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MAPS IN LOCALLY ORIENTABLE SURFACES, THE DOUBLE COSET ALGEBRA, AND ZONAL POLYNOMIALS

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ABSTRACT. The *genus* series is the generating series for the number of maps (inequivalent two-cell embeddings of graphs), in locally orientable surfaces, closed and without boundary, with respect to vertex- and face-degrees, number of edges and genus. A *hypermap* is a face two-colourable map. An expression for the genus series for (rooted) hypermaps is derived in terms of zonal polynomials by using a double coset algebra in conjunction with an encoding of a map as a triple of matchings. The expression is analogous to the one obtained for orientable surfaces in terms of Schur functions.

1. Introduction and preliminaries.

1.1. *Introduction.* The purpose of this paper is to show that the genus series for rooted maps (and its extension to hypermaps) in locally orientable surfaces has a concise expression in terms of a classical set of symmetric functions called *zonal polynomials*. Maps have been studied extensively in combinatorial theory (see, for example, Gross and Tucker [5] for an account), and appear in some significant applications (see, for example, [7, 9]). Previously ([10, 11]) the genus series for maps (and its extension to hypermaps) in orientable surfaces has been shown to have a concise expression in terms of another classical set of symmetric functions, the Schur functions.

The definition of a map and its rooting are given in Section 1.3. Section 2 gives the necessary properties of matchings, and their relationship to connection coefficients for the Hecke algebra associated with the hyperoctahedral group (hereinafter called the *double coset algebra*) and zonal polynomials. An encoding for rooted maps in locally orientable surfaces is given in terms of matchings in Section 3, which then leads to an expression for the genus series in terms of the connection coefficients for the double coset algebra. The approach is extended to hypermaps in Section 4. The main result (Corollary 4.4) expresses the genus series for hypermaps in locally orientable surfaces in terms of zonal polynomials. This contains the genus series for maps as a special case. Comments on a joint generalization of the genus series for rooted maps in orientable and locally orientable surfaces are given in Section 5.

Extensive use is made of results on zonal polynomials and the double coset algebra. Pertinent results have been included in Section 2 for completeness, and the reader is referred to [1], [6] and [16] for fuller details. Similarly, extensive use has been made

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of Tutte's axiomatization [17] for maps in locally orientable surfaces. However, it has been necessary to recast some of his results in Section 3 in a form that is adapted to enumerative purposes. The reader is directed to [17] for a fuller discussion of the axiomatization and related results, and to Gross and Tucker [5] for a discussion of the polygonal representation for maps and the embedding theorem.

1.2. *Notation.* Throughout, we have adopted Macdonald's [14] notational conventions for partitions and symmetric functions, and these are summarized below.

A *partition* $\theta = [\theta_1, \theta_2, \ldots]$ is a sequence of nonnegative integers in weakly decreasing order, with a finite number of positive entries, called its *parts*. The set of all partitions, including the empty partition, is $\mathcal{P}$. If the sum of the parts is n, we write $\theta \vdash n$ or $|\theta| = n$ to indicate that θ is a partition of n. The number of parts of θ is denoted by $l(\theta)$. The product of the hook-lengths of θ is denoted by H_θ. If m is a nonnegative integer, then $m\theta$ denotes the partition $[m\theta_1, m\theta_2, \ldots]$ of $m|\theta|$. If λ and μ are partitions, let $\lambda \cup \mu$ denote the partition obtained as the multiset union of the parts of λ and μ. In particular, $\theta \cup \theta$ is the partition $[\theta_1, \theta_1, \theta_2, \theta_2, \ldots]$.

Let $x_1, x_2, \ldots$ be commuting indeterminates and let $\mathbf{x} = (x_1, x_2, \ldots)$. Let $x_\theta = x_{\theta_1} x_{\theta_2} \cdots$, where $x_0 = 1$. The *power sum* symmetric function, *monomial* symmetric function and *Schur* symmetric function in $\mathbf{x}$ and indexed by θ are denoted by $p_\theta(\mathbf{x}), m_\theta(\mathbf{x}), s_\theta(\mathbf{x})$, respectively, where $p(\mathbf{x}) = (p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots)$, and $p_0(\mathbf{x}) = 1$. Let $\mathfrak{S}_n$ denote the symmetric group on n symbols. The identity in $\mathfrak{S}_n$ is denoted by ι. The conjugacy class of $\mathfrak{S}_n$ indexed by $\theta \vdash n$ is denoted by $\mathcal{C}_\theta$. Let $\langle \pi_1, \ldots, \pi_p \rangle$ be the subgroup generated by $\pi_1, \ldots, \pi_p \in \mathfrak{S}_n$, where $p \geq 1$.

Let $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{z} = (z_1, z_2, \ldots)$ be collections of commuting indeterminates. The coefficient of $x_\lambda y_\mu z_\nu$ in a power series f in $\mathbf{x}, \mathbf{y}, \mathbf{z}$ is denoted by $[x_\lambda y_\mu z_\nu] f$, where λ, μ, ν are partitions. We denote the vector $(0, 1, 0, \ldots)$ by $\mathbf{e}_2$.

1.3. *Maps and the genus series.* A *map* is a two-cell embedding of a connected graph in a surface that is assumed throughout to be closed, without boundary, and locally orientable (these are either orientable or nonorientable). The deletion of the edges of the graph separates the surface into regions homeomorphic to open discs, and these are called the *faces* of the map. Each edge has two ends and two sides, and therefore four *side-end positions*, each naturally associated with a side and end of the edge. A *rooted* map is a map with a distinguished side-end position, called the *root* of the map. Two rooted maps are *equivalent* if there is a homeomorphism of the surface that maps vertices to vertices, edges to edges, and preserves the root.

The number of edges incident with a vertex is the *degree* of the vertex (to which loops contribute 2). An edge can occur at most twice in the boundary of a face, and the number of edges, counted with respect to multiplicity, in the boundary of a face is the *degree* of the face. The *vertex-partition* of a map is the list of degrees of its vertices in weakly decreasing order. The *face-partition* is similarly defined.

