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STABLE MANIFOLDS OF A MAP AND A FLOW FOR A COMPACT MANIFOLD

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§ 0. Introduction

The purpose of this paper is to generalize the notion of the stable manifolds in Smale [5] and [6], in which the stable manifolds of flows or diffeomorphisms for a singular point or a closed orbit are defined in certain conditions. This generalization is concerned with Fenichel [1]. He considers the stable manifolds of flows and diffeomorphisms for a torus. Here, we consider the case of a compact manifold. But our argument does not exactly imply Fenichel's result.

It is interesting to investigate the conditions for the existence of stable manifolds of flows or diffeomorphisms. If the stable manifolds exist, then we can see to some extent the state of the orbits of flows or diffeomorphisms near the stable manifolds.

In § 1, we prove Theorem 1 by the method of successive approximations and we obtain a local stable manifold of a map for a compact manifold as a graph of the solution map. In Corollary of Theorem 1, we study the state of the orbits of a map. In § 2, we construct a local stable manifold of a flow by using the result of § 1 and we study the state of the orbits of the flow.

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§ 1. The stable manifold of a map.

First, we shall explain the notations.

Let T be a compact C^l-manifold $(1 \leq l < \infty)$ and E_i be a k_i dimensional Euclidean space for $i = 1, 2$. We denote by L_i a $k_i \times k_i$ non-singular matrix for $i = 1, 2$. Define the norm of a vector $z = (z_1, \cdots, z_{k_i}) \in E_i$ (resp. a matrix L_i) by $\|z\| = \max(|z_1|, \cdots, |z_{k_i}|)$ (resp. $\|L_i\| = \sup_{\|z\|=1} \|L_i z\|$). We suppose

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that the norms of L_1 and L_2^{-1} are less than one. If we put $\|L_1\| = a$ and $\|L_2^{-1}\| = \frac{1}{b}$, then $a < 1 < b$. Put $B_i(r_0) = \{z \in E_i \mid \|z\| \leq r_0\}$ for a positive constant r_0. Let ξ: $T \times B_1(r_0) \times B_2(r_0) \to T$ be of class C^l and let φ_i: $T \times B_1(r_0) \times B_2(r_0) \to E_i$ be of class C^l for $i = 1, 2$. Define F: $T \times B_1(r_0) \times B_2(r_0) \to T \times E_1 \times E_2$ by $F(\tau, x, y) = (\xi(\tau, x, y), L_1x + \varphi_1(\tau, x, y), L_2y + \varphi_2(\tau, x, y))$.

For defining the norm of $d\xi(\tau, x, y)$ and $\|\tau - \tau'\|$ for sufficiently close τ, τ' of T, we shall give the following remarks.

In general, let M and N be two compact C^1-manifolds and h: $M \to N$ be a C^1-map. Let $\{V_j\}_{j=1}^n$ be a finite open covering of N by *)nice coordinate neighborhoods V_j. Next, let $\{U_i\}_{i=1}^m$ be a finite open covering of M by *)nice coordinate neighborhoods U_i such that for each i $h(U_i)$ is contained in a suitable V_j. Let $\{h_i\}_{i=1}^m$ (resp. $\{h_j'\}_{j=1}^n$) be a coordinate system associated with $\{U_i\}_{i=1}^n$ (resp. $\{V_j\}_{j=1}^n$). We define the norm of dh by $\|dh\| = \max_{x \in U_i'} \|dh_j' h h_i^{-1}(x)\|$ for each (i, j) such that $h(U_i) \subset V_j$. When m and m' are sufficiently close in M, we put $\|m - m'\| = \max \|h_i(m) - h_i(m')\|$ for each i such that $m, m' \in U_i$.

By the first remark, we can define the norm of $d\xi(\tau, x, y)$ and we put $c = \max \{\|d\xi(\tau, x, y)\| \mid \tau \in T, x \in B_1(r_0), y \in B_2(r_0)\}$. Also, we can use the mean value theorem with respect to the first variable by the second remark.

Let f: $M_1 \times M_2 \times M_3 \to N$ be a C^l-map. We denote by $d_i f$ the partial derivative of f with respect to the i-th variable for $i = 1, 2, 3$ and by $d_{(2,3)}f$ the partial derivative of f with respect to the second and third variables.

THEOREM 1. *Suppose that*

$$\varphi_i(\tau, 0, 0) = 0, \quad d_{(2,3)}\varphi_i(\tau, 0, 0) = 0, \quad i = 1, 2, \tag{1.1}$$

for any $\tau \in T$ *and* $c, c^2, \cdots, c^{l'} < b$ $(1 \leq l' \leq l)$, *then there exist a positive number* δ *and a* $C^{l'}$ *map* g: $T \times B_1(\delta) \to E_2$ *such that*

$$g(\tau, 0) = 0, \quad d_2 g(\tau, 0) = 0 \quad \textit{for any} \quad \tau \in T, \tag{1.2}$$

$$g(\xi(\tau, x, g(\tau, x)), L_1x + \varphi_1(\tau, x, g(\tau, x))) = L_2 g(\tau, x) + \varphi_2(\tau, x, g(\tau, x)). \tag{1.3}$$

COROLLARY. *For a fixed* $(\tau_0, x_0, y_0) \in T \times B_1(r_0) \times B_2(r_0)$, *we put* $(\tau_1, x_1, y_1) = F(\tau_0, x_0, y_0)$. *If* $(\tau_m, x_m, y_m) \in T \times B_1(r_0) \times B_2(r_0)$, *then we define* $(\tau_{m+1}, x_{m+1}, y_{m+1}) = F(\tau_m, x_m, y_m)$.

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*) Let $\{U_i\}_{i=1}^n$ be a finite open covering of a compact manifold M by coordinate neighborhoods and $U_i'(\subset U_i)$ be compact and $\bigcup_{i=1}^n \operatorname{Int} U_i' = M$. We call these $\{(U_i, U_i')\}_{i=1}^n$ nice coordinate neighborhoods.

