we have obtained the singular solution, we investigate its relationship to regular solutions. Dancer [4] showed that for any $p > 0$ there exists an unbounded bifurcation branch $\mathcal{C} \subset \mathbb{R} \times L^\infty(B_1)$, which emanates from $(\lambda, u) = (0, 0)$. Let λ_1 be the first eigenvalue of the operator $-\Delta$ in B_1 with the Dirichlet boundary condition, and let ϕ_1 be the corresponding eigenfunction. By multiplying the equation in (1.1) by ϕ_1 and integrating the resulting equation, we see that if $(\lambda, u) \in \mathcal{C}$, we have $0 < \lambda < \lambda_1$. This yields that $\sup\{\|u\|_\infty | (\lambda, u) \in \mathcal{C}\} = \infty$. Moreover, from [12, theorem 2.1] (see also [15, proposition 6]), we see that the branch $\mathcal{C}$ can be parametrized by $\|u\|_\infty$. Namely, the branch $\mathcal{C}$ can be expressed as

$$\mathcal{C} = \{(\lambda(\gamma), u(x, \gamma)) \mid \gamma = \|u\|_{L^\infty},\ 0 < \gamma < \infty\}. \tag{1.3}$$

Then, we obtain the following.

Theorem 1.2. *Assume that $d \geqslant 3$ and $p > 0$. Let $(\lambda_{p,\infty}, W_p)$ be the singular solution to the equation (1.1) given by theorem 1.1 and $(\lambda(\gamma), u(x, \gamma)) \in \mathcal{C}$. Then, we have $\lambda(\gamma) \to \lambda_{p,\infty}$ and*

$$u(x, \gamma) \to W_p(x) \text{ in } C^1_{\text{loc}}(B_1 \setminus \{0\}) \quad \text{as } \gamma \to \infty.$$

From theorem 1.2, we can obtain the following result.

THEOREM 1.3. *Assume that $3 \leqslant d \leqslant 9$ and $p > 0$. Let $\lambda_{p,\infty} > 0$ be the eigenvalue given by theorem 1.1. Then, for any integer k, there exist at least k regular positive solutions to (1.1) if λ is sufficiently close to $\lambda_{p,\infty}$. In particular, there exist infinitely many regular solutions to (1.1) at $\lambda = \lambda_{p,\infty}$.*

Finally, we estimate the Morse index of the singular solution W_p in the $d \geqslant 11$ case. Here, we mean the Morse index by the maximal dimension of a subspace $X \subset H_0^1(B_1)$ such that the associated quadratic form is negative on $X \setminus \{0\}$, that is,

$$\max\{\dim X \mid X \subset H_0^1(B_1) \text{ is a subspace such that } \langle(-\Delta - pW_p^{p-1}\mathrm{e}^{W_p^p})v, v\rangle < 0 \text{ for } v \in X \setminus \{0\}\}.$$

It is well known that the Morse index plays an important role in the bifurcation analysis for nonlinear elliptic equations (see, for example, [2,8,11] and the references therein). In the $3 \leqslant d \leqslant 9$ case, by combining the argument in [8, proposition 2.1] with our proposition 4.1 we see that the Morse index of the singular solution W_p is infinite. However, for the $d \geqslant 11$ case, we find that the situation becomes different from that above. More precisely, we obtain the following result.

THEOREM 1.4. *Assume that $d \geqslant 11$ and $p > 0$. Let W_p be the singular solution to (1.1) obtained in theorem 1.1. Then, the Morse index of the singular solution W_p is finite.*

We prove theorem 1.1 in the spirit of Merle and Peletier [13]. We first transform (1.1) to a suitable equation: from the result in [7], we find that a positive solution to (1.1) is radially symmetric. Therefore, (1.1) can be transformed into the following ordinary differential equation (ODE):

$$\left.\begin{aligned} u_{rr} + \frac{d-1}{r}u_r + \lambda\exp(u^p) &= 0, \quad 0 < r < 1,\\ u(r) &= 0, \quad\quad r = 1. \end{aligned}\right\} \tag{1.4}$$

We set $s = \sqrt{\lambda}r$ and $\hat{u}(s) = u(r)$. Then, we see that $\hat{u}$ satisfies

$$\left.\begin{aligned} \hat{u}_{ss} + \frac{d-1}{s}\hat{u}_s + \mathrm{e}^{\hat{u}^p} &= 0, \quad 0 < s < \sqrt{\lambda},\\ \hat{u}(s) &= 0, \quad\quad s = \sqrt{\lambda}. \end{aligned}\right\} \tag{1.5}$$

We construct a local solution to the equation in (1.5), which has a singularity at the origin $s = 0$. To this end, we employ the Emden–Fowler transformation. Namely, we set $t = -\log s$ and $\bar{u}(t) = \hat{u}(s)$. This yields that $\bar{u}$ satisfies the following:

$$\left.\begin{aligned} \bar{u}_{tt} - (d-2)\bar{u}_t + \exp[-2t + \bar{u}^p] &= 0, \quad -\tfrac{1}{2}\log\lambda < t < \infty,\\ \bar{u}(t) &= 0, \quad\quad t = -\tfrac{1}{2}\log\lambda. \end{aligned}\right\} \tag{1.6}$$

We give an approximate form of a singular solution near $t = \infty$. Then, we make an error estimate for the approximation. The proof of theorem 1.2 is also based on that in [13]. We note that Dancer [4] proved that there exist infinitely many regular positive solutions to (1.1) by calculating the Morse index. Here, following [8,14,15], we shall show theorem 1.3 by counting the number of intersections between the singular solution and regular solutions. Thus, we can obtain a precise bifurcation diagram of solutions to (1.1). Let us explain this in detail. Let I be an interval in $\mathbb{R}$. For a function $v(s)$ on I, we define the number of zeros of v by

$$\mathcal{Z}_I[v(\cdot)] = \#\{s \in I \mid v(s) = 0\}.$$

We set $\hat{W}_p(s) = W_p(r)$, where $s = \sqrt{\lambda} r$ and W_p is the singular solution given by theorem 1.1. Let $(\lambda(\gamma), \hat{u}(s,\gamma))$ be a regular solution to (1.5) with $\hat{u}(0) = \gamma$. Then, we have

$$Z_{I_\lambda}[\hat{u}(\cdot,\gamma) - \hat{W}_p(\cdot)] \to \infty \quad \text{as } \gamma \to \infty.$$

See lemma 4.3 for the details. From this, we can show that the bifurcation branch $\mathcal{C}$ given by (1.3) has infinitely many turning points, which yields theorem 1.3.

Here, we remark that, in order to prove theorems 1.2 and 1.3, the asymptotic scaling invariance of the equation found by Dancer [4] plays an important role. Let us examine this in more detail. We denote by $\hat{u}(s,\gamma)$ a solution to the equation in (1.5) with $\hat{u}(0,\gamma) = \|\hat{u}\|_{L^\infty} = \gamma$. We set

$$\hat{u}(s,\gamma) = \gamma + \frac{\gamma^{1-p}}{p}\tilde{u}(\rho,\gamma), \qquad \rho = \sqrt{\gamma^{p-1}\exp(\gamma^p)}\,s. \tag{1.7}$$

It follows that $\tilde{u}(\rho,\gamma)$ satisfies

$$\left.\begin{aligned} \tilde{u}_{\rho\rho} + \frac{d-1}{\rho}\tilde{u}_\rho + p\exp\left[-\gamma^p + \gamma^p\left(1 + \frac{\gamma^{-p}}{p}\tilde{u}\right)^p\right] = 0,& \\ 0 < \rho < \sqrt{\lambda\gamma^{p-1}\exp(\gamma^p)},& \\ \tilde{u}(0) = 0,& \\ \tilde{u}(\rho) < 0, \quad 0 < \rho < \sqrt{\lambda\gamma^{p-1}\exp(\gamma^p)}.& \end{aligned}\right\} \tag{1.8}$$

We shall show that the function $\tilde{u}(\rho,\gamma)$ converges to U in $C^1_{\mathrm{loc}}([0,\infty))$ as γ tends to ∞, where U is a solution to

$$\left.\begin{aligned} U_{\rho\rho} + \frac{d-1}{\rho}U_\rho + p\exp[U] = 0, &\quad 0 < \rho < \infty, \\ U(\rho) = 0, &\quad \rho = 0, \\ U(\rho) < 0, &\quad 0 < \rho < \infty. \end{aligned}\right\} \tag{1.9}$$

We also use a similar transformation (see (3.6)). Since the equation in (1.9) is scale invariant, it is rather easy to handle. This is key for our analysis.

This paper is organized as follows: in § 2, we construct the singular solution to (1.1) in case of $d \geqslant 3$. In § 3, we investigate the asymptotic behaviour of the regular solutions $(\lambda(\gamma), u(r,\gamma))$ as $\gamma \to \infty$. In § 4, we count the number of intersections and prove theorem 1.3. In § 5, we show that the Morse index of the singular solution is finite when $d \geqslant 11$.

2. Existence of a singular solution

To prove theorem 1.1, we first consider (1.6) and restrict ourselves to the case where $t > 0$ is sufficiently large. We seek a solution to (1.6) of the form

$$\bar{u}(t) = (\varphi(t) + \kappa)^{1/p} + \eta(t), \tag{2.1}$$

where

$$\varphi(t) = 2t - A_p \log t, \quad A_p = 1 - \frac{1}{p}, \qquad \kappa = \log \frac{(d-2)2^{1/p}}{p}. \tag{2.2}$$

Then, the function η solves

$$\eta_{tt} - (d-2)\eta_t + \exp[-2t + \bar{u}^p] - \frac{2(d-2)}{p}(\varphi + \kappa)^{-A_p} = f_1(t) \tag{2.3}$$

for sufficiently large $t > 0$, where

$$f_1(t) = \frac{(d-2)A_p(\varphi+\kappa)^{-A_p}}{pt} + \frac{1}{p}\left(1 - \frac{1}{p}\right)(\varphi+\kappa)^{1/p-2}(\varphi_t)^2 - \frac{1}{p}(\varphi+\kappa)^{-A_p}\varphi_{tt}. \tag{2.4}$$

Then, we show the following.