Figure 1 gives a map in terms of the polygonal representation of the projective plane (a nonorientable surface), with the usual convention that directed lines with the same

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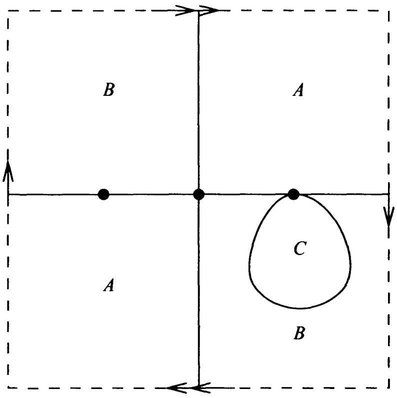

FIGURE 1: A map in the projective plane

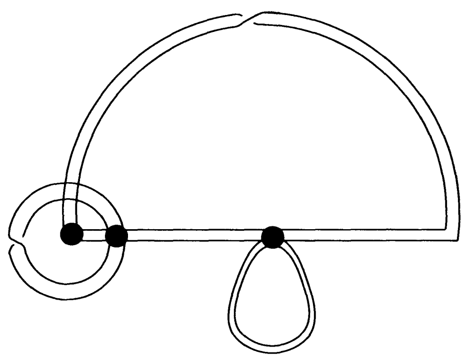

FIGURE 2: A band diagram for a map in the projective plane

label (single headed arrows and double headed arrows) in the polygonal boundary are to be identified. The map has three vertices, five edges and three faces. The latter have been inscribed with A, B, and C to assist in their recognition. We can also specify a map by a *band diagram.* This gives the local incidence structure of as map by displaying edges as bands that can be "twisted" at most once (direction of twist is immaterial). Figure 2 gives a band diagram that corresponds to the map given in Figure 1.

Let $l_{\mu,\nu}^{[2m]}$ be the number of inequivalent rooted maps, in locally orientable surfaces, with m edges, vertex- and face-partitions μ and ν, respectively. Then the generating series for rooted maps in locally orientable surfaces is

$$M(x,\mathbf{y},\mathbf{z}) = \sum_{m\geq 1} x^m \sum_{\mu,\nu\vdash 2m} l_{\mu,\nu}^{[2m]} y_\mu z_\nu.$$

Let $M_O(x,\mathbf{y},\mathbf{z})$ be the corresponding series for maps in orientable surfaces.

These series enable us to enumerate rooted maps in orientable and nonorientable surfaces with respect to an additional statistic, namely genus, since genus is completely determined by the combinatorial information in M and M_O; the *genus* $g(\mathcal{M})$ of a map

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$\mathcal{M}$ is the genus of the the unique surface in which it embeds as a two-cell embedding. By the Euler-Poincaré polyhedral formula, this is equal to $\frac{1}{2}(2 - n_v + n_e - n_f)$, where n_v, n_e, n_f are the numbers of vertices, edges and faces of $\mathcal{M}$. Thus we abuse terminology and refer to M and M_O as *genus series* for brevity.

Although our treatment in this paper of maps in locally orientable surfaces differs substantially from the treatment in [10, 11] of the orientable case, there are some general points at which a parallel may be drawn. The Embedding Theorem [5] can be used to encode a map in an orientable surface as a pair of permutations, called a *rotation system* for the map. The conjugacy classes of these permutations and their product specify the vertex- and face-partitions of the encoded map, and so the centre of the group algebra of the symmetric group can be used to count rotation systems (and therefore maps). It follows from this procedure [10] (Theorem 3.2 and Lemma 3.3) that $M_O(x, \mathbf{y}, \mathbf{z}) = H_O(x\mathbf{e}_2, \mathbf{y}, \mathbf{z}, 1)$, where

$$H_O\big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t\big) = t\frac{\partial}{\partial t}\log\Big\{\sum_{\theta\in\mathcal{P}} H_\theta s_\theta(\mathbf{x})s_\theta(\mathbf{y})s_\theta(\mathbf{z})t^{|\theta|}\Big\} \tag{1}$$

and H_O is the genus series for hypermaps in locally orientable surfaces. This is a representation of the genus series for maps in orientable surfaces in terms of Schur functions. Since $p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots$ are algebraically independent (and similarly for power sums in $\mathbf{y}$ and $\mathbf{z}$), $H_O(\mathbf{x}, \mathbf{y}, \mathbf{z}, t)$ is derivable from $H_O(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t)$.

In seeking an analogous representation for the genus series for maps in locally orientable surfaces, we show in this paper that a map in a locally orientable surface can be encoded as a triple of matchings, called a *corner-system* for the map. The superposition of these matchings in pairs specify the vertex- and face-partitions of the encoded map, and the appropriate algebra for counting corner-systems, and consequently maps, is the double coset algebra. This leads to a representation for the genus series for maps in locally orientable surfaces in terms of zonal polynomials. Again, a power sum basis in the three sets of variables is used.

2. Matchings, the double coset algebra and zonal polynomials.

2.1. *Matchings.* Let $\mathcal{F}_S$ be the set of all (perfect) matchings on a set $\mathcal{S}$ of even cardinality. If $F_1, \ldots, F_p \in \mathcal{F}_S$, let $\mathcal{G}(F_1, \ldots, F_p)$ be the multigraph with vertex-set $\mathcal{S}$ whose edges are formed by the pairs in $F_1, \ldots, F_p$. The components of $\mathcal{G}(F_1, F_2)$ are even cycles. Let the list of their lengths in weakly decreasing order be $(2\theta_1, 2\theta_2, \ldots) = 2\theta$, and define Λ by $\Lambda(F_1, F_2) = \theta$. Let $\mathcal{F}_n$ denote the set of all matchings on $\mathcal{N}_{2n} = \{1, \ldots, 2n\}$.

Let $d^\lambda_{\mu,\nu}$ be the number of $(f_1, f_2, f_3) \in \mathcal{F}_n^3$ such that $\Lambda(f_1, f_2) = \lambda$, $\Lambda(f_1, f_3) = \mu$, $\Lambda(f_2, f_3) = \nu$, and let $c^\lambda_{\mu,\nu}$ be the number of these triples for which $\mathcal{G}(f_1, f_2, f_3)$ is connected. We introduce the generating series

$$D(\mathbf{x}, \mathbf{y}, \mathbf{z}, t) = \sum_{n\geq 1}\frac{t^n}{(2n)!}\sum_{\lambda,\mu,\nu\vdash n} d^\lambda_{\mu,\nu} x_\lambda y_\mu z_\nu$$

and

$$C(\mathbf{x}, \mathbf{y}, \mathbf{z}, t) = \sum_{n\geq 1}\frac{t^n}{(2n)!}\sum_{\lambda,\mu,\nu\vdash n} c^\lambda_{\mu,\nu} x_\lambda y_\mu z_\nu.$$

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2.2. *The double coset algebra.* The series $C(\mathbf{x},\mathbf{y},\mathbf{z},t)$ can be expressed in terms of the connection coefficients for the algebra of double cosets of the hyperoctahedral group embedded in the symmetric group. A self-contained treatment of this algebra has been given by Hanlon, Stanley and Stembridge [6]. The hyperoctahedral group $\mathcal{B}_n$ is the subgroup of the symmetric group $\mathfrak{S}_{2n}$ that fixes any particular matching. Thus $|\mathcal{B}_n| = 2^n n!$. The double cosets of $\mathfrak{S}_{2n}$ with $\mathcal{B}_n$ have partitions of n as a natural index. Let $\mathcal{K}_\lambda$ be the double coset indexed by $\lambda \vdash n$. The $\mathcal{K}_\lambda$ can be determined, for example, using the chain-decomposition of [2].