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If $y_0 = g(\tau_0, x_0)$, *then* $y_m = g(\tau_m, x_m)$ *for any* m *and* $\|(x_m, y_m)\| = O((a+2\theta_1)^m)$ *as* $m \to \infty$, *where* θ_1 *is a sufficiently small positive number.*

If $y_0 \neq g(\tau_0, x_0)$, *then* $y_m \neq g(\tau_m, x_m)$ *as far as* (τ_m, x_m, y_m) *is defined. Actually, there exists a constant* $d > 1$ *such that* $\|y_m - g(\tau_m, x_m)\| \geq d^m \|y_0 - g(\tau_0, x_0)\|$ *and also* $\|(x_m, y_m)\| \geq \frac{1}{2} d^m \|y_0 - g(\tau_0, x_0)\|$ *as far as* (τ_m, x_m, y_m) *is defined.*

Let $V = \{(\tau, x, g(\tau, x)) \mid \tau \in T, \ \|x\| < \delta\}$, then V is a $C^{l'}$-manifold and (1.3) implies $F(V) \subset V$. By Corollary of Theorem 1, V consists of the point (τ, x, y) such that $\lim_{n\to\infty} F^n(\tau, x, y) \in T \times 0 \times 0$. We call V the local stable manifold of a map F for a compact manifold T.

If F: $T \times B_1(r_0) \times B_2(r_0) \to T \times E_1 \times E_2$ is an imbedding, $\bigcup_{n=0}^{\infty} F^{-n}(V)$ is a $C^{l'}$-manifold and is called the stable manifold of a map F for a compact manifold T. We can define the unstable manifold by a similar method.

Proof of Theorem 1. Let θ_1 be a sufficiently small positive number, which will be determined later so that the following arguments hold. (1.1) implies that for any $\theta_1 > 0$ there exists a positive number $r(\leq r_0)$ such that

(1.4) $$\|d\varphi_1\|, \ \|d\varphi_2\| \leq \theta_1 \quad \text{for } \tau \in T \text{ and } \|x\| \leq r.$$

Since φ_1, φ_2 and ξ are C^l-maps, there exists a positive number θ_m $(m = 2, 3, \cdots, l)$ such that

(1.5) $$\|d^m\varphi_1\|, \ \|d^m\varphi_2\|, \ \|d^m\xi\| \leq \theta_m \quad \text{for } \tau \in T, \ \|x\| \leq r \text{ and } \|y\| \leq r.$$

Denote by Γ the set of the map f with the following properties:

(1.6) $$y = f(\tau, x) : T \times B_1(\delta) \to E_2 \quad \text{is of class } C^{l'},$$

where $\delta = \dfrac{\theta_1}{1 - a - \theta_1} r$. Let $\theta_1 < \dfrac{1-a}{2}$. Then $0 < \delta \leq r$.

(1.7) $$f(\tau, 0) = 0, \ d_2 f(\tau, 0) = 0 \quad \text{for any } \tau \in T.$$

(1.8) If $\tau \in T$ and $\|x\| \leq \delta$, then

(a) $\|f(\tau, x)\| \leq r$,

(b) $\|L_1 x + \varphi_1(\tau, x, f(\tau, x))\| \leq \delta$,

(c) $\|L_2^{-1}[f(\xi(\tau, x, f(\tau, x)), L_1 x + \varphi_1(\tau, x, f(\tau, x))) - \varphi_2(\tau, x, f(\tau, x))]\| \leq r$.

(1.9) $$\|df(\tau, x)\| \leq \sigma_1 \quad \text{for any } \tau \in T \text{ and } \|x\| \leq \delta,$$

where $\sigma_1 = \dfrac{\theta_1}{b - \max(c, 1)}$. Let $\theta_1 < b - \max(c, 1)$. Then $0 < \sigma_1 < 1$.

Define a map Φ: $\Gamma \to \Gamma$ by $\Phi(f) = \psi$, where $\psi(\tau, x) = L_2^{-1}[f(\xi(\tau, x, f(\tau, x)), L_1 x + \varphi_1(\tau, x, f(\tau, x))) - \varphi_2(\tau, x, f(\tau, x))]$. We shall show $\psi \in \Gamma$. It is trivial that ψ satisfies (1.6), $\psi(\tau, 0) = 0$ and (a) of (1.8). First, we shall show $d_2(\tau, 0) = 0$. Since

$$d_2\psi(\tau, x) = L_2^{-1}[d_1 f(\xi(\tau, x, f(\tau, x)), L_1 x + \varphi_1(\tau, x, f(\tau, x)))\cdot\{d_2\xi(\tau, x, f(\tau, x)) + d_3\xi(\tau, x, f(\tau, x))\cdot d_2 f(\tau, x)\} + d_2 f(\xi(\tau, x, f(\tau, x)), L_1 x + \varphi_1(\tau, x, f(\tau, x)))\cdot\{L_1 + d_2\varphi_1(\tau, x, f(\tau, x)) + d_3\varphi_1(\tau, x, f(\tau, x))\cdot d_2 f(\tau, x)\} - d_2\varphi_2(\tau, x, f(\tau, x)) - d_3\varphi_2(\tau, x, f(\tau, x))\cdot d_2 f(\tau, x)],$$

$d_2\psi(\tau, 0) = L_2^{-1}[d_1 f((\tau, 0, 0), 0)\cdot d_2\xi(\tau, 0, 0)]$ by (1.1) and (1.7). Since $f(\tau, 0) = 0$, $d_1 f(\tau, 0) = 0$. Therefore, $d_2\psi(\tau, 0) = 0$.

Next, we shall prove (b) of (1.8). Let $\tau\in T$ and $\|x\|\leq\delta$. Then

$$\begin{aligned}
&\|L_1 x + \varphi_1(\tau, x, \psi(\tau, x))\|\\
&\leq a\delta + \|\varphi_1(\tau, x, \psi(\tau, x)) - \varphi_1(\tau, x, 0)\| + \|\varphi_1(\tau, x, 0) - \varphi_1(\tau, 0, 0)\|\\
&\leq a\delta + \theta_1\|\psi(\tau, x)\| + \theta_1\|x\| && \text{(mean value theorem and (1.4))}\\
&\leq a\delta + \theta_1 r + \theta_1\delta && \text{((a) of (1.8) for } \psi)\\
&\leq \delta && (1.6)
\end{aligned}$$

Now, we shall prove (c) of (1.8). Let $\tau\in T$ and $\|x\|\leq\delta$. Then

$$\begin{aligned}
&\|L_2^{-1}[\psi(\xi(\tau, x, \psi(\tau, x)), L_1 x + \varphi_1(\tau, x, \psi(\tau, x))) - \varphi_2(\tau, x, \psi(\tau, x))]\|\\
&\leq \frac{1}{b}[\|\psi(\xi(\tau, x, \psi(\tau, x)), L_1 x + \varphi_1(\tau, x, \psi(\tau, x)))\| + \|\varphi_2(\tau, x, \psi(\tau, x))\|]\\
&\leq \frac{1}{b}(r + \theta_1 r + \theta_1\delta) && \text{((a) and (b) of (1.8) for } \psi)\\
&\leq \frac{1}{b}(1 + 2\theta_1)r.
\end{aligned}$$

Let $\theta_1 < \frac{b-1}{2}$. Then $\frac{1}{b}(1 + 2\theta_1)r \leq r$.