THEOREM 2.1. *Let $d \geqslant 3$ and $p > 0$. There exist $T_\infty > 0$ and a solution $\eta_\infty \in C([T_\infty, \infty), \mathbb{R})$ to (2.3) satisfying $\lim_{t\to\infty} \varphi^{A_p}\eta_\infty(t) = 0$.*

We show theorem 2.1 by using the contraction-mapping principle. To this end, we transform (2.3). First, we have

$$\begin{aligned}
\exp[-2t + \bar{u}^p] &= \exp[-2t + \{(\varphi+\kappa)^{1/p} + \eta\}^p] \\
&= \exp[-2t + (\varphi+\kappa) + (\varphi+\kappa)\{(1 + (\varphi+\kappa)^{-1/p}\eta)^p - 1\}] \\
&= \frac{(d-2)2^{1/p}}{p} t^{-A_p} \exp[(\varphi+\kappa)\{(1 + (\varphi+\kappa)^{-1/p}\eta)^p - 1\}].
\end{aligned} \tag{2.5}$$

Furthermore, we obtain

$$(\varphi+\kappa)\{(1 + (\varphi+\kappa)^{-1/p}\eta)^p - 1\} = p(\varphi+\kappa)^{A_p}\eta + (\varphi+\kappa)g_1(t,\eta), \tag{2.6}$$

where

$$g_1(t,\eta) = \{1 + (\varphi+\kappa)^{-1/p}\eta\}^p - 1 - p(\varphi+\kappa)^{-1/p}\eta. \tag{2.7}$$

This yields

$$\begin{aligned}
&\exp[(\varphi+\kappa)\{(1 + (\varphi+\kappa)^{-1/p}\eta)^p - 1\}] \\
&\quad = \exp[p(\varphi+\kappa)^{A_p}\eta + (\varphi+\kappa)g_1(t,\eta)] \\
&\quad = \exp[p(\varphi+\kappa)^{A_p}\eta] + \exp[p(\varphi+\kappa)^{A_p}\eta]\{\exp[(\varphi+\kappa)g_1(t,\eta)] - 1\}.
\end{aligned} \tag{2.8}$$

By (2.5), (2.6) and (2.8), we have

$$\begin{aligned}
\exp[-2t + \bar{u}^p] = {} & \frac{(d-2)2^{1/p}}{p} t^{-A_p} \exp[p(\varphi+\kappa)^{A_p}\eta] \\
& + \frac{(d-2)2^{1/p}}{p} t^{-A_p} \exp[p(\varphi+\kappa)^{A_p}\eta]\{\exp[(\varphi+\kappa)g_1(t,\eta)] - 1\}.
\end{aligned}$$

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Therefore, (2.3) can be written as

$$
\begin{aligned}
&\eta_{tt} - (d-2)\eta_t + 2(d-2)\eta \\
&\qquad = f_1(t) - \frac{(d-2)2^{1/p}}{p} t^{-A_p} + \frac{2(d-2)}{p}(\varphi+\kappa)^{-A_p} \\
&\qquad\quad + 2(d-2)\eta - \frac{(d-2)2^{1/p}}{p} t^{-A_p} \times p(\varphi+\kappa)^{A_p}\eta \\
&\qquad\quad - \frac{(d-2)2^{1/p}}{p} t^{-A_p} \exp[p(\varphi+\kappa)^{A_p}\eta]\{\exp[(\varphi+\kappa)g_1(t,\eta)] - 1\} \\
&\qquad\quad - \frac{(d-2)2^{1/p}}{p} t^{-A_p} \{\exp[p(\varphi+\kappa)^{A_p}\eta] - 1 - p(\varphi+\kappa)^{A_p}\eta\} \\
&\qquad = f_1(t) + f_2(t) + f_3(t,\eta) + f_4(t,\eta) + f_5(t,\eta),
\end{aligned}
$$

where

$$
\begin{aligned}
f_2(t) &= -\frac{(d-2)2^{1/p}}{p} t^{-A_p} + \frac{2(d-2)}{p}(\varphi+\kappa)^{-A_p} \\
&= \frac{(d-2)2^{1/p}}{p} t^{-A_p}(1 - (2t)^{A_p}(\varphi+\kappa)^{-A_p}), \qquad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
f_3(t,\eta) &= 2(d-2)\eta - \frac{(d-2)2^{1/p}}{p} t^{-A_p} \times p(\varphi+\kappa)^{A_p}\eta \\
&= 2(d-2)\{1 - (2t)^{-A_p}(\varphi+\kappa)^{A_p}\}\eta, \qquad (2.10)
\end{aligned}
$$

$$
f_4(t,\eta) = -\frac{(d-2)2^{1/p}}{p} t^{-A_p} \exp[p(\varphi+\kappa)^{A_p}\eta]\{\exp[(\varphi+\kappa)g_1(t,\eta)] - 1\}, \qquad (2.11)
$$

$$
f_5(t,\eta) = -\frac{(d-2)2^{1/p}}{p} t^{-A_p}\{\exp[p(\varphi+\kappa)^{A_p}\eta] - 1 - p(\varphi+\kappa)^{A_p}\eta\}. \qquad (2.12)
$$

Thus, we seek a solution to the following equation:

$$
\eta_{tt} - (d-2)\eta_t + 2(d-2)\eta = f_1(t) + f_2(t) + f_3(t,\eta) + f_4(t,\eta) + f_5(t,\eta).
$$

We estimate the inhomogeneous terms $f_i(t)$ $(1 \leqslant i \leqslant 5)$ and obtain the following.

Lemma 2.2.

(i) $f_1(t) = O(t^{-A_p-1})$ *and* $f_2(t) = O(t^{-A_p-1}\log t)$ *as* $t \to \infty$.

(ii) *If* η *satisfies* $|\eta(t)| \leqslant \varepsilon t^{-A_p}$ *for sufficiently small* $\varepsilon > 0$, *we have*

$$
f_3(t,\eta) = O(t^{-A_p-1}\log t), \qquad f_4(t) = O(t^{-A_p-1}), \qquad |f_5(t)| \leqslant \varepsilon^2 t^{-A_p}
$$

for sufficiently large $t > 0$.

Proof. By (2.4) and (2.9), we obtain (i). It follows from (2.2) that

$$
|1 - (2t)^{-A_p}(\varphi+\kappa)^{A_p}| \lesssim \frac{\log t}{t} \qquad (2.13)
$$

for sufficiently large $t > 0$. Thus, by (2.10), we have

$$|f_3(t,\eta)| = |2(d-2)\{1-(2t)^{-A_p}(\varphi+\kappa)^{A_p}\}\eta| \lesssim t^{-A_p-1}\log t.$$

From (2.7), we have

$$|g_1(t,\eta)| \lesssim |\varphi+\kappa|^{-2/p}\eta^2. \tag{2.14}$$

This yields

$$|(\varphi+\kappa)g_1(t,\eta)| \lesssim t^{-1}.$$

It follows that

$$|\exp[(\varphi+\kappa)g_1(t,\eta)] - 1| \lesssim |(\varphi+\kappa)g_1(t,\eta)| \lesssim t^{-1}. \tag{2.15}$$

From (2.11), we have $f_4(t) = O(t^{-A_p-1})$. Similarly, we see that

$$|\exp[p(\varphi+\kappa)^{A_p}\eta] - 1 - p(\varphi+\kappa)^{A_p}\eta| \lesssim (\varphi+\kappa)^{2A_p}\eta^2 \lesssim \varepsilon^2.$$

Thus, we obtain $|f_5(t)| \leqslant \varepsilon^2 t^{-A_p}$ from (2.12). □

We are now in a position to prove theorem 2.1.

Proof of theorem 2.1. We set

$$F(t,\eta) = f_1(t) + f_2(t) + f_3(t,\eta) + f_4(t,\eta) + f_5(t,\eta).$$

In order to prove theorem 2.1, it is enough to solve the following final-value problem:

$$\left.\begin{aligned} \eta_{tt} - (d-2)\eta_t + 2(d-2)\eta = F(t,\eta), \quad T < t < +\infty,\\ \varphi^{A_p}(t)\eta(t) \to 0 \quad \text{as} \quad t \to +\infty. \end{aligned}\right\} \tag{2.16}$$

for some $T > 0$. We note that

$$(d-2)^2 - 8(d-2) = \begin{cases} (d-2)(d-10) < 0 & \text{if } 3 \leqslant d \leqslant 9,\\ (d-2)(d-10) = 0 & \text{if } d = 10,\\ (d-2)(d-10) > 0 & \text{if } d \geqslant 11. \end{cases}$$

We consider only the $3 \leqslant d \leqslant 9$ case, as the proof is similar in the other cases. Let $\mu = \sqrt{-(d-2)(d-10)}$. Then, the final-value problem (2.16) is transformed into the following integral equation:

$$\eta(t) = \mathcal{T}[\eta](t)$$

in which

$$\mathcal{T}[\eta](t) = \frac{\mathrm{e}^{(d-2)t/2}}{\mu}\int_t^\infty \mathrm{e}^{-(d-2)\sigma/2}\sin(\mu(\sigma-t))F(\sigma,\eta)\,\mathrm{d}\sigma.$$