LEMMA 2.1. *Let* $\lambda \vdash n$. *Then* $|\mathcal{K}_\lambda| = |\mathcal{B}_n|\,|\mathcal{C}_\lambda| 2^{n-l(\lambda)}$.

If $\mathsf{K}_\lambda \in \mathbb{C}\mathfrak{S}_{2n}$ is the formal sum of all of the elements in $\mathcal{K}_\lambda$, then $\{\mathsf{K}_\theta : \theta \vdash n\}$ is a basis of a commutative subalgebra of $\mathbb{C}\mathfrak{S}_{2n}$, namely the double coset algebra (it is identified as the Hecke algebra of the Gel'fand pair $(\mathfrak{S}_{2n}, \mathcal{B}_n)$). Let $[\mathsf{K}_\lambda]\mathsf{K}_\mu\mathsf{K}_\nu$ denote the coefficient of K_λ in the expansion of $\mathsf{K}_\mu\mathsf{K}_\nu$ with respect to this basis, and let $b^\lambda_{\mu,\nu}$ denote this connection coefficient. The connection coefficients are related to the size of sets of matchings in a straightforward way.

LEMMA 2.2 (LEMMA 3.2, [6]). *Let* f_1, f_2 *be fixed matchings in* $\mathcal{F}_n$ *such that* $\Lambda(f_1,f_2) = \lambda$ *where* $\lambda \vdash n$. *Then, for* $\mu,\nu \vdash n$,

$$b^\lambda_{\mu,\nu} = 2^n n!\,\left|\{f_3 \in \mathcal{F}_n : \Lambda(f_1,f_3) = \mu, \Lambda(f_2,f_3) = \nu\}\right|.$$

The generating series $C(\mathbf{x},\mathbf{y},\mathbf{z},t)$ can now be determined in terms of the connection coefficients using Lemma 2.2 and an exponential generating series argument. This series will be of direct use in finding the genus series for maps in Section 3 (and the extension to hypermaps in Section 4).

COROLLARY 2.3.

$$C(\mathbf{x},\mathbf{y},\mathbf{z},t) = \log\left\{1 + \sum_{n\geq 1}\frac{t^n}{n!}\sum_{\lambda,\mu,\nu\vdash n}\frac{|\mathcal{C}_\lambda|}{2^{l(\lambda)}}\frac{b^\lambda_{\mu,\nu}}{n!\,2^n}x_\lambda y_\mu z_\nu\right\}.$$

PROOF. For $(f_1,f_2,f_3) \in \mathcal{F}_n^3$, $\mathcal{G}(f_1,f_2,f_3)$ consists of p components for some $p \geq 1$. Let the vertex-sets of these components be $\mathcal{V}^{(i)}$, $i = 1,\ldots,p$, so $\bigcup_{i=1}^p \mathcal{V}^{(i)} = \mathcal{N}_{2n}$, and let the edges be $f_1^{(i)}, f_2^{(i)}, f_3^{(i)}$, so $\mathcal{G}(f_1^{(i)}, f_2^{(i)}, f_3^{(i)})$ is a connected graph on the vertex-set $\mathcal{V}^{(i)}$. Note that $\bigcup_{i=1}^p \Lambda(f_1^{(i)},f_2^{(i)}) = \Lambda(f_1,f_2)$, $\bigcup_{i=1}^p \Lambda(f_1^{(i)},f_3^{(i)}) = \Lambda(f_1,f_3)$ and $\bigcup_{i=1}^p \Lambda(f_2^{(i)},f_3^{(i)}) = \Lambda(f_2,f_3)$. Thus, by the exponential formula for exponential generating series of labelled graphs (see, for example, Chapter 3 [4]) we have $1 + D(\mathbf{x},\mathbf{y},\mathbf{z},t) = \exp C(\mathbf{x},\mathbf{y},\mathbf{z},t)$. But Lemma 2.2 can be adapted in a straightforward way to give

$$d^\lambda_{\mu,\nu} = \frac{(2n)!\,|\mathcal{C}_\lambda|}{(n!)^2 2^{n+l(\lambda)}} b^\lambda_{\mu,\nu},$$

and the result follows. ∎

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2.3. *Zonal polynomials.* The generating series for the connection coefficients $b^{\lambda}_{\mu,\nu}$ with respect to the power sums can be expressed compactly in terms of zonal polynomials. We begin with a description of a set of orthogonal idempotents E_λ such that $\{\mathsf{E}_\theta : \theta \vdash n\}$ is a basis for the double coset algebra. These can be used to determine the connection coefficient $b^{\lambda}_{\mu,\nu}$. Let $\chi^{2\lambda}(\sigma)$ be the character of the ordinary irreducible representation of $\mathfrak{S}_{2n}$ indexed by $\lambda \vdash n$, evaluated at $\sigma \in \mathfrak{S}_{2n}$, and let $\phi^{\lambda}(\mu) = \sum_{\sigma\in\mathcal{K}_\mu} \chi^{2\lambda}(\sigma)$. Then [6]

$$\mathsf{E}_\mu = \frac{1}{H_{2\mu}} \sum_{\nu\vdash n} \frac{1}{|\mathcal{K}_\nu|} \phi^{\mu}(\nu) \mathsf{K}_\nu, \tag{2}$$

for $\mu \vdash n$ have the property that $\mathsf{E}_\mu \mathsf{E}_\nu = \delta_{\mu,\nu}\mathsf{E}_\mu$ and, moreover,

$$\mathsf{K}_\mu = \sum_{\nu\vdash n} \phi^{\nu}(\mu)\mathsf{E}_\nu. \tag{3}$$

It follows directly from (2) and (3) that

$$b^{\lambda}_{\mu,\nu} = \frac{1}{|\mathcal{K}_\lambda|} \sum_{\theta\vdash n} \frac{1}{H_{2\theta}} \phi^{\theta}(\mu)\phi^{\theta}(\nu)\phi^{\theta}(\lambda). \tag{4}$$