Finally, we shall show (1.9). Since

$$d\psi(\tau, x) = L_2^{-1}[df(\xi(\tau, x, f(\tau, x)), L_1 x + \varphi_1(\tau, x, f(\tau, x)))\cdot(d\xi(\tau, x, f(\tau, x))\cdot(1, 1, df(\tau, x)), L_1 + d\varphi_1(\tau, x, f(\tau, x))\cdot(1, 1, df(\tau, x))) - d\varphi_2(\tau, x, f(\tau, x))\cdot(1, 1, df(\tau, x))],$$

$$\|d\psi(\tau, x)\| \leq \frac{1}{b}(\sigma_1 \max(c, a + \theta_1) + \theta_1).$$

Let $a + \theta_1 < 1$. Then $\frac{1}{b}(\sigma_1 \max(c, a + \theta_1) + \theta_1) \leq \frac{1}{b}(\sigma_1 \max(c, 1) + \theta_1)$. By the definition of σ_1, it is easy to see that the above right term equals to σ_1.

Thus, we have $\psi \in \Gamma$ and the map Φ is well defined.

Define $g_0(\tau, x) \equiv 0$ for any $\tau \in T$ and $\|x\| \leq \delta$, then it is easy to see $g_0 \in \Gamma$. Put $\Phi(g_0) = g_1$, $\Phi(g_1) = g_2, \cdots$, then $g_1, g_2, \cdots \in \Gamma$.

It will be verified by induction that there exist a positive number $k(<1)$ and M such that

$$\|g_m(\tau, x) - g_{m-1}(\tau, x)\| \leq Mk^m \quad \text{for } \tau \in T \text{ and } \|x\| \leq \delta. \tag{1.10}$$

For $m = 1$,

$$\|g_1(\tau, x) - g_0(\tau, x)\| \leq \|L_2^{-1}(-\varphi_2(\tau, x, 0))\| \leq \frac{\theta_1}{b}\delta.$$

Put $Mk = \frac{\theta_1}{b}\delta$, then (1.10) holds for $m = 1$. Assume that (1.10) holds if m is replaced by $m-1$. Then

$$\begin{aligned}
&\|g_m(\tau, x) - g_{m-1}(\tau, x)\| \\
&\leq \frac{1}{b}[\|g_{m-1}(\xi(\tau, x, g_{m-1}(\tau, x)), L_1x + \varphi_1(\tau, x, g_{m-1}(\tau, x))) \\
&\quad - g_{m-2}(\xi(\tau, x, g_{m-1}(\tau, x)), L_1x + \varphi_1(\tau, x, g_{m-1}(\tau, x)))\| \\
&\quad + \|g_{m-2}(\xi(\tau, x, g_{m-1}(\tau, x)), L_1x + \varphi_1(\tau, x, g_{m-1}(\tau, x))) \\
&\quad - g_{m-2}(\xi(\tau, x, g_{m-1}(\tau, x)), L_1x + \varphi_1(\tau, x, g_{m-2}(\tau, x)))\| \\
&\quad + \|g_{m-2}(\xi(\tau, x, g_{m-1}(\tau, x)), L_1x + \varphi_1(\tau, x, g_{m-2}(\tau, x))) \\
&\quad - g_{m-2}(\xi(\tau, x, g_{m-2}(\tau, x)), L_1x + \varphi_1(\tau, x, g_{m-2}(\tau, x)))\| \\
&\quad + \|\varphi_2(\tau, x, g_{m-1}(\tau, x)) - \varphi_2(\tau, x, g_{m-2}(\tau, x))\|].
\end{aligned}$$

By the assumption of induction and the mean value theorem,

$$\begin{aligned}
&\|g_m(\tau, x) - g_{m-1}(\tau, x)\| \\
&\leq \frac{1}{b}(Mk^{m-1} + \sigma_1\theta_1 Mk^{m-1} + \sigma_1 c Mk^{m-1} + \theta_1 Mk^{m-1}) \\
&\leq \frac{1}{b}Mk^{m-1}(1 + \sigma_1\theta_1 + \sigma_1 c + \theta_1).
\end{aligned}$$

We put $k = \frac{1}{b}(1 + \sigma_1\theta_1 + \sigma_1 c + \theta_1)$. For a sufficiently small θ_1 the following inequality holds: $\sigma_1\theta_1 + \sigma_1 c + \theta_1 < b - 1$ $\left(\text{Note } \sigma_1 = \frac{\theta_1}{b - \max(c, 1)}\right)$. For such a θ_1 we have $0 < k < 1$ and $\|g_m(\tau, x) - g_{m-1}(\tau, x)\| \leq Mk^m$. Therefore, $g(\tau, x) = \lim_{m\to\infty} g_m(\tau, x)$ exists uniformly for $\tau \in T$ and $\|x\| \leq \delta$. It is trivial that g is continuous and g satisfies $g(\tau, 0) = 0$ and (1.3).

Next, it will be proved that the solution map g is of class C^1. For any map $f = f(u)$ (resp. $f = f(v, w)$), let $\Delta f = f(u + \Delta u) - f(u)$ (resp. $\Delta f = f(v + \Delta v, w + \Delta w) - f(v, w)$). Put $v_1(\varepsilon) = \sup (\|\Delta d\xi\|, \|\Delta d\varphi_1\|, \|\Delta d\varphi_2\|)$ for $\|\Delta\tau\|$, $\|\Delta x\|$, $\|\Delta y\| \leq \varepsilon$. We shall show that

$$\|\Delta df\| \leq u_1(\varepsilon) \quad \text{implies} \quad \|\Delta d\psi\| \leq u_1(\varepsilon), \tag{1.11}$$

where $f \in \Gamma$, $\Phi(f) = \psi$ and $u_1(\varepsilon) = K_1 v_1(\varepsilon)$ and $K_1 = \dfrac{1 + \sigma_1}{b - \max(c, 1) - \sigma_1 c - \theta_1}$. For a sufficiently small θ_1, $b - \max(c, 1) > \sigma_1 c + \theta_1$ $\left(\text{Note } \sigma_1 = \dfrac{\theta_1}{b - \max(c, 1)}\right)$. For such a θ_1 we have $K_1 > 0$. Assume $\|\Delta df\| \leq u_1(\varepsilon)$. By using the analogue of $\Delta[f_1(u)f_2(u)] = f_1(u + \Delta u)\Delta f_2 + (\Delta f_1)f_2(u)$, we have the following equation:

$$\begin{aligned}\Delta d\psi = L_2^{-1}[&\Delta df(d\xi(1, 1, df), L_1 + d\varphi_1(1, 1, df)) + df(\Delta d\xi(1, 1, df)\\ &+ d\xi(0, 0, \Delta df), \Delta d\varphi_1(1, 1, df) + d\varphi_1(0, 0, \Delta df)) - \Delta d\varphi_2(1, 1, df)\\ &- d\varphi_2(0, 0, \Delta df)].\end{aligned}$$