Fix $T > 0$ to be sufficiently large and let X be a space of continuous functions on (T,∞) equipped with the following norm:

$$\|\xi\| = \sup\{|t|^{A_p}|\xi(t)| \mid t > T\}.$$

We fix an arbitrary $\varepsilon > 0$ and set

$$\Sigma = \{\xi \in X \,|\, \|\xi\| < \varepsilon\}. \tag{2.17}$$

First, we shall show that $\mathcal{T}$ maps from Σ to itself. It follows from lemma 2.2 that $|F(t,\eta)| \leqslant \varepsilon^2 t^{-A_p}$ for sufficiently large $t>0$. This yields

$$\begin{aligned}|\mathcal{T}[\eta](t)| &\lesssim \mathrm{e}^{(d-2)t/2}\int_t^\infty \mathrm{e}^{-(d-2)\sigma/2}\varepsilon^2\sigma^{-A_p}\,\mathrm{d}\sigma\\ &\leqslant \varepsilon^2 t^{-A_p}\mathrm{e}^{(d-2)t/2}\int_t^\infty \mathrm{e}^{-(d-2)\sigma/2}\,\mathrm{d}\sigma\\ &\lesssim \varepsilon^2 t^{-A_p} \end{aligned}\tag{2.18}$$

for $\eta\in\Sigma$. It follows that $T[\eta]\in\Sigma$. Thus, we have proved the claim.

Next, we shall show that $\mathcal{T}$ is a contraction mapping. For $\eta_1,\eta_2\in\Sigma$, we have

$$|\mathcal{T}[\eta_1](t)-\mathcal{T}[\eta_2](t)| \leqslant C\mathrm{e}^{(d-2)t/2}\sum_{i=3}^{5}\int_t^\infty \mathrm{e}^{-(d-2)\sigma/2}|f_i(\sigma,\eta_1)-f_i(\sigma,\eta_2)|\,\mathrm{d}\sigma.$$

From the definition, we obtain

$$|f_3(t,\eta_1)-f_3(t,\eta_2)| \lesssim t^{-1}\log t|\eta_1-\eta_2| \lesssim t^{-A_p-1}\log t\|\eta_1-\eta_2\|.\tag{2.19}$$

Thus, we see that

$$|f_3(t,\eta_1)-f_3(t,\eta_2)| \leqslant \varepsilon t^{-A_p}\|\eta_1-\eta_2\|.\tag{2.20}$$

Next, we estimate the term $|f_5(t,\eta_1)-f_5(t,\eta_2)|$. It follows that

$$\begin{aligned}&|f_5(t,\eta_1)-f_5(t,\eta_2)|\\ &\quad\lesssim t^{-A_p}|\exp[p(\varphi+\kappa)^{A_p}\eta_1]-\exp[p(\varphi+\kappa)^{A_p}\eta_2]-p(\varphi+\kappa)^{A_p}(\eta_1-\eta_2)|\\ &\quad= t^{-A_p}|\exp[p(\varphi+\kappa)^{A_p}\eta_2]\{\exp[p(\varphi+\kappa)^{A_p}(\eta_2-\eta_1)]-1\}-p(\varphi+\kappa)^{A_p}(\eta_1-\eta_2)|\\ &\quad\lesssim t^{-A_p}|\exp[p(\varphi+\kappa)^{A_p}\eta_2]\{\exp[p(\varphi+\kappa)^{A_p}(\eta_2-\eta_1)]-1-p(\varphi+\kappa)^{A_p}(\eta_1-\eta_2)\}|\\ &\qquad+t^{-A_p}|\exp[p(\varphi+\kappa)^{A_p}\eta_2]-1|p(\varphi+\kappa)^{A_p}|\eta_1-\eta_2|\\ &\quad\lesssim t^{-A_p}|p(\varphi+\kappa)^{A_p}(\eta_1-\eta_2)|^2+t^{-A_p}|p(\varphi+\kappa)^{A_p}\eta_2|\|\eta_1-\eta_2\|\\ &\quad\lesssim \varepsilon t^{-A_p}\|\eta_1-\eta_2\|.\end{aligned}$$

Therefore, for sufficiently large $t>0$, we have

$$|f_5(t,\eta_1)-f_5(t,\eta_2)| \leqslant C\varepsilon t^{-A_p}\|\eta_1-\eta_2\|.\tag{2.21}$$

Finally, we estimate the term $|f_4(t,\eta_1)-f_4(t,\eta_2)|$. We can compute that

$$\begin{aligned}|f_4(t,\eta_1)-f_4(t,\eta_2)| \lesssim\ & t^{-A_p}|\exp[p\varphi^{A_p}\eta_1]-\exp[p\varphi^{A_p}\eta_2]|\,|\exp[g_1(t,\eta_2)]-1|\\ &+t^{-A_p}\exp[p\varphi^{A_p}\eta_2]|\exp[g_1(t,\eta_1)]-\exp[g_1(t,\eta_2)]|\\ =:\ & \mathrm{I}+\mathrm{II}.\end{aligned}\tag{2.22}$$

By the Taylor expansion together with (2.15), we have

$$\begin{aligned}\mathrm{I} &\lesssim t^{-A_p-2}\exp[p\varphi^{A_p}\eta_2]|\exp[p\varphi^{A_p}(\eta_1-\eta_2)]-1|\\ &\lesssim t^{-A_p-2}\exp[p\varepsilon]|\varphi^{A_p}(\eta_2-\eta_1)|\\ &\lesssim t^{-A_p-2}\varphi^{A_p}|\eta_1-\eta_2|\\ &\lesssim t^{-A_p-2}\|\eta_1-\eta_2\|.\end{aligned}\tag{2.23}$$

Similarly, by (2.14), we obtain

$$\begin{aligned} \mathrm{II} &\lesssim t^{-A_p} \exp[p\varphi^{A_p}\eta_2]|\exp[g_1(t,\eta_1)] - \exp[g_1(t,\eta_2)]| \\ &\lesssim t^{-A_p} \exp[p\varphi^{A_p}\eta_2] \exp[g_1(t,\eta_2)]|\exp[g_1(t,\eta_1) - g_1(t,\eta_2)] - 1| \\ &\lesssim t^{-A_p}|g_1(t,\eta_1) - g_1(t,\eta_2)|. \end{aligned} \tag{2.24}$$

From (2.7), we obtain

$$\begin{aligned} &|g_1(t,\eta_1) - g_1(t,\eta_2)| \\ &\quad\lesssim |\{1 + p(\varphi+\kappa)^{-1/p}\eta_1\}^p - \{1 + p(\varphi+\kappa)^{-1/p}\eta_2\}^p| + (\varphi+\kappa)^{-1/p}|\eta_1 - \eta_2| \\ &\quad\lesssim |\varphi+\kappa|^{-1/p}|\eta_1 - \eta_2| \\ &\quad\lesssim t^{-1}\|\eta_1 - \eta_2\|. \end{aligned} \tag{2.25}$$

It follows from (2.22)–(2.25) that

$$|f_4(t,\eta_1) - f_4(t,\eta_2)| \leqslant \varepsilon t^{-A_p}\|\eta_1 - \eta_2\|. \tag{2.26}$$

By (2.19), (2.21) and (2.26), we see that

$$|\mathcal{T}[\eta_1](t) - \mathcal{T}[\eta_2](t)| \leqslant C\varepsilon t^{-A_p}\|\eta_1 - \eta_2\| \leqslant \tfrac{1}{2}t^{-A_p}\|\eta_1 - \eta_2\|. \tag{2.27}$$

Thus, we find that $\mathcal{T}$ is a contraction mapping. This completes the proof. □

We are now in a position to prove theorem 1.1.

Proof of theorem 1.1. It follows from theorem 2.1 that there exist a constant $T_\infty > 0$ and a solution $\eta_\infty(t)$ of (2.3) for $t \in (T_\infty, +\infty)$ satisfying $|t|^{A_p}|\eta_\infty(t)| \leqslant \varepsilon$. For such a solution η_∞, we set

$$\bar{u}_\infty(t) = (\varphi(t) + \kappa)^{1/p} + \eta_\infty(t).$$

Then we see that $\bar{u}_\infty(t)$ satisfies

$$\bar{u}_{tt} - (d-2)\bar{u}_t + \exp[-2t + \bar{u}^p] = 0 \tag{2.28}$$

for $t \in (T_\infty, +\infty)$. We shall show that $\bar{u}_\infty(t)$ has a zero for some $T_0 \in (-\infty, \infty)$. Suppose on the contrary that $\bar{u}_\infty(t)$ is positive for all $t \in (-\infty, \infty)$. Then, we see that $\bar{u}_\infty$ is monotone increasing. Indeed, otherwise, there exists a local minimum point $t_* \in (-\infty, \infty)$. It follows that $(\mathrm{d}^2\bar{u}_\infty/\mathrm{d}t^2)(t_*) \geqslant 0$ and $(\mathrm{d}\bar{u}_\infty/\mathrm{d}t)(t_*) = 0$. Then, from (2.28), we obtain

$$0 \leqslant \frac{\mathrm{d}^2\bar{u}_\infty}{\mathrm{d}t^2}(t_*) - (d-2)\frac{\mathrm{d}\bar{u}_\infty}{\mathrm{d}t}(t_*) = -\exp[-2t_* + \bar{u}_\infty^p(t_*)] < 0,$$

which is a contradiction. Since $\bar{u}_\infty$ is positive and monotone increasing, there exists a constant $C \geqslant 0$ such that $\bar{u}_\infty(t) \to C$ as $t \to -\infty$. This, together with (2.28), yields

$$0 = \lim_{t\to-\infty}\left\{\frac{\mathrm{d}^2\bar{u}_\infty}{\mathrm{d}t^2}(t) - (d-2)\frac{\mathrm{d}\bar{u}_\infty}{\mathrm{d}t}(t)\right\} = \lim_{t\to-\infty} -\exp[-2t + \bar{u}_\infty^p(t)] = -\infty,$$