For an indeterminate α, let the inner product $\langle \cdot, \cdot \rangle_\alpha$ be defined on the ring of symmetric functions in $x_1, x_2, \ldots$ by $\langle p_\lambda, p_\mu \rangle_\alpha = (\alpha^{l(\lambda)} |C_\lambda| / |\lambda|!) \delta_{\lambda,\mu}$. Then the zonal polynomials $Z_\lambda(\mathbf{x})$, where $\lambda \vdash n$, are defined uniquely [8], with $\alpha = 2$, by the orthogonality condition $\langle Z_\lambda, Z_\mu \rangle_2 = 0$ for $\lambda \neq \mu$, the triangularity condition $[m_\mu]Z_\lambda = 0$ if $\lambda \prec \mu$, and the normalization condition $[m_{[1^n]}]Z_\lambda = n!$ where $\lambda \vdash n$, in which $\prec$ denotes reverse lexicographic order. It can be shown [12] that

$$Z_\lambda = \frac{1}{|\mathcal{B}_n|} \sum_{\mu\vdash n} \phi^{\lambda}(\mu) p_\mu, \qquad p_\mu = \frac{1}{|\mathcal{K}_\mu|} \sum_{\nu\vdash n} \frac{|\mathcal{B}_n|}{H_{2\nu}} \phi^{\nu}(\mu) Z_\nu. \tag{5}$$

These relationships allow us to determine a generating series for the connection coefficients with respect to the power sums in the three sets of variables.

LEMMA 2.4.

$$\sum_{\lambda,\mu,\nu\vdash n} \frac{|C_\lambda|}{(n!)^2 2^{n+l(\lambda)}} b^{\lambda}_{\mu,\nu} p_\lambda(\mathbf{x}) p_\mu(\mathbf{y}) p_\nu(\mathbf{z}) = \sum_{\theta\vdash n} \frac{1}{H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) Z_\theta(\mathbf{z}).$$

PROOF. From (4)

$$\begin{aligned}\sum_{\lambda,\mu,\nu\vdash n} |\mathcal{K}_\lambda| b^{\lambda}_{\mu,\nu} p_\lambda(\mathbf{x}) p_\mu(\mathbf{y}) p_\nu(\mathbf{z}) &= \sum_{\theta\vdash n} \frac{1}{H_{2\theta}} \sum_{\lambda\vdash n} \phi^{\theta}(\lambda) p_\lambda(\mathbf{x}) \sum_{\mu\vdash n} \phi^{\theta}(\mu) p_\mu(\mathbf{y}) \sum_{\nu\vdash n} \phi^{\theta}(\nu) p_\nu(\mathbf{z}) \\ &= |\mathcal{B}_n|^3 \sum_{\theta\vdash n} \frac{1}{H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) Z_\theta(\mathbf{z})\end{aligned}$$

from (5), and the result follows from Lemma 2.1. ∎

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3. Corner-systems and the genus series for maps. To determine the genus series for maps in locally orientable surfaces, it is necessary, in the argument that is used here, to construct each rooted map a predictable number of times. This motivates the transition made in this section from premaps (defined in Section 3.1) that are used in the axiomatization of maps, to corner-systems (defined in Section 3.2) that facilitate the application of the double coset algebra.

3.1. *Premaps and Tutte's permutation axiomatization for maps.* There are several axiomatizations for maps on locally orientable surfaces [5, 13, 17, 18, 19]. In this paper we use Tutte's permutation axiomatization [17] in terms of premaps, given by the following definition.

DEFINITION 3.1. A *premap* is an ordered triple (σ, ρ, τ) of permutations acting on a set $\mathcal{S}$ of $4m$ elements, such that

1. $\rho^2 = \sigma^2 = \iota$, and $\rho\sigma = \sigma\rho$,
2. for any $a \in \mathcal{S}$, then $a, \sigma a, \rho a, \sigma\rho a$ are distinct,
3. $\tau\sigma = \sigma\tau^{-1}$,
4. for each $a \in \mathcal{S}$, the orbits of τ through a and σa are distinct.

Let premap($\mathcal{S}$) denote the set of all premaps on $\mathcal{S}$. For a premap (σ, ρ, τ) on $\mathcal{S}$, the maximal subsets of $\mathcal{S}$ on which the group $\langle\sigma, \rho\rangle$ acts transitively form a partition of $\mathcal{S}$ into subsets $\mathcal{E}^{(1)}, \dots, \mathcal{E}^{(m)}$ of four elements. The maximal subsets of $\mathcal{S}$ on which the group $\langle\sigma, \rho, \tau\rangle$ acts transitively form a partition of $\mathcal{S}$ into subsets $\mathcal{S}^{(1)}, \dots, \mathcal{S}^{(p)}$, for some $p \geq 1$. Moreover, each $\mathcal{S}^{(i)}$ is a union of some of the $\mathcal{E}^{(1)}, \dots, \mathcal{E}^{(m)}$ so its size is divisible by four. Let σ_i, ρ_i, τ_i be the restrictions of σ, ρ, τ to $\mathcal{S}^{(i)}$ for $i = 1, \dots, p$. Then $(\sigma_i, \rho_i, \tau_i)$ is a premap on $\mathcal{S}^{(i)}$, and we call it a *component* of the premap (σ, ρ, τ). When $p = 1$, we call the premap a *connected premap* (Tutte calls it a map). Thus premap($\mathcal{S}$) is constructed from connected premaps by taking connected premaps $(\sigma_i, \rho_i, \tau_i)$ on $\mathcal{S}^{(i)}$ and letting $\sigma = \sigma_1 \cdots \sigma_p$, $\rho = \rho_1 \cdots \rho_p$, and $\tau = \tau_1 \cdots \tau_p$ as permutations on $\mathcal{S}$.

Premaps provide an axiomatization for maps, as stated in the following result of Tutte [17].

THEOREM 3.2. *Each map corresponds to a connected premap, and each connected premap corresponds to a unique map.*

The next result gives the details of the above correspondence between maps and premaps, and follows from the commentary given by Tutte (Chapter X, [17]). At certain points we have departed from his terminology.

THEOREM 3.3. *Let $\mathcal{M}$ be a map on a locally orientable surface and let the symbols of $\mathcal{S}$ be assigned bijectively among the side-end positions of $\mathcal{M}$. Let σ be the permutation that interchanges symbols at the same end but on different sides of an edge, for each edge. Let ρ be the permutation that interchanges symbols on the same side but at different ends of an edge, for each edge.*

1. Vertices: Let v be a vertex of $\mathcal{M}$ and let $(a_1, a_2, \dots, a_{2k})$ be the list of symbols encountered in a tour of the side-end positions incident with v starting at an arbitrary

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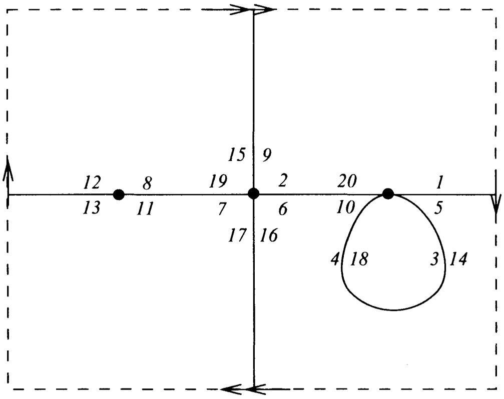

FIGURE 3: An $\mathcal{N}_{20}$-labelled map

symbol a_1, in the unique (local) direction such that $a_2 = \sigma a_1$. Then τ is the permutation whose disjoint cycles are associated in pairs with each vertex v, and have the form $(a_{2k}a_{2k-2}\cdots a_2)$ and $(a_1a_3\cdots a_{2k-1})$. The degree of v is k.

