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THE BRUN–HOOLEY SIEVE FOR $\mathbb{F}_2[X]$ AND SQUAREFREE SHIFTS OF INTEGER POLYNOMIALS

PRADIPTO BANERJEE AND AMIT KUNDU

Department of Mathematics, Indian Institute of Technology Hyderabad, Sangareddy, Telangana, India

Corresponding author: Pradipto Banerjee, email: pradipto@math.iith.ac.in

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Abstract Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{F}_2[x]$ with deg $f = n$. It is shown that for $n \gg 1$, there is an $g_1(x) \in \mathbb{F}_2[x]$ with deg $g_1 \leqslant \max\{\deg g, 6.7 \log n\}$ and $g(x) - g_1(x)$ having $< 6.7 \log n$ terms such that $\gcd(f(x), g_1(x)) = 1$. As an application, it is established using a result of Dubickas and Sha that given $f(x) \in \mathbb{F}_2[x]$ of degree $n \geqslant 1$, there is a separable $g(x) \in 2[x]$ with deg g = deg f and satisfying that $f(x) - g(x)$ has $\leqslant 6.7 \log n$ terms. As a simple consequence, the latter result holds in $\mathbb{Z}[x]$ after replacing 'number of terms' by the L_1-norm of a polynomial and $6.7 \log n$ by $6.8 \log n$. This improves the bound $(\log n)^{\log 4+\varepsilon}$ obtained by Filaseta and Moy.

Keywords: Turán's conjecture; squarefree polynomials; function fields; Brun–Hooley sieve

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1. Introduction

For $f(x) \in \mathbb{Z}[x]$ of degree n, let $L_1(f)$ denote the sum of the absolute values of the coefficients of $f(x)$. This is the L_1-norm on the $(n + 1)$-dimensional real vector space U_n of real polynomials of degree $\leqslant n$. Let $V_n = U_n \cap \mathbb{Z}[x]$. Further, let $I_n \subset V_n$ be the set of polynomials in V_n that are irreducible over the rationals. It is well-known that asymptotically, a 100% polynomials in V_n are irreducible over the rationals in the sense that

$$\lim_{B\to\infty} \frac{\#\{f(x) \in I_n : L_1(f) \leqslant B\}}{\#\{f(x) \in V_n : L_1(f) \leqslant B\}} = 1$$

Thus, given $f(x) \in \mathbb{Z}[x]$ of degree n, one can naturally expect to be able to find a polynomial $g(x) \in I_n$, such that $L_1(f - g)$ is 'small'. Let $C(n)$ denote the *smallest* positive integer such that for every $f(x) \in \mathbb{Z}[x]$ with deg $f = n$, there is an $g(x) \in I_n$ such that $L_1(f-g) \leqslant C(n)$. It is easy to see that Eisenstein's criterion with $p=2$ implies that $C(n)$ exists and that $C(n) \leqslant n + 2$. Pál Turán proposed the problem of showing that $C(n)$ is absolutely bounded. For each odd $n > 1$, the example $f(x) = x^n$ shows that $C(n) \geqslant 2$. Similarly, for every even $n > 2$, the polynomial $x^{n-2}(x^2 - x - 1)$ suggests that $C(n) \geqslant 2$. Filaseta [5] conjectured that $C(n) \leqslant 5$ for all n. In the same paper, he alludes to the possibility that $C(n) \leqslant 2$ cannot be ruled out.

Turán's conjecture remains open for $n > 40$. Bérczes and Hajdu [1, 2] have verified Turán's conjecture with $C(n) \leqslant 4$, for all polynomials $f(x) \in \mathbb{Z}[x]$ with $\deg f \leqslant 24$. Filaseta and Mossinghoff [6] have extended their results to all $f(x) \in \mathbb{Z}[x]$ with $\deg f \leqslant 40$ and with $C(n) \leqslant 5$.

Turán's conjecture is believed to be difficult. For instance, whether it is possible to do better than $C(n) \leqslant n + 2$ is unknown. The present paper is a byproduct of our attempts to improve this bound. Although we fell short in this pursuit, our approach considerably improved the corresponding bound in the *squarefree* analogue of Turán's conjecture. We discuss them next.

We begin with our initial idea to improve the bound on $C(n)$. For $f(x) \in 2[x]$, let $L(f)$ denote the number of terms of $f(x)$. Now, consider Turán's problem in $2[x]$, where the distance between $f(x)$ and $g(x)$ is now taken to be $L(f - g)$. Let $C_2(n)$ denote the counterpart for $C(n)$ in this case. We claim that $C(n) \leqslant C_2(n) + 1$ provided that $\deg g = \deg f = n$. To see this, for an $f(x) \in \mathbb{Z}[x]$ with $\deg f = n$, let $\delta \in \{0, 1\}$ be such that $f_\delta(x) = \delta x^n + f(x)$ has an odd leading coefficient. Let $\overline{f_\delta}(x) \in \mathbb{F}_2[x]$ denote the polynomial obtained by reducing the coefficients of $f_\delta(x)$ modulo 2. Observe that $\deg \overline{f_\delta} = n$. Now, suppose that there is an $g(x) \in \mathbb{F}_2[x]$, irreducible in $\mathbb{F}_2[x]$ with $\deg g = n$, such that $L(\overline{f_\delta} - g) \leqslant C_2(n)$. Consider the polynomial

$$g_\delta(x) = f_\delta(x) - \overline{f_\delta}(x) + g(x) = f(x) - \overline{f}(x) + g(x) \in \mathbb{Z}[x]$$

where, by abuse of notation, we now consider $\overline{f_\delta}(x)$, $\overline{f}(x)$ and $g(x)$ as polynomials in $\mathbb{Z}[x]$. If a denotes the leading coefficient of $f(x)$, then the leading coefficient of $g_\delta(x)$ is

$$a - \overline{a} + 1 \equiv 1 \pmod{2}.$$

In particular, $g_\delta(x)$ has degree n. Additionally, $g_\delta(x) \equiv g(x) \pmod 2$ implies that $g_\delta(x)$ is irreducible over the rationals. Furthermore,

$$L_1(f - g_\delta) \leqslant 1 + L_1(\overline{f_\delta} - g) = 1 + L(\overline{f_\delta} - g) \leqslant 1 + C_2(n).$$

The assertion follows.

In view of the last observation above, it suffices to bound $C_2(n)$. For $n \geqslant 1$, let $C_2'(n)$ denote the smallest positive integer such that given $f(x)$ and $g(x)$ in $2[x]$ with $\deg f = n$, there is a polynomial $g_1(x) \in 2[x]$ with

$$\deg g_1 \leqslant \max\{\deg g, C_2'(n)\}, \quad L(g - g_1) \leqslant C_2'(n)$$

such that $\gcd(f(x), g_1(x)) = 1$. The better part of the paper is devoted to developing a method to establishing that $C_2'(n) \ll \log n$.

Now, suppose for the moment that we have achieved $C_2'(n) \leqslant \theta \log n$ for some $\theta > 0$. Let $\deg g = m \geqslant 1$, and set $\ell = \lfloor m/2 \rfloor$. Take $f(x)$ to be the product of all irreducible

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polynomials of degree $\leqslant \ell$ in $2[x]$. By Lemma 3.2, [7], we have deg $f \leqslant 2^{\ell+1}$. The hypothesis on $C_2'(n)$ would then imply that there is a polynomial $g_1(x) \in 2[x]$ with deg $g_1 \leqslant$ deg g and satisfies

$$L(g - g_1) \leqslant C_2'(\deg f) \leqslant \theta(\ell+1)\log 2 \leqslant \frac{\theta(m+2)\log 2}{2}$$

such that $\gcd(f(x), g_1(x)) = 1$. The last condition implies that $g_1(x)$ has no irreducible factor of degree $\leqslant$ deg $g/2$. Since deg $g_1 \leqslant$ deg g, it would then follow that $g_1(x)$ is irreducible in $2[x]$. A suitably small θ would then give a better bound on $C(m)$ than $m+2$. In fact, any $\theta < 2/\log 2 = 2.885\ldots$ would give the first non-trivial improvement on $C(m)$. Our main result establishes that $C_2'(n) \ll \log n$.