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which is absurd. Therefore, we see that $\bar{u}_\infty$ has a zero for some $T_0 \in (-\infty, \infty)$. Then, $\bar{u}_\infty$ satisfies

$$\begin{aligned} \bar{u}_{tt} + (d-2)\bar{u}_t &= -\exp(-2t + \bar{u}^p), \quad && t \in (T_0, \infty), \\ \bar{u}(t) &= 0, && t = T_0, \\ \bar{u}(t) &> 0, && t \in (T_0, \infty). \end{aligned}$$

If we choose $\lambda_{p,\infty} > 0$ so that $-\log \lambda_{p,\infty} = 2T_0$, that is, $\lambda_{p,\infty} = \mathrm{e}^{-2T_0}$, we find that $\bar{u}_\infty(s)$ is a solution to (1.6) with $\lambda = \lambda_{p,\infty}$. This completes the proof. □

3. Asymptotic behaviour of a regular solution

In this section, we give a proof of theorem 1.2. We denote by $\hat{u}(s,\gamma)$ a positive solution to (1.5) with $\hat{u}(0) = \|\hat{u}\|_{L^\infty} = \gamma$. (Where there is no possibility of confusion, we just denote this by $\hat{u}(s)$.) As mentioned in § 1, we set

$$\hat{u}(s,\gamma) = \gamma + \frac{\gamma^{1-p}}{p}\tilde{u}(\rho,\gamma), \qquad \rho = \sqrt{\gamma^{p-1}\exp(\gamma^p)}\,s. \tag{3.1}$$

Then, we see that $\tilde{u}(\rho,\gamma)$ satisfies

$$\left.\begin{aligned} \tilde{u}_{\rho\rho} + \frac{d-1}{\rho}\tilde{u}_\rho + p\exp\left[-\gamma^p + \gamma^p\left(1 + \frac{\gamma^{-p}}{p}\tilde{u}\right)^p\right] = 0, \\ 0 < \rho < \sqrt{\lambda(\gamma)\gamma^{p-1}\exp(\gamma^p)}, \\ \tilde{u}(0) = 0, \\ \tilde{u}(\rho) < 0, \quad 0 < \rho < \sqrt{\lambda(\gamma)\gamma^{p-1}\exp(\gamma^p)}, \end{aligned}\right\} \tag{3.2}$$

where $\lambda(\gamma) > 0$. Concerning the solutions to (3.2), the following lemma holds.

Lemma 3.1. *Let $\tilde{u}(\rho,\gamma)$ be a solution to (3.2). Then, we have $\tilde{u}(\cdot,\gamma) \to U(\cdot)$ in $C^1_{\mathrm{loc}}([0,\infty))$ as $\gamma \to \infty$, where $U(\rho)$ is a solution to the following:*

$$\left.\begin{aligned} U_{\rho\rho} + \frac{d-1}{\rho}U_\rho + p\exp[U] = 0, \quad 0 < \rho < \infty, \\ U(\rho) = 0, \quad \rho = 0, \\ U(\rho) < 0, \quad 0 < \rho < \infty. \end{aligned}\right\} \tag{3.3}$$

Remark 3.2. Dancer [4] proved lemma 3.1 in more general situations. Here, using an ODE approach, we shall give an alternative proof.

Proof of lemma 3.1. First, it follows from [4, p. 155] that $\lambda(\gamma)\gamma^{p-1}\exp(\gamma^p) \to \infty$ as $\gamma \to \infty$. For each $\rho_0 > 0$, we shall show that $\tilde{u}(\rho,\gamma)$ is uniformly bounded for $\rho \in [0,\rho_0)$. Since $\gamma = \|\hat{u}\|_{L^\infty}$, and $\hat{u}(\rho,\gamma)$ is positive, (3.1) yields

$$-p\gamma^p < \tilde{u}(\rho,\gamma) \leqslant 0. \tag{3.4}$$

By (3.4), we have

$$0 < 1 + \frac{\gamma^{-p}}{p}\tilde{u} \leqslant 1.$$

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This yields

$$\exp\left[-\gamma^p + \gamma^p\left(1 + \frac{\gamma^{-p}}{p}\tilde{u}\right)^p\right] \leqslant \exp[-\gamma^p + \gamma^p] = 1.$$

It follows from the first equation in (3.2) that

$$\tilde{u}_{\rho\rho} + \frac{d-1}{\rho}\tilde{u}_\rho \geqslant -p.$$

This yields that

$$(\rho^{d-1}\tilde{u}_\rho)_\rho \geqslant -p\rho^{d-1}.$$

Integrating the above inequality from 0 to ρ, we have $\rho^{d-1}\tilde{u}_\rho(\rho) \geqslant -p\rho^d/d$. Thus, we obtain $\tilde{u}_\rho(\rho) \geqslant -p\rho/d$ for $\rho \in [0, \rho_0)$. Integrating the inequality yields

$$\tilde{u}(\rho) \geqslant \tilde{u}(0) - \frac{p}{d}\int_0^\rho \tau \,\mathrm{d}\tau = -\frac{p}{2d}\rho^2.$$

Therefore, for $\rho \in [0, \rho_0)$, we have

$$-\frac{p}{2d}\rho_0^2 \leqslant \tilde{u}(\rho) \leqslant 0. \tag{3.5}$$

This, together with the equation in (3.2), gives the uniform boundedness of $\tilde{u}_\rho$ and $\tilde{u}_{\rho\rho}$ for $\rho \in [0, \rho_0)$. Then, by the Ascoli–Arzelà theorem, there exists a function U such that $\tilde{u}(\rho, \gamma)$ converges to U in $C^1_{\mathrm{loc}}([0, \rho_0))$ as γ goes to ∞. Moreover, by the Taylor expansion, there exists $\theta \in (0, 1)$ such that

$$\begin{aligned}
&\left|\exp\left[-\gamma^p + \gamma^p\left(1 + \frac{\gamma^{-p}}{p}\tilde{u}(\rho,\gamma)\right)^p\right] - \exp[U]\right| \\
&\qquad = \left|\exp\left[\tilde{u} + \frac{p-1}{2p}\left(1 + \theta\frac{\gamma^{-p}}{p}\tilde{u}\right)^{p-2}\gamma^{-p}\tilde{u}^2\right] - \exp[U]\right| \\
&\qquad \leqslant \exp[\tilde{u}]\left|\exp\left[\frac{p-1}{2p}\left(1 + \theta\frac{\gamma^{-p}}{p}\tilde{u}\right)^{p-2}\gamma^{-p}\tilde{u}^2\right] - 1\right| + |\exp[\tilde{u}] - \exp[U]|.
\end{aligned}$$

Therefore, by (3.5), we have

$$\left|\exp\left[-\gamma^p + \gamma^p\left(1 + \frac{\gamma^{-p}}{p}\tilde{u}(\rho,\gamma)\right)^p\right] - \exp[U]\right| \to 0 \quad \text{as } \gamma \to \infty.$$

Thus, U satisfies (3.3). This completes the proof. □

Next, we set $t = -\log s$. We define $y(t, \gamma)$ using

$$\hat{u}(s, \gamma) = \varphi^{1/p}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + y(t, \gamma)), \tag{3.6}$$

and see that $y(t, \gamma)$ satisfies the following:

$$\begin{aligned}
y_{tt} &- \{(d-2) + 2A_p\varphi^{-1}\varphi_t\}y_t - 2(d-2) \\
&+ p\varphi^{A_p}\exp\left[-2t + \varphi\left(1 + \frac{\varphi^{-1}}{p}(\kappa + y)\right)^p\right] = f_6(t, y)
\end{aligned} \tag{3.7}$$

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for sufficiently large $t > 0$, where

$$f_6(t,y) = A_p\varphi^{-1}(\varphi_t)^2 - \varphi_{tt} - A_p(A_p+1)\varphi^{-2}(\varphi_t)^2(\kappa+y) + A_p\varphi^{-1}\varphi_{tt}(\kappa+y) - (d-2)A_p\varphi^{-1}\varphi_t(\kappa+y) + \frac{(d-2)A_p}{t}. \quad (3.8)$$

For the function $y(t,\gamma)$, we make the following spatial translation:

$$\tau = -\log\rho = t - \frac{\gamma^p}{2} - \frac{(p-1)\log\gamma}{2}, \qquad \hat{y}(\tau,\gamma) = y(t,\gamma), \qquad \hat{\varphi}(\tau) = \varphi(t). \quad (3.9)$$

Let U be the solution to (3.3). We set $U_*(\tau) = U(\rho)$ and

$$Y(\tau) = U_*(\tau) - 2\tau - \log\frac{2(d-2)}{p}. \quad (3.10)$$

Then, Y satisfies

$$\left.\begin{aligned} Y_{\tau\tau} - (d-2)Y_\tau + 2(d-2)\{\exp[Y]-1\} = 0, \quad -\infty < \tau < \infty, \\ \lim_{\tau\to\infty}\left\{Y(\tau) + 2\tau + \log\frac{2(d-2)}{p}\right\} = 0, \\ Y(\tau) + 2\tau + \log\frac{2(d-2)}{p} < 0, \quad -\infty < \tau < \infty, \end{aligned}\right\} \quad (3.11)$$

and the following lemma holds.