Then
$$\begin{aligned}\|\Delta d\psi\| &\leq \frac{1}{b}(u_1(\varepsilon) \max(c, a + \theta_1) + \sigma_1 \max(v_1(\varepsilon) + cu_1(\varepsilon), v_1(\varepsilon) + \theta_1 u_1(\varepsilon))\\ &\quad + v_1(\varepsilon) + \theta_1 u_1(\varepsilon))\\ &\leq \frac{1}{b}(u_1(\varepsilon) \max(c, a + \theta_1) + \sigma_1 v_1(\varepsilon) + \sigma_1 c u_1(\varepsilon) + v_1(\varepsilon) + \theta_1 u_1(\varepsilon))\\ &\qquad\qquad (\text{let } \theta_1 < c)\\ &\leq \frac{1}{b}(u_1(\varepsilon) \max(c, 1) + \sigma_1 v_1(\varepsilon) + \sigma_1 c u_1(\varepsilon) + v_1(\varepsilon) + \theta_1 u_1(\varepsilon))\\ &= \frac{1}{b}(u_1(\varepsilon)(\max(c, 1) + \sigma_1 c + \theta_1) + v_1(\varepsilon)(1 + \sigma_1)).\end{aligned}$$

By the definition of $u_1(\varepsilon)$, it is easy to see that the above term equals to $u_1(\varepsilon)$. Therefore, $\|\Delta d\psi\| \leq u_1(\varepsilon)$. Since $g_0 \in \Gamma$ and $\|\Delta dg_0\| \leq u_1(\varepsilon)$, we have $\|\Delta dg_m\| \leq u_1(\varepsilon)$ for $m = 1, 2, \cdots$, that is, $\{dg_0, dg_1, \cdots\}$ is equi-continuous. Since $g_m \in \Gamma$, $\{dg_0, dg_1, \cdots\}$ is uniformly bounded. Therefore, g is of class C^1 and $d_2 g(\tau, 0) = 0$ since $d_2 g_m(\tau, 0) = 0$.

Next, we shall show that the solution map g is of class $C^{l'}$. Define $\zeta_j(\tau, x) = \varphi_j(\tau, x, f(\tau, x))$ for $j = 1, 2, 3$, where $\zeta_3(\tau, x) = \xi(\tau, x, f(\tau, x))$. By induction, it is easy to see that

$$d^m\zeta_j = d^m\varphi_j(1, 1, df)^m + P_1 + d\varphi_j(0, 0, d^m f) \tag{1.12}$$

for $m = 2, 3, \cdots, l'$, where P_1 is a polynomial of $d^k\varphi_j$, $(1, 1, df)^k$ and $d^k f$ for $k \leq m - 1$. Also, put $\eta(\tau, x) = f(\zeta_3(\tau, x), L_1 x + \zeta_1(\tau, x))$. By induction, it is easy to see that

(1.13) $$d^m\eta = d^m f(d\zeta_3, L_1 + d\zeta_1)^m + P_2 + df(d^m\zeta_3, d^m\zeta_1)$$

for $m = 2, 3, \cdots, l'$, where P_2 is a polynomial of $d^k f$, $(d\zeta_3, L_1 + d\zeta_1)^k$ and $(d^k\zeta_3, d^j\zeta_1)$ for $j, k \leq m-1$. Since $\psi = L_2^{-1}(\eta - \zeta_2)$,

(1.14) $$d^m\psi = L_2^{-1}(d^m\eta - d^m\zeta_2) \text{ for } m = 2, 3, \cdots, l'.$$

We shall show by induction that there exists a positive number σ_m ($m = 2, 3, \cdots, l'$) such that

(1.15) $$\|d^m f\| \leq \sigma_m \quad \text{implies} \quad \|d^m\psi\| \leq \sigma_m,$$

where $f \in \Gamma$ and $\Phi(f) = \psi$. For $m = 2$, put $\sigma_2 = \dfrac{1+\sigma_1}{b - [\max(c, 1)]^2 - \sigma_1 c - \theta_1}\theta_2$. Since $c^2 < b$, $\sigma_2 > 0$ for a small θ_1 such that $b - [\max(c, 1)]^2 > \sigma_1 c + \theta_1$ $\left(\text{Note } \sigma_1 = \dfrac{\theta_1}{b - \max(c, 1)}\right)$. Assume $\|d^2 f\| \leq \sigma_2$. By a simple calculation, we have

(1.16) $$d^2\eta = d^2 f(d\zeta_3, L_1 + d\zeta_1)^2 + df(d^2\zeta_3, d^2\zeta_1),$$

(1.17) $$d^2\zeta_2 = d^2\varphi_2(1, 1, df)^2 + d\varphi_2(0, 0, d^2 f).$$

By (1.14), (1.16) and (1.17),

$$\begin{aligned}\|d^2\psi\| &\leq \frac{1}{b}(\sigma_2[\max(c, a+\theta_1)]^2 + \sigma_1\max(\theta_2 + c\sigma_2, \theta_2 + \sigma_1\theta_2) + \theta_2 + \theta_1\sigma_2)\\ &\leq \frac{1}{b}(\sigma_2[\max(c, 1)]^2 + \sigma_1\theta_2 + \sigma_1 c\sigma_2 + \theta_2 + \theta_1\sigma_2)\\ &= \frac{1}{b}(\sigma_2([\max(c, 1)]^2 + \sigma_1 c + \theta_1) + \theta_2(1+\sigma_1)).\end{aligned}$$