Theorem 1. *Let $f(x)$ and $g(x)$ be polynomials in $2[x]$ with* deg $f = n$. *For $n \gg 1$, there is a polynomial $g_1(x) \in \mathbb{F}_2[x]$ with* deg $g_1 \leqslant \max\{\deg g, 6.7\log n\}$ *and* $L(g - g_1) < 6.7\log n$ *such that* $\gcd(f(x), g_1(x)) = 1$.

Next, we discuss the squarefree analogue of Turán's conjecture. We refer to a polynomial $f(x) \in \mathbb{Z}[x]$ as *squarefree* if it has no multiple roots. For a positive integer n, let S_n denote the set of squarefree polynomials in V_n. Since $I_n \subset S_n$, it follows that the asymptotic density of squarefree polynomials in V_n is 1. Naturally, one is prompted to investigate the squarefree analogue Turán's problem. Dubickas and Sha [4] were the first to study this problem. For a positive integer n, let $D(n)$ denote the smallest positive integer such that given any $f(x) \in \mathbb{Z}[x]$ with deg $f = n$, there is an $g(x) \in S_n$ with $L_1(f - g) \leqslant D(n)$. It is easily seen that $D(n) \leqslant C(n)$. Dubickas and Sha [4] conjecture that $D(n) \leqslant 2$. They further showed that $D(n) \geqslant 2$ for every $n \geqslant 15$ (in fact, their result is much more explicit). Thus, the conjectured value is $D(n) = 2$. In some contrast to $C(n) \leqslant n + 2$, Filaseta and Moy [7] have obtained the bound

$$D(n) \leqslant (\log n)^{2\log 2+\varepsilon}$$

for $n \gg_\varepsilon 1$. As a simple application of Theorem 1, we will establish that $D(n) \ll \log n$.

Theorem 2. *For every $f(x) \in \mathbb{F}_2[x]$ of degree $n \gg 1$, there is a squarefree $g(x) \in 2[x]$ satisfying* deg $g = n$ *and* $L(f - g) \leqslant 6.7\log n$.

Arguing as we did to establish that $C(n) \leqslant C_2(n)+1$ above (in the case that deg $g = n$), we obtain the following.

Corollary 1. *For every $f(x) \in \mathbb{Z}[x]$ of degree $n \gg 1$, there is a squarefree $g(x) \in \mathbb{Z}[x]$ satisfying* deg $g = n$ *and* $L_1(f - g) \leqslant 1 + 6.7\log n < 6.8\log n$.

The proof of Theorem 1 is based on a function field analogue of Brun–Hooley sieve (see Theorem 3, § 2). Although this is identical to the usual Brun–Hooley sieve in almost every aspect needing only minor adjustments, there is no evidence of a suitable reference in the existing literature. This prompted the authors to establish a function field analogue of the Brun–Hooley sieve in its full rigour. This is presented in § 2. For an exhaustive account of the usual Brun–Hooley sieve, the reader may refer to Halberstam–Richert [9] or Bateman–Diamond [3]. Apart from these references, the authors have found the nice exposition by Kevin Ford [8] particularly useful. For general arithmetic in function fields, we refer the reader to Rosen [10].

We clarify some of the basic notation to be followed in the remainder of the paper. Throughout, $\mathbf{A}$ denotes the ring $2[x]$. The set of non-zero elements of $\mathbf{A}$ will be denoted by $\mathbf{A}^*$. Typically, in our proofs, we will use uppercase letters A, D, F and G to denote the elements of $\mathbf{A}$ where $D \in \mathbf{A}^*$, generally, will denote a divisor of some element in $\mathbf{A}$. The letter P is reserved for a non-zero prime (irreducible) in $\mathbf{A}$. Following [10], we define the *norm* $|A|$ of $A \in \mathbf{A}^*$ as

$$|A| = 2^{\deg A}.$$

As it turns out, $|A|$ is the correct analogue for the size of an integer in $\mathbb{Z}$. Sometimes, for A and A' in $\mathbf{A}$, we will use (A, A') to denote $\gcd(A, A')$. The function $\nu(A)$ will denote the number of distinct prime factors of $A \in \mathbf{A}^*$ with $\nu(1) = 0$. For a squarefree $A \in \mathbf{A}^*$, the Möbius function $\mu(A) = (-1)^{\nu(A)}$. Otherwise, $\mu(A) = 0$. For a real number $x > 0$, we will denote by $\log_2 x$ the base-2 logarithm of x, and $\log x$ denotes the natural logarithm of x.

The paper is organized as follows. We develop the necessary technical details, namely the Brun–Hooley sieve for $\mathbf{A}$, in § 2. Theorem 1 and Theorem 2 are respectively proved in § 3 and § 4.

2. Brun–Hooley sieve for $\mathbb{F}_2[x]$

Let $\mathcal{A} \subset \mathbf{A}$ with $\#\mathcal{A} = X$. Let z be a real number satisfying $2 \leqslant z \leqslant X$. Let

$$\mathscr{P} = \mathscr{P}(z) := \{P \in \mathbf{A}^* \text{ is prime} : |P| \leqslant z\}, \tag{2.1}$$

and define

$$\Pi = \Pi(z) := \prod_{P \in \mathscr{P}} P. \tag{2.2}$$

We fix a total order $\prec$ on $\mathbf{A}$. For instance, for F and G in $\mathbf{A}$, we say that $F \prec G$ if $F(2) < G(2)$ when $F(x)$ and $G(x)$ are considered as polynomials in $\mathbb{R}[x]$. Observe that F and G, when considered as polynomials in $\mathbb{R}[x]$, have coefficients in $\{0, 1\}$, so that

$$F(2) \neq G(2) \quad \Longleftrightarrow \quad F(x) \neq G(x),$$

as polynomials in $\mathbb{R}[x]$. Hence, if $F \neq G$ in $\mathbf{A}$, then exactly one of $F \prec G$ and $G \prec F$ holds. It is easy to see that $\prec$ thus defined is a total order in $\mathbf{A}$. In particular, every squarefree $A \neq 1$ can be uniquely expressed as the product

$$A = P_1 P_2 \cdots P_r,$$

where P_1, P_2, $\ldots$, P_r are primes in $\mathbf{A}^*$ satisfying

$$P_1 \prec P_2 \prec \cdots \prec P_r.$$

Additionally, for A as above, define $p^-(A) = P_1$ and $p^+(A) = P_r$. We also set $p^-(1) = 1 = p^+(1)$.

For each $D \in \mathbf{A}^*$, let

$$\mathcal{A}_D := \{A \in \mathcal{A} : D \mid A\},$$

with the understanding that $\mathcal{A}_1 = \mathcal{A}$. We suppose that there is a real-valued function ω satisfying

$$\omega(1) = 1, \quad 0 \leqslant \omega(P) \leqslant 1 \tag{Ω}$$

for every prime $P \in \mathbf{A}^*$. Next, extend ω multiplicatively to all of $\mathbf{A}^*$ by defining

$$\omega(D) := \prod_{P|D} \omega(P).$$

For a $D \in \mathbf{A}^*$, we denote by r_D the quantity

$$r_D := \#\mathcal{A}_D - \frac{\omega(D)}{|D|}X.$$

We assume that

$$|r_D| \leqslant \omega(D), \quad D \in \mathbf{A}^*. \tag{r}$$

Further, define

$$W = W(z) := \prod_{P \in \mathscr{P}} \left(1 - \frac{\omega(P)}{|P|}\right), \tag{2.3}$$

and let

$$S(\mathcal{A}; z) := \#\{A \in \mathcal{A} : (A, \Pi) = 1\}.$$

Our main result in this section is the following.