LEMMA 3.3. *Let $\hat{y}$ and Y be the functions defined by (3.9) and (3.10), respectively. Then, we have $\hat{y}(\tau,\gamma) \to Y(\tau)$ in $C^1_{\mathrm{loc}}((-\infty,\infty))$ as $\gamma\to\infty$.*

Proof. It follows from (3.1) and (3.6) that

$$\begin{aligned} \tilde{u}(\rho,\gamma) &= -p\gamma^p + p\gamma^{p-1}\hat{u}(s,\gamma) \\ &= -p\gamma^p + p\gamma^{p-1}\left\{\varphi^{1/p}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + y(t,\gamma))\right\} \\ &= p(-\gamma^p + \gamma^{p-1}\hat{\varphi}^{1/p}(\tau)) + \gamma^{p-1}\hat{\varphi}^{-A_p}(\tau)(\kappa + \hat{y}(\tau,\gamma)). \end{aligned} \quad (3.12)$$

By (2.2), (3.9) and the Taylor expansion, we have

$$\begin{aligned} &-\gamma^p + \gamma^{p-1}\hat{\varphi}^{1/p}(\tau) \\ &\quad = -\gamma^p + \gamma^{p-1}\left\{2\tau + \gamma^p + (p-1)\log\gamma - A_p\log\left(\tau + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma\right)\right\}^{1/p} \\ &\quad = -\gamma^p + \gamma^p\left\{\frac{2\tau}{\gamma^p} + 1 - \frac{A_p}{\gamma^p}\log\gamma^{-p} - \frac{A_p}{\gamma^p}\log\left(\tau + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma\right)\right\}^{1/p} \\ &\quad = -\gamma^p + \gamma^p\left\{1 + \frac{2\tau}{\gamma^p} - \frac{A_p}{\gamma^p}\log\left(\frac{\tau}{\gamma^p} + \frac{1}{2} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right\}^{1/p} \end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{p}\left(2\tau - A_p \log\left(\frac{1}{2} + \frac{\tau}{\gamma^p} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right) \\
&\quad + \frac{p-1}{2p^2\gamma^p}\left(1 + \theta\left(\frac{2\tau}{\gamma^p} - \frac{A_p}{\gamma^p}\log\left(\frac{1}{2} + \frac{\tau}{\gamma^p} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right)\right)^{1/p-2} \\
&\quad \times \left(2\tau - A_p \log\left(\frac{1}{2} + \frac{\tau}{\gamma^p} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right)^2 \qquad (3.13)
\end{aligned}
$$

for some $\theta \in (0,1)$. This yields that

$$
-\gamma^p + \gamma^{p-1}\hat{\varphi}^{1/p}(\tau) \to \frac{2\tau}{p} + \frac{A_p}{p}\log 2 \quad \text{as } \gamma \to \infty \qquad (3.14)
$$

for each $\tau \in (-\infty, \infty)$. Similarly, we obtain

$$
\gamma^{p-1}\hat{\varphi}^{-A_p}(\tau) = \left\{1 + \frac{2\tau}{\gamma^p} - \frac{A_p}{\gamma^p}\log\left(\frac{1}{2} + \frac{\tau}{\gamma^p} + \frac{(p-1)\log\gamma}{2\gamma^p}\right)\right\}^{-A_p} \to 1 \quad \text{as } \gamma \to \infty. \qquad (3.15)
$$

Formulae (3.12)–(3.15) imply that

$$
\lim_{\gamma\to\infty} \tilde{u}(\rho,\gamma) = 2\tau + A_p \log 2 + \kappa + \lim_{\gamma\to\infty} \hat{y}(\tau,\gamma). \qquad (3.16)
$$

It follows from lemma 3.1 that $\lim_{\gamma\to\infty} \tilde{u}(\rho,\gamma) = U(\rho) = U_*(\tau)$. Thus, by (2.2), (3.10) and (3.16), we see that

$$
\begin{aligned}
\lim_{\gamma\to\infty} \hat{y}(\tau,\gamma) &= -2\tau - A_p \log 2 - \kappa + U_*(\tau) \\
&= -2\tau - A_p \log 2 - \kappa + Y(\tau) + 2\tau + \log\frac{2(d-2)}{p} \\
&= Y(\tau) - \kappa + \log\frac{(d-2)2^{1/p}}{p} \\
&= Y(\tau).
\end{aligned}
$$

This completes the proof. □

LEMMA 3.4. *Let Y be a solution to (3.11). Then, Y satisfies $(Y, Y_\tau) \to (0,0)$ as $\tau \to -\infty$.*

Proof. We set $Z_1(\tau) = Y(\tau)$ and $Z_2(\tau) = Y_\tau(\tau)$. Then, the pair of functions (Z_1, Z_2) satisfies

$$
\left.
\begin{aligned}
\frac{\mathrm{d}Z_1}{\mathrm{d}\tau} &= Z_2, \\
\frac{\mathrm{d}Z_2}{\mathrm{d}\tau} &= (d-2)Z_2 - 2(d-2)[\exp[Z_1] - 1].
\end{aligned}
\right\} \qquad (3.17)
$$

We define an energy E by

$$
E(\tau) = \tfrac{1}{2}(Z_2)^2 + 2(d-2)[\exp[Z_1] - 1 - Z_1].
$$

From (3.17), we have

$$
\frac{\mathrm{d}E}{\mathrm{d}\tau}(\tau) = (d-2)(Z_2)^2 > 0.
$$

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Moreover, $(0,0)$ is an equilibrium point of (3.17) and a minimum of the energy E. This yields that $(Z_1(\tau), Z_2(\tau)) \to (0,0)$ as $\tau \to -\infty$. □

We set

$$z_1(t,\gamma) = y(t,\gamma), \qquad z_2(t,\gamma) = y_t(t,\gamma), \tag{3.18}$$

where $y(t,\gamma)$ is the function defined in (3.6). Then, $(z_1(t,\gamma), z_2(t,\gamma))$ satisfies

$$\left.\begin{aligned} &\frac{\mathrm{d}z_1}{\mathrm{d}t} = z_2 \quad \text{for } t \in (-\tfrac{1}{2}\log\lambda(\gamma), \infty),\\ &\frac{\mathrm{d}z_2}{\mathrm{d}t} = (d-2+2A_p\varphi^{-1}\varphi_t)z_2 + 2(d-2) + f_6(t,z_1)\\ &\qquad\qquad - p\varphi^{A_p}\exp\left[-2t + \varphi\left(1 + \frac{\varphi^{-1}}{p}(\kappa + z_1(t))\right)^p\right]\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in (-\tfrac{1}{2}\log\lambda(\gamma), \infty). \end{aligned}\right\} \tag{3.19}$$

From lemma 3.4, we see that for any $\varepsilon > 0$ there exists $\tau_\varepsilon \in (-\infty, 0)$ such that $|(Z_1(\tau_\varepsilon), Z_2(\tau_\varepsilon))| < \frac{1}{2}\varepsilon$, where (Z_1, Z_2) is a solution to (3.17). We fix $\tau_\varepsilon \in (-\infty, 0)$ and set

$$t_\varepsilon = \tau_\varepsilon + \tfrac{1}{2}\gamma^p + \tfrac{1}{2}((p-1)\log\gamma).$$

Then, by lemma 3.3, we have

$$|(z_1(t_\varepsilon,\gamma), z_2(t_\varepsilon,\gamma))| < \varepsilon \tag{3.20}$$

for sufficiently large $\gamma > 0$. We shall show the following.

LEMMA 3.5. *Let $(z_1(t,\gamma), z_2(t,\gamma))$ be the function defined by (3.18). For arbitrary $\varepsilon > 0$, we set*

$$\Gamma_\varepsilon = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid 2(d-2)\{\exp[\xi_1] - 1 - \xi_1\} + \tfrac{1}{2}\xi_2^2 < \varepsilon\}.$$

There exists a T_ε that does not depend on γ and t_ε but depends on ε such that $(z_1(t,\gamma), z_2(t,\gamma)) \in \Gamma_{2\varepsilon}$ for $t \in (T_\varepsilon, t_\varepsilon)$.

Proof. We define an energy by

$$E_1(t) = \tfrac{1}{2}z_2^2 + 2(d-2)\{\exp[z_1] - 1 - z_1\}.$$

By (3.19), we have

$$\begin{aligned} \frac{\mathrm{d}E_1}{\mathrm{d}t}(t) &= z_2 z_{2t} + 2(d-2)\{\exp[z_1] - 1\}z_2\\ &= (d-2+2A_p\varphi^{-1}\varphi_t)z_2^2\\ &\qquad - p\varphi^{A_p}\exp\left[-2t + \varphi\left(1 + \frac{\varphi^{-1}}{p}(\kappa + z_1)\right)^p\right]z_2 + f_6(t,z_1)z_2\\ &\qquad + 2(d-2)\exp[z_1]z_2. \end{aligned}$$

Analogously to (2.5), by the Taylor expansion, we obtain

$$
\begin{aligned}
&p\varphi^{A_p}\exp\left[-2t+\varphi\left(1+\frac{\varphi^{-1}}{p}(\kappa+z_1)\right)^p\right]\\
&\quad=(d-2)2^{1/p}\varphi^{A_p}t^{-A_p}\exp[z_1]\exp[\tilde g_1(t,z_1)]\\
&\quad=2(d-2)\exp[z_1]\\
&\qquad-(2(d-2)\exp[z_1]-(d-2)2^{1/p}\varphi^{A_p}t^{-A_p}\exp[z_1]\exp[\tilde g_1(t,z_1)]),
\end{aligned}
$$

where

$$\tilde g_1(t,z_1)=\varphi\left(1+\frac{\varphi^{-1}}{p}(\kappa+z_1)\right)^p-\varphi(t)-\kappa-z_1.$$