By the definition of σ_2, it is easy to see that the above term equals to σ_2. Thus, we have $\|d^2\psi\| \leq \sigma_2$. Next, assume that (1.15) holds for $m = 2, 3, \cdots, p-1$. Therefore, we can assume that $\sigma_1, \sigma_2, \cdots, \sigma_{p-1}$ are defined. By (1.12), (1.13) and (1.14), we have the following inequality:

$$\begin{aligned}\|d^p\psi\| &\leq \frac{1}{b}(\sigma_p[\max(c, a+\theta_1)]^p + \|P_2\|\\ &\quad + \sigma_1\max(\theta_p + \|P_1\| + c\sigma_p, \theta_p + \|P_1'\| + \theta_1\sigma_p) + \theta_p + \|P_3\| + \theta_1\sigma_p)\\ &\leq \frac{1}{b}(\sigma_p[\max(c, 1)]^p + \sigma_1 c + \theta_1) + \alpha),\end{aligned}$$

where α is a positive constant. Put $\sigma_p = \dfrac{\alpha}{b - [\max(c, 1)]^p - \sigma_1 c - \theta_1}$, then the above term equals to σ_p. Since $c^p < b$, $\sigma_p > 0$ for a small θ_1 such that $b - [\max(c, 1)]^p > \sigma_1 c + \theta_1$ $\left(\text{Note } \sigma_1 = \dfrac{\theta_1}{b - \max(c, 1)}\right)$. Thus, we prove (1.15).

Since $g_0 \in \Gamma$ and $\|d^m g_0\| \leq \sigma_m$ for $m = 2, \cdots, l'$, we have $\|d^m g_k\| \leq \sigma_m$ for $k = 1, 2, \cdots$. That is, $\{d^m g_0, d^m g_1, \cdots\}$ is uniformly bounded. Next, put $v_m(\varepsilon) = \sup_{\|\Delta\tau\|, \|\Delta x\|, \|\Delta y\| \leq \varepsilon} (\|\Delta d^m \xi\|, \|\Delta d^m \varphi_1\|, \|\Delta d^m \varphi_2\|)$ for $m = 2, 3, \cdots, l'$. By induction, it is easy to see that

$$\Delta d^m \zeta_j = \Delta d^m \varphi_j (1, 1, df)^m + P + d\varphi_j (0, 0, \Delta d^m f), \tag{1.18}$$

$$\Delta d^m \eta = \Delta d^m f (d\zeta_3, L_1 + d\zeta_1)^m + Q + df(\Delta d^m \zeta_3, \Delta d^m \zeta_1), \tag{1.19}$$

where P and Q are suitable polynomials. Also,

$$\Delta d^m \psi = L_2^{-1}(\Delta d^m \eta - \Delta d^m \zeta_2). \tag{1.20}$$

Replacing (1.12), (1.13) and (1.14) by (1.18), (1.19) and (1.20) respectively, we can prove similary as above that there exists a positive number K_m for a small θ_1 such that

$$\|\Delta d^m f\| \leq K_m v_m(\varepsilon) \quad \text{implies} \quad \|\Delta d^m \psi\| \leq K_m v_m(\varepsilon) \tag{1.21}$$

for $m = 2, 3, \cdots, l'$, where $f \in \Gamma$ and $\Phi(f) = \psi$. Since $g_0 \in \Gamma$ and $\|\Delta d^m g_0\| \leq K_m v_m(\varepsilon)$ for $m = 2, 3, \cdots, l'$, we have $\|\Delta d^m g_k\| \leq K_m v_m(\varepsilon)$ for $k = 1, 2, \cdots$. That is, $\{d^m g_0, d^m g_1, \cdots\}$ is equi-continuous. Thus, g is of class $C^{l'}$ and $g \in \Gamma$.

This completes the proof of Theorem 1.

Proof of Corollary.

If $y = g(\tau, x)$ for $\tau \in T$ and $\|x\| \leq \delta$, then

$$\|y\| \leq \|g(\tau, x) - g(\tau, 0)\| \leq \sigma_1 \|x\| \leq \delta. \tag{1.22}$$

It is trivial that

$$y_0 = g(\tau_0, x_0) \quad \text{implies} \quad y_1 = g(\tau_1, x_1). \tag{1.23}$$

By (1.22) and (1.23), we see that $y_0 = g(\tau_0, x_0)$ implies $(\tau_m, x_m, y_m) \in T \times B_1(\delta) \times B_2(\delta)$ and $y_m = g(\tau_m, x_m)$ for any m. Suppose $y_0 = g(\tau_0, x_0)$. From the equations $x_1 = L_1 x_0 + \varphi_1(\tau_0, x_0, y_0)$ and $y_1 = g(\tau_1, x_1)$, we have

$$\begin{aligned} \|y_1\| \leq \|x_1\| &\leq a\|x_0\| + \|\varphi_1(\tau,{}_0 x_0, y_0) - \varphi_1(\tau_0, x_0, 0)\| \\ &+ \|\varphi_1(\tau_0, x_0, 0) - \varphi_1(\tau_0, 0, 0)\| \leq a\|x_0\| + \theta_1\|y_0\| + \theta_1\|x_0\| \leq (a + 2\theta_1)\|x_0\|. \end{aligned}$$

Therefore, $\|y_m\| \leq \|x_m\| \leq (a + 2\theta_1)^m \|x_0\|$. Let $a + 2\theta_1 < 1$. Then we prove the first assertion of the corollary.

Ino rder to see that $y_0 \neq g(\tau_0, x_0)$ implies $y_1 \neq g(\tau_1, x_1)$, it is enough to consider the case of $\|x_1\| \leq \delta$ and $\|y_1\| \leq r_0$. From the equations

$$\tau_1 = \xi(\tau_0, x_0, y_0),$$
$$x_1 = L_1 x_0 + \varphi_1(\tau_0, x_0, y_0),$$
$$y_1 = L_2 y_0 + \varphi_2(\tau_0, x_0, y_0),$$
$$g(\tau_0, x_0) = L_2^{-1}[g(\xi(\tau_0, x_0, g(\tau_0, x_0)), L_1 x_0 + \varphi_1(\tau_0, x_0, g(\tau_0, x_0))) - \varphi_2(\tau_0, x_0, g(\tau_0, x_0))],$$

we have the following inequalities:

$$\begin{aligned}
&\|y_0 - g(\tau_0, x_0)\| \\
&\leq \|L_2^{-1}\| \|y_1 - \varphi_2(\tau_0, x_0, y_0) - g(\xi(\tau_0, x_0, g(\tau_0, x_0)), L_1 x_0 + \varphi_1(\tau_0, x_0, g(\tau_0, x_0))) \\
&\quad + \varphi_2(\tau_0, x_0, g(\tau_0, x_0))\| \\
&\leq \frac{1}{b}[\|y_1 - g(\tau_1, x_1)\| + \|g(\tau_1, L_1 x_0 + \varphi_1(\tau_0, x_0, y_0)) - g(\tau_1, L_1 x_0 \\
&\quad + \varphi_1(\tau_0, x_0, g(\tau_0, x_0)))\| + \|g(\xi(\tau_0, x_0, y_0), L_1 x_0 + \varphi_1(\tau_0, x_0, g(\tau_0, x_0))) \\
&\quad - g(\xi(\tau_0, x_0, g(\tau_0, x_0)), L_1 x_0 + \varphi_1(\tau_0, x_0, g(\tau_0, x_0)))\| + \|\varphi_2(\tau_0, x_0, g(\tau_0, x_0)) \\
&\quad - \varphi_2(\tau_0, x_0, y_0)\|] \\
&\leq \frac{1}{b}[\|y_1 - g(\tau_1, x_1)\| + \sigma_1\theta_1\|y_0 - g(\tau_0, x_0)\| + \sigma_1 c\|y_0 - g(\tau_0, x_0)\| \\
&\quad + \theta_1\|y_0 - g(\tau_0, x_0)\|].
\end{aligned}$$

Then $(b - \sigma_1\theta_1 - \sigma_1 c - \theta_1)\|y_0 - g(\tau_0, x_0)\| \leq \|y_1 - g(\tau_1, x_1)\|$. Put $d = b - \sigma_1\theta_1 - \sigma_1 c - \theta_1$. Then $d = 1 + b(1-k) > 1$ since $k < 1$. If $\|x_k\| \leq \sigma$ and $\|y_k\| \leq r_0$ for $k = 1, 2, \cdots, m$, then $d^m\|y_0 - g(\tau_0, x_0)\| \leq \|y_m - g(\tau_m, x_m)\|$. Next,

$$\begin{aligned}
\|(x_m, y_m)\| \geq \|y_m\| &\geq \|y_m - g(\tau_m, x_m)\| - \|g(\tau_m, x_m)\| \\
&\geq \|y_m - g(\tau_m, x_m)\| - \|x_m\| \\
&\geq \|y_m - g(\tau_m, x_m)\| - \|(x_m, y_m)\|.
\end{aligned}$$

Therefore, $\|(x_m, y_m)\| \geq \frac{1}{2}\|y_m - g(\tau_m, x_m)\| \geq \frac{1}{2}d^m\|y_0 - g(\tau_0, x_0)\|$. Thus, the second assertion of the corollary is proved.

§ 2. The stable manifold of a Flow.

Let T, E_i and $B_i(r_0)$ for $i = 1, 2$ be the same as in § 1. Let $\psi_i : T \times B_1(r_0) \times B_2(r_0) \to E_i$ be of class C^l such that $\psi_i(\tau, 0, 0) = 0$ and $d_{(2,3)}\psi_i(\tau, 0, 0) = 0$ for $i = 1, 2$. Let P_i be a $k_i \times k_i$ matrix for $i = 1, 2$. Assume that the real parts of the eigen values of P_1 (resp. P_2) are negative (resp. positive). If we put $\|e^{P_1}\| = a$ and $\|e^{-P_2}\| = \frac{1}{b}$, then $a < 1 < b$. We shall denote by T_τ the tangent space at a point τ of T. Let $\xi : T\times B_1(r_0)\times B_2(r_0) \to \bigcup_{\tau\in T} T_\tau$ be of class C^l such that $\xi(\tau, x, y)\in T_\tau$. Using a local coordinate neighborhood in T, we can express ξ as a map from $E^q\times B_1(r_0)\times B_2(r_0)$ to E^q, where E^q is a Euclidean space. We suppose $d\xi(\tau, 0, 0) = 0$ for any $\tau\in T$.

Let f be a function of class C^l from $E_1\times E_2$ to the set of all real numbers such that

(2.1) $$0\leq f(x, y)\leq 1 \quad \text{for any} \quad (x, y)\in E_1\times E_2,$$

(2.2) $$f(x, y) = 1 \quad \text{for} \quad \|x\|\leq \frac{r_0}{3} \quad \text{and} \quad \|y\|\leq\frac{r_0}{3},$$

(2.3) $$f(x, y) = 0 \quad \text{for} \quad \|x\|\geq\frac{2}{3}r_0 \quad \text{or} \quad \|y\|\geq\frac{2}{3}r_0.$$

We consider a vector field X on $T\times E_1\times E_2$ such that

(2.4) $$X = (f(x, y)\xi(\tau, x, y),\ P_1x + f(x, y)\psi_1(\tau, x, y),\ P_2y + f(x, y)\psi_2(\tau, x, y))$$

on $T\times B_1(r_0)\times B_2(r_0)$,

(2.5) $$X = (0, P_1x, P_2y) \quad \text{on} \quad \|x\|\geq\frac{2}{3}r_0 \quad \text{or} \quad \|y\|\geq\frac{2}{3}r_0.$$

This vector field X is an extension of a vector field $(\xi(\tau, x, y), P_1x + \psi_1(\tau, x, y), P_2y + \psi_2(\tau, x, y))$ on $T\times B_1\left(\frac{r_0}{3}\right)\times B_2\left(\frac{r_0}{3}\right)$. Let $(\Xi(t, \tau, x, y), \Phi_1(t, \tau, x, y), \Phi_2(t, \tau, x, y))$ be an integral curve for X with initial value (τ, x, y) for $-\alpha < t < \beta$ $(\alpha, \beta > 0)$, where the interval $(-\alpha, \beta)$ is the maximal interval for the existence of the solution for X. If $\beta < +\infty$, then a part of this integral curve is out of $T\times B_1\left(\frac{2}{3}r_0\right)\times B_2\left(\frac{2}{3}r_0\right)$ by p. 65 of Lang [2]. Since X equals to $(0, P_1x, P_2y)$ in the exterior of $T\times B_1\left(\frac{2}{3}r_0\right)\times B_2\left(\frac{2}{3}r_0\right)$, we can extend this integral curve. This contradicts the assumption $\beta < +\infty$. Thus, $\beta = +\infty$. And also $\alpha = +\infty$. Therefore, the vector field X is complete. Let F^t be the one-parameter group generated by X. Then $F^t(\tau, x, y) = (\Xi(t, \tau, x, y), \Phi_1(t, \tau, x, y), \Phi_2(t, \tau, x, y))$ for any t and any $(\tau, x, y)\in T\times E_1\times E_2$. Put $\Phi_1(t,\tau,x,y) = e^{P_1t}x + \varphi_1(t,\tau,x,y)$ and $\Phi_2(t,\tau,x,y) = e^{P_2t}y + \varphi_2(t,\tau,x,y)$. From the initial value condition (τ, x, y), we have $\Xi(0, \tau, x, y) = \tau$ and $\varphi_i(0, \tau, x, y) = 0$ for $i = 1, 2$. Also we have $\varphi_i(t, \tau, 0, 0) = 0$ since the solution for X with initial value $(\tau, 0, 0)$ belongs to $T\times 0\times 0$. Denote by $Y(t, \tau, x, y)$ the Jacobian of (Ξ, Φ_1, Φ_2) with respect to (τ, x, y) and denote by $A(\tau, x, y)$