2. Edges: The elements of $\mathcal{E}^{(i)}$ are the symbols assigned to the four side-end positions of the same edge. The number of edges of $\mathcal{M}$ is m.

3. Faces: Let f be a face of $\mathcal{M}$ and let $(b_1, b_2, \ldots, b_{2j})$ be the list of symbols encountered in a tour of the side-end positions of f, starting from an arbitrary symbol b_1, in the unique (local) direction such that $b_2 = \rho b_1$. Then the disjoint cycles of $\tau\rho\sigma$ are associated in pairs with each vertex f, and have the form $(b_{2j}b_{2j-2}\cdots b_2)$ and $(b_1b_3\cdots b_{2j-1})$. The degree of f is j.

Then (σ, ρ, τ) is a connected premap corresponding to $\mathcal{M}$, and the distinct (σ, ρ, τ) that arise in this way are precisely the connected premaps on $\mathcal{S}$.

An assignment of symbols in $\mathcal{S}$ to the side-end positions of a map $\mathcal{M}$, described in the above result, is called an *$\mathcal{S}$-labelling* of $\mathcal{M}$. Of course, the components of an arbitrary premap are obtained from an $\mathcal{S}^{(i)}$-labelling of a map as above for $i = 1, \ldots, p$.

EXAMPLE 3.4. Figure 3 gives an $\mathcal{N}_{20}$-labelling of the map displayed in Figure 1. In the corresponding connected premap on $\mathcal{N}_{20}$, we have

$$\sigma = (1\,5)(2\,6)(3\,14)(4\,18)(7\,19)(8\,11)(9\,15)(10\,20)(12\,13)(16\,17),$$
$$\rho = (1\,13)(2\,20)(3\,18)(4\,14)(5\,12)(6\,10)(7\,11)(8\,19)(9\,17)(15\,16),$$
$$\tau = (1\,14\,18\,10)(20\,4\,3\,5)(2\,16\,7\,15)(9\,19\,17\,6)(8\,13)(12\,11).$$

Note that $\tau \in \mathcal{C}_{[4^4\,2^2]}$ and $[4^4\,2^2] = [4^2\,2] \cup [4^2\,2]$, where $[4^2\,2]$ is the vertex-partition of the map. Also, if we compute $\tau\rho\sigma$, we get

$$\tau\rho\sigma = (5\,8\,15\,6\,4)(14\,10\,16\,19\,12)(1\,11\,17\,2)(20\,9\,7\,13)(3)(18).$$

Thus the cycles of $\tau\rho\sigma$ are indeed associated in pairs with the faces of the map, as described in (3) of Theorem 3.3; the first two cycles describe the same face B, but

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in opposite directions. The same holds for the third and fourth cycles, describing face A, and the fifth and sixth cycles describing face C. Note that $\tau\rho\sigma \in \mathcal{C}_{[5^2\,4^2\,1^2]}$ and $[5^2\,4^2\,1^2] = [5\,4\,1] \cup [5\,4\,1]$, where $[5\,4\,1]$ is the face-partition of the map. ■

To reverse this construction and determine the $\mathcal{S}$-labelled map corresponding to a connected premap (σ, ρ, τ), first use σ and τ to find the local incidence structure between vertices and half-edges. Then use ρ to join the half-edges, giving a band diagram of the $\mathcal{S}$-labelled map.

It is not easy in general to determine how many distinct connected premaps correspond to a given map because of automorphisms of the map. However, if we consider rooted maps, then the number of corresponding premaps becomes a predictable constant, and we are therefore able to give the following preliminary expression for the number of rooted maps, with given vertex- and face-partitions, in terms of premaps.

COROLLARY 3.5. *For* $\mu, \nu \vdash 2m$,

$$l_{\mu,\nu}^{[2^m]} = \frac{1}{(4m-1)!}\left|\{(\sigma,\rho,\tau) \in \mathsf{premap}(\mathcal{S}) : \tau \in \mathcal{C}_{\mu\cup\mu}, \tau\rho\sigma \in \mathcal{C}_{\nu\cup\nu}, (\sigma,\rho,\tau) \text{ connected}\}\right|.$$

PROOF. Each connected premap (σ, ρ, τ) with $\tau \in \mathcal{C}_{\mu\cup\mu}$ and $\tau\rho\sigma \in \mathcal{C}_{\nu\cup\nu}$ corresponds to a map with vertex-partition μ and face-partition ν, from Theorems 3.2 and 3.3. But the only automorphism that fixes the root side-end position of a rooted map is the identity. Thus all connected premaps are obtained from rooted maps which are $\mathcal{S}$-labelled with the restriction that a canonically chosen element of $\mathcal{S}$ is always assigned to the root side-end position. Moreover, all such premaps are distinct, and there are $(4m-1)!$ such $\mathcal{S}$-labellings of each rooted map on m edges, so the result follows. ■

We have been unable to exploit this expression directly to enumerate rooted maps because the cycle-pair restriction on τ proves awkward algebraically. This difficulty is avoided in the next section.

3.2. *Corner-systems for maps.* We now develop an axiomatization for maps in terms of matchings instead of permutations. For a related approach see, for example, Jones and Thornton [13].

DEFINITION 3.6. A *corner-system* is an ordered triple (F_e, F_s, F_c) of matchings on a set $\mathcal{S}$ with $4m$ elements such that $\Lambda(F_e, F_s) = [2^m]$. A corner-system (F_e, F_s, F_c) is said to be *connected* if $\mathcal{G}(F_e, F_s, F_c)$ is connected.

Let cs(S) denote the set of all corner-systems on $\mathcal{S}$. In the next result we prove that corner-systems are entirely equivalent to premaps, and in this sense we have an alternative axiomatization to Tutte's. The fact that the restriction on the triple of matchings is simple will enable us to enumerate maps straightforwardly by means of corner-systems.

For $\pi \in \mathcal{C}_{[2^n]}$, we define $\xi\pi$ to be the matching whose pairs are the elements that are interchanged in π. Clearly, ξ defines a bijection between $\mathcal{C}_{[2^n]}$ and $\mathcal{F}_n$.