Theorem 3. *(Brun–Hooley sieve for $2[x]$) Let $\mathcal{A}$, X, z, W and $S(\mathcal{A}; z)$ be as defined above. Let ω be a multiplicative function on $\mathbf{A}^*$ satisfying (Ω) and (r). Then for $z \gg 1$, one has*

(i) $S(\mathcal{A}; z) \geqslant 0.0001XW - z^{4.6385}$ *and*
(ii) $S(\mathcal{A}; z) \leqslant eXW + z^{3.6385}$.

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The proof of the next lemma is identical to its integer counterpart.

Lemma 1. *For every $A \in A^*$, one has*

$$\sum_{D|A} \mu(D) = \begin{cases} 1 & \textit{if} \quad A = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 2. *Let $\mathfrak{f}$ be a real-valued multiplicative function defined on $\mathbf{A}^*$, and let $A \in \mathbf{A}^*$ be squarefree. Then for every integer $k \geqslant 0$, one has*

$$\sum_{\substack{D|A \\ \nu(D)\leqslant k}} \mu(D)\mathfrak{f}(D) = \sum_{D|A} \mu(D)\mathfrak{f}(D) + (-1)^k \sum_{\substack{D|A \\ \nu(D)=k+1}} \mathfrak{f}(D) \prod_{\substack{P\in\mathscr{P} \\ P\prec p^-(D)}} (1-\mathfrak{f}(P)),$$

where an empty product is equal to 1.

Proof. Consider the terms in the sum on the right corresponding to D with $\nu(D) \geqslant k+1$. Every such D can be uniquely expressed as

$$D = D_1 D_2,$$

where $\nu(D_1) = k+1$, and D_2 is either 1 or $p^+(D_2) \prec p^-(D_1)$. It follows that

$$\begin{aligned} \sum_{D|A} \mu(D)\mathfrak{f}(D) - \sum_{\substack{D|A \\ \nu(D)\leqslant k}} \mu(D)\mathfrak{f}(D) &= \sum_{\substack{D|A \\ \nu(D)\geqslant k+1}} \mu(D)\mathfrak{f}(D) \\ &= \sum_{\substack{D_1|A \\ \nu(D_1)=k+1}} \mu(D_1)\mathfrak{f}(D_1) \sum_{\substack{D_2|A \\ p^+(D_2)\prec p^-(D_1)}} \mu(D_2)\mathfrak{f}(D_2) \\ &= (-1)^{k+1} \sum_{\substack{D|A \\ \nu(D)=k+1}} \mathfrak{f}(D) \prod_{\substack{P\in\mathscr{P} \\ P\prec p^-(D)}} (1-\mathfrak{f}(P)). \end{aligned}$$

The lemma follows. □

Corollary 2. *Let $\mathfrak{f}$ be a multiplicative function defined on $\mathbf{A}^*$ satisfying $0 \leqslant \mathfrak{f}(P) \leqslant 1$ for every prime P, and let $A \in \mathbf{A}^*$ be squarefree. Then for every even integer $k \geqslant 0$, one has*

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$$\sum_{D|A} \mu(D)\mathfrak{f}(D) \leqslant \sum_{\substack{D|A \\ \nu(D)\leqslant k}} \mu(D)\mathfrak{f}(D) \leqslant \sum_{D|A} \mu(D)\mathfrak{f}(D) + \sum_{\substack{D|A \\ \nu(D)=k+1}} \mathfrak{f}(D).$$

Let $z \geqslant 2$ be as defined earlier, and let $2 = z_{t+1} < z_t < \cdots < z_1 = z$. Partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \cdots \cup \mathscr{P}_t$ such that if $P \in \mathscr{P}_j$, then $z_{j+1} < |P| \leqslant z_j$ if $j < t$ and $z_{t+1} \leqslant |P| \leqslant z_t$ if $j = t$. Set

$$\Pi_j = \prod_{P \in \mathscr{P}_j} P,$$

so that

$$\prod_{j=1}^{t} \Pi_j = \Pi.$$

In proving Theorem 1, we will need both upper and lower bounds on $S(\mathcal{A}; z)$. As is usually the case, achieving a lower bound is relatively more difficult. We next embark on this pursuit. To this end, we begin with Hooley's lemma (for proof, see Lemma 12.6, [3]), which is the key step in the usual Brun–Hooley lower bound sieve.

Lemma 3. *Suppose that $0 \leqslant x_j \leqslant y_j$ for $1 \leqslant j \leqslant t$. Then one has*

$$x_1 x_2 \cdots x_t = y_1 y_2 \cdots y_t - \sum_{\ell=1}^{t} (y_\ell - x_\ell) \prod_{\substack{j=1 \\ j\neq \ell}}^{t} y_j.$$

Let $k_1, k_2, \ldots, k_t$ be a sequence of even non-negative integers. For each $j \in \{1, 2, \ldots, t\}$ and $A \in \mathcal{A}$, set

$$x_j = \sum_{D|(A,\Pi_j)} \mu(D), \quad y_j = \sum_{\substack{D|(A,\Pi_j) \\ \nu(D)\leqslant k_j}} \mu(D).$$

Setting $\mathfrak{f} \equiv 1$ and $A = (A, \Pi_j)$ in Corollary 2, we find that $x_j \leqslant y_j$ for every j. Furthermore, since k_j is even, setting $\mathfrak{f} \equiv 1$ in Corollary 2 again, we get

$$y_\ell - x_\ell \leqslant \sum_{\substack{D|(A,\Pi_\ell) \\ \nu(D)=k_\ell+1}} 1.$$

Thus, by Lemma 3, we have

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$$\sum_{D|(A,\Pi)} \mu(D) \geqslant \prod_{j=1}^{t} \sum_{\substack{D|(A,\Pi_j)\\ \nu(D)\leqslant k_j}} \mu(D) - \sum_{\ell=1}^{t} \sum_{\substack{D|(A,\Pi_\ell)\\ \nu(D)=k_\ell+1}} \left(\prod_{\substack{j=1\\ j\neq \ell}}^{t} \left(\sum_{\substack{D|(A,\Pi_j)\\ \nu(D)\leqslant k_j}}^{t} \mu(D) \right) \right)$$

$$= \sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|(A,\Pi_j)\\ \nu(D_j)\leqslant k_j}} \mu(D_1D_2\cdots D_t) - \sum_{\ell=1}^{t} \left(\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|(A,\Pi_j)\\ \nu(D_j)\leqslant k_j, j\neq \ell\\ \nu(D_\ell)=k_\ell+1}} \mu\left(\frac{D_1D_2\cdots D_t}{D_\ell}\right) \right).$$

Now, using Lemma 1 and the last lower bound above, we obtain

$$S(\mathcal{A};z) = \sum_{A\in\mathcal{A}} \sum_{D|(A,\Pi)} \mu(D)$$

$$\geqslant \sum_{A\in\mathcal{A}} \sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|(A,\Pi_j)\\ \nu(D_j)\leqslant k_j}} \mu(D_1D_2\cdots D_t) - \sum_{A\in\mathcal{A}} \sum_{\ell=1}^{t} \left(\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|(A,\Pi_j)\\ \nu(D_j)\leqslant k_j, j\neq \ell\\ \nu(D_\ell)=k_\ell+1}} \mu\left(\frac{D_1D_2\cdots D_t}{D_\ell}\right) \right)$$

$$= \sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j}} \mu(D_1D_2\cdots D_t)\#\mathcal{A}_{D_1D_2\cdots D_t}$$

$$- \sum_{\ell=1}^{t} \left(\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j, j\neq \ell\\ \nu(D_\ell)=k_\ell+1}} \mu\left(\frac{D_1D_2\cdots D_t}{D_\ell}\right) \#\mathcal{A}_{D_1D_2\cdots D_t} \right).$$