Therefore, we have

$$
\begin{aligned}
\frac{\mathrm{d}E_1}{\mathrm{d}t}(t)&=(d-2+2A_p\varphi^{-1}\varphi_t)z_2^2+f_6(t,z_1)z_2\\
&\quad+(2(d-2)\exp[z_1]-(d-2)2^{1/p}\varphi^{A_p}t^{-A_p}\exp[z_1]\exp[\tilde g_1(t,z_1)])z_2.
\end{aligned}\tag{3.21}
$$

Since $\varGamma_\varepsilon$ is a neighbourhood of $(0,0)$, we can take $\varepsilon>0$ to be so small that $\varGamma_{2\varepsilon}\subset\{(x_1,x_2)\mid |x_1|+|x_2|<1\}$. We choose $T_\varepsilon>0$ so that

$$0<\frac{C_*}{\sqrt{T_\varepsilon}}<\frac{\varepsilon}{2},\tag{3.22}$$

where the constant $C_*>0$, which does not depend on ε, is defined by (3.26). We shall show by contradiction that $(z_1(t),z_2(t))\in\varGamma_{2\varepsilon}$ for $t\in(T_\varepsilon,t_\varepsilon)$. Suppose to the contrary that $(z_1(t),z_2(t))\in\varGamma_{2\varepsilon}$ for $t\in(T_\varepsilon,t_\varepsilon]$ and $(z_1(T_\varepsilon),z_2(T_\varepsilon))\notin\varGamma_{2\varepsilon}$. Then, by (3.21), we have

$$
\begin{aligned}
&E_1(t_\varepsilon)-E_1(T_\varepsilon)\\
&\quad=\int_{T_\varepsilon}^{t_\varepsilon}(d-2+2A_p\varphi^{-1}\varphi_t)z_2^2\,\mathrm{d}s+\int_{T_\varepsilon}^{t_\varepsilon}f_6(s,z_1)z_2\,\mathrm{d}s\\
&\qquad+\int_{T_\varepsilon}^{t_\varepsilon}(2(d-2)\exp[z_1]-(d-2)2^{1/p}\varphi^{A_p}(s)s^{-A_p}\exp[z_1]\exp[\tilde g_1(s,z_1)])z_2\,\mathrm{d}s.
\end{aligned}\tag{3.23}
$$

Since $|z_1(t)|+|z_2(t)|<1$, we see from (3.8) that there exists a constant $C_1>0$ satisfying $|f_6(s,z_1)|\leqslant C_1/|s|$. Furthermore, from (2.2), we have

$$
\begin{aligned}
&|2(d-2)\exp[z_1]-(d-2)2^{1/p}\varphi^{A_p}(s)s^{-A_p}\exp[z_1]\exp[\tilde g_1(s,z_1)]|\\
&\qquad=2(d-2)\exp[z_1]\left|1-\left(\frac{\varphi(s)}{2}\right)^{A_p}s^{-A_p}\exp[\tilde g_1(s,z_1)]\right|\\
&\qquad=2(d-2)\exp[z_1]\left|1-\left(1-\frac{A_p}{2}\frac{\log s}{s}\right)^{A_p}\exp[\tilde g_1(s,z_1)]\right|\\
&\qquad\leqslant C|1-\exp[\tilde g_1(s,z_1)]|+C\left|1-\left(1-\frac{A_p}{2}\frac{\log s}{s}\right)^{A_p}\right|\exp[\tilde g_1(s,z_1)].
\end{aligned}\tag{3.24}
$$

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Similarly to the proof of lemma 2.2, there exists a constant $C > 0$ such that

$$\left|1 - \left(1 - \frac{A_p}{2}\frac{\log s}{s}\right)^{A_p}\right| \leqslant C\frac{\log s}{s}, \qquad |\tilde{g}_1(s, z_1)| \leqslant \frac{C}{s}$$

for sufficiently large $s > 0$. Together with (3.24), this yields that

$$|2(d-2)\exp[z_1] - (d-2)2^{1/p}\varphi^{A_p}(s)s^{-A_p}\exp[z_1]\exp[g_1(s, z_1)]| \leqslant \frac{C}{s^{3/4}}$$

for some constant $C > 0$. Therefore, by Young's inequality, we have

$$\begin{aligned}\left|\int_{T_\varepsilon}^{t_\varepsilon}(2(d-2)\exp[z_1] - (d-2)2^{1/p}\varphi^{A_p}s^{-A_p}\exp[z_1]\exp[g_1(s, z_1)])z_2\,\mathrm{d}s\right| \\ + \left|\int_{T_\varepsilon}^{t_\varepsilon} f_6(s, z_1)z_2\,\mathrm{d}s\right| \\ \leqslant \int_{T_\varepsilon}^{t_\varepsilon}\frac{C}{s^{3/4}}z_2\,\mathrm{d}s \\ \leqslant \frac{2C^2}{d-2}\int_{T_\varepsilon}^{t_\varepsilon}\frac{1}{s^{3/2}}\,\mathrm{d}s + \frac{(d-2)}{2}\int_{T_\varepsilon}^{t_\varepsilon}|z_2|^2\,\mathrm{d}s \\ \leqslant \frac{4C^2}{(d-2)\sqrt{T_\varepsilon}} + \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon}|z_2|^2\,\mathrm{d}s.\end{aligned} \tag{3.25}$$

We set

$$C_* = \frac{4C^2}{d-2}. \tag{3.26}$$

Then, it follows from (3.22) and (3.25) that

$$\begin{aligned}\left|\int_{T_\varepsilon}^{t_\varepsilon}(2(d-2)\exp[z_1] - (d-2)2^{1/p}\varphi^{A_p}s^{-A_p}\exp[z_1]\exp[g_1(s, z_1)])z_2\,\mathrm{d}s\right| \\ + \left|\int_{T_\varepsilon}^{t_\varepsilon} f_6(s, z_1)z_2\,\mathrm{d}s\right| \\ \leqslant \frac{C_*}{\sqrt{T_\varepsilon}} + \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon}|z_2|^2\,\mathrm{d}s \\ \leqslant \frac{\varepsilon}{2} + \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon}|z_2|^2\,\mathrm{d}s.\end{aligned} \tag{3.27}$$

Moreover, we take $T_\varepsilon > 0$ so that $|2A_p\varphi^{-1}(t)\varphi_t(t)| < \frac{1}{2}(d-2)$ for $t > T_\varepsilon$. Then, we have

$$\int_{T_\varepsilon}^{t_\varepsilon}(d - 2 + 2A_p\varphi^{-1}\varphi_t)z_2^2\,\mathrm{d}s \geqslant \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon}|z_2|^2\,\mathrm{d}s. \tag{3.28}$$

It follows from (3.23), (3.27) and (3.28) that

$$E_1(t_\varepsilon) - E_1(T_\varepsilon) \geqslant \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon}|z|_2^2\,\mathrm{d}s - \frac{\varepsilon}{2} - \frac{d-2}{2}\int_{T_\varepsilon}^{t_\varepsilon}|z_2|^2\,\mathrm{d}s > -\frac{\varepsilon}{2}.$$

This, together with (3.20) and $(z_1(T_\varepsilon), z_2(T_\varepsilon)) \notin \Gamma_{2\varepsilon}$, implies that

$$2\varepsilon \leqslant E_1(T_\varepsilon) < E_1(t_\varepsilon) + \frac{\varepsilon}{2} = \frac{3\varepsilon}{2},$$

which is a contradiction. Therefore, our assertion holds. □

We are now in a position to prove theorem 1.2.

Proof of theorem 1.2. Let $\{\gamma_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ be a sequence satisfying $\lim_{n\to\infty} \gamma_n = \infty$. Let $(z_1(t,\gamma_n), z_2(t,\gamma_n))$ be the function defined by (3.18). By lemma 3.5, we find that $(z_1(t,\gamma_n), z_2(t,\gamma_n))$ is uniformly bounded in the interval $(T_\varepsilon, t_\varepsilon)$. This, together with (3.7), implies that $y_{tt}(t,\gamma_n)$ is also uniformly bounded in the interval $(T_\varepsilon, t_\varepsilon)$. Differentiating (3.7) implies that $y_{ttt}(t,\gamma)$ is also uniformly bounded in $(T_\varepsilon, t_\varepsilon)$. This yields that $(z_1(t,\gamma_n), z_2(t,\gamma_n))$ and $(z_{1t}(t,\gamma_n), z_{2t}(t,\gamma_n))$ are equicontinuous. Thus, it follows from the Ascoli–Arzelà theorem that there exists a subsequence $\{(z_1(t,\gamma_n), z_2(t,\gamma_n))\}$ (which we still denote by the same letter) and a pair of functions $(z_{*,1}(t), z_{*,2}(t))$ in $(C^1(T_\varepsilon, t_\varepsilon))^2$ as $n \to \infty$, Since $t_\varepsilon (> T_\varepsilon)$ is arbitrary, we find that $(z_1(t,\gamma_n), z_2(t,\gamma_n))$ converges to $(z_{*,1}(t), z_{*,2}(t))$ in $(C^1(T_\varepsilon, \infty))^2$ as n goes to ∞. We note that $0 < \lambda(\gamma_n) < \lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ in B_1 with the Dirichlet boundary condition. Thus, there exists $\lambda_* \geqslant 0$ such that $\lambda(\gamma_n) \to \lambda_*$ as $n \to \infty$. By the result in [4], we see that $\lambda_* > 0$. From these results, we see that $(z_{*,1}, z_{*,2}, \lambda_*)$ satisfies

$$\begin{aligned}
\frac{\mathrm{d}z_1}{\mathrm{d}t} &= z_2 \quad \text{for } t \in (-\tfrac{1}{2}\log\lambda_*, \infty), \\
\frac{\mathrm{d}z_2}{\mathrm{d}t} &= (d-2-2A_p\varphi^{-1}\varphi_t)z_2 + 2(d-2) + f_6(t,z_1) \\
&\qquad - p\varphi^{A_p}\exp\left[-2t + \varphi\left(1 + \frac{\varphi^{-1}}{p}(\kappa + z_1(t))\right)^p\right] \quad \text{for } t \in (-\tfrac{1}{2}\log\lambda_*, \infty).
\end{aligned}$$