the Jacobian of $(\xi, P_1+\psi_1, P_2+\psi_2)$ with respect to (τ, x, y) for $\|x\|, \|y\| \leq r_0$. Put $B=\begin{pmatrix} 0 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & P_2 \end{pmatrix}$. By solving the following variation equations with respect to the initial value $(\tau, 0, 0)$,

$$\frac{d}{dt}Y(t, \tau, 0, 0) = A(\Xi(t, \tau, 0, 0), 0, 0)Y(t, \tau, 0, 0), \tag{2.6}$$

$$Y(0, \tau, 0, 0) = E \qquad (E \text{ is a unit matrix}), \tag{2.7}$$

we have $d_{(2,3)}\varphi_i(t, \tau, 0, 0)=0$ for $i=1, 2$ and a sufficiently small t. Using the compactness of T and the property of one-parameter group F^t, we have $d_{(2,3)}\varphi_i(t, \tau, 0, 0)=0$ for any t and $\tau \in T$. Moreover, we have $\Xi(t, \tau, x, y) \equiv$ *constant* and $\varphi_i(t, \tau, x, y) \equiv 0$ for $\|x\| \geq s_0$ or $\|y\| \geq s_0$, where s_0 is determined by r_0 and $s_0 \to 0$ as $r_0 \to 0$. We note that $\|d\varphi_i(t, \tau, x, y)\| \to 0$ as $s_0 \to 0$ for $0 \leq t \leq 1$.

Put $M_1 = \max\{\|A(\tau, x, y)\| \mid (\tau, x, y) \in T \times B_1(r_0) \times B_2(r_0)\}$ and $M_2 = \max_{0 \leq t \leq 1}\|e^{Bt}\|$. Let ε be a small positive number such that $c \equiv 1 + 3\varepsilon M_2 \dfrac{e^{M_1}-1}{M_1} < b$. For $M_1 = 0$, $\dfrac{e^{M_1}-1}{M_1}$ will be replaced by 1. From the properties of ξ, ψ_1 and ψ_2, we can choose a small positive number $r_1\left(\leq \dfrac{r_0}{3}\right)$ such that $\|d\psi_1(\tau, x, y)\| \leq \varepsilon$, $\|d\psi_2(\tau, x, y)\| \leq \varepsilon$ and $\|d\xi(\tau, x, y)\| \leq \varepsilon$ for $\tau \in T$, $\|x\| \leq r_1$ and $\|y\| \leq r_1$. Now we consider the following variation equations with respect to the initial value (τ, x, y).

$$\frac{d}{dt}Y(t, \tau, x, y) = A(F^t(\tau, x, y))Y(t, \tau, x, y), \tag{2.8}$$

$$Y(0, \tau, x, y) = E. \tag{2.9}$$

From the properties of Φ_1 and Φ_2, we can choose a small positive number $r_2(\leq r_1)$ such that $\|\Phi_i(t, \tau, x, y)\| \leq r_1$ for $\|x\| \leq r_2$ and $\|y\| \leq r_2$. Since

$$\|Be^{Bt} - A(F^t(\tau, x, y))e^{Bt}\|$$
$$\leq \left\| \begin{pmatrix} \frac{\partial}{\partial\tau}\xi & \frac{\partial}{\partial x}\xi & \frac{\partial}{\partial y}\xi \\ \frac{\partial}{\partial\tau}\psi_1 & \frac{\partial}{\partial x}\psi_1 & \frac{\partial}{\partial y}\psi_1 \\ \frac{\partial}{\partial\tau}\psi_2 & \frac{\partial}{\partial x}\psi_2 & \frac{\partial}{\partial y}\psi_2 \end{pmatrix}_{F(^t(\tau, x, y))} \right\| \cdot \|e^{Bt}\|$$

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$\leq 3\varepsilon M_2$ for $0 \leq t \leq 1$, $\|x\| \leq r_2$ and $\|y\| \leq r_2$,

e^{Bt} is an $3\varepsilon M_2$-approximate solution of (2.8). By p. 56 of Lang [2], we have the following inequality,

$$\|Y(t, \tau, x, y) - e^{Bt}\| \leq 3\varepsilon M_2 \frac{e^{M_1|t|} - 1}{M_1}$$

for $0 \leq t \leq 1$, $\|x\| \leq r_2$ and $\|y\| \leq r_2$. For $t = 1$, we have

$$\|Y(1, \tau, x, y) - e^{B}\| \leq 3\varepsilon M_2 \frac{e^{M_1} - 1}{M_1}.$$

That is, $\left\|\frac{\partial}{\partial \tau}\Xi(1, \tau, x, y) - E\right\| \leq 3\varepsilon M_2 \frac{e^{M_1} - 1}{M_1}$,

$$\left\|\frac{\partial}{\partial x}\Xi(1, \tau, x, y)\right\| \leq 3\varepsilon M_2 \frac{e^{M_1} - 1}{M_1} \text{ and}$$

$$\left\|\frac{\partial}{\partial y}\Xi(1, \tau, x, y)\right\| \leq 3\varepsilon M_2 \frac{e^{M_1} - 1}{M_1} \quad \text{for } \|x\|, \|y\| \leq r_2.$$