Setting above

$$\#\mathcal{A}_{D_1D_2\cdots D_t} = \frac{\omega(D_1D_2\cdots D_t)}{|D_1D_2\cdots D_t|}X + r_{D_1D_2\cdots D_t},$$

we get

$$S(\mathcal{A};z) \geqslant X\Sigma - R,$$

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where

$$\Sigma = \sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j}} \prod_{j=1}^{t} \mu(D_j)\frac{\omega(D_j)}{|D_j|} - \sum_{\ell=1}^{t}\left(\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j, j\neq \ell\\ \nu(D_\ell)=k_\ell+1}} \frac{\omega(D_\ell)}{|D_\ell|} \prod_{\substack{j=1\\ j\neq\ell}}^{t} \mu(D_j)\frac{\omega(D_j)}{|D_j|}\right), \tag{2.4}$$

and

$$R = \sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j}} |r_{D_1D_2\cdots D_t}| + \sum_{\ell=1}^{t}\left(\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j, j\neq \ell\\ \nu(D_\ell)=k_\ell+1}} |r_{D_1D_2\cdots D_t}|\right).$$

By assumptions (Ω) and (r), we have $|r_D| \leqslant \omega(D) \leqslant 1$. Therefore,

$$R \leqslant \sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j}} 1 + \sum_{\ell=1}^{t}\left(\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j, j\neq \ell\\ \nu(D_\ell)=k_\ell+1}} 1\right).$$

The above sum is over all D_1, D_2, …, D_t satisfying $D_j \mid \Pi_j$, and either $\nu(D_j) \leqslant k_j$ for all j, or $\nu(D_j) \leqslant k_j$ for all but one j for which $\nu(D_j) = k_j + 1$. This is bounded by

$$\sum_{|D|\leqslant z_1^{k_1+1} z_2^{k_2}\cdots z_t^{k_t}} \mu^2(D) < 2z_1^{k_1+1}z_2^{k_2}\cdots z_t^{k_t}.$$

Thus,

$$R < Z := 2z_1^{k_1+1}z_2^{k_2}\cdots z_t^{k_t}. \tag{2.5}$$

Next, for each $j \in \{1, 2, \ldots, t\}$, define

$$U_j := \sum_{\substack{D|\Pi_j\\ \nu(D)\leqslant k_j}} \mu(D)\frac{\omega(D)}{|D|}, \quad W_j := \sum_{D|\Pi_j} \mu(D)\frac{\omega(D)}{|D|} = \prod_{P\in\mathscr{P}_j}\left(1 - \frac{\omega(P)}{|P|}\right).$$

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Then

$$\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j}}\prod_{j=1}^{t}\mu(D_j)\frac{\omega(D_j)}{|D_j|}=U_1U_2\cdots U_t, \tag{2.6}$$

and

$$\sum_{\ell=1}^{t}\left(\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j, j\neq\ell\\ \nu(D_\ell)=k_\ell+1}}\frac{\omega(D_\ell)}{|D_\ell|}\prod_{\substack{j=1\\ j\neq\ell}}^{t}\mu(D_j)\frac{\omega(D_j)}{|D_j}\right)=U_1U_2\cdots U_t\sum_{\ell=1}^{t}\frac{1}{U_\ell}\sum_{\substack{D_\ell|\Pi_\ell\\ \nu(D_\ell)=k_\ell+1}}\frac{\omega(D_\ell)}{|D_\ell|}. \tag{2.7}$$

From Equations (2.4), (2.6) and (2.7), we have

$$\Sigma=U_1U_2\cdots U_t\left(1-\sum_{\ell=1}^{t}\frac{1}{U_\ell}\sum_{\substack{D_\ell|\Pi_\ell\\ \nu(D_\ell)=k_\ell+1}}\frac{\omega(D_\ell)}{|D_\ell|}\right). \tag{2.8}$$

By Corollary 2,

$$U_j\geqslant W_j,\quad j=1,2,\cdots,t,$$

so that

$$U_1U_2\cdots U_t\geqslant W_1W_2\cdots W_t:=W.$$

Next, in order to estimate the expression following the negative sign in Equation (2.8), we will make use of the following lemma.

Lemma 4. *We have*

$$\sum_{\substack{D|\Pi_\ell\\ \nu(D)=k_\ell+1}}\frac{\omega(D)}{|D|}\leqslant\frac{I_\ell^{k_\ell+1}}{(k_\ell+1)!},$$

where

$$I_\ell=\log\frac{1}{W_\ell}=-\sum_{P|\Pi_\ell}\log\left(1-\frac{\omega(P)}{|P|}\right).$$

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Proof. Let $\mathscr{P}_\ell = \{P_1, P_2, \ldots, P_T\}$ with

$$P_1 \prec P_2 \prec \cdots \prec P_T.$$

For $D \mid \Pi_\ell$, set $\mathfrak{f}(D) = \omega(D)/|D|$. Thus, $0 \leqslant \mathfrak{f}(D) < 1$. By the multinomial theorem, we have

$$\begin{aligned}\left(\sum_{P\in\mathscr{P}_\ell} \mathfrak{f}(P)\right)^{k_\ell+1} &= \sum_{\substack{m_1+m_2+\cdots+m_T=k_\ell+1\\ m_j\geqslant 0}} \frac{(k_\ell+1)!}{m_1!m_2!\cdots m_T!}\prod_{j=1}^{T}\mathfrak{f}(P_j)^{m_j}\\ &> (k_\ell+1)! \sum_{P_{e_1}\prec P_{e_2}\prec\cdots\prec P_{e_{k_\ell+1}}} \mathfrak{f}(P_{e_1})\mathfrak{f}(P_{e_2})\cdots\mathfrak{f}(P_{e_{k_\ell+1}})\\ &= (k_\ell+1)! \sum_{\substack{D\mid\Pi_\ell\\ \nu(D)=k_\ell+1}} \mathfrak{f}(D).\end{aligned}$$

On the other hand, since $0 \leqslant \mathfrak{f}(P) < 1$, we have

$$\sum_{P\in\mathscr{P}_\ell} \mathfrak{f}(P) \leqslant \sum_{P\in\mathscr{P}_\ell} -(\log(1-\mathfrak{f}(P))) = \log\frac{1}{W_\ell} = I_\ell.$$

This finishes the proof of the lemma. □

Now, by the estimate of Lemma 4, we have

$$\sum_{\substack{D\mid\Pi_\ell\\ \nu(D)=k_\ell+1}} \frac{\omega(D_\ell)}{|D_\ell|} \leqslant W_\ell\left(\frac{W_\ell^{-1}I_\ell^{k_\ell+1}}{(k_\ell+1)!}\right) = W_\ell\left(\frac{e^{I_\ell}I_\ell^{k_\ell+1}}{(k_\ell+1)!}\right).$$

Recalling that $U_\ell \geqslant W_\ell$, we get

$$\frac{1}{U_\ell}\sum_{\substack{d_\ell\mid\Pi_\ell\\ \nu(d_\ell)=k_\ell+1}} \frac{\omega(D_\ell)}{|D_\ell|} \leqslant \frac{e^{I_\ell}I_\ell^{k_\ell+1}}{(k_\ell+1)!}.$$

Observe that if $k_\ell = 0$ for some ℓ, then $U_\ell = 1$. Accordingly, in this case, the expression on the left side of the last display is then bounded by

$$W_\ell\left(\frac{e^{I_\ell}I_\ell^{k_\ell+1}}{(k_\ell+1)!}\right) = W_\ell e^{I_\ell}I_\ell = I_\ell.$$

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From these estimates, we deduce from Equation (2.8) that

$$\Sigma \geqslant (1-E)\,W, \quad E=\sum_{\ell=1}^{t}\frac{\psi(\ell)I_\ell^{k_\ell+1}}{(k_\ell+1)!}, \tag{2.9}$$

where

$$\psi(\ell)=\begin{cases} e^{I_\ell} & \text{if} \quad k_\ell\neq 0 \\ 1 & \text{if} \quad k_\ell=0. \end{cases} \tag{2.10}$$

As such,

$$S(\mathcal{A};z)\geqslant X(1-E)W-Z, \tag{2.11}$$

where Z is as defined in Equation (2.5). Next, we obtain an upper bound on $S(\mathcal{A};z)$. In this case, from Corollary 2, we have

$$\sum_{D|(a,\Pi)}\mu(D)=\prod_{j=1}^{t}\sum_{D_j|(a,\Pi_j)}\mu(D_j)\leqslant\prod_{j=1}^{t}\sum_{\substack{D_j|(a,\Pi_j)\\ \nu(D_j)\leqslant k_j}}\mu(D_j).$$