We shall show that

$$z_{*,1}(t) \to 0 \quad \text{as } t \to \infty. \tag{3.29}$$

Let us admit (3.29) for a moment and continue to the proof. We set

$$\eta_*(t) = \varphi^{1/p}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + z_*(t)) - (\varphi(t)+\kappa)^{1/p}.$$

Then, we see that η_* satisfies (2.3). Moreover, it follows that

$$\begin{aligned}
\eta_*(t) &= \varphi^{1/p}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + z_*(t)) - \varphi^{1/p}(t) - \kappa\frac{\varphi^{-A_p}(t)}{p} \\
&\qquad - \frac{1}{2p}\left(\frac{1}{p} - 1\right)(1 + \theta_*\kappa\varphi^{-1}(t))^{1/p-2}(\kappa\varphi^{-1}(t))^2 \\
&= \frac{\varphi^{-A_p}(t)}{p}z_*(t) - \frac{1}{2p}\left(\frac{1}{p} - 1\right)(1 + \theta_*\kappa\varphi^{-1}(t))^{1/p-2}(\kappa\varphi^{-1}(t))^2
\end{aligned} \tag{3.30}$$

for some $\theta_* \in (0,1)$. This, together with (3.29), implies that $\eta_* \in \Sigma$, where the function space Σ is defined by (2.17). From theorem 2.1, there exists a unique

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solution η_∞ to (2.3) in Σ. Therefore, we have $\eta_*(t) = \eta_\infty(t)$. This yields that $\lambda_* = \lambda_{p,\infty}$.

Thus, all we have to do is to prove (3.29). Suppose to the contrary that there exist $\delta > 0$ and $\{t_k\} \subset \mathbb{R}_+$ such that $|z_{*,1}(t_k)| \geqslant \delta$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Then, there exists $k_0 \in \mathbb{N}$ such that $t_{k_0} > T_\varepsilon$. Then, we see that $|z_1(t_{k_0}, \gamma)| \geqslant \frac{1}{2}\delta$ for sufficiently large $\gamma > 0$. We choose $\varepsilon = \frac{1}{4}\delta$. It follows from (3.20) that

$$\left(z_1\left(\tau_\varepsilon + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma, \gamma\right), z_2\left(\tau_\varepsilon + \frac{\gamma^p}{2} + \frac{p-1}{2}\log\gamma, \gamma\right)\right) \in \Gamma_\varepsilon.$$

By lemma 3.5, we see that $(z_1(t,\gamma), z_2(t,\gamma)) \in \Gamma_{2\varepsilon} = \Gamma_{\delta/2}$ for $t \in (T_\varepsilon, \tau_\varepsilon + \frac{1}{2}\gamma^p + \frac{1}{2}(p-1)\log\gamma)$. We can take $\gamma > 0$ sufficiently large that $t_{k_0} \in (T_\varepsilon, \tau_\varepsilon + \frac{1}{2}\gamma^p + \frac{1}{2}(p-1)\log\gamma)$, which is a contradiction. This completes the proof. □

4. Infinitely many regular solutions in the $3 \leqslant d \leqslant 9$ case

In this section, following [8, 14, 15], we shall give a proof of theorem 1.3. More precisely, we count the number of intersections between the singular and regular solutions. Let I be an interval in $\mathbb{R}$. For a function $v(s)$ on I, we define the number of zeros of v by

$$\mathcal{Z}_I[v(\cdot)] = \#\{s \in I \mid v(s) = 0\}.$$

Then the following result is known.

Proposition 4.1. *Let $U(\rho)$ be a solution to (3.3). We define a function V by*

$$V(\rho) = -2\log\rho + \log\frac{2(d-2)}{p}. \tag{4.1}$$

Then, in the $3 \leqslant d \leqslant 9$ case, we have

$$Z_{[0,\infty)}[U(\rho) - V(\rho)] = \infty.$$

See [17] or [15] for a proof of proposition 4.1.

Remark 4.2. We can easily check that the V defined by (4.1) is a singular solution to the equation in (3.3).

We set

$$\hat{W}_p(s) = \varphi^{1/p}(t) + \frac{\varphi^{-A_p}}{p}(\kappa + y_\infty(t)), \tag{4.2}$$

where $t = -\log s$ and

$$y_\infty(t) = p\varphi^{A_p}((\varphi+\kappa)^{1/p} - \varphi^{1/p}) + p\varphi^{A_p}\eta_\infty - \kappa.$$

Here, η_∞ is the solution to (2.3) given by theorem 2.1. Then, it follows from theorem 2.1 that $\lim_{t\to\infty} y_\infty(t) = 0$. Thus, we see that $\hat{W}_p$ is a singular solution to (1.5) with $\lambda = \lambda_{p,\infty}$. Using proposition 4.1, we shall show the following.

Lemma 4.3. *Let $\hat{u}(s,\gamma)$ be a regular solution to (1.5) with $\hat{u}(0) = \gamma$. Then, we have*

$$Z_{I_\gamma}[\hat{u}(\cdot,\gamma) - \hat{W}_p(\cdot)] \to \infty \quad \text{as } \gamma \to \infty, \tag{4.3}$$

where $I_\gamma = [0, \min\{\sqrt{\lambda_{p,\infty}}, \sqrt{\lambda(\gamma)}\})$.

Proof. We set

$$\tilde{u}_*(\rho,\gamma) = -p\gamma^p + p\gamma^{p-1}\hat{W}_p(s), \quad \rho = \sqrt{\gamma^{p-1}\exp(\gamma^p)}s, \tag{4.4}$$

where $\hat{W}_p$ is defined by (4.2). We claim that

$$\tilde{u}_*(\rho,\gamma) \to V(\rho) \text{ in } C^1_{\text{loc}}([0,\infty)) \quad \text{as } \gamma \to \infty. \tag{4.5}$$

It follows from (4.2) and (4.4) that

$$\tilde{u}_*(\rho,\gamma) = -p\gamma^p + p\gamma^{p-1}\hat{W}_p(s) = -p\gamma^p + p\gamma^{p-1}\varphi^{1/p}(t) + \gamma^{p-1}\varphi^{-A_p}(t)(\kappa + y_\infty(t)).$$

We fix $\rho > 0$. Then, it follows that

$$t = -\log s = -\log\rho + \frac{\gamma^p}{2} + \frac{(p-1)\log\gamma}{2} \to \infty \quad \text{as } \gamma \to \infty.$$

This implies that

$$y_\infty(t) \to 0 \quad \text{as } \gamma \to \infty. \tag{4.6}$$

As in (3.14), (3.15), together with (4.6), we similarly obtain

$$\begin{aligned}\tilde{u}_*(\rho,\gamma) &= -p\gamma^p + p\gamma^{p-1}\varphi^{1/p}(t) + \gamma^{p-1}\varphi^{-A_p}(t)(\kappa + y_\infty(t)) \\ &\to -2\log\rho + \log\frac{2(d-2)}{p} = V(\rho) \quad \text{as } \gamma \to \infty.\end{aligned}$$

Therefore, (4.5) holds.

It follows from (3.1) and (4.4) that

$$Z_{I_\gamma}[\hat{u}(s,\gamma) - \hat{W}_p(s)] = Z_{J_\gamma}[\tilde{u}(\rho,\gamma) - \tilde{u}_*(\rho,\gamma)], \tag{4.7}$$

where

$$J_\gamma = [0, \sqrt{\gamma^{p-1}\exp(\gamma^p)\min\{\sqrt{\lambda_{p,\infty}}, \sqrt{\lambda(\gamma)}\}}).$$

Combining lemma 3.1, proposition 4.1 and (4.5), we find that

$$\lim_{\gamma\to\infty} Z_{J_\gamma}[\tilde{u}(\rho,\gamma) - \tilde{u}_*(\rho,\gamma)] = Z_{[0,\infty)}[U(\rho) - V(\rho)] = \infty. \tag{4.8}$$

From (4.7) and (4.8), we obtain the desired result. □

Once we have obtained lemma 4.3, we can prove theorem 1.3 by employing the same argument as in [15, lemma 5]. However, for the reader's convenience, we shall give a proof.