Therefore, we have $\|d\Xi(1, \tau, x, y)\| \leq 1 + 3\varepsilon M_2 \frac{e^{M_1} - 1}{M_1} = c < b$ for $\|x\|, \|y\| \leq r_2$. The map $F^1(\tau, x, y) = (\Xi(1, \tau, x, y),\ e^{P_1}x + \varphi_1(1, \tau, x, y),\ e^{P_2}y + \varphi_2(1, \tau, x, y))$: $T \times B_1(r_2) \times B_2(r_2) \to T \times E_1 \times E_2$ satisfies the assumption of Theorem 1. Therefore, there exist $\delta > 0$ and a C^1-map g defined on $T \times B_1(\delta) \times B_2(\delta)$ to E_2 such that

$$g(\tau, 0) = 0,\ d_2 g(\tau, 0) = 0 \quad \text{for any } \tau \in T, \tag{2.10}$$

$$\begin{aligned} g(\Xi(1, \tau, x, g(\tau, x)),\ e^{P_1}x + \varphi_1(1, \tau, x, g(\tau, x))) \\ = e^{P_2}g(\tau, x) + \varphi_2(1, \tau, x, g(\tau, x)). \end{aligned} \tag{2.11}$$

We note that g is a $C^{l'}$-map if $c, c^2, \cdots, c^{l'} < b$ $(l' \leq l)$. Let V be the local stable manifold of a diffeomorphism F^1.

THEOREM 2. *Let* $(\tau_t, x_t, y_t) = F^t(\tau_0, x_0, y_0)$ *for* $(\tau_0, x_0, y_0) \in T \times E_1 \times E_2$.

If $(\tau_0, x_0, y_0) \in V$, *then there exists* $n_0 > 0$ *such that* $(\tau_t, x_t, y_t) \in V$ *for* $t \geq n_0$ *and* $\|(x_t, y_t)\| = O((a + 2\theta_0)^t)$ *as* $t \to \infty$, *where* θ_0 *is a sufficiently small positive number.*

If $(\tau_0, x_0, y_0) \notin V$ *and* $\|x_s\|, \|y_s\| \leq \delta$ *for* $0 \leq s \leq t + n_0 + 1$, *then* $(\tau_s, x_s, y_s) \notin V$ *and* $\|(x_s, y_s)\| \geq K d^s$ *for* $s \leq t$, *where* K *is a positive constant and* $d > 1$.

Proof. We shall first verify that if $n \leq t \leq n + 1$, then there exist positive constants c_1, c_2 independent of n such that

$$c_1\|(x_n,\ y_n)\| \leq \|(x_t,\ y_t)\| \leq c_2\|(x_n,\ y_n)\|. \tag{2.12}$$

In order to see this, note that $F^t = F^{t-n}\circ F^n$, that is,

$$x_t = e^{P_1(t-n)}x_n + \varphi_1(t-n,\ \tau_n,\ x_n,\ y_n),$$
$$y_t = e^{P_2(t-n)}y_n + \varphi_2(t-n,\ \tau_n,\ x_n,\ y_n).$$

From this equations, we have the following inequalities:

$$\|x_t - e^{P_1(t-n)}x_n\| \leq \eta\|(x_n,\ y_n)\|,$$
$$\|y_t - e^{P_2(t-n)}y_n\| \leq \eta\|(x_n,\ y_n)\|,$$

where η is a small positive number under a small r_0. Hence, we have $\|x_t\| \leq \eta\|(x_n,\ y_n)\| + \|e^{P_1(t-n)}\|\ \|x_n\|$. On the other hand,

$$\|x_t\| \geq \|e^{P_1(t-n)}x_n\| - \|x_t - e^{P_1(t-n)}x_n\|$$
$$\geq \frac{1}{\|e^{-P_1(t-n)}\|}\|x_n\| - \eta\|(x_n,\ y_n)\|.$$

For y_t, we have similar inequalities. Thus, we have (2.12).

There exists a natural number n_0 such that $c_2(a+2\theta_0)^n \leq 1$ for $n \geq n_0$, where $\theta_0 < \frac{1-a}{2}$. If $(\tau_0,\ x_0,\ y_0)\in V$, then $(\tau_n,\ x_n,\ y_n)\in V$ and $\|(x_n,\ y_n)\| \leq (a+2\theta_0)^n\|(x_0,\ y_0)\|$ for any n from Corollary of Theorem 1. For $n_0 \leq n \leq t \leq n+1$, $\|(x_t,\ y_t)\| \leq c_2\|(x_n,\ y_n)\| \leq c_2(a+2\theta_0)^n\|(x_0,\ y_0)\| \leq \delta$. Therefore, we have $\|(x_t,\ y_t)\| \leq \delta$ for $t \geq n_0$ and $\lim_{t\to\infty}\|(x_t,\ y_t)\| = 0$. Suppose that there exists t_0 such that $(\tau_{t_0},\ x_{t_0},\ y_{t_0})\notin V$ and $t_0 \geq n_0$. By Corollary of Theorem 1,

$$\|(x_{t_0+k},\ y_{t_0+k})\| \geq \frac{1}{2}d^k\|(x_{t_0},\ y_{t_0})\| \text{ for any } k.$$

This is a contradiction. Therefore, we prove the first assertion of Theorem 2.

If $(\tau_0,\ x_0,\ y_0)\notin V$ and $\|x_s\|,\ \|y_s\| \leq \delta$ for $0 \leq s \leq t+n_0+1$, then we have $(\tau_n,\ x_n,\ y_n)\notin V$ and $\|(x_n,\ y_n)\| \geq \frac{1}{2}d^n\|(x_0,\ y_0)\|$ for $n \leq t+n_0+1$ by Corollary of Theorem 1. Suppose that there exists $t_1(\leq t)$ such that $(\tau_{t_1},\ x_{t_1},\ y_{t_1})\in V$. By the above argument, we have $(\tau_m,\ x_m,\ y_m)\in V$ for $t_1+t_0 \leq m \leq t+t_0+1$. This is a contradiction. The last inequality is trivial by (2.12).

By Theorem 2, we have

$$V = \{(\tau_0,\ x_0,\ y_0)\in T\times B_1(\delta)\times B_2(\delta)\,|\lim_{t\to\infty}F^t(\tau_0,\ x_0,\ y_0)\in T\times 0\times 0\}.$$

We call this V the local stable manifold of a flow F^t. If we put $W =$

$\bigcup_{t \geq 0} F^{-t}(V)$, then W is a C^1-manifold. We call this W the stable manifold of a vector field X. The stable manifold W is characterized as the set of $(\tau_0, x_0, y_0) \in T \times E_1 \times E_2$ such that $\lim_{t \to \infty} F^t(\tau_0, x_0, y_0) \in T \times 0 \times 0$. Similarly, we can define the unstable manifolds of a vector field X.

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