Accordingly, we have, using Lemma 1, that

$$\begin{aligned} S(\mathcal{A};z) &= \sum_{A\in\mathcal{A}}\sum_{D|(A,\Pi)}\mu(D) \\ &\leqslant \sum_{A\in\mathcal{A}}\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|(A,\Pi_j)\\ \nu(D_j)\leqslant k_j}}\mu(D_1D_2\cdots D_t) \\ &= \sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j}}\mu(D_1D_2\cdots D_t)\#\mathcal{A}_{D_1D_2\cdots D_t} \\ &= X\Sigma+R, \end{aligned}$$

where

$$\Sigma=\sum_{\substack{D_1,D_2,\ldots,D_t\\ D_j|\Pi_j\\ \nu(D_j)\leqslant k_j}}\prod_{j=1}^{t}\mu(D_j)\frac{\omega(D_j)}{|D_j|}=\prod_{j=1}^{t}\sum_{\substack{D|\Pi_j\\ \nu(D)\leqslant k_j}}\mu(D)\frac{\omega(D)}{|D|},$$

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and

$$R = \sum_{\substack{D_1, D_2, \ldots, D_t \\ D_j \mid \Pi_j \\ \nu(D_j) \leqslant k_j}} \mu(D_1 D_2 \cdots D_t) r_{D_1 D_2 \cdots D_t}.$$

Working as before,

$$|R| \leqslant \sum_{\substack{D_1, D_2, \ldots, D_t \\ D_j \mid \Pi_j \\ \nu(D_j) \leqslant k_j}} 1 \leqslant 2 z_1^{k_1} z_2^{k_2} \cdots z_t^{k_t} = \frac{Z}{z_1} = \frac{Z}{z}.$$

Appealing again to Corollary 2, we have

$$\sum_{\substack{D_j \mid \Pi_j \\ \nu(D_j) \leqslant k_j}} \mu(D_j) \frac{\omega(D_j)}{|D_j|} \leqslant \sum_{D_j \mid \Pi_j} \mu(D_j) \frac{\omega(D_j)}{|D_j|} + \sum_{\substack{D_j \mid \Pi_j \\ \nu(D_j) = k_j + 1}} \mu(D_j) \frac{\omega(D_j)}{|D_j|}$$
$$= W_j + \sum_{\substack{D_j \mid \Pi_j \\ \nu(D_j) = k_j + 1}} \mu(D_j) \frac{\omega(D_j)}{|D_j|}.$$

Proceeding as in the proof of Lemma 4, we get

$$\sum_{\substack{D_j \mid \Pi_j \\ \nu(D_j) = k_j + 1}} \mu(D_j) \frac{\omega(D_j)}{|D_j|} \leqslant \frac{I_j^{k_j + 1}}{(k_j + 1)!}.$$

Therefore,

$$\sum_{\substack{D_j \mid \Pi_j \\ \nu(D_j) \leqslant k_j}} \mu(D_j) \frac{\omega(D_j)}{|D_j|} \leqslant W_j \left(1 + \frac{e^{I_j} I_j^{k_j + 1}}{(k_j + 1)!}\right).$$

However, if $k_j = 0$ for some j, then the left side of the last display is equal to 1, and consequently, it is bounded by $W_j(1 + I_j)$ since $W_j \geqslant 1$. Thus,

$$\Sigma \leqslant \prod_{j=1}^{t} W_j \left(1 + \frac{\psi(j) I_j^{k_j + 1}}{(k_j + 1)!}\right) \leqslant W \prod_{j=1}^{t} \exp\left(\frac{\psi(j) I_j^{k_j + 1}}{(k_j + 1)!}\right) = W \exp(E),$$

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where E and $\psi(j)$ are as defined by Equations (2.9) and (2.10), respectively. In conclusion,

$$S(\mathcal{A}; z) \leqslant XW \exp(E) + \frac{Z}{z}, \tag{2.12}$$

where Z is as defined in Equation (2.5).

Next, we choose the parameters z_2, z_3, ..., z_t and k_1, k_2, ..., k_t optimally to obtain explicit upper and lower bounds on $S(\mathcal{A}; z)$ suitable for our purposes. Set $c = 0.26249$. For each $j \in \{1, 2, \dots, t\}$, set

$$\alpha_j = \exp\left(c(j-1)^2\right),$$

and $z_j = z^{1/\alpha_j}$. Let t be the maximal positive integer such that

$$z^{1/\alpha_t} > 2.$$

That is,

$$t = \left\lceil \sqrt{\frac{1}{c} \log \log_2 z} \right\rceil,$$

where, for a real number x, we denote by $\lceil x \rceil$, the integer m satisfying $m - 1 < x \leqslant m$. Next, set $k_j = 2(j-1)$. In order to make the bounds (2.11) and (2.12) explicit, we need to find suitable upper bounds on E and Z. To this end, we begin by estimating I_ℓ.

Lemma 5. *We have*

$$I_\ell \leq \begin{cases} \sum_{\log_2 z_{\ell+1} < \deg P \leqslant \log_2 z_\ell} \frac{1}{|P|} + \frac{1}{|P|^2} & \text{if} \quad \ell < t \\ \sum_{1 \leqslant \deg P \leqslant \log_2 z_\ell} \frac{1}{|P|} + \frac{1}{|P|^2} & \text{if} \quad \ell = t. \end{cases}$$

Proof. Since $|P| \geqslant 2$ for every $P \in \mathscr{P}_\ell$, we have

$$\log\left(1 - \frac{\omega(P)}{|P|}\right)^{-1} = \omega(P) \sum_{j=1}^{\infty} \frac{1}{j|P|^j} \leqslant \frac{1}{|P|} + \frac{1}{|P|^2}.$$

The lemma follows after recalling the definition of I_ℓ. □

For an integer $d \geqslant 1$, recall that M_d, the number of irreducible polynomials in $\mathbf{A}$ of degree d, satisfies $M_d \leqslant 2^d/d$. Since $z_{\ell+1} \geqslant 2$, it follows from Lemma 5 that for every $\ell < t$,

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$$\begin{aligned}
I_\ell &\leqslant \sum_{\log_2 z_{\ell+1} < d \leqslant \log_2 z_\ell} \left(\frac{1}{d} + \frac{1}{d2^d}\right) \\
&\leqslant \log \frac{\alpha_{\ell+1}}{\alpha_\ell} + \sum_{\log_2 z_{\ell+1} < d \leqslant \log_2 z_\ell} \int_0^{1/2} x^{d-1}\,\mathrm{d}x \\
&\leqslant c(2\ell-1) + \sum_{\log_2 z_{\ell+1} < d \leqslant \log_2 z_\ell} \int_0^{1/2} x^{d-1}\,\mathrm{d}x.
\end{aligned}$$

We estimate the second sum above as follows. For $x \in (0, 1/2]$, one has

$$\sum_{\log_2 z_{\ell+1} < d \leqslant \log_2 z_\ell} x^{d-1} \leqslant 2x^{\log_2 z_{\ell+1}-1}.$$