Proof of theorem 1.3. Let $\hat{u}(s,\gamma)$ be a solution to (1.5) with $\hat{u}(0) = \gamma$, and let $\hat{W}_p(s)$ be the singular solution defined by (4.2). We set $\hat{v}(s,\gamma) = \hat{u}(s,\gamma) - \hat{W}_p(s)$. Then, $\hat{v}(s,\gamma)$ satisfies the following ODE:

$$\hat{v}_{ss} + \frac{d-1}{s}\hat{v}_s + \mathrm{e}^{(\hat{v}+W_p)^p} - \mathrm{e}^{W_p^p} = 0, \quad 0 < s < \hat{\lambda}(\gamma),$$

where $\hat{\lambda}(\gamma) = \min\{\sqrt{\lambda_{p,\infty}}, \sqrt{\lambda(\gamma)}\}$. Then, if $\hat{v}(s,\gamma)$ has a zero at s_0, we have

$$\hat{v}(s_0,\gamma) = 0, \qquad \hat{v}_s(s_0,\gamma) \neq 0 \tag{4.9}$$

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from the uniqueness of the solution. Moreover, for each $\gamma > 0$, $\hat{v}(s,\gamma)$ has at most finitely many zeros in $(0,\hat{\lambda}(\gamma))$. Indeed, if it does not, there exist a sequence of $\{s_n\} \subset [0,\hat{\lambda}(\gamma)]$ and $s_* > 0$ such that $\lim_{n\to\infty} s_n = s_*$. Then, we see that $\hat{v}(s_*,\gamma) = \hat{v}_s(s_*,\gamma) = 0$, which is a contradiction. In addition, it follows from (4.9) and the implicit function theorem that each zero depends continuously on γ. Therefore, we find that the number of zeros of $\hat{v}(s,\gamma)$ does not change unless another zero enters from the boundary of the interval $[0,\hat{\lambda}(\gamma)]$. We note that $\hat{v}(0,\gamma) = \hat{u}(0,\gamma) - \hat{W}_p(0) = -\infty$. From this, we find that the zero of $\hat{v}(s,\gamma)$ enters the interval $[0,\hat{\lambda}(\gamma)]$ from $s = \hat{\lambda}(\gamma)$ only.

In order to prove theorem 1.3, it is enough to show that the function $\lambda(\gamma)$ oscillates infinitely many times around $\lambda_{p,\infty}$ as $\gamma \to \infty$. Suppose that there exists $\gamma_0 > 0$ such that $\lambda(\gamma) > \lambda_{p,\infty}$ for all $\gamma > \gamma_0$. Then, we have $\hat{\lambda}(\gamma) = \sqrt{\lambda_{p,\infty}}$ for all $\gamma > \gamma_0$ and we see that

$$\hat{v}(\sqrt{\lambda_{p,\infty}}) = \hat{u}(\sqrt{\lambda_{p,\infty}},\gamma) - W_p(\sqrt{\lambda_{p,\infty}}) = \hat{u}(\sqrt{\lambda_{p,\infty}},\gamma) > 0.$$

This implies that the number of zeros cannot increase. This contradicts (4.3). Next, suppose that there exists $\gamma_1 > 0$ such that $\lambda(\gamma) < \lambda_{p,\infty}$ for all $\gamma > \gamma_1$. By the same argument as above, we can derive a contradiction. These results imply that the function $\lambda(\gamma)$ oscillates infinitely many times around $\lambda_{p,\infty}$. □

5. Finiteness of the Morse index when $d \geqslant 11$

In this section, we investigate the Morse index of the singular solution in the $d \geqslant 11$ case. It is enough to restrict ourselves to radially symmetric functions. Let $\hat{W}_p$ be the singular solution to (1.5). The following lemma is key to the proof of theorem 1.4.

LEMMA 5.1. *Assume that $d \geqslant 11$ and $p > 0$. Then, there exists $\rho_1 > 0$ such that*

$$p\hat{W}_p^{p-1}(s)\exp(\hat{W}_p^p(s)) < \frac{(d-2)^2}{4s^2} \quad \textit{for } 0 < s < \rho_1. \tag{5.1}$$

Proof. We set $\bar{W}_p(t) = \hat{W}_p(s)$ and $t = -\log s$. By the proof of theorems 1.1 and 1.2, the singular solution $\bar{W}_p(t)$ can be written as

$$\bar{W}_p(t) = \varphi^{1/p}(t) + \frac{\varphi^{-A_p}(t)}{p}(\kappa + y_*(t)),$$

where $\lim_{t\to\infty} y_*(t) = 0$. Then, for any $\varepsilon > 0$, there exists $t_1 = t_1(\varepsilon) > 0$ such that

$$\bar{W}_p^p(t) \leqslant 2t - A_p \log t + \kappa + \varepsilon, \qquad \bar{W}_p^{p-1}(t) \leqslant (2t)^{A_p}(1+\varepsilon) \quad \text{for } t \geqslant t_1.$$

This yields that

$$\begin{aligned} p\bar{W}_p^{p-1}(t)\mathrm{e}^{\bar{W}_p^p(t)} &\leqslant p(2t)^{A_p}(1+\varepsilon)\exp(2t - A_p\log t + \kappa + \varepsilon) \\ &= p2^{A_p}(1+\varepsilon)\mathrm{e}^{2t}\frac{(d-2)2^{1/p}}{p}\mathrm{e}^{\varepsilon} \\ &= 2(d-2)(1+\varepsilon)\mathrm{e}^{\varepsilon}\mathrm{e}^{2t}. \end{aligned} \tag{5.2}$$

We note that $2(d-2) < \frac{1}{4}(d-2)^2$ if $d \geqslant 11$. Therefore, we can take $\varepsilon > 0$ sufficiently small so that

$$p\bar{W}_p^{p-1}(t)\mathrm{e}^{\bar{W}_p^p(t)} < \frac{(d-2)^2}{4}\mathrm{e}^{2t}.$$

Thus, we see that (5.1) holds for $0 < s < \rho_1$ with $\rho_1 = \mathrm{e}^{-t_1}$. □

We are now in a position to prove theorem 1.4.

Proof of theorem 1.4. It is enough to show that the number of negative eigenvalues of the operator L_∞ on $H^1_{0,\mathrm{rad}}(B_{\sqrt{\lambda_*}})$ is finite, where $L_\infty = -\Delta - p\hat{W}_p^{p-1}(s)\mathrm{e}^{\hat{W}_p^p}$. We define smooth functions χ_1 and χ_2 on $[0, \sqrt{\lambda_*})$ by

$$\chi_1(s) = \begin{cases} 1, & 0 \leqslant s < \frac{1}{2}\rho_1, \\ 0, & \rho_1 < s < \sqrt{\lambda_*}, \end{cases} \qquad 0 \leqslant \chi_1(s) \leqslant 1,\ 0 \leqslant s \leqslant \sqrt{\lambda_*},$$

and $\chi_2(s) = 1 - \chi_1(s)$. For each $\hat{\phi} \in H^1_{0,\mathrm{rad}}(B_{\sqrt{\lambda_*}})$, we have

$$\begin{aligned}
\langle L_\infty\hat{\phi}, \hat{\phi}\rangle &= \omega_{d-1}\int_0^{\sqrt{\lambda_*}} \left\{\left|\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}s}\right|^2 - p\hat{W}_p^{p-1}\exp(\hat{W}_p^p(s))|\hat{\phi}|^2\right\}s^{d-1}\,\mathrm{d}s \\
&= \omega_{d-1}\int_0^{\sqrt{\lambda_*}} \left\{\left|\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}s}\right|^2 - p(\chi_1(s)+\chi_2(s))\hat{W}_p^{p-1}\exp(\hat{W}_p^p(s))|\hat{\phi}|^2\right\}s^{d-1}\,\mathrm{d}s \\
&\geqslant \omega_{d-1}\int_0^{\rho_1} \left\{\left|\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}s}\right|^2 - p\hat{W}_p^{p-1}\exp(\hat{W}_p^p(s))|\hat{\phi}|^2\right\}s^{d-1}\,\mathrm{d}s \\
&\quad + \omega_{d-1}\int_0^{\sqrt{\lambda_*}} \left\{\left|\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}s}\right|^2 - p\chi_2(s)\hat{W}_p^{p-1}\exp(\hat{W}_p^p(s))|\hat{\phi}|^2\right\}s^{d-1}\,\mathrm{d}s,
\end{aligned} \tag{5.3}$$

where ω_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$. By (5.1) and the Hardy inequality, we obtain

$$\begin{aligned}
&\int_0^{\rho_1} \left\{\left|\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}s}\right|^2 - p\hat{W}_p^{p-1}\exp(\hat{W}_p^p(s))|\hat{\phi}|^2\right\}s^{d-1}\,\mathrm{d}s \\
&\qquad \geqslant \int_0^{\rho_1} \left\{\left|\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}s}\right|^2 - \frac{(d-2)^2}{4s^2}|\hat{\phi}|^2\right\}s^{d-1}\,\mathrm{d}s \\
&\qquad \geqslant 0.
\end{aligned}$$

This together with (5.3) yields that

$$\langle \hat{L}\hat{\phi}, \hat{\phi}\rangle \geqslant \omega_{d-1}\int_0^{\sqrt{\lambda_*}} \left\{\left|\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}s}\right|^2 - p\chi_2(s)\hat{W}_p^{p-1}\exp(\hat{W}_p^p(s))|\hat{\phi}|^2\right\}s^{d-1}\,\mathrm{d}s. \tag{5.4}$$

We note that the potential $p\chi_2(s)\hat{W}_p^{p-1}\exp(\hat{W}_p^p(s))$ is bounded. Therefore, we find that

$$\inf_{\substack{\phi\in H^1_{0,\mathrm{rad}}(B_{\sqrt{\lambda_*}}),\\ \|\phi\|_{L^2}=1}} \left\{\omega_{d-1}\int_0^{\sqrt{\lambda_*}} \left[\left|\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}s}\right|^2 - p\chi_2(s)\hat{W}_p^{p-1}\exp(\hat{W}_p^p(s))|\hat{\phi}|^2\right]s^{d-1}\,\mathrm{d}s\right\} > -\infty.$$

This, together with (5.4), implies that the number of the negative eigenvalues of the operator L_∞ is finite. This completes the proof. □

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