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LEMMA 3.7.

$$\mathsf{premap}(\mathcal{S}) \xrightarrow{\sim} \mathsf{cs(S)}\colon (\sigma,\rho,\tau) \longmapsto (F_e,F_s,F_c)$$

where $F_e = \xi\sigma$, $F_s = \xi\rho$, $F_c = \xi(\tau\sigma)$. *Moreover, if* (σ,ρ,τ) *is connected and corresponds as in Theorem 3.3 to an* $\mathcal{S}$*-labelled map* $\mathcal{M}$ *with vertex-partition* μ *and face-partition* ν, *then*

1. (a) *the pairs in* F_e *are the symbols at the same end but on opposite sides of an edge,*
   (b) *the pairs in* F_s *are the symbols at the same side but at opposite ends of an edge,*
   (c) *the pairs in* F_c *are the union of* $\{\{a_2,a_3\},\{a_4,a_5\},\dots,\{a_{2k},a_1\}\}$ *over all vertices* v *of* $\mathcal{M}$ *(and are thus the symbols in the corners of the faces).*

Also

2. (a) (F_e,F_s,F_c) *is connected,*
   (b) $\Lambda(F_e,F_c)=\mu$,
   (c) $\Lambda(F_s,F_c)=\nu$.

PROOF. Consider $(\sigma,\rho,\tau)\in \mathsf{premap}(\mathcal{S})$. Parts (1) and (2) of Definition 3.1 imply that $\sigma,\rho\in\mathcal{C}_{[2^{2m}]}$ with $\Lambda(\xi\sigma,\xi\rho)=[2^m]$. Moreover, from part (3) of Definition 3.1, $\tau\sigma=\sigma\tau^{-1}$. Thus

$$(\tau\sigma)^2=\sigma^2=\iota \tag{6}$$

from part (1) of Definition 3.1, so $\tau\sigma$ is an involution. But, for each $a\in\mathcal{S}$, $\tau\sigma a$ is on the orbit of τ through σa and is therefore distinct from a, by part (4) of Definition 3.1, so $\tau\sigma$ has no fixed points. Thus $\tau\sigma\in\mathcal{C}_{[2^{2m}]}$.

We now show that, for each σ and ρ, all elements π of $\mathcal{C}_{[2^{2m}]}$ can be realized in this way. With π and σ given, let $\tau=\pi\sigma$, so $\tau\sigma=\pi$. Then, from (6), τ satisfies part (3) of Definition 3.1. Now consider the graph $\mathcal{G}(\xi\pi,\xi\sigma)$ and $a\in\mathcal{S}$. The orbit of $\tau=\pi\sigma$ through a consists of the vertices that are an even distance from a on the (even-length) cycle containing a in the graph. The orbit of $\tau\sigma=\pi$ through a consists of the vertices that are an odd distance from a on the cycle. Thus τ satisfies part (4) of Definition 3.1 and $(\sigma,\rho,\tau)\longmapsto(\sigma,\rho,\pi)$ defines a bijection between $\mathsf{premap}(\mathcal{S})$ and triples of permutations in $\mathcal{C}_{[2^{2m}]}$ with $\Lambda(\xi\sigma,\xi\rho)=[2^m]$. The bijection between premaps and corner-systems follows from the natural bijection ξ between $\mathcal{C}_{[2^{2m}]}$ and matchings on a set of size $4m$, with matchings given by $F_e=\xi\sigma$, $F_s=\xi\rho$, $F_c=\xi(\tau\sigma)$.

The remaining statements in the result give the relationship between corner-systems and maps and are obtained straightforwardly from Theorem 3.3 and the above bijection. ∎

Of course, an arbitrary corner-system is obtained by $\mathcal{S}^{(i)}$-labelling a map for $i=1,\dots,p$ and taking the disjoint union of the p F_e's, F_s's and F_c's to get F_e,F_s,F_c respectively. In this case, $\mathcal{G}(F_e,F_s,F_c)$ has p components, with vertex sets $\mathcal{S}^{(i)}$, for $i=1,\dots,p$.

EXAMPLE 3.8. The connected corner-system on $\mathcal{N}_{20}$ corresponding to the $\mathcal{N}_{20}$-labelled map in Figure 3 has

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$$F_e = \{\{1,5\},\{2,6\},\{3,14\},\{4,18\},\{7,19\},\{8,11\},\{9,15\},\{10,20\},\{12,13\},\{16,17\}\},$$
$$F_s = \{\{1,13\},\{2,20\},\{3,18\},\{4,14\},\{5,12\},\{6,10\},\{7,11\},\{8,19\},\{9,17\},\{15,16\}\},$$
$$F_c = \{\{1,20\},\{2,9\},\{3,18\},\{4,10\},\{5,14\},\{6,16\},\{7,17\},\{8,12\},\{11,13\},\{15,19\}\}.$$

It is straightforward to verify that $\mathcal{G}(F_e, F_s, F_c)$ is connected, and that $\Lambda(F_e, F_s) = [2^5]$, $\Lambda(F_e, F_c) = [4^2\,2]$, and $\Lambda(F_s, F_c) = [5\,4\,1]$, where $[4^2\,2]$ is the vertex-partition of the map and $[5\,4\,1]$ is the face-partition of the map. ∎

We can immediately translate the expression for the number of rooted maps given in Corollary 3.5 to an equivalent expression involving matchings.

COROLLARY 3.9. *For* $\mu, \nu \vdash 2m$,

$$l_{\mu,\nu}^{[2^m]} = \frac{1}{(4m-1)!} c_{\mu,\nu}^{[2^m]}.$$

PROOF. From Corollaries 3.5 and 3.7 we have $l_{\mu,\nu}^{[2^m]} = w/(4m-1)!$ where w is the number of $(F_e, F_s, F_c) \in \mathcal{F}_{\mathcal{S}}^3$ such that $\Lambda(F_e, F_s) = [2^m]$, $\Lambda(F_e, F_c) = \mu$, $\Lambda(F_s, F_c) = \nu$ and $\mathcal{G}(F_e, F_s, F_c)$ is connected. The result follows immediately since $\mathcal{S}$ is a set of size $4m$. ∎

We are now able to determine the genus series for rooted maps in locally orientable surfaces in terms of connection coefficients for the double coset algebra.