Thus,

$$\begin{aligned}
\sum_{\log_2 z_{\ell+1} < d \leqslant \log_2 z_\ell} \int_0^{1/2} x^{d-1}\,\mathrm{d}x &= \int_0^{1/2} \left(\sum_{\log_2 z_{\ell+1} < d \leqslant \log_2 z_\ell} x^{d-1}\right)\mathrm{d}x \\
&\leqslant 2\int_0^{1/2} x^{\log_2 z_{\ell+1}-1}\,\mathrm{d}x \\
&= \frac{2}{z_{\ell+1}\log_2 z_{\ell+1}} \\
&\leqslant \frac{2}{z_{\ell+1}},
\end{aligned}$$

since $z_{\ell+1} \geqslant 2$. It follows that

$$I_\ell \leqslant c(2\ell-1) + \frac{2}{z_{\ell+1}}. \tag{2.13}$$

Working similarly, for $\ell = t$, we obtain from Lemma 5 that

$$\begin{aligned}
I_t &\leqslant \sum_{1 \leqslant \deg P \leqslant \log_2 z_t} \left(\frac{1}{|P|} + \frac{1}{|P|^2}\right) \\
&\leqslant \sum_{1 \leqslant d \leqslant \frac{\log_2 z}{\alpha_t}} \left(\frac{1}{d} + \frac{1}{d2^d}\right) \\
&< 2 + (\log\log_2 z - \log\alpha_t) \\
&= 2 + \left(\log\log_2 z - c(t-1)^2\right).
\end{aligned}$$

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Next, recall that

$$t = \left\lceil \sqrt{\frac{1}{c}\log\log_2 z} \right\rceil \geqslant \sqrt{\frac{1}{c}\log\log_2 z},$$

so that

$$ct^2 > \log\log_2 z.$$

Using the last estimate, we deduce that

$$I_t \leqslant 2 + c(2t-1) < 0.27(2t-1),$$

for $t \gg 1$. Thus, for $z \gg 1$ (so that $t \gg 1$), the contribution of $\ell = t$ in the sum for E in Equation (2.9) is bounded by

$$\frac{e^{0.27(2t-1)}\left(0.27(2t-1)\right)^{2t-1}}{(2t-1)!} < \left(0.27e^{1.27}\right)^{2t-1} < (0.97)^{2t-1}, \tag{2.14}$$

where, we have used that $(2t-1)^{2t-1}/(2t-1)! < e^{2t-1}$.

We will next estimate E by separately considering the contributions from terms corresponding to $\ell < t$ for which $\alpha_{\ell+1} \leqslant \sqrt{\log z}$ and $\alpha_{\ell+1} > \sqrt{\log z}$. First, consider the case that $\alpha_{\ell+1} \leqslant \sqrt{\log z}$. In this case,

$$z_{\ell+1} = z^{1/\alpha_{\ell+1}} \geqslant z^{1/\sqrt{\log z}} = e^{\sqrt{\log z}}.$$

Thus, from Equation (2.13) and the above, we get that

$$I_\ell \leqslant c(2\ell-1) + \frac{2}{e^{\sqrt{\log z}}}.$$

Additionally, $\alpha_{\ell+1} \leqslant \sqrt{\log z}$ implies that $c\ell^2 \leqslant (\log\log z)/2$. That is,

$$\ell \leqslant 1.5\sqrt{\log\log z}.$$

Let ψ be as defined in Equation (2.10). Note that $\psi(1) = 1$. For $1 < \ell \leqslant 1.5\sqrt{\log\log z}$ and for $z \gg 1$, using the estimates for I_ℓ from Equation (2.13), we have

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$$\begin{aligned}\frac{\psi(\ell)I_\ell^{2\ell-1}}{(2\ell-1)!} &\leqslant e^{2/\exp(\sqrt{\log z})}e^{c(2\ell-1)}\frac{\left(c(2\ell-1)+\frac{2}{e^{\sqrt{\log z}}}\right)^{2\ell-1}}{(2\ell-1)!}\\ &\leqslant e^{c(2\ell-1)}\left(1+O(e^{-\sqrt{(\log z)}})\right)\frac{\left((c(2\ell-1))^{2\ell-1}+(2\ell-1)^{2\ell-1}e^{-\sqrt{\log z}}3^{2\ell-1}\right)}{(2\ell-1)!}\\ &\leqslant e^{c(2\ell-1)}\left(1+O(e^{-\sqrt{(\log z)}})\right)\left(\frac{(c(2\ell-1))^{2\ell-1}}{(2\ell-1)!}+O(e^{3(2\ell-1)}e^{-\sqrt{\log z}})\right)\\ &= e^{c(2\ell-1)}\left(1+O(e^{-\sqrt{(\log z)}})\right)\left(\frac{(c(2\ell-1))^{2\ell-1}}{(2\ell-1)!}+O(e^{-\sqrt{\log z}/2})\right)\\ &= \frac{e^{c(2\ell-1)}(c(2\ell-1))^{2\ell-1}}{(2\ell-1)!}+O(e^{-\sqrt{\log z}/3}),\end{aligned}$$

where, to obtain the bound in the second line above, we have used the binomial theorem as follows:

$$\begin{aligned}\left(c(2\ell-1)+\frac{2}{e^{\sqrt{\log z}}}\right)^{2\ell-1} &\leqslant (c(2\ell-1))^{2\ell-1}+\sum_{j=1}^{2\ell-1}\binom{2\ell-1}{j}(c(2\ell-1))^{2\ell-1-j}2^j\\ &< (c(2\ell-1))^{2\ell-1}+(2\ell-1)^{2\ell-1}(2+c)^{2\ell-1}\\ &< (c(2\ell-1))^{2\ell-1}+(2\ell-1)^{2\ell-1}3^{2\ell-1}.\end{aligned}$$

Thus, the contribution to the sum E from the terms corresponding to $\alpha_{\ell+1}\leqslant\sqrt{\log z}$ is bounded above by

$$\begin{aligned}&I_1+\sum_{\ell>1}\frac{e^{c(2\ell-1)}(c(2\ell-1))^{2\ell-1}}{(2\ell-1)!}+O(\sqrt{\log\log z}e^{-\sqrt{\log z}/3}) \qquad (2.15)\\ &< c+\sum_{\ell>1}\frac{e^{c(2\ell-1)}(c(2\ell-1))^{2\ell-1}}{(2\ell-1)!}+O(e^{-\sqrt{\log z}/4})\\ &< 0.9997+O(e^{-\sqrt{\log z}/4}).\end{aligned}$$

Next, consider the case that $\alpha_{\ell+1}>\sqrt{\log z}$. In this case, $\ell>\sqrt{\log\log z}$. Since $z_{\ell+1}\geqslant 2$, for $\sqrt{\log\log z}<\ell<t$, we have from Equation (2.13) that

$$I_\ell\leqslant c(2\ell-1)+\frac{2}{z_{\ell+1}}\leqslant c(2\ell-1)+1,$$

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since $z_{\ell+1} \geqslant 2$. Thus, for ℓ as above and z sufficiently large, we have

$$\begin{aligned}\frac{\psi(\ell) I_\ell^{2\ell-1}}{(2\ell-1)!} &\leqslant e^{c(2\ell-1)+1}\frac{(c(2\ell-1)+1)^{2\ell-1}}{(2\ell-1)!}\\ &< e^{0.27(2\ell-1)}\frac{(0.27(2\ell-1))^{2\ell-1}}{(2\ell-1)!}\\ &< (0.27e^{1.27})^{2\ell-1} < 0.97^{2\ell-1}.\end{aligned}$$

Thus, the contribution to the sum E from the terms corresponding to the ℓ under consideration is less than

$$\sum_{\ell > \sqrt{\log\log z}} (0.97)^{2\ell-1} = O(0.97^{\sqrt{\log\log z}}).$$

From the last estimate above and Equation (2.15), we deduce that

$$E < 0.9999$$

for $z \gg 1$.

It remains to estimate

$$Z := 2z_1^{k_1+1} z_2^{k_2} \cdots z_t^{k_t} = 2\exp\left(\log z\left(\frac{1}{\alpha_1} + \frac{2}{\alpha_2} + \cdots + \frac{2(t-1)}{\alpha_t}\right)\right).$$

The exponent of z above is bounded by

$$1 + \sum_{n=1}^{\infty} \frac{2n}{\exp(0.26249n^2)} < 4.63833.$$

We now obtain (i) and (ii) of Theorem 3 by putting the estimates $E < 0.9999$ and $Z < z^{4.6385}$ (for $z \gg 1$) in Equations (2.11) and (2.12), respectively.

3. A proof of Theorem 1

Let $f(x)$ and $g(x)$ be as stated in Theorem 1 with deg $f = n$. Let $t := \lfloor 4.64 \log_2 n \rfloor$, and set $X := 2^t$. Observe that $t \leqslant 4.64 \log_2 n < 6.7 \log n$. For future reference, we make a note of the fact that

$$n < 2^{\frac{t+1}{4.64}} < 2X^{\frac{1}{4.64}}.$$

Let

$$\mathcal{A} := \{g + u : u \in \mathbf{A}, \deg\, u < t\}.$$

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Thus, $\#\mathcal{A} = X$. We will establish that for $n \gg 1$, there is some $g_1 \in \mathcal{A}$ satisfying $\gcd(f, g_1) = 1$. If $g_1 = g + u$, then

$$\deg\, g_1 \leqslant \max\{\deg\, g, \deg\, u\} \leqslant \max\{\deg\, g, 6.7 \log n\},$$

and

$$L(g - g_1) = L(u) \leqslant \deg\, u + 1 \leqslant t < 6.7 \log n,$$

as is required to be shown.

Let $P \in \mathbf{A}^*$ be irreducible. If $P \mid f$, and $\deg\, P > t$, then P divides at most one polynomial in $\mathcal{A}$. Thus, at most n polynomials in $\mathcal{A}$ have a common prime factor of degree greater than t with f.

For every irreducible $P \in \mathbf{A}^*$ with $\deg\, P \leqslant t$, we define $\omega(P) = 1$ if P divides some element of $\mathcal{A}$, and $\omega(P) = 0$, otherwise. We extend ω multiplicatively to all of $\mathbf{A}^*$ by defining

$$\omega(D) := \prod_{P \mid D} \omega(P), \quad D \in \mathbf{A}^*.$$

For $D \in \mathbf{A}^*$, let

$$\mathcal{A}_D := \{A \in \mathcal{A} : D \mid A\}.$$

Observe that if $\deg\, D \leqslant t$, then $\omega(D) = 1$ implies that

$$\#\mathcal{A}_D = 2^{t - \deg\, D} = \frac{\omega(D)}{|D|} X.$$

If $\deg\, D > t$ and $\omega(D) = 1$, then $\#\mathcal{A}_D = 1$; while, $\omega(D) = 0$ implies $\#\mathcal{A}_D = 0$. Define

$$r_D := |\mathcal{A}_D| - \frac{\omega(D)}{|D|} X.$$

Then $r_D = 0$ if either $\deg\, D \leqslant t$ or $\omega(D) = 0$. If $\deg\, D > t$ and $\omega(D) = 1$, then

$$r_D = 1 - 2^{t - \deg\, D} < 1.$$

Thus, in any case, $0 \leqslant r_D \leqslant \omega(D)$. In particular, $\omega(D)$ and r_D satisfy (Ω) and (r). Let

$$\mathcal{P}_f = \{P \,\text{is irreducible} : P \mid f, \omega(P) = 1\},$$

and

$$\Pi_f = \prod_{P \in \mathcal{P}_f} P.$$

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Note that $\deg \Pi_f \leqslant \deg f$, and if $A \in \mathcal{A}$, then $(f, A) = 1$ if and only if $(A, \Pi_f) = 1$. So, without loss of any generality, we may and do assume that $f = \Pi_f$. Specifically, $\omega(P) = 1$ for every $P \mid f$.

Next, set $z = X^{\frac{1}{4.64}}$ in Theorem 3. We have

$$z = 2^{\frac{t}{4.64}} = 2^{\frac{\lfloor 4.64 \log_2 n \rfloor}{4.64}} \leqslant n.$$

Let $\mathscr{P}$, Π and W have the same meaning as implied in Equations (2.1), (2.2) and (2.3), respectively. Then the conclusion (i) of Theorem 3 implies that

$$S(\mathcal{A}; X^{\frac{1}{4.64}}) \geqslant 0.0001XW - X^{\frac{4.6385}{4.64}}, \tag{3.1}$$

for $n \gg 1$. Let $\mathcal{A}'$ denote the set $\{A \in \mathcal{A} : (A, \Pi) = 1\}$. Thus, the norm of each irreducible factor of every polynomial in $\mathcal{A}'$ is $\geqslant X^{\frac{1}{4.64}}$, and $\#\mathcal{A}' = S(\mathcal{A}; X^{\frac{1}{4.64}})$.

If $A \in \mathcal{A}'$ has a common prime factor P with f, then

$$\deg P \geqslant \log_2 X^{\frac{1}{4.64}} = \frac{\log_2 X}{4.64}.$$

Let S_1 denote the number of elements in $\mathcal{A}'$ that have a common prime factor of degree $\geqslant \frac{2\log_2 X}{4.64}$ with f, and S_2 the same for prime factors having degrees in $\left[\frac{\log_2 X}{4.64}, \frac{2\log_2 X}{4.64}\right)$. If n_d denotes the number of distinct irreducible factors of f of degree d, then

$$\begin{aligned} S_1 &\leqslant \sum_{\substack{\deg P \geqslant \frac{2\log_2 X}{4.64} \\ P \mid f}} \#\mathcal{A}_P \\ &= \sum_{\substack{\frac{2\log_2 X}{4.64} \leqslant \deg P \leqslant t \\ P \mid f}} \#\mathcal{A}_P + \sum_{\substack{\deg P > t \\ P \mid f}} \#\mathcal{A}_P \\ &\leqslant X \sum_{\substack{\frac{2\log_2 X}{4.64} \leqslant \deg P \leqslant t \\ P \mid f}} \frac{1}{|P|} + n \\ &\leqslant X \sum_{d \geqslant \frac{2\log_2 X}{4.64}} \frac{n_d}{2^d} + n \\ &\leqslant \frac{X}{2^{\frac{2\log_2 X}{4.64}}} \sum_{d \geqslant \frac{2\log_2 X}{4.64}} n_d + n \\ &\leqslant X^{\frac{2.64}{4.64}} \frac{4.64n}{2\log_2 X} + n \\ &< \frac{4.64X^{\frac{3.64}{4.64}}}{\log_2 X} + 2X^{\frac{1}{4.64}} < \frac{5X^{\frac{3.64}{4.64}}}{\log_2 X}, \end{aligned} \tag{3.2}$$

for $n \gg 1$.

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We now turn to estimating S_2. We begin by observing that

$$\frac{2\log_2 X}{4.64} = \frac{2t}{4.64} < t,$$

so that if deg $P < \frac{2\log_2 X}{4.64}$, then

$$\#\mathcal{A}_P = \frac{\omega(P)}{|P|}X.$$

We will apply Theorem 3, (ii) to the sets $\mathcal{A}_P$ where P is a prime factor of f with deg P in $\left[\frac{\log_2 X}{4.64}, \frac{2\log_2 X}{4.64}\right)$. In what follows, we assume that $P \mid f$ with deg $P \in \left[\frac{\log_2 X}{4.64}, \frac{2\log_2 X}{4.64}\right)$. Observe that for every P under consideration, we have $\omega(P) = 1$ so that

$$\#\mathcal{A}_P = \frac{X}{|P|} > X^{\frac{2.64}{4.64}} > z.$$

Let $\omega(D)$ be as defined earlier in this section. For $D \in \mathbf{A}^*$, define

$$r'_D := \#\mathcal{A}_{DP} - \frac{\omega(D)}{|D|}\#\mathcal{A}_P = \#\mathcal{A}_{DP} - \frac{\omega(D)}{|D|}\frac{X}{|P|}.$$

If $P \mid D$, then $\omega(DP) = \omega(D)$ whence, $r'_D = r(DP)$. Next, consider that $P \nmid D$. If $\omega(D) = 1$, then since $\omega(P) = 1$, we have

$$\omega(DP) = \omega(D)\omega(P) = 1 = \omega(D).$$

Conversely, if $\omega(DP) = 1$, then obviously $\omega(D) = 1$. It follows that $\omega(DP) = \omega(D)$, and as such,

$$r'_D = r_{DP}.$$

Thus,

$$|r'_D| = |r_{DP}| \leqslant \omega(DP) = \omega(D).$$

Thus, $\mathcal{A}_P$ and ω satisfy all the assumptions of Theorem 3. By Theorem 3 (ii), we now have for $n \gg 1$ that

$$S(\mathcal{A}_P; X^{\frac{1}{4.64}}) \leqslant e\frac{X}{|P|}W + X^{\frac{3.6385}{4.64}}.$$