THEOREM 3.10.

$$M(x, \mathbf{y}, \mathbf{z}) = 4x\frac{\partial}{\partial x} \log\left\{1 + \sum_{m\geq 1} \frac{x^m}{4^m m!} \sum_{\mu,\nu\vdash 2m} \frac{b_{\mu,\nu}^{[2^m]}}{2^{2m}(2m)!} y_\mu z_\nu\right\}.$$

PROOF. From Corollary 3.9 we obtain

$$M(x, \mathbf{y}, \mathbf{z}) = 2\frac{\partial}{\partial t} C(x\mathbf{e}_2, \mathbf{y}, \mathbf{z}, t)|_{t=1}$$

and the result follows directly from Corollary 2.3 since $|\mathcal{C}_{[2^m]}| = (2m)!/2^m m!$. ∎

4. **Extension to hypermaps.** The form of the genus series given in Theorem 3.10 can be generalized to one with greater symmetry by considering maps, each of whose faces can be coloured with one of two colours so that edges separate only faces of different colours. Such face two-colourable maps are called *hypermaps* in a locally orientable surface, when the following identifications have been made. The *hyperedges*

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of a hypermap are the faces of one specified colour in the face two-colourable map, and the *hyperedge-partition* specifies the degrees of these faces. The *faces* of the hypermap are the faces of the other colour in the face two-colourable map, and their degrees give the *face-partition* of the hypermap. The vertices of the hypermap are the vertices of the face two-colourable map, but the degree of a vertex in the hypermap is the number of incident hyperedges, and is therefore exactly half the degree of the vertex in the face two-colourable map. The *vertex-partition* of the hypermap specifies the degrees of the vertices in the hypermap.

Maps are recoverable from hypermaps as follows. Each map corresponds to a unique hypermap obtained by duplicating each edge of the map to form a digon. This map is clearly face two-colourable, with each digon constructed in this way being of one colour class, and the remaining faces being in the other colour class. Conversely, a hypermap such that each face of one colour class is a digon corresponds to a unique map obtained by identifying the bounding edges of each digon.

A *rooted hypermap* is a face two-colourable map with a distinguished side-end position (called the *root*). The rooting specifies which colour class of faces of the face two-colourable map gives the hyperedges and which gives the faces; our convention is that the colour class containing the root side-end position gives the faces of the hypermap.

Let $l^{\lambda}_{\mu,\nu}$ be the number of rooted hypermaps, with hyperedge-partition λ, vertex-partition μ and face-partition ν, in which $\lambda, \mu, \nu \vdash m$ and $m \geq 1$ is the number of edges in the underlying face two-colourable map. Then the genus series for hypermaps in locally orientable surfaces is

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}, t) = \sum_{m\geq 1} t^m \sum_{\lambda,\mu,\nu\vdash m} l^{\lambda}_{\mu,\nu} x_\lambda y_\mu z_\nu .$$

If H_O is the genus series for orientable surfaces, it follows from Lemma 3.1 of [11] that H_O is given by (1). From the above identification of maps as a special class of hypermaps, we immediately have $M(x, \mathbf{y}, \mathbf{z}) = H(x\mathbf{e}_2, \mathbf{y}, \mathbf{z}, 1)$.

We now count hypermaps by considering the corner-systems of face two-colourable maps. We adopt the convention that R is the colour of the face containing the root and Q is the other colour. The colours of the other faces are then determined. Consider an $\mathcal{S}$-labelling of a rooted face two-colourable map on m edges. Each side-end position lies in a single face and we let the sets of symbols in faces of colour R and Q be $\mathcal{R}$ and $\mathcal{Q}$, respectively. Thus $\mathcal{Q} \cup \mathcal{R} = \mathcal{S}$ with $|\mathcal{Q}| = |\mathcal{R}| = 2m$, and by the convention of Corollary 3.5, $\mathcal{R}$ contains the canonical element of $\mathcal{S}$ specifying the root side-end position.

In the corresponding connected corner-system (F_e, F_s, F_c), each pair in F_e contains one element of $\mathcal{R}$ and one element of $\mathcal{Q}$, but the pairs in F_s and F_c are either both in $\mathcal{Q}$ or both in $\mathcal{R}$, and we say that such a pair is coloured Q or R, respectively. The set of pairs coloured Q and R in F_s are denoted by $Q(F_s)$ and $R(F_s)$, respectively, and $Q(F_c)$ and $R(F_c)$ are defined similarly. We can therefore define the three matchings M_1, M_2, M_3 on $\mathcal{R}$ by

$$M_1 = R(F_s), \quad M_2 = R(F_c), \quad M_3 = f_e\big(Q(F_c)\big),$$

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where the permutation $f_e = \xi^{-1} F_e$ acts on the symbols in the pairs of $Q(F_c)$.

EXAMPLE 4.1. Consider the $\mathcal{N}_{20}$-labelled map in Figure 3, and suppose that the symbol 1 specifies the root side end-position. This map is face two-colourable, with faces A and C coloured R and B coloured Q. Thus, as a hypermap, it has hyperedge-partition [5], face-partition [4 1] and vertex-partition $[2^2\ 1]$. The corner-system for this $\mathcal{N}_{20}$-labelled map is given in Example 3.8, and we obtain $\mathcal{Q} = \{4,5,6,8,10,12,14,15,16,19\}$, and $\mathcal{R} = \{1,2,3,7,9,11,13,17,18,20\}$. Then

$$M_1 = R(F_s) = \big\{\{1,13\},\{2,20\},\{3,18\},\{7,11\},\{9,17\}\big\},$$
$$M_2 = R(F_c) = \big\{\{1,20\},\{2,9\},\{3,18\},\{7,17\},\{11,13\}\big\},$$
$$M_3 = \big\{\{18,20\},\{1,3\},\{2,17\},\{11,13\},\{9,7\}\big\}$$

are all matchings on $\mathcal{R}$. Note that $\Lambda(M_1,M_3) = [5]$, $\Lambda(M_1,M_2) = [4\ 1]$, $\Lambda(M_2,M_3) = [2^2\ 1]$, and that $\mathcal{G}(M_1,M_2,M_3)$ is connected. Note also that

$$Q(F_s) = \big\{\{4,14\},\{5,12\},\{6,10\},\{8,19\},\{15,16\}\big\},$$
$$Q(F_c) = \big\{\{4,10\},\{5,14\},\{6,16\},\{8,12\},\{15,19\}\big\}$$

are the sets of the complementary colour. ∎

In the next result, a generalization of Corollary 3.9, we express the number of rooted maps in terms of matchings by analysing the properties of M_1, M_2, M_3.

THEOREM 4.2. *For* $\lambda,\mu,\nu \vdash m$, $l^{\lambda}_{\mu,\nu} = c^{\lambda}_{\mu,\nu}/(2m-1)!$.

PROOF. We begin with the following general observation. Consider $\mathcal{R}$, $\mathcal{Q}$ and F_e as above. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the edge-sets of any graphs on vertex-sets $\mathcal{R}$ and $\mathcal{Q}$, respectively. Let G be the graph on vertex-set $\mathcal{N}_{4m}$ with edge-set $F_e \cup \mathcal{E}_1 \cup \mathcal{E}_2$ and let H be the graph on vertex-set $\mathcal{R}$ with edge-set $\mathcal{E}_1 \cup f_e\mathcal{E}_2$. Then H is obtained from G by identifying the pairs of vertices joined by an edge of $\mathcal{G}(F_e)$. Thus, in particular, H and G have the same number of components.

Now consider the matchings M_1, M_2, M_3, as above, when the hypermap has hyperedge-partition λ, vertex-partition μ and face-partition ν. In the notation of the previous paragraph, let $\mathcal{E}_1 = R(F_e) \cup R(F_c)$ and $\mathcal{E}_2 = Q(F_e) \cup Q(F_c)$. But $f_e\big(Q(F_s)\big) = R(F_s)$, since each edge of the map has two sides, one of each colour, so

$$\mathcal{E}_1 \cup f_e \mathcal{E}_2 = M_1 \cup M_2 \cup M_3.$$

Thus, in this case, G and H are the multigraphs $\mathcal{G}(F_e,F_s,F_c)$ and $\mathcal{G}(M_1,M_2,M_3)$ with multiple edges identified so we conclude that $\mathcal{G}(M_1,M_2,M_3)$ is connected precisely when (F_e,F_s,F_c) is a connected corner-system.

Now $\Lambda(M_1,M_2) = \nu$ since the cycles in $\mathcal{G}(M_1,M_2)$ correspond to faces coloured R in the map. Moreover, $\mathcal{G}(M_1,M_3) \cong \mathcal{G}\big(f_e(M_1), f_e(M_3)\big) = \mathcal{G}\big(Q(F_s), Q(F_c)\big)$, so we get

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$\Lambda(M_1, M_3) = \lambda$ by considering the faces coloured Q in the map. Finally, $\Lambda(M_2, M_3) = \mu$ since each cycle in $\mathcal{G}(M_2, M_3)$ arises from touring the corners of the faces about a vertex.

Now $\mathcal{R}$, F_s and F_c are determined uniquely by M_1, M_2, M_3, but there are $(2m)!$ choices of F_e since F_e is a bijection between $\mathcal{R}$ and $\mathcal{Q} = \mathcal{N}_{4m} - \mathcal{R}$. Thus we obtain

$$l^{\lambda}_{\mu,\nu} = \frac{1}{(4m-1)!}\binom{4m-1}{2m-1}(2m)!\, c^{\lambda}_{\mu,\nu}$$

where $c^{\lambda}_{\mu,\nu}$ is the number of (M_1, M_2, M_3) for each $\mathcal{R}$, $\binom{4m-1}{2m-1}$ is the number of choices for $\mathcal{R}$, $(2m)!$ is the number of choices for F_e, and the division by $(4m-1)!$ accounts for the number of times each rooted map corresponds to a connected corner-system (F_e, F_s, F_c), as in Corollaries 3.5 and 3.9. The result follows immediately. ∎

This enables us to determine the genus series for rooted hypermaps in locally orientable surfaces in terms of the connection coefficients for the double coset algebra.

THEOREM 4.3.

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}, t) = 2t\frac{\partial}{\partial t}\log\left\{1 + \sum_{n\geq 1}\frac{t^n}{n!}\sum_{\lambda,\mu,\nu\vdash n}\frac{|C_\lambda|}{2^{l(\lambda)}}\frac{b^{\lambda}_{\mu,\nu}}{n!\,2^n}x_\lambda y_\mu z_\nu\right\}.$$

PROOF. From Theorem 4.2 we have $H(\mathbf{x}, \mathbf{y}, \mathbf{z}, t) = 2t\frac{\partial}{\partial t}C(\mathbf{x}, \mathbf{y}, \mathbf{z}, t)$, and the result follows from Corollary 2.3. ∎

Note that this specializes to $M(x, \mathbf{y}, \mathbf{z})$, given in Theorem 3.10, by setting $\mathbf{x} = x\mathbf{e}_2$ and $t = 1$.

We are now in a position to state the main result, that expresses the genus series for rooted hypermaps in terms of zonal polynomials. The proof is immediate from Theorem 4.3 and Lemma 2.4.

COROLLARY 4.4.

$$H\big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t\big) = 2t\frac{\partial}{\partial t}\log\left\{\sum_{\theta\in\mathcal{P}}\frac{1}{H_{2\theta}}Z_\theta(\mathbf{x})Z_\theta(\mathbf{y})Z_\theta(\mathbf{z})t^{|\theta|}\right\}.$$

5. **Concluding comments.** It has been observed in Section 2.3 that the orthogonality, triangularity and normalization conditions uniquely define the zonal polynomials when $\alpha = 2$. When $\alpha = 1$, these conditions define the Schur functions. Moreover, with general α, these conditions define [8] the Jack symmetric functions. Thus, the striking similarity of (1) and Corollary 4.4, giving the genus series for hypermaps in orientable and locally orientable surfaces respectively, suggests that they have a common generalization in terms of Jack symmetric functions. This is examined elsewhere [3].

The SF (symmetric function) package [15] has been invaluable for computing zonal polynomials and thence the first few terms of $H(\mathbf{x}, \mathbf{y}, \mathbf{z}, t)$. Zonal polynomials can be computed from the three conditions of normalization, orthogonality and triangularity with respect to the monomial symmetric function basis. This is straightforward but laborious, so we have given in Table 1 the zonal polynomials indexed by partitions of at

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θ	$Z_\theta(p_1, p_2, \ldots)$
[1]	p_1
[2]	$p_1^2 + 2p_2$
$[1^2]$	$p_1^2 - p_2$
[3]	$p_1^3 + 6p_1p_2 + 8p_3$
[2 1]	$p_1^3 + p_1p_2 - 2p_3$
$[1^3]$	$p_1^3 - 3p_1p_2 + 2p_3$
[4]	$p_1^4 + 12p_1^2p_2 + 32p_1p_3 + 12p_2^2 + 48p_4$
[3 1]	$p_1^4 + 5p_1^2p_2 + 4p_1p_3 - 2p_2^2 - 8p_4$
$[2^2]$	$p_1^4 + 2p_1^2p_2 - 8p_1p_3 + 7p_2^2 - 2p_4$
$[2\,1^2]$	$p_1^4 - p_1^2p_2 - 2p_1p_3 - 2p_2^2 + 4p_4$
$[1^4]$	$p_1^4 - 6p_1^2p_2 + 8p_1p_3 + 3p_2^2 - 6p_4$

TABLE 1: Zonal polynomials Z_θ in terms of power sums for partitions θ of at most 4

most four. These are presented in terms of power sums. The first few terms of the genus series can be computed with the aid of these.

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