Since $|P| > 2^{\frac{\log_2 X}{4.64}}$, hence

$$S(\mathcal{A}_P; X^{\frac{1}{4.64}}) \leqslant eX^{\frac{3.64}{4.64}}W + X^{\frac{3.6385}{4.64}}. \tag{3.3}$$

Thus,

$$S_2 = \sum_{\substack{P|f \\ \frac{\log_2 X}{4.64} \leqslant \deg P < \frac{2\log_2 X}{4.64}}} S(\mathcal{A}_P; X^{\frac{1}{4.64}}) \tag{3.4}$$
$$\leqslant \left(eX^{\frac{3.64}{4.64}}W + X^{\frac{3.6385}{4.64}}\right) \sum_{\substack{P|f \\ \frac{\log_2 X}{4.64} \leqslant \deg P < \frac{2\log_2 X}{4.64}}} 1$$
$$\leqslant \left(eX^{\frac{3.64}{4.64}}W + X^{\frac{3.6385}{4.64}}\right) \frac{4.64n}{\log_2 X}$$
$$\leqslant 10e\frac{XW}{\log_2 X} + 10\frac{X^{\frac{4.6385}{4.64}}}{\log_2 X},$$

since $n \leqslant 2X^{\frac{1}{4.64}}$. Now, from Equations (3.2) and (3.4), we have

$$S_1 + S_2 \leqslant 10e\frac{XW}{\log_2 X} + 15\frac{X^{\frac{4.6385}{4.64}}}{\log_2 X}. \tag{3.5}$$

If every polynomial in $\mathcal{A}'$ has a non-trivial gcd with f, then

$$S_1 + S_2 \geqslant \#\mathcal{A}' = S(\mathcal{A}; X^{\frac{1}{4.64}}).$$

Substituting from Equations (3.1) and (3.5) in the last estimate above, we get

$$10e\frac{XW}{\log_2 X} + 15\frac{X^{\frac{4.6385}{4.64}}}{\log_2 X} \geqslant 0.0001XW - X^{\frac{4.6385}{4.64}}.$$

Rearranging terms, we have

$$XW\left(0.0001 - \frac{10e}{\log_2 X}\right) \leqslant 16X^{\frac{4.6385}{4.64}}. \tag{3.6}$$

Observe that

$$W \geqslant V := \prod_{P\in\mathscr{P}}\left(1 - \frac{1}{|P|}\right).$$

Now, if M_d denotes the number of irreducible polynomials in $\mathbf{A}$ of degree d, then

$$\begin{aligned} -\log V &= \sum_{\substack{P-\text{a prime} \\ |P|\leqslant z}} -\log\left(1-\frac{1}{|P|}\right) \\ &= \sum_{\substack{P-\text{a prime} \\ |P|\leqslant z}} \sum_{j=1}^{\infty} \frac{1}{j|P|^j} \\ &= \sum_{d\leqslant \log_2 z} M_d \sum_{j=1}^{\infty} \frac{1}{j2^{dj}}. \end{aligned}$$

Using an earlier estimate that $M_d \leqslant 2^d/d$, we get

$$\begin{aligned} -\log V &\leqslant \sum_{d\leqslant \log_2 z} \frac{2^d}{d} \sum_{j=1}^{\infty} \frac{1}{j2^{dj}} \\ &= \sum_{d\leqslant \log_2 z} \frac{1}{d} + E', \end{aligned}$$

where

$$\begin{aligned} E' &= \sum_{d\leqslant \log_2 z} \frac{2^d}{d} \sum_{j=2}^{\infty} \frac{1}{j2^{dj}} \\ &< \sum_{d\leqslant \log_2 z} \frac{2^d}{2d} \sum_{j=2}^{\infty} \frac{1}{2^{dj}} \\ &= \sum_{d\leq \log_2 z} \frac{2^d}{2d} \frac{1}{2^d(2^d-1)} \\ &< \sum_{d\leqslant \log_2 z} \frac{1}{d2^d} < 1. \end{aligned}$$

Therefore,

$$-\log V < \sum_{d\leqslant \log_2 z} \frac{1}{d} + 1 < \log(\log_2 z) + 2.$$

Upon exponentiating, we get

$$V > \frac{1}{e^2 \log_2 z} = \frac{4.64}{e^2 \log_2 X} > \frac{0.6}{\log_2 X}.$$

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Now, using the above estimate in Equation (3.6), we obtain

$$\frac{0.6X}{\log_2 X}\left(0.0001 - \frac{10e}{\log_2 X}\right) \leqslant 16X^{\frac{4.6385}{4.64}}.$$

The last inequality is impossible for $n \gg 1$ (whence $X \gg 1$). Therefore, for $n \gg 1$, there is an $g_1 = g + u$ in $\mathcal{A}$ such that $\gcd(f, g_1) = 1$, as asserted. This concludes the proof of Theorem 1.

4. A proof of Theorem 2

Let $f(x) \in \mathbb{F}_2[x]$ with $\deg f = n$. There are unique polynomials $f_e(x)$ and $f_o(x)$ in $2[x]$ such that $f(x)$ can be expressed as

$$f(x) = f_e(x^2) + xf_o(x^2).$$

Let $m := \max\{\deg f_e, \deg f_o\} = \lfloor n/2 \rfloor$. The proof of Theorem 2 rests upon the following result (Lemma 5.1) from [4] (also see Lemma 3.1, [7]).

Lemma 6. *Let $h(x) \in \mathbb{F}_2[x]$ be of degree at least 2. Then h(x) is squarefree if and only if* $\gcd(h_e(x), h_o(x)) = 1$.

Let $u(x) \in \{f_e(x), f_o(x)\}$ be defined as

$$u(x) = \begin{cases} f_e(x) & \text{if} \quad \deg f \equiv 0 \pmod 2 \\ f_o(x) & \text{if} \quad \deg f \equiv 1 \pmod 2). \end{cases}$$

Thus, $\deg u = m$. Let $v(x) \in \{f_e(x), f_o(x)\}$ denote the other polynomial. By Theorem 1, for $n \gg 1$, there is an $v_1(x) \in \mathbb{F}_2[x]$ with $\deg v_1 \leqslant \max\{\deg v, 6.7 \log n\}$ and $L(v - v_1) < 6.7 \log m$ such that $\gcd(u(x), v_1(x)) = 1$. In particular, $\deg v_1 \leqslant \deg v \leqslant \deg u = m$. Set

$$g(x) = \begin{cases} u(x^2) + xv_1(x^2) & \text{if} \quad u(x) = f_e(x) \\ v_1(x^2) + xu(x^2) & \text{if} \quad u(x) = f_o(x). \end{cases}$$

Then $g(x)$ is squarefree by Lemma 6. Furthermore,

$$L(f - g) = L(v - v_1) < 6.7 \log m < 6.7 \log n,$$

as required. We conclude by clarifying that $\deg g = \deg f$. Assuming $\deg f = 2m$ is even, we have $u(x) = f_e(x)$ with $\deg f_e = m$. Furthermore, $\deg v < m$ in this case.

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Consequently deg $v_1 < m$ (for $n \gg 1$). It follows that

$$\deg\, g = \max\{2\deg\, u, 1 + 2\deg\, v_1\} = \max\{2m, 1 + 2\deg\, v_1\} = 2m.$$

Similarly, if deg f is odd, say, deg $f = 2m + 1$, then $u(x) = f_o(x)$ with deg $f_o = m$. Then,

$$\deg\, g = \max\{2\deg\, v_1, 1 + 2\deg\, u\} = 2m + 1 = \deg\, f.$$

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